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On the Johnson Filtration of the Basis-Conjugating Automorphism Group of a Free Group

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1. Introduction

For $n \geq 2$, let F_n be a free group of rank n with basis $x_1, x_2, \dots, x_n$ and let $F_n = \Gamma_n(1), \Gamma_n(2), \dots$ be its lower central series. We denote by $\mathrm{Aut}\, F_n$ the group of automorphisms of F_n. For each $k \geq 0$, let $\mathcal{A}_n(k)$ be the group of automorphisms of F_n that induce the identity on the nilpotent quotient group $F_n/\Gamma_n(k+1)$. The group $\mathcal{A}_n(1)$ is called the IA-automorphism group and is also denoted by IA_n. Then we have a descending filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of $\mathrm{Aut}\, F_n$, called the *Johnson filtration* of $\mathrm{Aut}\, F_n$.

The Johnson filtration of $\mathrm{Aut}\, F_n$ was originally introduced in 1963 through the remarkable pioneering work by Andreadakis [1], who showed that $\mathcal{A}_n(1), \mathcal{A}_n(2), \dots$ is a descending central series of $\mathcal{A}_n(1)$ and that, for each $k \geq 1$, the graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. Andreadakis [1] also computed the rank of $\mathrm{gr}^1(\mathcal{A}_n)$. More recently, the independent work of Cohen and Pakianathan [2; 3], Farb [5], and Kawazumi [6] has shown that $\mathrm{gr}^1(\mathcal{A}_n)$ is isomorphic to the abelianization of IA_n. The $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$ has been determined by Pettet [14] and Satoh [16] for $k = 2$ and 3, respectively. For $k \geq 4$, however, there are few results for the structure of $\mathrm{gr}^k(\mathcal{A}_n)$.

When studying the Johnson filtration, we often face a problem of how to find a generating set of $\mathcal{A}_n(k)$. Although each of the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ is a finitely generated free abelian group, it has not yet been determined whether each of the $\mathcal{A}_n(k)$ is finitely generated or not for $k \geq 2$. Neither do we know whether the abelianization $\mathcal{A}_n(k)^{\mathrm{ab}}$ of $\mathcal{A}_n(k)$ is finitely generated for $k \geq 2$.

In the study of the Johnson filtration of $\mathrm{Aut}\, F_n$, it is also interesting to determine whether or not $\mathcal{A}_n(1), \mathcal{A}_n(2), \dots$ coincides with the lower central series $\mathrm{IA}_n^{(1)}, \mathrm{IA}_n^{(2)}, \dots$ of IA_n. Andreadakis [1] showed that $\mathcal{A}_2(k) = \mathrm{IA}_2^{(k)}$ and $\mathcal{A}_3(3) = \mathrm{IA}_3^{(3)}$. By [2; 3; 5; 6] we have $\mathcal{A}_n(2) = \mathrm{IA}_n^{(2)}$ for $n \geq 3$. Furthermore, in [14] it was shown that $\mathrm{IA}_n^{(3)}$ has finite index in $\mathcal{A}_n(3)$. Andreadakis [1] conjectured that $\mathcal{A}_n(k) = \mathrm{IA}_n^{(k)}$ for any $n \geq 3$ and $k \geq 3$.

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In this paper we consider the problems just mentioned for a certain subgroup of $\mathrm{Aut}\, F_n$. An automorphism σ of F_n such that x_i^σ is conjugate to x_i, $1 \le i \le n$, is called a *basis-conjugating* automorphism of F_n. Let $\mathrm{P\Sigma}_n$ be the subgroup of $\mathrm{Aut}\, F_n$ consisting of the basis-conjugating automorphisms. The group $\mathrm{P\Sigma}_n$ is called the basis-conjugating automorphism group of F_n or a *McCool group*. It is easily checked that $\mathrm{P\Sigma}_n \subset \mathrm{IA}_n$. By the work of McCool [9] it is known that, in general, $\mathrm{P\Sigma}_n$ has a finite presentation (see Section 2.4).

In our previous paper [17] we studied the image of the Johnson filtration of IA_n by the Burau representation τ_B. In particular, we were interested in some problems for $\tau_B(\mathrm{IA}_n)$ that correspond to the open problems for IA_n mentioned here. For example, we considered whether or not $\tau_B(\mathrm{IA}_n)$ is finitely presentable, whether or not the $\tau_B(\mathcal{A}_n(k))$ are finitely generated, and so on. In general, however, these problems are still difficult to handle. Yet one notable finding was that the image of IA_n by τ_B coincides with that of $\mathrm{P\Sigma}_n$. Hence it is meaningful to study the structure of the Johnson filtration of $\mathrm{P\Sigma}_n$ from the viewpoint of research on the Burau representation.

Now, let $\mathcal{P}_n(k) := \mathrm{P\Sigma}_n \cap \mathcal{A}_n(k)$ for each $k \ge 1$. Then we have the descending central filtration

$$\mathrm{P\Sigma}_n = \mathcal{P}_n(1) \supset \mathcal{P}_n(2) \supset \mathcal{P}_n(3) \supset \cdots$$

of $\mathrm{P\Sigma}_n$, which we call the *Johnson filtration* of $\mathrm{P\Sigma}_n$. That being said, if $\mathrm{P\Sigma}_n^{(1)} \supset \mathrm{P\Sigma}_n^{(2)} \supset \cdots$ is the lower central series of $\mathrm{P\Sigma}_n$ then $\mathrm{P\Sigma}_n^{(1)} = \mathcal{P}_n(1)$ by definition. Because the Johnson filtration is central, it follows that $\mathrm{P\Sigma}_n^{(k)} \subset \mathcal{P}_n(k)$ for each $k \ge 1$.

