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Pullback of Parabolic Bundles and Covers of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

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1. Introduction

We work over an algebraically closed ground field k of characteristic 0. If G is a finite group then, by [8], a G-torsor $f: X \to Y$ in the category of algebraic varieties can be viewed as a tensor functor Rep-$G \to \text{Vect}(Y)$. More concretely, the associated tensor functor sends the representation V to the vector bundle $f_*(V \otimes \mathcal{O})^G$. When the cover ramifies, as was observed in [9], we need to put tensor functors in the category of vector bundles with appropriate parabolic structure.

In the case where $Y = \mathbb{P}^1$ we have $f_*(V \times \mathcal{O})^G = \bigoplus \mathcal{O}(s_i)$. The integers s_i are difficult to compute, and one of our results is to find an upper bound on them when there is ramification at 0, 1, and ∞ only. The bound described in Theorem 8.4 and Example 8.6 improves the known bound in [3]. There is one case in which it is easy to compute the integers s_i—namely, when the group G is cyclic. Our method is a type of reduction to the cyclic case by removing ramification at 0. More precisely, the endomorphism $z \mapsto z^n$ of $\mathbb{P}^1$ algebraically de-loops loops around the origin. Pulling back a cover along this morphism removes ramification of order n at the origin. For our method to work we must define a pullback morphism for parabolic bundles. As in [6] and [3], this entails using the equivalence of categories (due to Biswas [2]) between parabolic bundles of a certain kind and vector bundles on an associated root stack. The pullback operation is difficult to reverse—that is, given a morphism $f: X \to Y$ of smooth projective curves and a parabolic bundle $\mathcal{F}_\bullet$ on X, to construct a parabolic bundle on Y that pulls back to $\mathcal{F}_\bullet$. In fact, the difficulty in reversing the parabolic pullback gives a new explanation for why it is difficult to compute the s_i.

The interest in computing these s_i can be explained as follows. A finite quotient $q: F_2 \twoheadrightarrow G$ of the free group on two letters produces a cover $X_q \to \mathbb{P}^1$ ramified at three points. The absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$ acts faithfully on such covers. For a given q, however, the Galois action is difficult to understand; and it is not known what finite quotient of $G_{\mathbb{Q}}$ acts in sending the cover to some other nonisomorphic cover. One way of addressing this question is to give a more algebraic construction of the cover. The theory of tannakian categories allows one to do this. One should view the cover as a tensor functor into parabolic bundles and then understand the Galois action on such tensor functors. This work should be

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seen as a first step toward understanding these tensor functors. In this paper we identify their parabolic pullbacks. To understand the original functor amounts to describing faithfully flat descent for parabolic bundles; this is a topic of future work.

In Section 2 we recall some results of Nori on principal bundles and tensor functors. Section 3 recalls the notion of root stack introduced in [4], and Section 4 introduces parabolic bundles in our context. The definition here is equivalent to the one in [7]; we also recall from [11] the construction of tensor product and internal Hom for parabolic bundles. Section 5 is devoted to proving the orbifold–parabolic correspondence in our context. This result is not new and goes back to [2], though the formulation here is based on the results of [3].

The new results begin in Section 6, where we describe a construction on parabolic bundles that corresponds to the pullback of orbifold bundles. In Section 7 we use some combinatorics to describe the case of cyclic covers. Finally, Section 8 gives an upper bound on the integers s_i described previously in the case of a G-cover of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$; here, the group G need not be abelian.

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NOTATION AND CONVENTIONS.

(i) k is an algebraically closed field of characteristic 0.
(ii) X is a connected smooth projective curve over k.
(iii) For $x \in \mathbb{R}$ we use $\lfloor x \rfloor$ to denote the *floor* of x (i.e., the largest integer smaller than x).

2. Some Results of Nori

In this section we recall some results from [8] and [9]. We begin by recalling the notion of a tannakian category. For a more detailed formulation the reader may refer to [10] or [5].

Let L be a field. We denote by $\mathrm{Vect}(L)$ the category of finite-dimensional L-vector spaces.

DEFINITION 2.1. For any field L, a *tannakian category* over L consists of a quadruple $(\mathbf{C}, \otimes, F, U)$, where:

T1. $\mathbf{C}$ is a small, L-linear, abelian category.
T2. $F : \mathbf{C} \to \mathrm{Vect}(L)$ is an L-linear additive faithful exact functor known as the *fiber functor*;
T3. $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ is an associative and commutative functor that is L-linear in each variable; and
T4. U is a unit for $\otimes$.

This data is subject to the following constraints:

C1. F preserves $\otimes$;
C2. F preserves the associativity and commutativity constraints;
C3. $FU \xrightarrow{\sim} k$; and
C4. $\dim FV = 1$ if and only if there exists a $V^{-1} \in \mathrm{Objects}(\mathbf{C})$ such that $V \otimes V^{-1} \cong U$.

REMARK 2.2. One can use [5, Prop. 1.20] to show that the category $\mathbf{C}$ is necessarily rigid.

If G is an affine group scheme over k, then the category Rep-G of finite-dimensional left representations of G is a tannakian category over k. In fact, we have the following theorem.

THEOREM 2.3. *Any tannakian category over k is equivalent to* Rep-G *for some affine group scheme G over k. Under this correspondence, a homomorphism of affine group schemes corresponds to a tensor functor that commutes with the fiber functor and preserves units.*

For a scheme X over k, denote by $\mathrm{Vect}(X)$ the category of algebraic vector bundles over X. The category $\mathrm{Vect}(X)$ is a k-linear tensor category. The tensor product is associative and commutative and has a unit. Taking the fiber over a k-point gives it the structure of a tannakian category.

DEFINITION 2.4. A rigid tensor G-functor on X is a k-linear exact $\otimes$-functor $F\colon \text{Rep-}G \to \mathrm{Vect}(X)$ such that:

F1. F commutes with $\otimes$;
F2. F preserves the associativity and commutativity constraint;
F3. $\mathrm{rk}\, FV = \dim V$; and
F4. $F(V_{\mathrm{triv}}) = \mathcal{O}_X$.

We denote the category of such functors by $\mathrm{Func}^{\otimes}(\text{Rep-}G, \mathrm{Vect}(X))$. A morphism in this category is a natural transformation $\eta\colon F \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} \bigotimes_{i\in I} F(X_i) & \xrightarrow{\sim} & F\big(\bigotimes_{i\in I} X_i\big) \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\ \bigotimes_{i\in I} G(X_i) & \xrightarrow{\sim} & G\big(\bigotimes_{i\in I} X_i\big). \end{array}$$

Such a natural transformation is necessarily an isomorphism by [5, Prop. 1.13].

Given $P \to X$ a G-torsor, we obtain the natural functor

$$F_P \in \mathrm{Func}^{\otimes}(\text{Rep-}G, \mathrm{Vect}(X))$$

given by $V \mapsto P \times_G V$.

We denote by $\mathrm{Bun}_{G,X}$ the category of G-torsors over X. Notice that all the morphisms in this category are isomorphisms.

THEOREM 2.5. *There is an equivalence of categories*

$$\mathrm{Bun}_{G,X} \xrightarrow{\sim} \mathrm{Func}^{\otimes}(\text{Rep-}G, \mathrm{Vect}(X)).$$

Proof. See [8]. □

We will mostly be interested in the case when G is a finite group and $X = \mathbb{P} \setminus \{0, 1, \infty\}$. To make our setup more useful in this case, we need a ramified version of Theorem 2.5. Such a theorem already exists in [9], but we wish to restate matters in terms of stacks. For now, we record a relevant corollary.

COROLLARY 2.6. *Let H be another finite group acting on X. Denote by $\mathrm{Bun}_{G,X}^H$ the category of G-torsors with an action of H that commutes with the action of G. Then we have an equivalence of categories*

$$\mathrm{Bun}_{G,X}^H \xrightarrow{\sim} \mathrm{Func}^{\otimes}(\text{Rep-}G, \mathrm{Vect}_H(X)),$$

where $\mathrm{Vect}_H(X)$ is the category of H-vector bundles on X.

Proof. Given a G-torsor $P \to X$ with a commuting H-action, for each $h \in H$ a tensor functor we obtain

$$F_h \colon \text{Rep-}G \to \mathrm{Vect}(X).$$

Yet because the pullbacks $P \times_{X,h} X$ are all isomorphic, the functors described here are all isomorphic by the theorem; hence we obtain a functor into $\mathrm{Vect}_H(X)$.

Conversely, suppose that we have a tensor functor

$$F \colon \text{Rep-}G \to \mathrm{Vect}_H(X).$$

Ignoring the H-action, we obtain a torsor $P \to X$. But now the pullbacks $P \times_{X,h} X$ are all isomorphic because the original bundles were H-bundles. □

3. Root Stacks

In this section we recall some constructions from [4].

We shall implicitly make use of the following fact throughout this section: giving a morphism from a scheme S to the quotient stack $[\mathbb{A}^k/\mathbb{G}_m^k]$ is the same as giving a tuple $(\mathcal{L}_i, s_i)_{i=1}^k$ of line bundles $\mathcal{L}_i$ on S and sections $s_i \in \Gamma(S, \mathcal{L}_i)$; see [4, Lemma 2.1.1].

Given a k-tuple $\vec{r} = (r_1, \ldots, r_k)$ of positive integers, there is a morphism of quotient stacks

$$\theta_{\vec{r}} \colon [\mathbb{A}^k/\mathbb{G}_m^k] \to [\mathbb{A}^k/\mathbb{G}_m^k]$$

induced by the morphism

$$\mathbb{A}^k \to \mathbb{A}^k,$$
$$(x_1, \ldots, x_k) \mapsto (x_1^{r_1}, \ldots, x_k^{r_k}).$$

DEFINITION 3.1. Let $\mathbb{D} = (D_1, \ldots, D_k)$ be a k-tuple of effective Cartier divisors on a scheme S. These data define a morphism $S \to [\mathbb{A}^k/\mathbb{G}_m^k]$. Define the root stack $S_{\mathbb{D},\vec{r}}$ to be

$$S_{\mathbb{D},\vec{r}} = S \times_{[\mathbb{A}^k/\mathbb{G}_m^k],\theta_{\vec{r}}} [\mathbb{A}^k/\mathbb{G}_m^k].$$

REMARK 3.2. Let $f\colon T \to S$ be a morphism. A lift of f to a T-point of $S_{\mathbb{D},\vec{r}}$ is the same as giving

$$(M_1, \ldots, M_k, t_1, \ldots, t_k, \phi_1, \ldots, \phi_k);$$

here the M_i are line bundles on T, the ϕ_i are isomorphisms $M_i^{r_i} \xrightarrow{\sim} f^*\mathcal{O}(D_i)$, and the t_i are global sections of M_i such that

$$\phi_i(t_i^{r_i}) = s_{D_i},$$

where s_{D_i} denotes the tautological section of $\mathcal{O}(D_i)$ vanishing along D_i.

