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On Canonical Bundle Formulas and Subadjunctions

Osamu Fujino & Yoshinori Gongyo

1. Introduction

The following lemma, which is missing in the literature, is one of the main results of this paper. It is a canonical bundle formula for generically finite proper surjective morphisms.

Lemma 1.1 (Main Lemma). *Let X and Y be normal varieties and let $f: X \to Y$ be a generically finite proper surjective morphism. Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Suppose there exists an effective $\mathbb{K}$-divisor Δ on X such that (X, Δ) is log canonical and $K_X + \Delta \sim_{\mathbb{K}, f} 0$. Then there exists an effective $\mathbb{K}$-divisor Γ on Y such that (Y, Γ) is log canonical and*

$$K_X + \Delta \sim_{\mathbb{K}} f^*(K_Y + \Gamma).$$

Moreover, if (X, Δ) is kawamata log terminal, then we can choose Γ such that (Y, Γ) is kawamata log terminal.

As an application of Lemma 1.1, we prove a subadjunction formula for minimal log-canonical (lc) centers. It is a generalization of Kawamata's subadjunction formula [K3, Thm. 1]. For a local version, see Theorem 7.2.

Theorem 1.2 (Subadjunction formula for minimal lc centers). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log-canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

The paper proceeds as follows. Section 2 is devoted to the proof of Lemma 1.1. In Section 3, we discuss Ambro's canonical bundle formula for projective kawamata log-terminal (klt) pairs with a generalization for $\mathbb{R}$-divisors (Theorem 3.1), which is one of the key ingredients in the proof of Theorem 1.2. Although Theorem 3.1 is sufficient for applications in the subsequent sections, we treat slight

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generalizations of Ambro's canonical bundle formula for projective log-canonical pairs. In Section 4, we prove a subadjunction formula for minimal log-canonical centers (Theorem 1.2), which is a generalization of Kawamata's subadjunction formula [K3, Thm. 1]. In Section 5, we treat images of log-Fano varieties by generically finite surjective morphisms as an application of Lemma 1.1. Theorem 5.1 answers the question raised by Schwede [SSm, Rem. 6.5]. In Section 6, we give a quick proof of the nonvanishing theorem for log-canonical pairs as an application of Theorem 1.2, which is the main theorem of [F4]. In Section 7 we prove a local version of our subadjunction formula for minimal log-canonical centers (Theorem 7.2), which is useful for local studies of singularities of pairs. This local version does not directly follow from the global version (i.e., from Theorem 1.2). The reason is that we do not know how to compactify log-canonical pairs.

We conclude this introduction by summarizing the notation. We also use the standard notation in [KoM].

NOTATION. Let $\mathbb{K}$ be the real number field $\mathbb{R}$ or the rational number field $\mathbb{Q}$.

Let X be a normal variety and let B be an effective $\mathbb{K}$-divisor such that $K_X + B$ is $\mathbb{K}$-Cartier. Then we can define the *discrepancy* $a(E, X, B) \in \mathbb{K}$ for every prime divisor E *over* X. If $a(E, X, B) \geq -1$ (resp. > -1) for every E, then (X, B) is called *log canonical* (resp. *kawamata log terminal*).

Assume that (X, B) is log canonical. If E is a prime divisor over X such that $a(E, X, B) = -1$, then $c_X(E)$ is called a *log-canonical center* (or lc center) of (X, B), where $c_X(E)$ is the closure of the image of E on X. For the basic properties of log-canonical centers, see [F4, Thm. 2.4] or [F5, Sec. 9].

We use $\sim_{\mathbb{K}}$ to denote $\mathbb{K}$-*linear equivalence* of $\mathbb{K}$-divisors. Let $f : X \to Y$ be a morphism between normal varieties and let D be a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on X. Then D is $\mathbb{K}$-linearly f-trivial, denoted by $D \sim_{\mathbb{K}, f} 0$, if and only if there is a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor B on Y such that $D \sim_{\mathbb{K}} f^*B$.