In this paper we concentrate on the subgroups $\mathrm{P\Sigma}_n^{(k)}$ and $\mathcal{P}_n(k)$ for $k \ge 1$. We show that the abelianization of $\mathrm{P\Sigma}_n$ is a free abelian group of rank $n(n-1)$ by analyzing the restriction of the first Johnson homomorphism to $\mathrm{P\Sigma}_n$ (see Section 2.4). In particular, we see that the Magnus generators of type K_{ij} form a basis of $\mathrm{P\Sigma}_n^{\mathrm{ab}}$. As a corollary, we obtain $\mathrm{P\Sigma}_n^{(2)} = \mathcal{P}_n(2)$ (see Corollary 2.2). We remark that some properties of the associated graded Lie algebra and the (co)homological structure of $\mathrm{P\Sigma}_n$ have been studied by Cohen, Pakianathan, Vershinin, and Wu [4]. Here, by determining the image of the second Johnson homomorphism restricted to $\mathrm{P\Sigma}_n$, we obtain the following statement.

LEMMA 1 (= Corollary 2.4). *For $n \ge 3$, $\mathrm{P\Sigma}_n^{(3)} = \mathcal{P}_n(3)$.*

From this lemma it immediately follows that $\tau_B(\mathrm{IA}_n^{(3)}) = \tau_B(\mathcal{A}_n(3))$, which was obtained previously in [17].

Next, using the Reidemeister–Schreier method, we obtain an infinite presentation for $\mathrm{P\Sigma}_n^{(2)} = \mathcal{P}_n(2)$. Then we construct a surjective homomorphism,

$$\Psi \colon \mathrm{P\Sigma}_n^{(2)} \to A,$$

whose target is a free abelian group of infinite rank (see Section 3.2). Observing Ψ and its restriction to the subgroups $\mathrm{P\Sigma}_n^{(k)}$ for $k \ge 3$, we obtain our main result of the paper as follows.

THEOREM 1 (= Theorems 3.1 and 3.5). *For $n \ge 3$ and $k \ge 2$, $(\mathrm{P\Sigma}_n^{(k)})^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

As a corollary, we have the following result.

COROLLARY 1 (= Corollaries 3.2 and 3.7). *For $n \geq 3$ and $k \geq 2$, $\mathcal{P}_n(k)^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

These results show that $\mathrm{P}\Sigma_n^{(k)}$ and $\mathcal{P}_n(k)$ are not finitely generated for $k \geq 2$.

The balance of the paper proceeds as follows. In Section 2 we recall the IA-automorphism group as well as the Johnson filtration of $\mathrm{Aut}\, F_n$ and of the basis-conjugating automorphism group of a free group. In Section 3, we give an infinite presentation for $\mathrm{P}\Sigma_n^{(2)}$ and detect infinitely many linearly independent elements in $\mathcal{P}_n(k)^{\mathrm{ab}}$ for $k \geq 2$.

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2. Preliminaries

In this section, after fixing notation and conventions, we briefly recall the definition and some properties of the IA-automorphism group, the Johnson filtration of $\mathrm{Aut}\, F_n$, and the Johnson filtration of the basis-conjugating automorphism group of a free group.

2.1. Notation and Conventions

Throughout the paper, we use the following notation and conventions. Let G be a group and N a normal subgroup of G.

- The abelianization of G is denoted by G^{ab}.
- The group $\mathrm{Aut}\, G$ of G acts on G from the right. For any $\sigma \in \mathrm{Aut}\, G$ and $x \in G$, the action of σ on x is denoted by x^σ.
- For an element $g \in G$, we also denote the coset class of g by $g \in G/N$ if there is no confusion.
- For elements x and y of G, the commutator bracket $[x, y]$ of x and y is defined to be $[x, y] := xyx^{-1}y^{-1}$.
- For elements $g_1, \ldots, g_k \in G$, a commutator of weight k of the type

  $$[[\cdots[[g_1, g_2], g_3], \ldots], g_k],$$

  with all of its brackets to the left of all the elements occurring, is called a *simple k-fold commutator* and is denoted by $[g_{i_1}, g_{i_2}, \ldots, g_{i_k}]$.

2.2. IA-Automorphism Group

We fix a basis $x_1, \ldots, x_n$ of a free group F_n of rank n. Let $H := F_n^{\mathrm{ab}}$ be the abelianization of F_n and let $\rho\colon \mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$ be the natural homomorphism induced by the abelianization of F_n. We shall identify $\mathrm{Aut}\, H$ with the general linear group $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H induced from the basis $x_1, \ldots, x_n$ of F_n. The kernel IA_n of ρ is called the *IA-automorphism group* of F_n. It is clear that the inner automorphism group Inn F_n of F_n is contained in IA_n. In general, however, IA_n for $n \geq 3$ is much larger than Inn F_n. In fact, Magnus [8] showed that for any $n \geq 3$, IA_n is finitely generated by the automorphisms

$$K_{ij} : x_t \mapsto \begin{cases} x_j^{-1} x_i x_j, & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i, j \in \{1, 2, \ldots, n\}$ and by

$$K_{ijl} : x_t \mapsto \begin{cases} x_i [x_j, x_l], & t = i, \\ x_t, & t \neq i \end{cases}$$

for distinct $i, j, l \in \{1, 2, \ldots, n\}$ such that $j < l$.