PROPOSITION 3.3. *Let Y be a smooth projective curve with an action of a finite group G. Let $\psi\colon Y \to Y/G = X$ be the projection, and assume that the action is generically free. Let the ramification divisor of ψ be $p_1 + \cdots + p_k$ with ramification indices $r_1, \ldots, r_k$. Set $\mathbb{D} = (p_1, \ldots, p_k)$ and $\vec{r} = (r_1, \ldots, r_k)$. Then*

$$[Y/G] \xrightarrow{\sim} X_{\mathbb{D},\vec{r}}.$$

Proof. Let $\pi : X_{\mathbb{D},\vec{r}} \to X$ be the canonical morphism, and write

$$\psi^*(p_i) = r_i D_i.$$

Then the D_i produce a G-equivariant morphism

$$\alpha\colon Y \to X_{\mathbb{D},\vec{r}}.$$

Hence the question of whether we have an isomorphism is a local one.

We consider an open affine Spec $A \subset X$ with preimage Spec $B \subset Y$. We may assume that $p_1 \in$ Spec A and $p_i \notin$ Spec A for $i > 1$. Let s_{p_1} be a parameter at p_1. Then π^{-1}(Spec A) is the quotient stack

$$[\mathrm{Spec}(A[t]/(t^{r_1} - s_{p_1}))/\mu_{r_1}]$$

(see [4, Exam. 2.4.1]). We have the diagram

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A[t]/(t^{r_1} - s_{p_1})) & \longrightarrow & X, \end{array}$$

where $\tilde{Y}$ is the normalization of Y restricted to $\mathrm{Spec}(A[t]/(t^{r_1} - s_{p_1}))$. By Abhyankar's lemma, $\tilde{Y}$ is a G-torsor and so we obtain a morphism

$$\mathrm{Spec}(A[t]/(t^{r_1} - s_{p_1})) \to [Y/G].$$

Because the torsor $\tilde{Y}$ has a μ_r-action, we see that this morphism gives the morphism

$$\beta\colon [\mathrm{Spec}(A[t]/(t^{r_1} - s_{p_1}))/\mu_{r_1}] \to [Y/G].$$

Now we need only show that $\alpha \cdot \beta$ and $\beta \cdot \alpha$ are automorphisms, and this is easily checked. □

Consider a pair $(\mathbb{D}, \vec{r})$ with $\mathbb{D} = (n_1 p_1, \ldots, n_k p_k)$ and $\vec{r} = (r_1, \ldots, r_k)$. We define

$$(\mathbb{D}, \vec{r})_{\text{red}} = \left((p_1, \ldots, p_k), \left(\frac{r_1}{d_1}, \ldots, \frac{r_k}{d_k} \right) \right),$$

where $d_i = \gcd(n_i, r_i)$.

PROPOSITION 3.4. *There is a morphism*

$$X_{(\mathbb{D}, \vec{r})_{\text{red}}} \to X_{(\mathbb{D}, \vec{r})}.$$

Proof. Consider a scheme $f\colon S \to X$. A lift of f to a point of $X_{(\mathbb{D}, \vec{r})_{\text{red}}}$ corresponds to the tuple

$$(M_1, \ldots, M_k, t_1, \ldots, t_k, \phi_1, \ldots, \phi_k),$$

where the M_i are line bundles, with global sections t_i and isomorphisms

$$\phi_i\colon M_i^{r_i/d_i} \xrightarrow{\sim} f^* \mathcal{O}_X(p_i), \quad \phi_i t_i^{r_i/d_i} = s_{p_i}.$$

Here s_{p_i} is a section vanishing at p_i.

Now, by [4, Rem. 2.2.2], the lifting of a morphism of stacks $X_{(\mathbb{D}, \vec{r})_{\text{red}}} \to X$ to $X_{(\mathbb{D}, \vec{r})}$ is similar to the lifting of a morphism of schemes in that it entails the same data as given in Remark 3.2. Observe that

$$M_i^{n_i/d_i},\ t_i^{n_i/d_i},\ \phi_i^{n_i}$$

give the data of a morphism to $X_{(\mathbb{D}, \vec{r})}$. □

PROPOSITION 3.5. *We work in the situation of Proposition 3.3. Suppose that*

$$[Y/G] = X_{(\mathbb{D}, \vec{r})}.$$

Consider $f\colon Z \to X$ with Z a smooth projective curve. Denote by $\widetilde{f^*Y}$ the normalization of the fibered product

$$Z \times_X Y.$$

Then

$$[\widetilde{f^*Y}/G] = Z_{(f^*\mathbb{D}, \vec{r})_{\text{red}}}.$$

Proof. By the proof of Proposition 3.3, this result will follow once we have computed the ramification indices of the morphism

$$\widetilde{f^*Y} \to Z.$$

Infinitesimally locally, the morphism $Y \to X$ is of the form $y \mapsto y^n$ and the morphism $Z \to X$ is of the form $z \mapsto z^m$. The pullback is the high-order cusp $y^n = z^m$, which has $d = \gcd(n, m)$ branches in its resolution; a local calculation then gives the result. □

We shall later need the following result.

PROPOSITION 3.6. *Every vector bundle on $X_{(\mathbb{D}, \vec{r})}$ is locally a direct sum of line bundles. Furthermore, if $X = \operatorname{Spec}(R)$ with R local, then $\operatorname{Pic}(X_{p,r})$ is cyclic of order r and is generated by the canonical root line bundle.*

Proof. See [3, Prop. 3.12] and its proof. □

NOTATION 3.7. We will denote the canonical root line bundles on $X_{(\mathbb{D},\vec{r})}$ by

$$\mathcal{N}_1, \ldots, \mathcal{N}_k.$$

4. Parabolic Bundles

Let $D = n_1p_1 + \cdots + n_kp_k$ be an effective divisor on X with $p_i \neq p_j$ for $i \neq j$ and $n_i \geq 0$. We denote by $\mathbb{D}$ the tuple $(n_1p_1, n_2p_2, \ldots, n_kp_k)$. Fix a tuple of integers $\vec{r} = (r_1, \ldots, r_k)$ with $r_i \geq 1$. The set

$$\frac{1}{r_1}\mathbb{Z} \times \cdots \times \frac{1}{r_k}\mathbb{Z}$$

has a natural partial ordering with

$$\left(\frac{x_1}{r_1}, \ldots, \frac{x_k}{r_k}\right) \leq \left(\frac{y_1}{r_1}, \ldots, \frac{y_k}{r_k}\right)$$

if and only if

$$\frac{x_i}{r_i} \leq \frac{y_i}{r_i}$$

for all i. We shall often denote the poset

$$\frac{1}{r_1}\mathbb{Z} \times \cdots \times \frac{1}{r_k}\mathbb{Z}$$

by

$$\frac{1}{\vec{r}}\mathbb{Z}.$$

If $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \frac{1}{\vec{r}}\mathbb{Z}$, then there is a natural shift functor $[\vec{\alpha}]$ on the category of functors

$$\left(\frac{1}{r_1}\mathbb{Z} \times \cdots \times \frac{1}{r_k}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X)$$

given by precomposition with the addition functor

$$+\vec{\alpha}\colon \frac{1}{\vec{r}}\mathbb{Z} \to \frac{1}{\vec{r}}\mathbb{Z}.$$

DEFINITION 4.1. A parabolic bundle supported on $\mathbb{D}$ with $\vec{r}$-divisible weights is a functor

$$\mathcal{F}_\bullet\colon \left(\frac{1}{r_1}\mathbb{Z} \times \cdots \times \frac{1}{r_k}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X)$$

with natural isomorphisms

$$j_{\mathcal{F}_\bullet,i}\colon \mathcal{F}_\bullet \otimes \mathcal{O}(-n_ip_i) \xrightarrow{\sim} \mathcal{F}_\bullet[0,\ldots,0,1,0,\ldots,0]$$

(with 1 in the ith position) that make the following diagram commute:

$$\begin{array}{ccc} \mathcal{F}_\bullet(-n_i p_i) & \longrightarrow & \mathcal{F}_\bullet[0,\dots,0,1,0,\dots,0] \\ & \searrow \quad \swarrow & \\ & \mathcal{F}_\bullet\,. & \end{array}$$

These data are required to satisfy the following axioms.

(i) If $\alpha_i \leq \alpha_i' \leq \alpha_i + 1$ for all i, then $\operatorname{coker}(\mathcal{F}_{\vec{\alpha}'} \hookrightarrow \mathcal{F}_{\vec{\alpha}})$ is a locally free $\mathcal{O}_D$-module; here $\vec{\alpha} = (\alpha_1, \dots, \alpha_k)$ and $\vec{\alpha}' = (\alpha_1', \dots, \alpha_k')$.

(ii) For every $\vec{\alpha} = (\alpha_1, \dots, \alpha_k) \in \frac{1}{\vec{r}}\mathbb{Z}$ we have that $\mathcal{F}_{\vec{\alpha}}$ is the fibered product of $\mathcal{F}_{(\lfloor\alpha_1\rfloor, \dots, \lfloor\alpha_{i-1}\rfloor, \alpha_i, \lfloor\alpha_{i+1}\rfloor, \dots, \lfloor\alpha_k\rfloor)}$ over $\mathcal{F}_{(\lfloor\alpha_1\rfloor, \dots, \lfloor\alpha_k\rfloor)}$; that is,

$$\mathcal{F}_{\vec{\alpha}} = \mathop{\times}_{\mathcal{F}_{(\lfloor\alpha_1\rfloor, \dots, \lfloor\alpha_k\rfloor)}} \mathcal{F}_{(\lfloor\alpha_1\rfloor, \dots, \lfloor\alpha_{i-1}\rfloor, \alpha_i, \lfloor\alpha_{i+1}\rfloor, \dots, \lfloor\alpha_k\rfloor)}.$$

When the context is clear, we write $j_{\mathcal{F}_\bullet, i} = j_i$. The morphisms making up the functor

$$\mathcal{F}_{\vec{\beta}} \to \mathcal{F}_{\vec{\alpha}}, \quad \vec{\alpha} \leq \vec{\beta},$$

are necessarily injective, so the second axiom merely asserts that

$$\mathcal{F}_{\vec{\alpha}} = \bigcap \mathcal{F}_{(0,\dots,0,\alpha_i,0,\dots,0)}$$

when $\alpha_i > 0$ and the intersection is as submodules of $\mathcal{F}_{(0,0,\dots,0)}$.

REMARK 4.2. When the underlying divisor is reduced, this definition is equivalent to the original one of Mehta and Seshadri in [7]. In other words, a Mehta–Seshadri parabolic bundle with $\vec{r}$-divisible weights and parabolic structure along $\mathbb{D}$ consists of a vector bundle $\mathcal{E}$ and, for each p_i, a filtration of

$$\mathcal{E}_{n_i p_i} := \mathcal{E}_{p_i} \otimes \mathcal{O}_{X,p_i}/\mathfrak{m}_{p_i}^{n_i}$$

given by

$$\mathcal{E}_{n_i p_i} = F_{1,i}(\mathcal{E}_{n_i p_i}) \supsetneq \cdots \supsetneq F_{m_{p_i},i}(\mathcal{E}_{n_i p_i}) \supsetneq F_{m_{p_i}+1,i}(\mathcal{E}_{n_i p_i}) = 0$$

and rational numbers $(\alpha_{i,j})_{1 \leq j \leq m_{p_i}}$ of the form l/r_i satisfying

$$0 \leq \alpha_{i,1} < \cdots < \alpha_{i,m_{p_i}} < 1,$$

subject to the condition that

$$F_{j,i}(\mathcal{E}_{n_i p_i})/F_{j+1,i}(\mathcal{E}_{n_i p_i})$$

is locally free as modules over $\mathcal{O}_{X,p_i}/\mathfrak{m}_{p_i}^{n_i}$.