The base locus of the linear system Λ is denoted by $\operatorname{Bs} \Lambda$.

We will work over $\mathbb{C}$, the complex number field, throughout this paper.

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2. Main Lemma

In this section, we prove Lemma 1.1.

Proof of Lemma 1.1. Let

$$f : X \xrightarrow{g} Z \xrightarrow{h} Y$$

be the Stein factorization. By replacing (X, Δ) with $(Z, g_*\Delta)$, we can assume that $f : X \to Y$ is finite. Let D be a $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on Y such that $K_X + \Delta \sim_{\mathbb{K}} f^*D$. We consider the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\nu} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\mu]{} & Y, \end{array}$$

where the following statements hold.

(i) μ is a resolution of singularities of Y.

(ii) There exists an open set $U \subseteq Y$ such that μ is isomorphic over U and f is étale over U; moreover, $\mu^{-1}(Y-U)$ has a simple normal crossing support and $Y-U$ contains Supp $f_*\Delta$.

(iii) X' is the normalization of the irreducible component of $X \times_Y Y'$ that dominates Y'; in particular, f' is finite.

Let $\Omega = \sum_i \delta_i D_i$ be a $\mathbb{K}$-divisor on X' such that

$$K_{X'} + \Omega = \nu^*(K_X + \Delta).$$

We consider the ramification formula

$$K_{X'} = f'^* K_{Y'} + R,$$

where $R = \sum_i (r_i - 1) D_i$ is an effective $\mathbb{Z}$-divisor such that r_i is the ramification index of D_i for every i. Note that it suffices to show the above ramification formula *outside* codimension-2 closed subsets of X'. Then it holds that

$$(\mu \circ f')^* D \sim_{\mathbb{K}} f'^* K_{Y'} + R + \Omega.$$

By pushing forward this expression by f', we see that

$$\deg f' \cdot \mu^* D \sim_{\mathbb{K}} \deg f' \cdot K_{Y'} + f'_*(R + \Omega).$$

Let

$$\Gamma := \frac{1}{\deg f'} \mu_* f'_*(R + \Omega)$$

on Y. Then Γ is effective because

$$\mu_* f'_*(R + \Omega) = f_* \nu_*(R + \Omega) = f_*(\nu_* R + \Delta).$$

Now let $Y' \setminus \mu^{-1} U = \bigcup_j E_j$ be the irreducible decomposition, where $\sum_j E_j$ is a simple normal crossing divisor, and put

$$I_j := \{ i \mid f'(D_i) = E_j \}.$$

The coefficient of E_j in $\frac{1}{\deg f'} f'_*(R + \Omega)$ is

$$\frac{\sum_{i \in I_j} (r_i + \delta_i - 1) \deg(f'|_{D_i})}{\deg f'}.$$

Since $\delta_i \leq 1$, it follows that

$$\sum_{i \in I_j} (r_i + \delta_i - 1) \deg(f'|_{D_i}) \leq \sum_{i \in I_j} r_i \deg(f'|_{D_i}) = \deg f'.$$

Thus (Y, Γ) is log canonical because $K_{Y'} + \frac{1}{\deg f'} f'_*(R + \Omega) = \mu^*(K_Y + \Gamma)$. Moreover, if (X, Δ) is kawamata log terminal then $\delta_i < 1$. Hence (Y, Γ) is kawamata log terminal. □

3. Ambro's Canonical Bundle Formula

Theorem 3.1 is Ambro's canonical bundle formula for projective klt pairs [A2, Thm. 4.1] with a generalization for $\mathbb{R}$-divisors. We need this theorem for the proof of our subadjunction formula (Theorem 1.2).