Several authors (see [2; 3; 5; 6]) have independently demonstrated that

$$\mathrm{IA}_n^{\mathrm{ab}} \cong H^* \otimes_{\mathbf{Z}} \Lambda^2 H \tag{1}$$

as a $\mathrm{GL}(n, \mathbf{Z})$-module, where $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$ is the $\mathbf{Z}$-linear dual group of H. From this result it follows, in particular, that $\mathrm{IA}_n^{\mathrm{ab}}$ is a free abelian group with basis the coset classes of the Magnus generators K_{ij} and K_{ijl}.

2.3. *Johnson Filtration*

Here we recall the Johnson filtration of Aut F_n. Let $\Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of a free group F_n defined by the rule

$$\Gamma_n(1) := F_n, \quad \Gamma_n(k) := [\Gamma_n(k-1), F_n], \ k \geq 2.$$

We denote by $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ the graded quotient of the lower central series of F_n for each $k \geq 1$. A well-known classical result due to Witt [20] is that each $\mathcal{L}_n(k)$ is a $\mathrm{GL}(n, \mathbf{Z})$-equivariant free abelian group of rank

$$\frac{1}{k} \sum_{d|k} \mu(d) n^{k/d}, \tag{2}$$

where μ is the Möbius function. For example, $\mathcal{L}_n(1) = H$ and $\mathcal{L}_n(2) \cong \Lambda^2 H$, the degree-2 exterior product of H. The graded sum $\mathcal{L}_n := \bigoplus_{k \geq 1} \mathcal{L}_n(k)$ naturally has a graded Lie algebra structure induced by the commutator bracket on F_n and called the *free Lie algebra generated by* H. (See [15] for basic material concerning the free Lie algebra.)

For each $k \geq 0$, the action of Aut F_n on the nilpotent quotient group $F_n/\Gamma_n(k+1)$ of F_n induces the homomorphism

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}(F_n/\Gamma_n(k+1)),$$

whose kernel we denote by $\mathcal{A}_n(k)$. Then the groups $\mathcal{A}_n(k)$ define a descending central filtration

$$\mathrm{Aut}\, F_n = \mathcal{A}_n(0) \supset \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of Aut F_n, where $\mathcal{A}_n(1) = \mathrm{IA}_n$ (see [1] for details). It is called the Johnson filtration of Aut F_n. Let $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ for each $k \geq 1$. Then the graded sum $\mathrm{gr}(\mathcal{A}_n) := \bigoplus_{k \geq 1} \mathrm{gr}^k(\mathcal{A}_n)$ has a graded Lie algebra structure induced by the commutator bracket on IA_n. The subgroups $\mathcal{A}_n(k)$ and the graded quotients $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ play an important role in the various studies of the IA-automorphism group IA_n. It is known (see [1]) that each of $\mathrm{gr}^k(\mathcal{A}_n)$ is a free abelian group of finite rank. However, this rank has not yet been determined in general.

In order to study the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$, the Johnson homomorphisms are often used. For each $k \geq 1$, we can define a homomorphism $\tilde{\tau}_k \colon \mathcal{A}_n(k) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1))$ by

$$\sigma \mapsto (x \mapsto x^{-1}x^{\sigma}), \quad x \in H.$$

Then the kernel of this homomorphism is exactly $\mathcal{A}_n(k+1)$. Hence it induces an injective homomorphism

$$\tau_k \colon \mathrm{gr}^k(\mathcal{A}_n) \hookrightarrow \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)).$$

The homomorphism τ_k is called the kth *Johnson homomorphism* of $\mathrm{Aut}\, F_n$. Thus, studying the graded quotients $\mathrm{gr}^k(\mathcal{A}_n)$ is equivalent to studying the images of the Johnson homomorphisms. For the Magnus generators of IA_n, these images by τ_1 are

$$\tau_1(K_{ij}) = x_i^* \otimes [x_i, x_j], \qquad \tau_1(K_{ijl}) = x_i^* \otimes [x_j, x_l], \tag{3}$$

where $x_1^*, \ldots, x_n^* \in H^*$ is the dual basis of $x_1, \ldots, x_n \in H$. We remark that τ_1 is an isomorphism and nothing but the abelianization of IA_n (see [2; 3; 5; 6]).