Let $\mathcal{F}_\bullet$ be a parabolic bundle as in Definition 4.1. The quotients

$$\mathcal{F}_{(0,\dots,0,l/r_i,0,\dots,0)}/\mathcal{F}_{(0,\dots,0,1,0,\dots,0)}$$

for $0 \leq l/r_i < 1$ define a filtration

$$F_{1,i}(\mathcal{F}_\bullet) \supsetneq F_{2,i}(\mathcal{F}_\bullet) \supsetneq \cdots \supsetneq F_{n_i,i}(\mathcal{F}_\bullet) \supsetneq 0$$

of $\mathcal{F}_{(0,\dots,0)}/\mathcal{F}_{(0,\dots,0,1,0,\dots,0)} = \mathcal{F}_{(0,\dots,0)} \otimes \mathcal{O}(-n_i p_i)$. We attach weights $\alpha_{i,j}$ to $F_{j,i}(\mathcal{F}_\bullet)$ by setting $\alpha_{i,j} = l/r_i$, where l is maximal such that

$$F_{j,i}(\mathcal{F}_\bullet) = \mathcal{F}_{(0,\ldots,0,l/r_i,0,\ldots,0)}/\mathcal{F}_{(0,\ldots,0,1,0,\ldots,0)}.$$

This process is clearly reversible.

DEFINITION 4.3. A morphism of parabolic bundles is a natural transformation

$$\phi\colon \mathcal{F}_\bullet \to \mathcal{F}_\bullet'$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_\bullet(-n_i p_i) & \xrightarrow{\sim} & \mathcal{F}_\bullet[0,\ldots,0,1,0,\ldots,0] \\ \downarrow & & \downarrow \\ \mathcal{F}_\bullet'(-n_i p_i) & \xrightarrow{\sim} & \mathcal{F}_\bullet'[0,\ldots,0,1,0,\ldots,0]. \end{array}$$

Denote by $\mathrm{Vect}_{\mathrm{par}}(\mathbb{D},\vec{r})$ the category of $\vec{r}$-divisible parabolic bundles with parabolic structure along $\mathbb{D}$. By modifying constructions and arguments given in [11], it is possible to endow this category with the structure of a rigid tensor category. This entails defining a suitable tensor product and internal Hom, which we describe next.

We have an addition bifunctor

$$+\colon \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \times \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \to \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}}.$$

DEFINITION 4.4. Let $\mathcal{E}_\bullet$, $\mathcal{F}_\bullet$, and $\mathcal{P}_\bullet$ be parabolic bundles. Then there is a functor

$$2\mathcal{E}_\bullet \oplus \mathcal{F}_\bullet\colon \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \times \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X).$$

A *bilinear* morphism from $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ to $\mathcal{P}_\bullet$ is a natural transformation

$$\eta\colon \mathcal{E}_\bullet \oplus \mathcal{F}_\bullet \to \mathcal{P}_\bullet \circ +$$

such that, for every local section $f \in F_{\vec{\alpha}}$ (resp., $e \in E_{\vec{\alpha}}$), there is a parabolic morphism induced from η:

$$\mathcal{E}_\bullet \to \mathcal{P}[\vec{\alpha}]_\bullet \quad (\text{resp., } \mathcal{F}_\bullet \to \mathcal{P}[\vec{\alpha}]_\bullet).$$

As before, let $\vec{\alpha}$ denote $(\alpha_1,\ldots,\alpha_k)$ and similarly for $\vec{\beta}$ and $\vec{\gamma}$.

DEFINITION 4.5. Given parabolic bundles $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ in $\mathrm{Ob}(\mathrm{Vect}_{\mathrm{par}}(\mathbb{D},\vec{r}))$, define a functor

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_\bullet\colon \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X)$$

by setting

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_{\vec{\alpha}} := \frac{\left(\bigoplus_{\beta+\gamma=\alpha} \mathcal{E}_{\vec{\beta}} \otimes_{\mathcal{O}_X} \mathcal{F}_{\vec{\gamma}}\right)}{R_{\vec{\alpha}}},$$

where $R_{\vec{\alpha}}$ is the $\mathcal{O}_X$ submodule of the direct sum, which is locally generated by the sections

$$[\mathcal{E}_\bullet(\vec{\beta} \to \vec{\beta}')]x \otimes y - x \otimes [\mathcal{F}_\bullet(\vec{\gamma}' \to \vec{\gamma})]y$$

for any $\vec{\beta}+\vec{\gamma}=\vec{\beta}'+\vec{\gamma}'=\vec{\alpha}$. Here $x\in\mathcal{E}_{\vec{\beta}}$ and $y\in\mathcal{F}_{\vec{\gamma}'}$; $[\mathcal{E}_\bullet(\vec{\beta}\to\vec{\beta}')]$ denotes the morphism in $\mathrm{Vect}(X)$, which is the image of the morphism $\vec{\beta}\to\vec{\beta}'$ in $\left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}}$ under the functor $\mathcal{E}_\bullet$ (and similarly for $[\mathcal{F}_\bullet(\vec{\gamma}'\to\vec{\gamma})]$); and

$$x-j_i^{\vec{\beta},\vec{\gamma}}x$$

for $i=1,\dots,k$, where $j_i^{\vec{\beta},\vec{\gamma}}$ denotes the morphism

$$(1\otimes j_{\mathcal{F}_\bullet,i}(\vec{\gamma}))\circ(j_{\mathcal{E}_\bullet,i}(\vec{\beta}-(0,\dots,0,1,0,\dots,0))^{-1}\otimes 1)$$

mapping

$$\begin{aligned}\mathcal{E}_{\vec{\beta}}\otimes\mathcal{F}_{\vec{\gamma}}&\to\mathcal{E}_{(\beta_1,\dots,\beta_{i-1},\beta_i-1,\beta_{i+1},\dots,\beta_k)}\otimes\mathcal{O}(-n_ip_i)\otimes\mathcal{F}_{\vec{\gamma}}\\&\to\mathcal{E}_{(\beta_1,\dots,\beta_{i-1},\beta_i-1,\beta_{i+1},\dots,\beta_k)}\otimes\mathcal{F}_{(\gamma_1,\dots,\gamma_{i-1},\gamma_i+1,\gamma_{i+1},\dots,\gamma_k)}.\end{aligned}$$

Also define the morphism $\psi^{\vec{\alpha},\vec{\alpha}'}_{(\mathcal{E}\otimes\mathcal{F})_\bullet}:=(\mathcal{E}\otimes\mathcal{F})_\bullet(\vec{\alpha}\to\vec{\alpha}')$ from $(\mathcal{E}\otimes\mathcal{F})_{\vec{\alpha}}$ to $(\mathcal{E}\otimes\mathcal{F})_{\vec{\alpha}'}$ in $\mathrm{Vect}(X)$ by specifying, for local sections $x\in\mathcal{E}_{\vec{\beta}}$ and $y\in\mathcal{F}_{\vec{\gamma}}$ with $\vec{\beta}+\vec{\gamma}=\vec{\alpha}$, that

$$\begin{aligned}\psi^{\vec{\alpha},\vec{\alpha}'}_{(\mathcal{E}\otimes\mathcal{F})_\bullet}(x\otimes y \bmod R_{\vec{\alpha}})&=([\mathcal{E}_\bullet(\vec{\beta}\to\vec{\alpha}'-\vec{\gamma})]x)\otimes y \bmod R_{\vec{\alpha}'}\\&=x\otimes([\mathcal{F}_\bullet(\vec{\gamma}\to\vec{\alpha}'-\vec{\beta})]y)\bmod R_{\vec{\alpha}'}.\end{aligned}$$

It is now possible to define, for each i, the isomorphism j_i associated to the functor $(\mathcal{E}\otimes\mathcal{F})_\bullet$ as follows. For $i=1,\dots,k$, consider

$$J^i_{\vec{\alpha}}:=\bigoplus_{\vec{\gamma}}(1\otimes j_{\mathcal{F}_\bullet,i}(\vec{\gamma}))$$

mapping

$$\bigoplus_{\vec{\gamma}}\mathcal{E}_{(\vec{\alpha}-\vec{\gamma})}\otimes\mathcal{F}_{\vec{\gamma}}\otimes\mathcal{O}(-n_ip_i)\to\bigoplus_{\vec{\gamma}}\mathcal{E}_{(\vec{\alpha}-\vec{\gamma})}\otimes\mathcal{F}_{(\gamma_1,\dots,\gamma_{i-1},\gamma_i+1,\gamma_{i+1},\dots,\gamma_k)}.$$

Then $J^i_{\vec{\alpha}}(R_{\vec{\alpha}}\otimes\mathcal{O}(-n_ip_i))=R_{(\alpha_1,\dots,\alpha_i+1,\dots,\alpha_k)}$. Hence $J^i_\bullet$ descends to the quotient, and we denote this morphism $j_{(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_\bullet,i}$.

LEMMA 4.6. *With these data, $(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_\bullet$ is a parabolic bundle with a bilinear morphism*

$$\mathcal{E}_\bullet\oplus\mathcal{F}_\bullet\to(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_\bullet\circ+$$

that is universal for all bilinear morphisms.

Proof. It is easy to check that $((\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_\bullet, j_{(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_\bullet,i})\in\mathrm{Ob}(\mathrm{Vect}_{\mathrm{par}}(\mathbb{D},\vec{r}))$.