THEOREM 3.1 (Ambro's canonical bundle formula for projective klt pairs). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a projective kawamata log-terminal pair and let $f : X \to Y$ be a projective surjective morphism onto a normal projective variety Y with connected fibers. Assume that*

$$K_X + B \sim_{\mathbb{K}, f} 0.$$

Then there exists an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is klt and

$$K_X + B \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

Proof. If $\mathbb{K} = \mathbb{Q}$, then the statement is nothing but [A2, Thm. 4.1]. From now on, we assume that $\mathbb{K} = \mathbb{R}$. Let $\sum_i B_i$ be the irreducible decomposition of Supp B. We put $V = \bigoplus_i \mathbb{R} B_i$. Then it is well known that

$$\mathcal{L} = \{\Delta \in V \mid (X, \Delta) \text{ is log canonical}\}$$

is a rational polytope in V. We can also check that

$$\mathcal{N} = \{\Delta \in \mathcal{L} \mid K_X + \Delta \text{ is } f\text{-nef}\}$$

is a rational polytope (cf. [B, Prop. 3.2(3)]) and $B \in \mathcal{N}$. We remark that $\mathcal{N}$ is known as Shokurov's polytope and that, by [F5, Thm. 18.2], the proof of [B, Prop. 3.2(3)] works for our setting without any changes. Therefore, we can write

$$K_X + B = \sum_{i=1}^{k} r_i (K_X + \Delta_i)$$

such that:

(i) $\Delta_i \in \mathcal{N}$ is an effective $\mathbb{Q}$-divisor on X for every i;
(ii) (X, Δ_i) is klt for every i; and
(iii) $0 < r_i < 1$, $r_i \in \mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i = 1$.

Since $K_X + B$ is numerically f-trivial and since $K_X + \Delta_i$ is f-nef for every i, it follows that $K_X + \Delta_i$ is numerically f-trivial for every i. Thus, by [N, Chap. V, Cor. 2.9] we have

$$\kappa(X_\eta, (K_X + \Delta_i)_\eta) = \nu(X_\eta, (K_X + \Delta_i)_\eta) = 0$$

for every i, where η is the generic point of Y (see also [A2, Thm. 4.2]). Therefore, $K_X + \Delta_i \sim_{\mathbb{Q}, f} 0$ for every i by [F3, Thm. 1.1]. By the case when $\mathbb{K} = \mathbb{Q}$, we can find an effective $\mathbb{Q}$-divisor Θ_i on Y such that (Y, Θ_i) is klt and

$$K_X + \Delta_i \sim_{\mathbb{Q}} f^*(K_Y + \Theta_i)$$

for every i. Putting $B_Y = \sum_{i=1}^k r_i \Theta_i$, we obtain that

$$K_X + B \sim_{\mathbb{R}} f^*(K_Y + B_Y)$$

and that (Y, B_Y) is klt. □

Corollary 3.2 is a direct consequence of Theorem 3.1.

COROLLARY 3.2. *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a log-canonical pair and let $f : X \to Y$ be a projective surjective morphism between normal projective varieties. Assume that*

$$K_X + B \sim_{\mathbb{K}, f} 0$$

and that every lc center of (X, B) is dominant onto Y. Then we can find an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is kawamata log terminal and

$$K_X + B \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

Proof. By taking a dlt blow-up (cf. [F5, Thm. 10.4]), we can assume that (X, B) is dlt. By replacing (X, B) with its minimal lc center and then taking the Stein factorization, we can assume that (X, B) is klt and that f has connected fibers (cf. Lemma 1.1). Hence we can choose the required B_Y by Theorem 3.1. □

From now on, we treat Ambro's canonical bundle formula for projective log-canonical pairs. We note that Theorem 3.1 is sufficient for applications in subsequent sections.