Let $\mathrm{Der}(\mathcal{L}_n)$ be the graded Lie algebra of derivations of $\mathcal{L}_n$. The degree-k part of $\mathrm{Der}(\mathcal{L}_n)$ is considered to be $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1)$, which we identify in this paper. Then the sum of the Johnson homomorphisms

$$\tau := \bigoplus_{k \geq 1} \tau_k \colon \mathrm{gr}(\mathcal{A}_n) \to \mathrm{Der}(\mathcal{L}_n)$$

is a graded Lie algebra homomorphism. In fact, if we denote by $\partial\xi$ the element of $\mathrm{Der}(\mathcal{L}_n)$ corresponding to an element $\xi \in H^* \otimes_{\mathbf{Z}} \mathcal{L}_n$ and write the action of $\partial\xi$ on $X \in \mathcal{L}_n$ as $X^{\partial\xi}$, then

$$\tau_{k+l}([\sigma, \sigma']) = \tau_k(\sigma)^{\partial\tau_l(\sigma')} - \tau_l(\sigma')^{\partial\tau_k(\sigma)}$$

for any $\sigma \in \mathcal{A}_n(k)$ and $\sigma' \in \mathcal{A}_n(l)$. In general, this formula is useful for calculating the image of the Johnson homomorphism inductively. In this paper we use it in the proof of Lemma 2.3 in order to give some elements in the image of the second Johnson homomorphism.

Now, for $1 \leq k \leq 3$, the $\mathrm{GL}(n, \mathbf{Z})$-module structure of $\mathrm{gr}^k(\mathcal{A}_n) \otimes_{\mathbf{Z}} \mathbf{Q}$ has been completely determined by Andreadakis [1], Pettet [14], and Satoh [16] for $k = 1$, 2, and 3 (respectively). Yet it seems that studying $\mathrm{gr}^k(\mathcal{A}_n)$ for $k \geq 4$ is a difficult problem in general because not even its generating set has been obtained.

Let $\mathrm{IA}_n^{(k)}$ be the lower central series of IA_n with $\mathrm{IA}_n^{(1)} = \mathrm{IA}_n$. Since the Johnson filtration is central, $\mathrm{IA}_n^{(k)} \subset \mathcal{A}_n(k)$ for each $k \geq 1$. Andreadakis conjectured that $\mathrm{IA}_n^{(k)} = \mathcal{A}_n(k)$ for each $k \geq 1$, and he showed in [1] that $\mathrm{IA}_2^{(k)} = \mathcal{A}_2(k)$ for each $k \geq 1$ and that $\mathrm{IA}_3^{(3)} = \mathcal{A}_3(3)$. From [2; 3; 5; 6] we know that $\mathrm{IA}_n^{(2)} = \mathcal{A}_n(2)$, and $\mathrm{IA}_n^{(3)}$ has at most finite index in $\mathcal{A}_n(3)$ by [14]. Because IA_n is finitely generated, so is each of the graded quotients $\mathrm{gr}^{(k)}(\mathrm{IA}_n) := \mathrm{IA}_n^{(k)}/\mathrm{IA}_n^{(k+1)}$.

A restriction of $\tilde{\tau}_k$ to $\mathrm{IA}_n^{(k)}$ induces the $\mathrm{GL}(n,\mathbf{Z})$-equivariant homomorphism

$$\tau_{(k)}\colon \mathrm{gr}^{(k)}(\mathrm{IA}_n) \to H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(k+1).$$

By an abuse of terminology we also call this the Johnson homomorphism. Since each term of the graded quotients $\mathrm{gr}^{(k)}(\mathrm{IA}_n)$ is finitely generated, we can directly investigate the image of $\tau_{(k)}$ for $k \geq 1$. In particular, in [16; 18; 19] we studied the $\mathrm{GL}(n,\mathbf{Z})$-module structure of the cokernel of the rational Johnson homomorphism $\tau_{(k),\mathbf{Q}} = \tau_{(k)} \otimes \mathrm{id}_{\mathbf{Q}}$. Although the $\mathrm{GL}(n,\mathbf{Z})$-module structure of $\mathrm{gr}^{(k)}(\mathrm{IA}_n)$ is today considerably clarified by virtue of these works, the group structure of each of $\mathrm{IA}_n^{(k)}$ is still not well understood. For example, it has not been determined whether each of the $\mathrm{IA}_n^{(k)}$ is finitely generated.

2.4. Basis-Conjugating Automorphism Group

Here we recall the basis-conjugating automorphisms of F_n. In general, an automorphism σ of F_n such that x_i^σ is conjugate to x_i for each $1 \leq i \leq n$ is called a basis-conjugating automorphism of F_n. Let $\mathrm{P}\Sigma_n$ be the subgroup of $\mathrm{Aut}\, F_n$ consisting of the basis-conjugating automorphisms. The group $\mathrm{P}\Sigma_n$ is called the basis-conjugating automorphism group of F_n, or the McCool group. It is easily checked that $\mathrm{P}\Sigma_n \subset \mathrm{IA}_n$ and that an IA-automorphism K_{ij} for $i \neq j$ belongs to $\mathrm{P}\Sigma_n$. McCool obtained a finite presentation for $\mathrm{P}\Sigma_n$ as follows.

THEOREM 2.1 [9]. *The group $\mathrm{P}\Sigma_n$ has a finite presentations with generators K_{ij} for $1 \leq i \neq j \leq n$, subject to the relations*

(R1) $[K_{ij}, K_{kj}] = 1$ *for* $i < k$,
(R2) $[K_{ij}, K_{kl}] = 1$ *for* $i < k$, *and*
(R3) $[K_{ik}, K_{ij}K_{kj}] = 1$;

here the subscripts i, j, k, l are distinct.