To see the universal property, observe (as in [11]) that the canonical maps

$$f_{\vec{\alpha},\vec{\beta}}\colon\mathcal{E}_{\vec{\alpha}}\otimes_{\mathcal{O}_X}\mathcal{F}_{\vec{\beta}}\to(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_{\vec{\alpha}+\vec{\beta}}$$

determine a canonical bilinear morphism

$$f_{\bullet,\bullet}\colon\mathcal{E}_\bullet\oplus\mathcal{F}_\bullet\to(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_\bullet\circ+$$

of $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ to $(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)_\bullet$ via the morphisms $f_{\bullet,\vec{\beta}}\colon\mathcal{E}_\bullet\to(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)[\vec{\beta}]_\bullet$ and $f_{\vec{\alpha},\bullet}\colon\mathcal{F}_\bullet\to(\mathcal{E}_\bullet\otimes\mathcal{F}_\bullet)[\vec{\alpha}]_\bullet$ defined, respectively, for each fixed local section $b\in\mathcal{F}_{\vec{\beta}}$ and $a\in\mathcal{E}_{\vec{\alpha}}$. Since the latter morphisms are canonical embeddings, it follows that any bilinear morphism of $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ to some parabolic bundle $\mathcal{P}_\bullet$ factors uniquely through $(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_\bullet \circ +$. □

DEFINITION 4.7. Given parabolic bundles $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ in $\mathrm{Ob}(\mathrm{Vect}_{\mathrm{par}}(\mathbb{D}, \vec{r}))$, define a functor

$$\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{F}_\bullet)_\bullet \colon \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X)$$

by setting

$$\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{F}_\bullet)_{\vec{\alpha}} := \mathcal{H}om(\mathcal{E}_\bullet, \mathcal{F}[\vec{\alpha}]_\bullet),$$

the (vector bundle of) natural transformations from the functor $\mathcal{E}_\bullet$ to the shifted functor $\mathcal{F}[\vec{\alpha}]_\bullet$. The morphism $\vec{\alpha} \to \vec{\beta}$ in $\left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}}$ induces a natural transformation of $\mathcal{F}[\vec{\alpha}]_\bullet$ to $\mathcal{F}[\vec{\beta}]_\bullet$ (i.e., the shift $[\vec{\beta} - \vec{\alpha}]$) and thereby induces the natural transformation

$$\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{F}_\bullet)_{\vec{\alpha}} \to \mathcal{H}om(\mathcal{E}_\bullet, \mathcal{F}_\bullet)_{\vec{\beta}},$$

which we regard as the image of $\vec{\alpha} \to \vec{\beta}$ under the functor $\mathcal{H}om(\mathcal{E}_\bullet, \mathcal{F}_\bullet)_\bullet$.

LEMMA 4.8. *For a given* $\mathbb{D}$ *and* $\vec{r}$*, the bundle category* $\mathrm{Vect}_{\mathrm{par}}(\mathbb{D}, \vec{r})$ (*with the tensor product and internal Hom as in Definitions 4.5 and 4.7, respectively*) *is a rigid tensor category.*

Proof. This follows from the same arguments used to prove Lemmas 3.5 and 3.6 (eq. (3.2)) in [11], modified to accord with our definitions. □

An alternative description of the tensor product was given in [1]. This comes in handy for computations, so for later use we formulate it here. The definition hinges on the embedding $\tau \colon X \setminus D \to X$.

DEFINITION 4.9. The BBN tensor of the parabolic bundles $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ is the functor

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_\bullet^{\mathrm{BBN}} \colon \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X)$$

sending $\vec{\alpha}$ to the subsheaf of $\tau_* \tau^*(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)$ generated by (the canonical images of) $\mathcal{E}_{\vec{\beta}} \otimes \mathcal{F}_{\vec{\gamma}}$ for all $\vec{\beta} + \vec{\gamma} = \vec{\alpha}$.

Because $\mathcal{E}_\bullet$ and $\mathcal{F}_\bullet$ are parabolic, the requisite axioms are automatically satisfied. To show that the BBN tensor gives a parabolic bundle, one need only prove the existence of isomorphisms j_i. Instead, we prove the following statement.

LEMMA 4.10. *For any* $\vec{\alpha} \in \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}}$ *and any parabolic bundles* $\mathcal{E}_\bullet$ *and* $\mathcal{F}_\bullet$,

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_{\vec{\alpha}} \simeq (\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_{\vec{\alpha}}^{\mathrm{BBN}}.$$

Proof. Any bundle $\mathcal{E}_{\vec{\beta}} \otimes \mathcal{F}_{\vec{\gamma}}$ with $\vec{\beta} + \vec{\gamma} = \vec{\alpha}$ maps into $\tau_* \tau^*(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)$ and so yields a mapping

$$\phi \colon \bigoplus_{\vec{\beta} + \vec{\gamma} = \vec{\alpha}} \mathcal{E}_{\vec{\beta}} \otimes \mathcal{F}_{\vec{\gamma}} \to (\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)_\alpha^{\mathrm{BBN}},$$

which by construction is a surjection. We leave it to the reader to show that $R_{\vec{\alpha}} = \ker \phi$. □

We define a parabolic bundle $\mathcal{O}_{X_\bullet}\colon \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X)$ by setting

$$\mathcal{O}_{X(0,\dots,0)} = \mathcal{O}_X$$
$$\mathcal{O}_{X(0,\dots,0,t,0,\dots,0)} = \mathcal{O}_X(-np_i) \quad \text{for } t \in (0,1].$$

It is easily seen that this bundle is a unit for the tensor product.

5. The Parabolic–Orbifold Correspondence

Recall that $\mathcal{N}_1, \dots, \mathcal{N}_k$ denote the canonical line bundles on $X_{\mathbb{D},\vec{r}}$ that are roots of $\mathcal{O}(n_i p_i)$. Following [2] and [3], we now define a functor

$$\mathbf{F}_{\mathbb{D},\vec{r}}\colon \mathrm{Vect}(X_{\mathbb{D},\vec{r}}) \to \mathrm{Vect}_{\mathrm{par}}(\mathbb{D},\vec{r}),$$
$$\mathcal{F} \mapsto \left[\left(\frac{l_1}{r_1},\dots,\frac{l_k}{r_k}\right) \mapsto \pi_*(\mathcal{N}_1^{-l_1} \otimes \cdots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F})\right].$$

REMARK 5.1. This functor is actually a tensor functor, where the tensor product in the category of parabolic bundles is defined as in Section 4. In proving this we use the description of the tensor product in [1]. Given two vector bundles $\mathcal{F}_1$ and $\mathcal{F}_2$, we need to show that the two parabolic bundles $\mathbf{F}(\mathcal{F}_1 \otimes \mathcal{F}_2)$ and $\mathbf{F}(\mathcal{F}_1) \otimes \mathbf{F}(\mathcal{F}_2)$ are isomorphic. Away from the support of $\mathbb{D}$, the stack $X_{\mathbb{D},\vec{r}}$ is isomorphic to the curve X; hence both of these bundles are subbundles of $\tau_*\tau^*(\mathbf{F}(\mathcal{F}_1) \otimes \mathbf{F}(\mathcal{F}_2))$. We must establish that they are the same subbundle. This problem is local, so we reduce to the case of one parabolic point and $\mathcal{F}_i = \mathcal{N}^{a_i}$. This is now easily checked.

The main result of this section is our next theorem.

THEOREM 5.2. *The functor $\mathbf{F}_{\mathbb{D},\vec{r}}$ is an equivalence of categories.*

Proof. The proof given here is entirely analogous to the one given in [3].

We start with a canonical isomorphism

$$\pi^*\mathcal{O}^{\alpha}(n_i p_i) \to \mathcal{N}_i^{\alpha r_i}$$

and a section

$$s \in \Gamma(X_{\mathbb{D},\vec{r}}, \mathcal{N}_i).$$

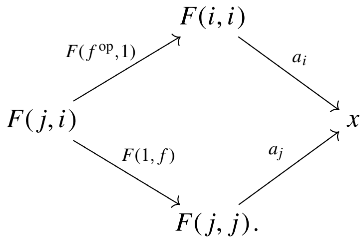
By adjointness, this produces the canonical morphism

$$\mathcal{O}(n_i p_i)^{\lfloor l/r_i \rfloor} \to \pi_*(\mathcal{N}_i^{l}). \tag{$*$}$$

PROPOSITION 5.3. *The morphism $(*)$ is an isomorphism.*

Proof. See [3, 3.11]. □

Before proceeding, we recall the notion of a universal wedge in category theory. Let **B** and **C** be categories and consider a functor $F\colon \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \to \mathbf{C}$. A *wedge* of F is an object x of **C** and a collection of morphisms $a_i\colon F(i,i) \to x$ that are *dinatural*; in other words, for every morphism $f\colon i \to j$ in **B**, the following diagram commutes:



A smallest such wedge is called a *universal* wedge. If it exists we will denote it by $\int^I F(I,I)$.

PROPOSITION 5.4. *Let* $\mathcal{F}_\bullet \in \mathrm{Vect}_{\mathrm{par}}(\mathbb{D}, \vec{r})$. *The universal wedge*

$$\int^{(1/\vec{r})\mathbb{Z}} \mathcal{N}_1^{l_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k} \otimes \pi^* \mathcal{F}_{(l_1/r_1,\ldots,l_k/r_k)}$$

exists in $\mathrm{Vect}(X_{(\mathbb{D},\vec{r})})$.

Proof. The problem is local because wedges are colimits, and proof in the local case has been given in [3]. □

We use $\mathbf{G}_{\mathbb{D},\vec{r}}$ to denote the functor arising from Proposition 5.4.

PROPOSITION 5.5. *Let* $\mathcal{F} \in \mathrm{Vect}(X_{\mathbb{D},\vec{r}})$. *The natural map*

$$\mathcal{N}_1^{l_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k} \otimes \pi^*\pi_*(\mathcal{N}_1^{-l_1} \otimes \cdots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F}) \to \mathcal{F}$$

is dinatural in $(l_1,\ldots,l_k)$.

Proof. The morphism in question is derived by tensoring the counit of adjunction,

$$\pi^*\pi_*(\mathcal{N}_1^{-l_1} \otimes \cdots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F}) \to \mathcal{N}_1^{-l_1} \otimes \cdots \otimes \mathcal{N}_k^{-l_k} \otimes \mathcal{F}.$$

It is relatively straightforward to show that the resulting morphism is dinatural. The details are spelled out in [3, Lemma 3.18]. □

COROLLARY 5.6.

$$\mathbf{G}_{\mathbb{D},\vec{r}} \circ \mathbf{F}_{\mathbb{D},\vec{r}} \simeq 1.$$

Proof. By the proposition, there exists a natural transformation

$$\mathbf{G}_{\mathbb{D},\vec{r}} \circ \mathbf{F}_{\mathbb{D},\vec{r}} \to 1.$$

To show that it is an isomorphism, we may argue locally. This argument can be found in [3, p. 18]. □

Finally, we need to show that

$$\mathbf{F}_{\mathbb{D},\vec{r}} \circ \mathbf{G}_{\mathbb{D},\vec{r}} \simeq 1.$$