3.3 (Observation). Let (X, B) be a log-canonical pair, and let $f : X \to Y$ be a projective surjective morphism between normal projective varieties with connected fibers. Assume that $K_X + B \sim_{\mathbb{Q}, f} 0$ and that (X, B) is kawamata log terminal over the generic point of Y. We can write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + M_Y + \Delta_Y),$$

where M_Y is the *moduli $\mathbb{Q}$-divisor* and Δ_Y is the *discriminant $\mathbb{Q}$-divisor* (see e.g. [A1] for details). It is conjectured that we can construct a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ \nu\ } & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ \mu\ ]{} & Y \end{array}$$

with the following properties:

(i) ν and μ are projective birational;
(ii) X' is normal and $K_{X'} + B_{X'} = \nu^*(K_X + B)$;
(iii) $K_{X'} + B_{X'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + M_{Y'} + \Delta_{Y'})$ such that Y' is smooth, the moduli $\mathbb{Q}$-divisor $M_{Y'}$ is semi-ample, and the discriminant $\mathbb{Q}$-divisor $\Delta_{Y'}$ has a simple normal crossing support.

For these properties, the nontrivial part is the semi-ampleness of $M_{Y'}$. We know that we can construct the desired commutative diagrams of $f'\colon X' \to Y'$ and $f\colon X \to Y$ when

(1) $\dim X - \dim Y = 1$ (cf. [K1, Thm. 5] and so on),
(2) $\dim Y = 1$ (cf. [A1, Thm. 0.1; A2, Thm. 3.3]),
(3) general fibers of f are Abelian varieties or smooth surfaces with $\kappa = 0$ (cf. [F2, Thms. 1.2 and 6.3]),

and so on. We take a general member $D \in |mM_{Y'}|$ of the free linear system $|mM_{Y'}|$, where m is a sufficiently large and divisible integer. We put

$$K_Y + B_Y = \mu_*\left(K_{Y'} + \frac{1}{m}D + \Delta_{Y'}\right).$$

Then it is easy to see that

$$\mu^*(K_Y + B_Y) = K_{Y'} + \frac{1}{m}D + \Delta_{Y'},$$

that (Y, B_Y) is log canonical, and that

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

Observation 3.3 yields Ambro's canonical bundle formula for projective log-canonical pairs under some special assumptions.

Theorem 3.4. *Let (X, B) be a projective log-canonical pair, and let $f\colon X \to Y$ be a projective surjective morphism onto a normal projective variety Y such that $K_X + B \sim_{\mathbb{Q},f} 0$. Assume that* $\dim Y \le 1$ *or* $\dim X - \dim Y \le 1$. *Then there exists an effective $\mathbb{Q}$-divisor B_Y on Y such that (Y, B_Y) is log canonical and*

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y).$$

Proof. By taking a dlt blow-up (as in [F5, Thm. 10.4]), we can assume that (X, B) is dlt. If necessary, by replacing (X, B) with a suitable lc center of (X, B) and by taking the Stein factorization (see Lemma 1.1), we can assume that $f\colon X \to Y$ has connected fibers and that (X, B) is kawamata log terminal over the generic point of Y. We can assume that $\dim Y = 1$ or $\dim X - \dim Y = 1$. By the arguments in Observation 3.3, we can find an effective $\mathbb{Q}$-divisor B_Y on Y such that (Y, B_Y) is log canonical and $K_X + B \sim_{\mathbb{Q}} f^*(K_Y + B_Y)$. □

4. Subadjunction for Minimal Log-Canonical Centers

The following theorem is a generalization of Kawamata's subadjunction formula [K3, Thm. 1]. Theorem 4.1 is new even for threefolds, and it answers Kawamata's question [K2, Ques. 1.8].

Theorem 4.1 (Subadjunction formula for minimal lc centers). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal projective variety, and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log-canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

REMARK 4.2. Kawamata [K3, Thm. 1] proved that

$$(K_X + D + \varepsilon H)|_W \sim_{\mathbb{Q}} K_W + D_W,$$

where H is an ample Cartier divisor on X and ε is a positive rational number, under the extra assumption that D is a $\mathbb{Q}$-divisor and there exists an effective $\mathbb{Q}$-divisor D^o such that $D^o < D$ and (X, D^o) is kawamata log terminal. Therefore, Kawamata's theorem claims nothing when $D = 0$.