In this paper we consider the restriction of the Johnson filtration to $\mathrm{P}\Sigma_n$. Let $\mathcal{P}_n(k) := \mathrm{P}\Sigma_n \cap \mathcal{A}_n(k)$ for each $k \geq 1$. Then we have the descending central filtration

$$\mathrm{P}\Sigma_n = \mathcal{P}_n(1) \supset \mathcal{P}_n(2) \supset \mathcal{P}_n(3) \supset \cdots$$

of $\mathrm{P}\Sigma_n$, which we call the Johnson filtration of $\mathrm{P}\Sigma_n$. Then each of the graded quotients $\mathrm{gr}^k(\mathcal{P}_n) := \mathcal{P}_n(k)/\mathcal{P}_n(k+1)$ is a $\mathbf{Z}$-submodule of $\mathrm{gr}^k(\mathcal{A}_n)$ for $k \geq 1$. We denote by $\tau_{P,k}$ the kth Johnson homomorphism τ_k restricted to $\mathrm{gr}^k(\mathcal{P}_n)$ for $k \geq 1$.

Let $\mathrm{P}\Sigma_n^{(1)} \supset \mathrm{P}\Sigma_n^{(2)} \supset \cdots$ be the lower central series of $\mathrm{P}\Sigma_n$. Then $\mathrm{P}\Sigma_n^{(1)} = \mathcal{P}_n(1)$ by definition. Since the Johnson filtration is central, we see that $\mathrm{P}\Sigma_n^{(k)} \subset \mathcal{P}_n(k)$ for each $k \geq 1$. Let $\mathrm{gr}^{(k)}(\mathrm{P}\Sigma_n) := \mathrm{P}\Sigma_n^{(k)}/\mathrm{P}\Sigma_n^{(k+1)}$ for $k \geq 1$. Then there is a natural homomorphism $\iota_k\colon \mathrm{gr}^{(k)}(\mathrm{P}\Sigma_n) \to \mathrm{gr}^k(\mathcal{P}_n)$, $k \geq 1$, induced by the inclusion $\mathrm{P}\Sigma_n^{(k)} \hookrightarrow \mathcal{P}_n(k)$. Now we define a homomorphism $\tau_{P,(k)}$ to be the composition of ι_k and the Johnson homomorphism $\tau_{P,k}$:

$$\tau_{P,(k)} := \tau_{P,k} \circ \iota_k\colon \mathrm{gr}^{(k)}(\mathrm{P}\Sigma_n) \to \mathrm{Hom}_{\mathbf{Z}}(H, \mathcal{L}_n(k+1)).$$

According to the conjecture of Andreadakis [1] mentioned previously, it would seem that $\mathrm{P}\Sigma_n^{(k)} = \mathcal{P}_n(k)$ for each $k \geq 1$. However, this is still an open problem in general. Here we observe that $\mathrm{P}\Sigma_n^{(2)} = \mathcal{P}_n(2)$ and show that $\mathrm{P}\Sigma_n^{(3)} = \mathcal{P}_n(3)$ by using the second Johnson homomorphism. We start by observing the first Johnson homomorphism $\tau_{P,1}$, which shows that the abelianization of $\mathrm{P}\Sigma_n = \mathcal{P}_n(1)$ is given by

$$\mathcal{P}_n(1)^{\mathrm{ab}} \cong \mathbf{Z}^{\oplus n(n-1)} \tag{4}$$

for $n \geq 3$. In particular, the Magnus generators K_{ij} for $1 \leq i \neq j \leq n$ form a basis of $\mathcal{P}_n(1)^{\mathrm{ab}}$. Hence, as a corollary, we have the following statement.

COROLLARY 2.2. *For $n \geq 3$, $\mathrm{P}\Sigma_n^{(2)} = \mathcal{P}_n(2)$.*

Proof. Because $\tau_{P,1}$ is the abelianization of $\mathcal{P}_n(1)$, the natural homomorphism $\iota_1 \colon \mathrm{gr}^{(1)}(\mathrm{P}\Sigma_n) \to \mathrm{gr}^1(\mathcal{P}_n)$ must be injective. Hence $\mathrm{P}\Sigma_n^{(2)} = \mathcal{P}_n(2)$, completing the proof. □

Next, we consider $\mathrm{P}\Sigma_n^{(3)}$. We begin by determining the image of the Johnson homomorphism $\tau_{P,2}$.