We have

$$
\begin{aligned}
&\pi_*\Big(\mathcal{N}_1^{-m_1} \otimes \cdots \otimes \mathcal{N}_k^{-m_k} \otimes \int \mathcal{N}_1^{l_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k} \otimes \pi^* \mathcal{F}_{(l_1/r_1,\ldots,l_k/r_k)}\Big) \\
&\quad \simeq \pi_*\Big(\int \mathcal{N}_1^{l_1-m_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k-m_k} \otimes \pi^* \mathcal{F}_{(l_1/r_1,\ldots,l_k/r_k)}\Big) \\
&\quad \simeq \int \pi_*(\mathcal{N}_1^{l_1-m_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k-m_k} \otimes \pi^* \mathcal{F}_{(l_1/r_1,\ldots,l_k/r_k)}) \quad (\pi_* \text{ is exact}) \\
&\quad \simeq \int \pi_*(\mathcal{N}_1^{l_1-m_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k-m_k}) \otimes \mathcal{F}_{(l_1/r_1,\ldots,l_k/r_k)} \quad (\text{projection formula}) \\
&\quad \simeq \int \mathcal{O}(n_1p_1)^{\lfloor (l_1-m_1)/r_1 \rfloor} \otimes \cdots \otimes \mathcal{O}(n_kp_k)^{\lfloor (l_k-m_k)/r_k \rfloor} \otimes \mathcal{F}_{(l_1/r_1,\ldots,l_k/r_k)} \\
&\quad \simeq \int \mathcal{F}_{(l_1/r_1 - \lfloor (l_1-m_1)/r_1 \rfloor, \ldots, l_k/r_k - \lfloor (l_k-m_k)/r_k \rfloor)} \\
&\quad \simeq \mathcal{F}_{(m_1/r_1,\ldots,m_k/r_k)},
\end{aligned}
$$

completing the proof of Theorem 5.2. □

6. The Parabolic Pullback

Consider a morphism $f\colon Y \to X$ of smooth projective curves. We obtain a diagram

$$
\begin{array}{ccc}
Y_{f^*\mathbb{D},\vec{r}} & \xrightarrow{g} & X_{\mathbb{D},\vec{r}} \\
\downarrow{\scriptstyle \pi_Y} & & \downarrow{\scriptstyle \pi_X} \\
Y & \xrightarrow{f} & X,
\end{array}
$$

and there are associated equivalences of categories

$$\mathbf{F}^X_{\mathbb{D},\vec{r}}\colon \mathrm{Vect}(X_{\mathbb{D},\vec{r}}) \to \mathrm{Vect}_{\mathrm{par}}(\mathbb{D},\vec{r})$$

and

$$\mathbf{F}^Y_{\mathbb{D},\vec{r}}\colon \mathrm{Vect}(Y_{\mathbb{D},\vec{r}}) \to \mathrm{Vect}_{\mathrm{par}}(\mathbb{D},\vec{r}).$$

There is also an obvious pullback functor:

$$f^*\colon \mathrm{Vect}_{\mathrm{par}}(\mathbb{D},\vec{r}) \to \mathrm{Vect}_{\mathrm{par}}(f^*\mathbb{D},\vec{r}).$$

PROPOSITION 6.1. *We have* $f^* \circ \mathbf{F}^X_{\mathbb{D},\vec{r}} = \mathbf{F}^Y_{f^*\mathbb{D},\vec{r}} \circ g^*$.

Proof. The identity follows by flat base change. □

In what follows, we will frequently apply the correspondence described in Remark 4.2.

Set $\vec{r} = (r_1,\ldots,r_k)$, $\mathbb{D} = (n_1p_1,\ldots,n_kp_k)$, and $\vec{n} = (n_1,\ldots,n_k)$. Consider an $\vec{r}$-divisible parabolic bundle $\mathcal{F}_\bullet$ with parabolic structure along $\mathbb{D}$. Using Remark 4.2 then yields the filtration

$$F_{i,1} \supset \cdots \supset F_{i,m_i} \supset F_{i,m_{i+1}} = 0$$

and weights

$$0 \le \alpha_{i,1} = \frac{s_{i1}}{r_i} < \cdots < \alpha_{i,m_i} = \frac{s_{im_i}}{r_i} < 1.$$

Write $n_i s_{ij} = a_{ij} r_i + e_{ij}$ with $0 \le e_{ij} < r_i$. We also denote by $\mathcal{F}_{ij}$ the preimage of F_{ij} in $\mathcal{F}_{(0,0,\dots,0)}$. For $x \in \frac{1}{r_i}\mathbb{Z} \cap [0, 1)$ define a subsheaf $W^x_{ij}(\mathcal{F}_\bullet)$ of $\mathcal{F}_{(0,\dots,0)}(n_i p_i)$ by

$$W^x_{ij}(\mathcal{F}_\bullet) = \begin{cases} \mathcal{F}_{(0,\dots,0)}(a_{ij} p_i) + \mathcal{F}_{i,j+1}(n_i p_i) & \text{if } x \le e_{ij}/r_i, \\ \mathcal{F}_{(0,\dots,0)}((a_{ij} - 1) p_i) + \mathcal{F}_{i,j+1}(n_i p_i) & \text{otherwise.} \end{cases}$$

We have a subsheaf

$$\mathcal{F}^x_i = \bigcap_j W^x_{ij}(\mathcal{F}_\bullet)$$

of $\mathcal{F}_{(0,\dots,0)}(n_i p_i)$.

When $x \ge 0$, we construct subsheaves $\sqrt[\vec{n}]{\mathcal{F}_\bullet}_{(0,\dots,0,x,0,\dots,0)}$ of

$$\mathcal{F}_{(0,\dots,0)}(n_1 p_1 + \cdots + n_k p_k)$$

by setting

$$\sqrt[\vec{n}]{\mathcal{F}_\bullet}_{(0,\dots,0,x,0,\dots,0)} = \left(\bigcap_j W^x_{ij}(\mathcal{F}_\bullet)\right) + \sum_{i \ne k} \mathcal{F}^0_k = \mathcal{F}^x_i + \sum_{i \ne k} \mathcal{F}^0_k,$$

where the nonzero entry of the tuple is in the ith position. If $a_{i(j+1)} = a_{ij}$ then $e_{i,j+1} > e_{ij}$; hence $x \le y$ implies

$$\sqrt[\vec{n}]{\mathcal{F}_\bullet}_{(0,\dots,0,x,0,\dots,0)} \supseteq \sqrt[\vec{n}]{\mathcal{F}_\bullet}_{(0,\dots,0,y,0,\dots,0)}.$$

This result extends uniquely to a parabolic bundle

$$\sqrt[\vec{n}]{\mathcal{F}_\bullet}_\bullet \colon \left(\frac{1}{\vec{r}}\mathbb{Z}\right)^{\mathrm{op}} \to \mathrm{Vect}(X).$$

Setting $\frac{\vec{r}}{\vec{d}} = \left(\frac{r_1}{d_1}, \dots, \frac{r_k}{d_k}\right)$ for $d_i = \gcd(r_i, n_i)$, we see that this parabolic bundle is really $\frac{\vec{r}}{\vec{d}}$-divisible!

Set $\mathbb{D}_{\mathrm{red}} = (p_1, \dots, p_k)$. We have the diagram

$$\begin{array}{ccc} X_{(\mathbb{D}_{\mathrm{red}}, \vec{r}/\vec{d})} & \xrightarrow{\ \alpha\ } & X_{(\mathbb{D}, \vec{r})} \\ & {}_{\pi}\searrow \quad \swarrow_{\pi_n} & \\ & X & \end{array}$$

as well as the associated equivalences

$$\mathbf{F} \colon \mathrm{Vect}(X_{\mathbb{D}_{\mathrm{red}}, \vec{r}/\vec{d}}) \rightleftarrows \mathrm{Vect}_{\mathrm{par}}(\mathbb{D}_{\mathrm{red}}, \vec{r}/\vec{d}) : \mathbf{G}$$

and

$$\mathbf{F}_n \colon \mathrm{Vect}(X_{\mathbb{D}, \vec{r}}) \rightleftarrows \mathrm{Vect}_{\mathrm{par}}(\mathbb{D}, \vec{r}) : \mathbf{G}_n.$$

The balance of this section will be devoted to proving that, for a vector bundle $\mathcal{F}$ on $X_{(\mathbb{D}, \vec{r})}$,

$$\sqrt[\vec{n}]{\mathbf{F}_n(\mathcal{F})} \cong \mathbf{F}(\alpha^*(\mathcal{F})).$$

In order to motivate the proof and to explicate our definition, we compute some examples.

EXAMPLE 6.2. Assume that there is only one parabolic point p with parabolic divisor np having r-divisable weights, and set $d = \gcd(r,n)$. Consider the root line bundle $\mathcal{N}^w$ with $0 < w < r$ on $X_{np,r}$. A calculation shows that

$$\mathbf{F}_n(\mathcal{N}^w)\colon \frac{l}{r} \mapsto \mathcal{O}(np)^{\lfloor (w-l)/r \rfloor},$$
$$\mathbf{F}(\alpha^*\mathcal{N}^w)\colon \frac{dl}{r} \mapsto \mathcal{O}(p)^{\lfloor (nw-dl)/r \rfloor}.$$

We begin our computation of $\sqrt[n]{\mathbf{F}_n(\mathcal{N}^w)}$ by writing $wn = ar + e$. The filtration of $\mathbf{F}_n(\mathcal{N}^w)_0$ is then given by

$$\mathcal{F}_1 = \mathcal{O}, \qquad \mathcal{F}_2 = \mathcal{O}(-np),$$

and the weight of $\mathcal{F}_1$ is w/r. Therefore,

$$W_1^x = \begin{cases} \mathcal{O}(ap), & 0 \le x \le e/r, \\ \mathcal{O}((a-1)p), & e/r < x < 1 \end{cases}$$

and so

$$\left(\sqrt[n]{\mathbf{F}_n(\mathcal{N}^w)}\right)_x = \begin{cases} \mathcal{O}(ap), & 0 \le x \le e/r, \\ \mathcal{O}((a-1)p), & e/r < x < 1, \end{cases}$$

which agrees with $\mathbf{F}(\alpha^*\mathcal{N}^w)$.

Now we compute a rank-2 example. Consider the bundle

$$\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2}$$

with $0 < w_1 < w_2 < r$. A calculation shows that

$$\mathbf{F}_n(\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2})\colon \frac{l}{r} \mapsto \mathcal{O}(np)^{\lfloor (w_1-l)/r \rfloor} \oplus \mathcal{O}(np)^{\lfloor (w_2-l)/r \rfloor},$$
$$\mathbf{F}(\alpha^*(\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2}))\colon \frac{dl}{r} \mapsto \mathcal{O}(p)^{\lfloor (nw_1-dl)/r \rfloor} \oplus \mathcal{O}(np)^{\lfloor (nw_2-dl)/r \rfloor}.$$

To compute $\sqrt[n]{\mathbf{F}_n(\mathcal{N}_n^{w_1} \oplus \mathcal{N}^{w_2})}$, we write $w_j n = a_j r + e_j$. The filtration of $\mathbf{F}_n(\mathcal{N}^w)_0$ is given by

$$\mathcal{F}_1 = \mathcal{O} \oplus \mathcal{O},$$
$$\mathcal{F}_2 = \mathcal{O}(-np) \oplus \mathcal{O},$$
$$\mathcal{F}_3 = \mathcal{O}(-np) \oplus \mathcal{O}(-np),$$

and the weight of $\mathcal{F}_j$ is w_j/r when $j = 1,2$. Hence

$$W_1^x = \begin{cases} \mathcal{O}(a_1 p) \oplus \mathcal{O}(np), & 0 \le x \le e_1/r, \\ \mathcal{O}((a_1-1)p) \oplus \mathcal{O}(np), & e_1/r < x < 1 \end{cases}$$

and

$$W_2^x = \begin{cases} \mathcal{O}(a_2 p) \oplus \mathcal{O}(a_2 p), & 0 \le x \le e_2/r, \\ \mathcal{O}((a_2-1)p) \oplus \mathcal{O}((a_2-1)p), & e_2/r < x < 1. \end{cases}$$

Notice that $a_1 \le a_2$ and equality implies $e_1 < e_2$. Thus $\sqrt[n]{\mathbf{F}\alpha^*(\mathcal{N}^{w_1} \oplus \mathcal{N}^{w_2})}$ agrees with $\mathbf{F}(\alpha^*\mathcal{N}^w)$.