Proof of Theorem 4.1. By taking a dlt blow-up (as in [F5, Thm. 10.4]), we can take a projective birational morphism $f: Y \to X$ from a normal projective variety Y with the following properties:

(i) $K_Y + D_Y = f^*(K_X + D)$;
(ii) (Y, D_Y) is a $\mathbb{Q}$-factorial dlt pair.

We can take a minimal lc center Z of (Y, D_Y) such that $f(Z) = W$. Observe that $K_Z + D_Z = (K_Y + D_Y)|_Z$ is klt because Z is a minimal lc center of the dlt pair (Y, D_Y). Let

$$f : Z \xrightarrow{g} V \xrightarrow{h} W$$

be the Stein factorization of $f: Z \to W$. By construction we can write

$$K_Z + D_Z \sim_{\mathbb{K}} f^*A,$$

where A is a $\mathbb{K}$-divisor on W such that $A \sim_{\mathbb{K}} (K_X + D)|_W$. Note that W is normal by [F4, Thm. 2.4(4)]. Since (Z, D_Z) is klt, by Theorem 3.1 we can take an effective $\mathbb{K}$-divisor D_V on V such that

$$K_V + D_V \sim_{\mathbb{K}} h^*A$$

and (V, D_V) is klt. By Lemma 1.1, we can find an effective $\mathbb{K}$-divisor D_W on W such that

$$K_W + D_W \sim_{\mathbb{K}} A \sim_{\mathbb{K}} (K_X + D)|_W$$

and (W, D_W) is klt. □

5. Log-Fano Varieties

In this section, we give an easy application of Lemma 1.1. Theorem 5.1 answers the question raised by Schwede in [SSm, Rem. 6.5]. For related topics, see [FG, Sec. 3].

THEOREM 5.1. *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is ample, and let $f: X \to Y$ be a generically finite surjective morphism to a normal projective variety Y. Then we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

Proof. Without loss of generality, we can assume that Δ is a $\mathbb{Q}$-divisor by perturbing the coefficients of Δ. Let H be a general very ample Cartier divisor on Y and let ε be a sufficiently small positive rational number. Then $K_X + \Delta + \varepsilon f^*H$ is anti-ample and $(X, \Delta + \varepsilon f^*H)$ is klt. We can take an effective $\mathbb{Q}$-divisor Θ on X such that $m\Theta$ is a general member of the free linear system $|-m(K_X+\Delta+\varepsilon f^*H)|$, where m is a divisible integer that is sufficiently large. Then

$$K_X + \Delta + \varepsilon f^*H + \Theta \sim_{\mathbb{Q}} 0.$$

Let δ be a positive rational number such that $0 < \delta < \varepsilon$. Then

$$K_X + \Delta + (\varepsilon - \delta) f^*H + \Theta \sim_{\mathbb{Q}} f^*(-\delta H).$$

By Lemma 1.1, we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that

$$K_Y + \Delta_Y \sim_{\mathbb{Q}} -\delta H$$

and (Y, Δ_Y) is klt. We note that

$$-(K_Y + \Delta_Y) \sim_{\mathbb{Q}} \delta H$$

is ample. □

By combining Theorem 5.1 with [FG, Thm. 3.1], we can easily obtain the following corollary.

COROLLARY 5.2. *Let (X, Δ) be a projective klt pair such that $-(K_X + \Delta)$ is ample, and let $f: X \to Y$ be a projective surjective morphism onto a normal projective variety Y. Then we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is klt and $-(K_Y + \Delta_Y)$ is ample.*

6. Nonvanishing Theorem for Log-Canonical Pairs

The following theorem is the main result of [F4]. It is almost equivalent to the base point free theorem for log-canonical pairs. For details, see [F4].

THEOREM 6.1 (Nonvanishing theorem). *Let X be a normal projective variety, and let B be an effective $\mathbb{Q}$-divisor on X such that (X, B) is log canonical. Let L be a nef Cartier divisor on X, and assume that $aL - (K_X + B)$ is ample for some $a > 0$. Then the base locus of the linear system $|mL|$ contains no lc centers of (X, B) for every $m \gg 0$; in other words, there is a positive integer m_0 such that $\mathrm{Bs}|mL|$ contains no lc centers of (X, B) for every $m \geq m_0$.*

Here we give a quick proof of Theorem 6.1 by using Theorem 4.1.