LEMMA 2.3. *For $n \geq 3$, $\mathrm{Im}(\tau_{P,2}) \cong \mathbf{Z}^{\oplus n(n-1)^2/2}$.*

Proof. By Corollary 2.2, $\mathrm{Im}(\tau_{P,2}) = \mathrm{Im}(\tau_{P,(2)})$. We show that $\mathrm{gr}^{(2)}(\mathrm{P}\Sigma_n)$ is generated by $n(n-1)^2/2$ elements:

$$\begin{aligned} S' := {}& \{[K_{ij}, K_{iq}] \mid 1 \leq i \leq n,\ 1 \leq j < q \leq n,\ j,q \neq i\} \\ & \cup \{[K_{ij}, K_{ji}] \mid 1 \leq i < j \leq n\}. \end{aligned}$$

In general, the graded quotient $\mathrm{gr}^{(2)}(\mathrm{P}\Sigma_n)$ is generated by commutators $[K_{ij}, K_{pq}]$. Let $N := \#\{i, j, p, q\}$. If $N = 4$, or if $N = 3$ and $q = j$, then $[K_{ij}, K_{pq}] = 1 \in \mathcal{P}_n(2)$ by the relations (R1) and (R2). It is also clear that (i) if $N = 3$ and $q = i$ then $[K_{ij}, K_{pi}] = [K_{pi}, K_{ij}]^{-1} \in \mathcal{P}_n(2)$, and (ii) if $N = 3$, $p = i$, and $j > q$ then $[K_{ij}, K_{iq}] = [K_{iq}, K_{ij}]^{-1}$. On the other hand, by (R3) we have

$$[K_{ij}, K_{iq}K_{jq}] = [K_{ij}, K_{iq}] + [K_{ij}, K_{jq}] = 0 \in \mathrm{gr}^{(2)}(\mathrm{P}\Sigma_n).$$

Hence we can reduce the generators to type $[K_{ij}, K_{jq}]$. Then we see that S' generates $\mathrm{gr}^{(2)}(\mathrm{P}\Sigma_n)$.

This shows that $\mathrm{Im}(\tau_{P,2})$ is generated by

$$\begin{aligned} S := {}& \{\tau_{P,2}([K_{ij}, K_{iq}]) \mid 1 \leq i \leq n,\ 1 \leq j < q \leq n,\ j,q \neq i\} \\ & \cup \{\tau_{P,2}([K_{ij}, K_{ji}]) \mid 1 \leq i < j \leq n\}, \end{aligned}$$

where

$$\tau_{P,2}([K_{ij}, K_{iq}]) = x_i^* \otimes [x_i, x_q, x_j] - x_i^* \otimes [x_i, x_j, x_q]$$

and

$$\tau_{P,2}([K_{ij}, K_{ji}]) = x_i^* \otimes [x_i, x_j, x_i] - x_j^* \otimes [x_j, x_i, x_j].$$

In order to show that S is linearly independent in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$, set

$$\sum{}' a_{i,j,q} \tau_{P,2}([K_{ij}, K_{iq}]) + \sum{}'' b_{i,j} \tau_{P,2}([K_{ij}, K_{ji}]) = 0;$$

here the first sum runs over $1 \le i \le n$ and $1 \le j < q \le n$ such that $j, q \neq i$ while the second sum runs over $1 \le i < j \le n$. Then, for any $1 \le i_0 \le n$,

$$\sum_{j<q}^{j,q \neq i_0} a_{i_0,j,q} \tau_{P,2}([K_{i_0,j}, K_{i_0,q}]) + \sum_{i_0<j} b_{i_0,j} x_{i_0}^* \otimes [x_{i_0}, x_j, x_{i_0}] \\ - \sum_{j<i_0} b_{j,i_0} x_{i_0}^* \otimes [x_{i_0}, x_j, x_{i_0}] = 0. \quad (5)$$

Now consider the composition of homomorphisms

$$\Phi\colon H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3) \to H^* \otimes_{\mathbf{Z}} H^{\otimes 3} \to H^{\otimes 2},$$

where the first homomorphism is induced from the natural embedding $\mathcal{L}_n(3) \hookrightarrow H^{\otimes 3}$ defined by $[X, Y] \mapsto X \otimes Y - Y \otimes X$ and the second isomorphism is the contraction map

$$x_i^* \otimes x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \mapsto x_i^*(x_{i_1}) \cdot x_{i_2} \otimes x_{i_3}.$$

Observing the image of (5) by Φ, we obtain

$$\sum_{j<q}^{j,q \neq i_0} a_{i_0,j,q}(x_q \otimes x_j - x_j \otimes x_q) + \sum_{i_0<j} b_{i_0,j}(x_j \otimes x_{i_0} - x_{i_0} \otimes x_j) \\ - \sum_{j<i_0} b_{j,i_0}(x_j \otimes x_{i_0} - x_{i_0} \otimes x_j) = 0 \in H^{\otimes 2}.$$

This expression shows that $a_{i_0,j,q} = b_{i_0,j} = b_{j,i_0} = 0$ for $1 \le j < q \le n$ and $j, q \neq i_0$, so S is linearly independent in $H^* \otimes_{\mathbf{Z}} \mathcal{L}_n(3)$. We thus obtain the required result, which completes the proof. □

COROLLARY 2.4. *For $n \ge 3$, $\mathrm{P}\Sigma_n^{(3)} = \mathcal{P}_n(3)$.*

Proof. Since the image of the generating set S' of $\mathrm{gr}^{(2)}(\mathrm{P}\Sigma_n)$ by $\tau_{P,(2)}$ is a basis of $\mathrm{Im}(\tau_{P,(2)}) = \mathrm{Im}(\tau_{P,2})$, it follows that $\mathrm{gr}^{(2)}(\mathrm{P}\Sigma_n)$ is a free abelian group with basis S' and that

$$\tau_{P,(2)}\colon \mathrm{gr}^{(2)}(\mathrm{P}\Sigma_n) \xrightarrow{\iota_2} \mathrm{gr}^2(\mathcal{P}_n) \xrightarrow{\tau_{P,2}} \mathrm{Im}(\tau_{P,2})$$

is an isomorphism. Therefore, ι_2 is also an isomorphism. This shows $\mathrm{P}\Sigma_n^{(3)} = \mathcal{P}_n(3)$, completing the proof. □