PROPOSITION 6.3. *Let $\mathcal{F}$ be a vector bundle on $X_{\mathbb{D},\vec{r}}$. Then we have the canonical inclusion*

$$\pi_*\alpha^*\mathcal{F} \subset \pi_{n*}\mathcal{F}(n_1p_1 + \cdots + n_kp_k).$$

Proof. We denote the canonical line bundles on $X_{\mathbb{D},\vec{r}}$ by

$$\mathcal{N}_{1,\vec{n}}, \mathcal{N}_{2,\vec{n}}, \ldots, \mathcal{N}_{k,\vec{n}}.$$

We have the diagram

$$\begin{array}{ccc} \alpha_*\alpha^*\mathcal{F} & \longrightarrow & \alpha_*\alpha^*(\mathcal{F} \otimes \mathcal{N}_{n_1}^{r_1} \otimes \cdots \otimes \mathcal{N}_{n_k}^{r_k}) \\ \uparrow & & \uparrow \\ \mathcal{F} & \longrightarrow & \mathcal{F} \otimes \mathcal{N}_{1,\vec{n}}^{r_1} \otimes \cdots \otimes \mathcal{N}_{k,\vec{n}}^{r_k}, \end{array}$$

and we apply $\pi_{\vec{n},*}$ to obtain the diagram

$$\begin{array}{ccc} \pi_*\alpha^*\mathcal{F} & \xrightarrow{\lambda} & \pi_*\alpha^*(\mathcal{F} \otimes \mathcal{N}_{n_1}^{r_1} \otimes \cdots \otimes \mathcal{N}_{n_k}^{r_k}) \\ \uparrow & & \uparrow \mu \\ \pi_{\vec{n},*}\mathcal{F} & \longrightarrow & \pi_{\vec{n},*}\mathcal{F}(n_1p_1 + n_2p_2 + \cdots + n_kp_k). \end{array}$$

The problem is now local and is easily checked. □

THEOREM 6.4. *We have*

$$\sqrt[\vec{n}]{(\mathbf{F}_n\mathcal{F})_{\bullet}} \simeq (\mathbf{F}\alpha^*\mathcal{F})_{\bullet}.$$

Proof. We use Remark 4.2. Both sides are then subbundles of $\mathbf{F}_n\mathcal{F}_{\bullet}(n_1p_1 + \cdots + n_kp_k)$, so the problem is once again local. We may assume that there is only one parabolic point. Applying Proposition 3.6 and Theorem 5.2, we can assume that $(\mathbf{F}_n\mathcal{F})_{\bullet}$ is of the form

$$\frac{l}{r} \mapsto (\mathcal{O}(p)^{n\lfloor (w_1 - l)/r \rfloor})^{\oplus \rho_1} \oplus \cdots \oplus (\mathcal{O}(p)^{n\lfloor (w_k - l)/r \rfloor})^{\oplus \rho_k}$$

with $0 \le w_1 < w_2 < \cdots < w_k < r$. Pulling back root line bundles along the morphism

$$\alpha\colon X_{p,r/d} \to X_{np,r}$$

yields $\alpha^*(\mathcal{N}_n) = \mathcal{N}_1^{(n/d)}$, where $d = \gcd(r, n)$. By Proposition 5.3, $(\mathbf{F}\alpha^*\mathcal{F})_{\bullet}$ is the parabolic bundle

$$\frac{l}{r} \mapsto (\mathcal{O}(p)^{\lfloor (nw_1 - l)/r \rfloor})^{\oplus \rho_1} \oplus \cdots \oplus (\mathcal{O}(p)^{\lfloor (nw_k - l)/r \rfloor})^{\oplus \rho_k}.$$

In order to evaluate $\sqrt[\vec{n}]{(\mathbf{F}_n\mathcal{F})_{\bullet}}$, we first compute the value at $l = 0$ (one can deduce the general result by shifting weights). Thus,

$$W_1^0((\mathbf{F}_n\mathcal{F})_\bullet) = (\mathcal{O}(p)^{\lfloor nw_1/r\rfloor})^{\oplus\rho_1} \oplus \mathcal{O}(np)^{\oplus\rho_3} \oplus \cdots \oplus \mathcal{O}(np)^{\oplus\rho_k},$$
$$W_2^0((\mathbf{F}_n\mathcal{F})_\bullet) = (\mathcal{O}(p)^{\lfloor nw_2/r\rfloor})^{\oplus\rho_1} \oplus (\mathcal{O}(p)^{\lfloor nw_2/r\rfloor})^{\oplus\rho_2} \oplus \mathcal{O}(np)^{\oplus\rho_4} \oplus \cdots \oplus \mathcal{O}(np)^{\oplus\rho_k},$$
$$\vdots$$

and taking the intersection yields

$$\bigcap W_j^0 = (\mathcal{O}(p)^{\lfloor nw_1/r\rfloor})^{\oplus\rho_1} \oplus \cdots \oplus (\mathcal{O}(p)^{\lfloor nw_k/r\rfloor})^{\oplus\rho_k},$$

which is what was needed. □

7. The Cyclic Case

Given a 1-dimensional representation V of $\mathbb{Z}/c\mathbb{Z}$, we call the integer j $(0 \le j \le c-1)$ the *weight* of the representation if the generator $1 + c\mathbb{Z}$ acts via multiplication by $\exp\{2\pi j\sqrt{-1}/c\}$.

Let $q\colon X \to Y$ be a G-cover that is ramified at points $p_1, \ldots, p_k$ of Y. Let the ramification index at p_i be r_i, and set $\vec{r} = (r_1, \ldots, r_k)$ and $\mathbb{D} = (p_1, \ldots, p_k)$. By combining the results of Corollary 2.6, Proposition 3.3, and Theorem 5.2, we may view the cover as a tensor functor

$$\mathcal{F}_q\colon \text{Rep-}G \to \text{Vect}_{\text{par}}(Y, \mathbb{D}, \vec{r}).$$

If we choose preimages $q_i \in X$ of the p_i, we obtain cyclic subgroups $\mathbb{Z}/r_i\mathbb{Z}$ of G that correspond to the stabilizers of q_i. We canonically identify the stabilizer with $\mathbb{Z}/r_i\mathbb{Z}$ by insisting that the stabilizer act on the fiber of the sheaf $\mathcal{O}(-q_i)$ at q_i with weight 1.

Fix an irreducible representation V of G. At each point p_i, we have a weight space decomposition of

$$V = \bigoplus_j W_j^i$$

derived from the induced action of the stabilizers $\mathbb{Z}/r_i\mathbb{Z}$. The spaces W_j^i are representations of $\mathbb{Z}/r_i\mathbb{Z}$, and the generator of the group $\mathbb{Z}/r_i\mathbb{Z}$ acts via multiplication by $\exp\{2\pi j\sqrt{-1}/r_i\}$. The numbers j do not depend upon the choice of preimage q_i.

PROPOSITION 7.1. *In the terminology of Remark 4.2, the weights of the $\mathcal{F}_q(V)_\bullet$ at p_i are j/r_i. In other words, consider tuples*

$$I = \Big(0, \ldots, 0, \underset{i\text{th}}{\frac{j}{r_i}}, 0, \ldots, 0\Big), \qquad I' = \Big(0, \ldots, 0, \underset{i\text{th}}{\frac{j+1}{r_i}}, 0, \ldots, 0\Big).$$

Then

$$\mathcal{F}_q(V)_I = \mathcal{F}_q(V)_{I'}$$

if and only if $W_j^i = 0$.

Proof. By Proposition 3.3 we have the diagram

$$\begin{array}{ccccc} X & \longrightarrow & [X/G] & \xrightarrow{\sim} & Y_{(\mathbb{D},\vec{r})} \\ & \searrow^{\pi'} & \downarrow{\scriptstyle \pi} & & \\ & & Y. & & \end{array}$$

If $\mathcal{E}$ is a G-equivariant bundle on X that is the pullback of some $\tilde{\mathcal{E}}$ on $[X/G]$, then $\pi_*(\tilde{\mathcal{E}}) = \pi'_*(\mathcal{E})^G$. Set $D_i = \pi^*(p_i)_{\mathrm{red}}$. Hence

$$\pi_*(\mathcal{N}_1^{l_1} \otimes \cdots \otimes \mathcal{N}_k^{l_k} \otimes \tilde{\mathcal{E}}) = \pi'_*(\mathcal{O}(l_1 D_1) \otimes \cdots \otimes \mathcal{O}(l_k D_k) \otimes \mathcal{E})^G.$$

The problem is now local. In formal neighborhoods of q_i and p_i, the morphism comes from a morphism of algebras of the form

$$k[[t]] \to k[[s]],$$
$$t \mapsto s^{r_i}.$$

The group action is via multiplication by roots of unity. Computing invariants gives the result. □

Denote by F_m a free group on the symbols $x_1, \ldots, x_m$. Consider the surjection $q\colon F_m \twoheadrightarrow \mathbb{Z}/c\mathbb{Z}$ that sends $x_i \mapsto 1$. There is an associated cover $X_q \to \mathbb{P}^1$ that is possibly ramified at $\{p_1, \ldots, p_m\} \cup \{\infty\}$ for some $p_i \in \mathbb{P}^1 \setminus \{\infty\}$. Set $\vec{c} = \left(c, \ldots, c, \frac{c}{\gcd\{c, m\}}\right) \in \mathbb{Z}^{m+1}$, $\mathbb{D} = (p_1, \ldots, p_m, \infty)$, and $D = p_1 + \cdots + p_m + \infty$. For the rest of this section, V_j will denote the 1-dimensional representation of $\mathbb{Z}/c\mathbb{Z}$ where $1 + c\mathbb{Z}$ acts via multiplication by $\exp\{2\pi j \sqrt{-1}/c\}$. Set

$$\mathcal{F}_{X_q}(V_j)_{(0,\ldots,0)} =: \mathcal{O}(s_j),$$

where s_j is some integer. Also, let w_j denote the rational number in $[0, 1)$ that differs from $-\frac{mj}{c}$ by an integer.

The purpose of this section is to describe the functor $\mathcal{F}_{X_q}$. Toward this end, in Proposition 7.1 take $X = X_q$, $Y = \mathbb{P}^1$, $G = \mathbb{Z}/c\mathbb{Z}$, $k = m + 1$, $D_j = p_j$ for $1 \le j \le m$, $D_{m+1} = \infty$, and $\mathcal{F}_q(V_j) = \mathcal{F}_{X_q}(V_j)_\bullet$. This gives the following result.