Proof of Theorem 6.1. Let W be any minimal lc center of the pair (X, B). It is sufficient to prove that W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$. By Theorem 4.1, we can find an effective $\mathbb{Q}$-divisor B_W on W such that (W, B_W) is klt and $K_W + D_W \sim_{\mathbb{Q}} (K_X + B)|_W$. Therefore, $aL|_W - (K_W + B_W) \sim_{\mathbb{Q}} (aL - (K_X + B))|_W$ is ample. By the Kawamata–Shokurov base point free theorem, $|mL|_W|$ is free for $m \gg 0$. By [F4, Thm. 2.2],

$$H^0(X, \mathcal{O}_X(mL)) \to H^0(W, \mathcal{O}_W(mL))$$

is surjective for $m \geq a$. Therefore, W is not contained in $\mathrm{Bs}|mL|$ for $m \gg 0$. □

REMARK 6.2. This proof of Theorem 6.1 is shorter than the original proof in [F4]. However, the proof of Theorem 4.1 depends on very deep results, such as the existence of dlt blow-ups, whereas the proof in [F4] depends only on various well-prepared vanishing theorems and standard techniques.

7. Subadjunction Formula: Local Version

In this section, we give a local version of our subadjunction formula for minimal log-canonical centers. Theorem 7.1 is a local version of Ambro's canonical bundle formula for kawamata log-terminal pairs (Theorem 3.1); it is essentially [F1, Thm. 1.2].

THEOREM 7.1. *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let (X, B) be a kawamata log-terminal pair, and let $f : X \to Y$ be a proper surjective morphism onto a normal* affine *variety Y with connected fibers. Assume that*

$$K_X + B \sim_{\mathbb{K}, f} 0.$$

Then there exists an effective $\mathbb{K}$-divisor B_Y on Y such that (Y, B_Y) is klt and

$$K_X + B \sim_{\mathbb{K}} f^*(K_Y + B_Y).$$

We shall simply explain how to modify the proof of [F1, Thm. 1.2].

Comments on the proof. First, we assume that $\mathbb{K} = \mathbb{Q}$. In this case, the proof of [F1, Thm. 1.2] works with some minor modifications. We remark that (i) M in the proof of [F1, Thm. 1.2] is μ-nef and (ii) we can assume $H = 0$ in [F1, Thm. 1.2] because Y is affine. Next, we assume that $\mathbb{K} = \mathbb{R}$. In this case, we can reduce the problem to the case when $\mathbb{K} = \mathbb{Q}$ as in the proof of Theorem 3.1. Thus we obtain the desired formula. □

By Theorem 7.1, we can obtain a local version of Theorem 4.1. The proof of Theorem 4.1 works without any modification.

THEOREM 7.2 (Subadjunction formula for minimal lc centers: Local version). *Let $\mathbb{K}$ be the rational number field $\mathbb{Q}$ or the real number field $\mathbb{R}$. Let X be a normal* affine *variety, and let D be an effective $\mathbb{K}$-divisor on X such that (X, D) is log canonical. Let W be a minimal log-canonical center with respect to (X, D). Then there exists an effective $\mathbb{K}$-divisor D_W on W such that*

$$(K_X + D)|_W \sim_{\mathbb{K}} K_W + D_W$$

and the pair (W, D_W) is kawamata log terminal. In particular, W has only rational singularities.

Theorem 7.2 does not follow directly from Theorem 4.1, since we do not know how to compactify log-canonical pairs.

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O. Fujino
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto 606-8502
Japan

fujino@math.kyoto-u.ac.jp

Y. Gongyo
Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro
Tokyo 153-8914
Japan

gongyo@ms.u-tokyo.ac.jp