The proofs of Lemma 2.3 and Corollary 2.4 reveal that, for each $k \ge 3$, if we determine the rank $r_n(k) := \mathrm{rank}_{\mathbf{Z}}(\mathrm{gr}^k(\mathcal{P}_n))$ of $\mathrm{gr}^k(\mathcal{P}_n)$ and show that $\mathrm{gr}^{(k)}(\mathrm{P}\Sigma_n)$ is generated by $r_n(k)$ elements, then we can demonstrate $\mathrm{P}\Sigma_n^{(k)} = \mathcal{P}_n(k)$ inductively. However, this approach is complicated because we must consider so many types of commutators among the K_{ij} in $\mathrm{gr}^{(k)}(\mathrm{P}\Sigma_n)$ for large k.

3. On the Abelianization of $\mathrm{P}\Sigma_n^{(k)}$ and $\mathcal{P}_n(k)$ for $k \ge 2$

In this section we show that the abelianization of $\mathrm{P}\Sigma_n^{(k)}$ contains a free abelian group of infinite rank for $n \ge 2$ and $k \ge 2$; we also prove that $\mathcal{P}_n(k)$ has the same property. Toward this end, we obtain a presentation for the commutator subgroup $\mathrm{P}\Sigma_n^{(2)}$ of $\mathrm{P}\Sigma_n$ by the Reidemeister–Schreier method. Then we construct a homomorphism from $\mathrm{P}\Sigma_n^{(2)}$ to some free abelian group that detects an infinitely generated free abelian quotient of $\mathrm{P}\Sigma_n^{(k)}$ for $k \geq 2$.

3.1. *A Presentation for* $\mathrm{P}\Sigma_n^{(2)}$

In this section we obtain a presentation for $\mathrm{P}\Sigma_n^{(2)}$ by applying the Reidemeister–Schreier method. (For details on this method, see e.g. [7, Chap. II, Prop. 4.1].)

Let F be a free group with basis $\{K_{ij} \mid 1 \leq i \neq j \leq n\}$, and let $\varphi\colon F \to \mathrm{P}\Sigma_n$ be the canonical map. Set $N = \varphi^{-1}(\mathrm{P}\Sigma_n^{(2)})$. Then a subset

$$T := \{K_{12}^{e_{12}} K_{13}^{e_{13}} \cdots K_{n(n-1)}^{e_{n(n-1)}} \mid e_{ij} \in \mathbf{Z}\} \subset F$$

is a Schreier transversal for N of F because $\mathrm{P}\Sigma_n^{\mathrm{ab}}$ is a free abelian group with basis $\{K_{ij} \mid 1 \leq i \neq j \leq n\}$. Here the order of the product among $K_{ij}^{e_{ij}}$ in $K_{12}^{e_{12}} K_{13}^{e_{13}} \cdots K_{n(n-1)}^{e_{n(n-1)}}$ is the usual lexicographic order with respect to the index set

$$I := \{(i, j) \mid 1 \leq i \neq j \leq n\}.$$

In other words, for any (i, j) and $(p, q) \in I$, we have that $(i, j) < (p, q)$ if and only if (a) $i < p$ or (b) $i = p$ and $j < q$.

Let

$$\gamma_{ij}(e_{12}, \ldots, e_{n(n-1)}) := (K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}}) K_{ij} (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \in F.$$

Then, after applying the Reidemeister–Schreier method to the McCool presentation of $\mathrm{P}\Sigma_n$ and the Schreier transversal T for N of F, we see that $\mathrm{P}\Sigma_n^{(2)}$ is generated by

$$\begin{aligned}\mathfrak{E} := \{\gamma_{ij}(e_{12}, \ldots, e_{n(n-1)}) \mid {} & (1,2) \leq (i, j) \leq (n, n-2), \\ & e_{pq} \neq 0 \text{ for some } (i, j+1) \leq (p, q) \leq (n, n-1)\}\end{aligned}$$

subject to the relators

$$\tau(trt^{-1}) \quad \text{for } t \in T \text{ and } r = \text{(R1), (R2), (R3)}$$

where τ denotes the “rewriting function”. Namely, for any word $w \in N$ among the K_{ij}, $\tau(w)$ ($= w$ in F) is a word among the $\gamma_{ij}(e_{12}, \ldots, e_{n(n-1)})$. In what follows, for any $t = K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}} \in T$ we write out $\tau(trt^{-1})$ explicitly.