Corollary 7.2. *Let* $t = \frac{a}{\gcd(m,c)}$ *and suppose* $0 \le t \le w_j$. *Then*

$$\mathcal{F}_{X_q}(V_j)_{(0,\ldots,0,t)} = \mathcal{O}(s_j)$$

and

$$\mathcal{F}_{X_q}(V_j)_{(0,\ldots,0,w_j+\gcd(m,c)/c)} = \mathcal{O}(s_j)(-\infty).$$

Moreover, if the nonzero entry of the tuple is at the ith *position for* $1 \le i \le m$, *then*

$$\mathcal{F}_{X_q}(V_j)_{(0,\ldots,0,(j+1)/c,0,\ldots,0)} = \mathcal{O}(s_j)(-p_i)$$

but

$$\mathcal{F}_{X_q}(V_j)_{(0,\ldots,0,j/c,0,\ldots,0)} = \mathcal{O}(s_j).$$

Let δ_{ij} denote the Kronecker delta function.

LEMMA 7.3. *If* $1 \leq w_1 + w_j$, *then*

$$(\mathcal{F}_{X_q}(V_1)_\bullet \otimes \mathcal{F}_{X_q}(V_j)_\bullet)_{(0,\dots,0)} = \mathcal{O}(s_1 + s_j + 1 + m\delta_{c-1,j});$$

otherwise,

$$(\mathcal{F}_{X_q}(V_1)_\bullet \otimes \mathcal{F}_{X_q}(V_j)_\bullet)_{(0,\dots,0)} = \mathcal{O}(s_1 + s_j + m\delta_{c-1,j}).$$

Proof. Consider $t \in \frac{\gcd(m,c)}{c}\mathbb{Z}$ and set

$$\vec{t} = (0,\dots,0,t).$$

Write $t = n + f$, where $f \in [0,1)$. We compute

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}).$$

The possibilities are

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \begin{cases} \mathcal{O}(s_1 + s_j + 1), \\ \mathcal{O}(s_1 + s_j), \\ \mathcal{O}(s_1 + s_j - 1), \\ \mathcal{O}(s_1 + s_j - 2). \end{cases}$$

We are interested in when the first possibility occurs. The second occurs at $t = 0$ and so, when we take the sheaf generated by all possible tensor products, the value will be at least this sheaf.

Suppose that $1 \leq w_1 + w_j$, and take $t = 1 - w_j$. Then

$$\mathcal{F}_{X_q}(V_j)_{-\vec{t}} = \mathcal{O}(s_j + 1)$$

and

$$\mathcal{F}_{X_q}(V_1)_{\vec{t}} = \mathcal{O}(s_1).$$

Conversely, suppose that

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \mathcal{O}(s_1 + s_j + 1);$$

then either

$$w_1 - 1 \leq w_j - 1 < w_1 \leq w_j$$

or

$$w_j - 1 \leq w_1 - 1 < w_j \leq w_1.$$

We conclude that $-f \leq w_j - 1$ and $f \leq w_1$ or we must have $-f \leq w_1 - 1$ and $f \leq w_j$. Hence there is a t for which

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \mathcal{O}(s_1 + s_j + 1)$$

if and only if $w_1 + w_j \geq 1$.

Now we turn our attention to the other parabolic points. We preserve the previous notation except to set

$$\vec{t} = (0,\dots,0,t,0,\dots,0),$$

where now $t \in \frac{1}{c}\mathbb{Z}$. We have the chain of inequalities

$$\frac{1}{c} - 1 \leq \frac{j}{c} - 1 < \frac{1}{c} \leq \frac{j}{c}.$$

Suppose first that $j < c-1$. If $-f \leq \frac{j}{c} - 1$ then $f \geq 1 - \frac{j}{c} > \frac{1}{c}$, and if $-f = \frac{1}{c} - 1$ then $f > \frac{j}{c}$. It follows that

$$(\mathcal{F}_{X_q}(V_1)_{\vec{t}} \otimes \mathcal{F}_{X_q}(V_j)_{-\vec{t}}) = \mathcal{O}(s_1 + s_j).$$

When $j < c-1$, the result follows by putting this together.

Now fix $j = c-1$. Set

$$\vec{u} = (u_1, \ldots, u_m, u_{m+1}),$$

where $u_i \in \frac{1}{c}\mathbb{Z}$ for $1 \leq i \leq m$ and $u_{m+1} \in \frac{\gcd(m,c)}{c}$, and write $u_i = n_i + f_i$ for $f_i \in [0,1)$.

When we compute

$$\mathcal{F}_{X_q}(V_1)_{\vec{u}} \otimes \mathcal{F}_{X_q}(V_{c-1})_{-\vec{u}}$$

the possibilities are

$$\mathcal{O}(s_1 + s_{c-1} + g(\vec{u})),$$

where $g(\vec{u})$ ranges over all integers from -2 to $m+1$. Indeed, as before, the parabolic point at infinity gives at most a contribution of $+1$ to $g(\vec{u})$ and at least -2 while each finite parabolic point contributes either 0 or $+1$.

At the same time,

$$\mathcal{F}_{X_q}(V_1)_{(1/c,\ldots,1/c,0)} \otimes \mathcal{F}_{X_q}(V_{c-1})_{(-1/c,\ldots,-1/c,0)} = \mathcal{O}(s_1 + s_{c-1} + m).$$

This means that

$$(\mathcal{F}_{X_q}(V_1)_{\bullet} \otimes \mathcal{F}_{X_q}(V_{c-1})_{\bullet})_{(0,\ldots,0)} \supseteq \mathcal{O}(s_1 + s_{c-1} + m)$$

by the definition of parabolic tensor product. Therefore, we need only determine when $g(\vec{u}) = m+1$.

Suppose that $1 \leq w_1 + w_{c-1}$. Then, if $\vec{u} = \left(\frac{1}{c}, \ldots, \frac{1}{c}, 1 - w_{c-1}\right)$, we have

$$\mathcal{F}_{X_q}(V_{c-1})_{-\vec{u}} = \mathcal{O}(s_{c-1} + m + 1)$$

and

$$\mathcal{F}_{X_q}(V_1)_{\vec{u}} = \mathcal{O}(s_1).$$

Conversely, suppose there exists a $\vec{u}$ such that

$$\mathcal{F}_{X_q}(V_1)_{\vec{u}} \otimes \mathcal{F}_{X_q}(V_{c-1})_{-\vec{u}} = \mathcal{O}(s_1 + s_{c-1} + m + 1).$$

By the same argument as before, this case occurs only when either $-f_{m+1} \leq w_{c-1} - 1$ and $f_{m+1} \leq w_1$ or $-f_{m+1} \leq w_1 - 1$ and $f_{m+1} \leq w_{c-1}$. Necessarily, then, $w_1 + w_{c-1} \geq 1$. □

REMARK 7.4. $\mathcal{F}_{X_q}(V_j)_{\bullet}$ is the jth parabolic tensor power of $\mathcal{F}_{X_q}(V_1)_{\bullet}$. Indeed, since $\mathcal{F}_{X_q}$ is a tensor functor, we must have $\mathcal{F}_{X_q}(V_1)_{\bullet}^{\otimes c} = \mathcal{F}_{X_q}(V_1^{\otimes c})_{\bullet} = \mathcal{F}_{X_q}(V_0)_{\bullet}$, the trivial parabolic bundle. Similarly, $\mathcal{F}_{X_q}(V_1)_{\bullet}^{\otimes l} = \mathcal{F}_{X_q}(V_j)_{\bullet}$ whenever $l \equiv j$ modulo c. Therefore, in order to determine $\mathcal{F}_{X_q}(V_j)_{\bullet}$, it suffices to compute s_1.

For each j with $1 \leq j \leq c-1$, set

$$\kappa_{m,c}^{(j)} = \begin{cases} 1 & \text{if } w_1 + w_j \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\kappa_{m,c} = \sum_{j=1}^{c-1} \kappa_{m,c}^{(j)} = |\{j : 1 \le j \le c-1,\ w_1 + w_j \ge 1\}|.$$

THEOREM 7.5. *With notation as before,*

$$s_1 = -\frac{m + \kappa_{m,c}}{c}.$$

Proof. Applying Lemma 7.3 iteratively along with Remark 7.4, one finds that

$$\mathcal{O}(s_{c-1}) = \mathcal{O}((c-1)s_1 + \kappa_{m,c} - \kappa_{m,c}^{(c-1)}).$$

Next, repeat the calculation once more (in the special case that $j = c-1$) to obtain

$$\mathcal{O}(s_c) = \mathcal{O}(cs_1 + \kappa_{m,c} + m).$$

The result now follows. □

The proof of Theorem 7.5 yields our next corollary.

COROLLARY 7.6. *For $1 \le j \le c-1$, the s_j of Corollary 7.2 are given in terms of s_1 by*

$$s_j = js_1 + \sum_{i=1}^{j-1} \kappa_{m,c}^{(i)} = -j\left(\frac{m + \kappa_{m,c}}{c}\right) + \sum_{i=1}^{j-1} \kappa_{m,c}^{(i)}.$$

COROLLARY 7.7. *We have $s_0 = 0$ and $s_j \le -1$ for $j > 0$.*

Proof. The assertion for s_0 is clear. The numbers are necessarily integers and so, by definition, we have $s_1 < 0$ and hence $s_1 \le -1$. The result now follows. □

By the preceding computation, $\kappa_{m,c}$ is necessarily congruent to $-m$ modulo c. This fact may be shown independently as follows.

LEMMA 7.8.

$$\kappa_{m,c} \equiv -m \text{ modulo } c.$$

Proof. When $m \equiv 0$ modulo c, it follows that $w_j = 0$ for all $1 \le j \le c-1$ and hence $\kappa_{m,c} = 0$.

Suppose now that $m \equiv -v$ modulo c for some $0 < v < c$. Then $w_1 = \frac{v}{c}$ and, for j with $1 \le j \le c-1$,

$$w_j = \begin{cases} \frac{vj}{c} & 0 < vj < c, \\ \vdots & \vdots \\ \frac{vj - tc}{c} & tc \le vj < (t+1)c, \\ \vdots & \vdots \\ \frac{vj-(v-1)c}{c} & (v-1)c \le vj < vc. \end{cases}$$

For t with $0 \le t \le c-1$ it follows that $tc \le vj < (t+1)c$ implies $0 \le vj - tc < c$. Now let j_t be the largest integer value of j satisfying this inequality. Then $v(j_t + 1) - tc \ge c$, so that

$$w_1 + w_{j_t} = \frac{v(1 + j_t) - tc}{c} \geq 1.$$

At the same time, for any integer j that satisfies the inequality and that is also less than j_t, we have $j + 1 \leq j_t$ and necessarily

$$w_1 + w_j \leq \frac{vj_t - tc}{c} < 1.$$

So among the integers j such that $tc \leq vj < (t + 1)c$, there is exactly one with $w_1 + w_j \geq 1$. Since there are exactly v such inequalities, it follows that $\kappa_{m,c} = v$. □

8. Reduction to the Cyclic Case

Suppose that $X_q \to \mathbb{P}^1$ is a Galois covering with $\mathrm{Deck}(X_q/\mathbb{P}^1) = G$ ramified at 0, 1, and ∞. Let $q\colon F_2 \twoheadrightarrow G$ denote the corresponding surjection and let $\mathbb{T} = (0, 1, \infty)$. Then, as before, by Corollary 2.6, Proposition 3.3, and Theorem 5.2 the cover may be viewed as a functor

$$F_{X_q}\colon \text{Rep-}G \to \mathrm{Vect}_{\mathrm{par}}(\mathbb{P}^1, \mathbb{T}).$$

Our goal in this section is to produce a bound on the u_j for which

$$F_{X_q}(V)_{(0,\dots,0)} = \mathcal{O}(u_1) \oplus \cdots \oplus \mathcal{O}(u_k)$$

for a fixed $V \in \mathrm{Ob}(\text{Rep-}G)$.

The idea is to reduce to the cyclic case by de-looping the ramification at 0 as follows. Suppose that the ramification index at 0 is m—in other words, that under the mapping q, the image of the generator of F_2 corresponding to a loop about 0 in $\pi_1(\mathbb{P}^1)$ has order m in G. Form the base change

$$\begin{array}{ccc} X_q \times_{\mathbb{P}^1} \mathbb{P}^1 & \longrightarrow & X_q \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{z \mapsto z^m} & \mathbb{P}^1 \end{array}$$

and denote the desingularization of $X_q \times_{\mathbb{P}^1} \mathbb{P}^1$ by Y. Now $Y \to \mathbb{P}^1$ ramifies at ∞ and the mth roots of unity, μ_m. Hence Y corresponds to a homomorphism $h\colon F_m \to G$, which factors through F_2 by mapping the generators of F_m corresponding to each root of unity to the generator σ_1 of F_2 corresponding to 1. Then the image of h is generated by $q(\sigma_1)$, which is a cyclic subgroup of G (say, $\mathbb{Z}/c\mathbb{Z}$).

We have a decomposition $Y = \coprod_{\tau \in G/\mathrm{Im}(h)} Y_\tau$, where the Y_τ are all cyclic covers. Using our argument at the start of Section 7, we obtain a tensor functor

$$F_Y\colon \text{Rep-}G \to \mathrm{Vect}_{\mathrm{par}}(\mathbb{P}^1, (\mu_m, \infty)).$$

Lemma 8.1. *The functor F_Y factors as*

$$\begin{array}{ccc} \text{Rep-}G & \xrightarrow{F_Y} & \mathrm{Vect}_{\mathrm{par}}(\mathbb{P}^1, (\mu_m, \infty)) \\ \downarrow & \nearrow_{F_{Y_e}} & \\ \text{Rep-}\mathbb{Z}/c\mathbb{Z}. & & \end{array}$$

Proof. The functors are computed by taking invariants as in the proof of Proposition 7.1. The result now follows from the disjoint union Y. □

We shall need the following statement.

PROPOSITION 8.2. *If* $\mathbb{D} = (p_1, \dots, p_k)$ *with* $\vec{r} = (r_1, \dots, r_k)$ *and if* $\mathbb{D}' = (p_0, p_1, \dots, p_k)$ *with* $\vec{r}' = (1, r_1, \dots, r_k)$, *then there exist natural equivalences of tensor categories*

$$\mathbf{F}' : \mathrm{Vect}_{\mathrm{par}}(\mathbb{D}', \vec{r}') \rightleftarrows \mathrm{Vect}_{\mathrm{par}}(\mathbb{D}, \vec{r}) : \mathbf{G}'.$$

Proof. The root stacks $X_{\mathbb{D},\vec{r}}$ and $X_{\mathbb{D}',\vec{r}'}$ are isomorphic. Now invoke Theorem 5.2. □

REMARK 8.3. Let ζ_m denote a primitive mth root of unity. Then, in the notation of Proposition 8.2, set $\mathbb{D} = (\zeta_m, \zeta_m^2, \dots, \zeta_m^{m-1}, 1, \infty)$ and $\vec{r} = \left(c, \dots, c, \frac{c}{\gcd(m,c)}\right)$. Also take $p_0 = 0$. By Proposition 3.5 and Theorem 6.4, $f_{\mathrm{par}}^*(F_{X_q}) = \mathbf{G}' F_Y$.

Since $\mathbf{G}'$ is an equivalence of tensor categories, the constants computed in Section 7 that pertain to F_Y are the same as those relating to $\mathbf{G}' F_Y$.

We denote by $\kappa_{m,c}$ and $\kappa_{m,c}^{(i)}$ the numbers defined before Theorem 7.5 for the cover $Y_e \to \mathbb{P}^1$. We will also make use of the notation set up after Proposition 6.1. In particular, let a_1 denote the minimum among the a_{i1}. We also use a_0 and a_∞ to denote a_{i1} for the index i corresponding to the points 0 and ∞, respectively.

The representation V, when viewed as a representation of $\mathbb{Z}/c\mathbb{Z}$, decomposes into weight spaces:

$$V = V_{j_1} \oplus \cdots \oplus V_{j_k}.$$

We have

$$F_{Y_e}(V)_{(0,\dots,0)} = \mathcal{O}(t_1) \oplus \cdots \oplus \mathcal{O}(t_k),$$

where the t_i are as computed in Theorem 7.5 and Corollary 7.6. We may re-index so that

$$t_1 \leq t_2 \leq \cdots \leq t_k \leq 0.$$

The last inequality follows from Corollary 7.7.

THEOREM 8.4. *With notation as before, consider*

$$F_{X_q}(V)_{(0,\dots,0)} = \mathcal{O}(u_1) \oplus \cdots \oplus \mathcal{O}(u_k).$$

We re-index so that

$$u_1 \leq u_2 \leq \cdots \leq u_k.$$

Then the u_j *are bounded above as follows*:

$$u_j \leq \frac{t_j}{m} - \frac{a_0}{m} - \frac{a_\infty}{m}.$$

(*Hence, by Corollary 7.7, the* u_j *are negative.*)

Proof. We have

$$f^*(F_{X_q}(V)_{(0,\dots,0)}) = \mathcal{O}(mu_1) \oplus \cdots \oplus \mathcal{O}(mu_k).$$

With ζ_m denoting a primitive mth root of unity as before, the curve Y ramifies over

$$p_1 = \zeta_m, \ldots, p_m = \zeta_m^m = 1, \qquad p_{m+1} = \infty.$$

By Remark 8.3, the parabolic pullback of $F_{X_q}(V)_\bullet$ also has 1-divisibility at $p_0 := 0$.

Now, by the definition of parabolic pullback, $f_{\mathrm{par}}^* F_{X_q}(V)_{(0,\ldots,0)}$ contains the intersection $\bigcap_j W_{ij}^0$. Hence

$$f_{\mathrm{par}}^* F_{X_q}(V)_{(0,\ldots,0)} \supseteq (f^*(F_{X_q}(V)_{(0,\ldots,0)})(a_{i1}))$$

because $a_{i1} \leq a_{ij}$. Note that

$$a_{11} = \cdots = a_{m1} = a_1.$$

Therefore,

$$\begin{aligned}
&\mathcal{O}(mu_1) \oplus \cdots \oplus \mathcal{O}(mu_k)\Big(a_0.0 + a_\infty.\infty + \sum a_1 p_i\Big) \\
&\quad \simeq \mathcal{O}(mu_1 + a_0 + ma_1 + a_\infty) \oplus \cdots \oplus \mathcal{O}(mu_k + a_0 + ma_1 + a_\infty) \\
&\quad \subseteq f_{\mathrm{par}}^* F_{X_q}(V)_{(0,\ldots,0)} \\
&\quad = \mathcal{O}(t_1) \oplus \cdots \oplus \mathcal{O}(t_k).
\end{aligned}$$

The result now follows from Lemma 8.5 after we observe that $a_1 = 0$. □

LEMMA 8.5. *If* $\mathcal{O}(s_1) \oplus \cdots \oplus \mathcal{O}(s_u) \subseteq \mathcal{O}(t_1) \oplus \cdots \oplus \mathcal{O}(t_u)$, *then there exists a* $\sigma \in S_u$ *such that* $s_{\sigma(j)} \leq t_j$ *for all* j *with* $1 \leq j \leq u$.

Proof. When $u = 1$, this is well known. Proceeding by induction, suppose that the assertion is known to be valid for all $u \leq N - 1$. Then consider an injection

$$\phi\colon \mathcal{O}(s_1) \oplus \cdots \oplus \mathcal{O}(s_N) \hookrightarrow \mathcal{O}(t_1) \oplus \cdots \oplus \mathcal{O}(t_N),$$

where the s_j and t_j may be taken to be ordered (i.e., $s_1 \leq \cdots \leq s_N$ and $t_1 \leq \cdots \leq t_N$). Necessarily, $s_N \leq t_L$ for some L, but if $s_N \leq t_1$ then we are done. So suppose there exists an i such that $t_{i-1} < s_N \leq t_i$. For j with $i \leq j \leq N$, consider the mapping

$$\phi_j\colon \mathcal{O}(s_1) \oplus \cdots \oplus \mathcal{O}(s_{N-1}) \to \mathcal{O}(t_1) \oplus \cdots \oplus \widehat{\mathcal{O}(t_j)} \oplus \cdots \oplus \mathcal{O}(t_N)$$

induced from ϕ. If there exist j for which ϕ_j is injective, then we are done by the inductive hypothesis. Suppose to the contrary that, for every j, ϕ_j is not injective; then we can show that this implies the original ϕ could not have been injective. Indeed, $s_N > t_{i-1}$ implies that, under ϕ, the restricted morphism $\mathcal{O}(s_N) \to \mathcal{O}(t_1) \oplus \cdots \oplus \mathcal{O}(t_{i-1})$ is zero.

Passing to the generic point of the curve, we find that the morphism ϕ is given by an $N \times N$ matrix whose last row begins with $i-1$ zero entries. Computing the determinant of ϕ by cofactor expansion along this row yields

$$\det\phi = 0 + \det\phi_i \cdot \gamma_i + \cdots + \det\phi_N \cdot \gamma_N$$

for some constants γ_j. Hence the morphism at the generic point is not injective. This is a contradiction, since pullback to the generic point is flat. □

EXAMPLE 8.6. Denote by Q_8 the quaternion group of order 8; it has a 2-dimensional representation given (in terms of matrices) by

$$i \mapsto \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix},$$
$$j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$k \mapsto \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Consider the quotient $F_2 \twoheadrightarrow Q_8$ with $x_0 \mapsto j$ and $x_1 \mapsto i$. Since x_1 has a weight-3 eigenspace, it follows that $t_1 = -3$. Both a_1 and a_∞ are 1, so $u_1 \leq -2$.

It follows from the lower bound in [3, Thm. 5.12] that u_1 must be -2.

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