3.1.1. $r = [K_{ij}, K_{kj}]$ *for distinct* $1 \leq i, j, k \leq n$ *and* $i < k$

When r is so defined, we have

$$\begin{aligned} trt^{-1} = {} & (K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}}) K_{ij} (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\ & \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) K_{kj} (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\ & \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) \\ & \cdot K_{ij}^{-1} (K_{12}^{e_{12}} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\ & \cdot (K_{12}^{e_{12}} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) K_{kj}^{-1} (K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1}. \end{aligned}$$

If $e_{pq} \neq 0$ for some $(k, j+1) \le (p,q) \le (n, n-1)$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \dots, e_{n(n-1)})\gamma_{kj}(e_{12}, \dots, e_{ij}+1, \dots, e_{n(n-1)}) \\ \cdot \gamma_{ij}(e_{12}, \dots, e_{kj}+1, \dots, e_{n(n-1)})^{-1}\gamma_{kj}(e_{12}, \dots, e_{n(n-1)})^{-1}.$$

If $\{(p,q) \in I \mid (k, j+1) \le (p,q) \le (n, n-1), e_{pq} \neq 0\} = \phi$, then

$$trt^{-1} = (K_{12}^{e_{12}} \cdots K_{kj}^{e_{kj}})K_{ij}(K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kj}^{e_{kj}})^{-1} \\ \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kj}^{e_{kj}+1})K_{ij}^{-1}(K_{12}^{e_{12}} \cdots K_{kj}^{e_{kj}+1})^{-1}.$$

Therefore, the following statements hold.

(i) If $e_{pq} \neq 0$ for some $(i, j+1) \le (p,q) \le (k, j-1)$ or if $e_{kj} \neq 0, -1$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \dots, e_{kj}, 0, \dots, 0)\gamma_{ij}(e_{12}, \dots, e_{kj}+1, 0, \dots, 0)^{-1}.$$

(ii) If $\{(p,q) \in I \mid (i, j+1) \le (p,q) \le (k, j-1), e_{pq} \neq 0\} = \phi$ and $e_{kj} = 0$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \dots, e_{ij}, 0, \dots, 1, \dots, 0)^{-1},$$

where 1 appears in the (k, j) entry.

(iii) If $\{(p,q) \in I \mid (i, j+1) \le (p,q) \le (k, j-1), e_{pq} \neq 0\} = \phi$ and $e_{kj} = -1$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \dots, e_{ij}, 0, \dots, -1, \dots, 0),$$

where -1 appears in the (k, j) entry.

The exposition in Sections 3.1.2 and 3.1.3 proceeds along similar lines.

3.1.2. $r = [K_{ij}, K_{kl}]$ *for distinct* $1 \le i, j, k, l \le n$ *and* $i < k$
For such r, we have

$$trt^{-1} = (K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}})K_{ij}(K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\ \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})K_{kl}(K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kl}^{e_{kl}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\ \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kl}^{e_{kl}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) \\ \cdot K_{ij}^{-1}(K_{12}^{e_{12}} \cdots K_{kl}^{e_{kl}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\ \cdot (K_{12}^{e_{12}} \cdots K_{kl}^{e_{kl}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})K_{kl}^{-1}(K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1}.$$

If $e_{pq} \neq 0$ for some $(k, l+1) \le (p,q) \le (n, n-1)$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \dots, e_{n(n-1)})\gamma_{kl}(e_{12}, \dots, e_{ij}+1, \dots, e_{n(n-1)}) \\ \cdot \gamma_{ij}(e_{12}, \dots, e_{kl}+1, \dots, e_{n(n-1)})^{-1}\gamma_{kl}(e_{12}, \dots, e_{n(n-1)})^{-1}.$$

If $\{(p,q) \in I \mid (k, l+1) \le (p,q) \le (n, n-1), e_{pq} \neq 0\} = \phi$, then the following statements hold.

(i) If $e_{pq} \neq 0$ for some $(i, j+1) \le (p,q) \le (k, l-1)$ or if $e_{kl} \neq 0, -1$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \dots, e_{kl}, 0, \dots, 0)\gamma_{ij}(e_{12}, \dots, e_{kl}+1, 0, \dots, 0)^{-1}.$$

(ii) If $\{(p,q) \in I \mid (i,j+1) \leq (p,q) \leq (k,l-1), e_{pq} \neq 0\} = \phi$ and $e_{kl} = 0$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \ldots, e_{ij}, 0, \ldots, 1, \ldots, 0)^{-1},$$

where 1 appears in the (k,l) entry.

(iii) If $\{(p,q) \in I \mid (i,j+1) \leq (p,q) \leq (k,l-1), e_{pq} \neq 0\} = \phi$ and $e_{kl} = -1$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12}, \ldots, e_{ij}, 0, \ldots, -1, \ldots, 0),$$

where -1 appears in the (k,l) entry.

3.1.3. $r = [K_{ik}, K_{ij}K_{kj}]$ *for distinct* $1 \leq i, j, k \leq n$

Case 1: $i < k$ and $j < k$. Then $(i,j) < (i,k) < (k,j)$. In this case, we have

$$\begin{aligned}
trt^{-1} = {} & (K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}}) K_{ik} (K_{12}^{e_{12}} \cdots K_{ik}^{e_{ik}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\
& \cdot (K_{12}^{e_{12}} \cdots K_{ik}^{e_{ik}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) K_{ij} (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{ik}^{e_{ik}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\
& \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{ik}^{e_{ik}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) \\
& \cdot K_{kj} (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}} \cdots K_{ik}^{e_{ik}+1} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\
& \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{ik}^{e_{ik}+1} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) \\
& \cdot K_{ik}^{-1} (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\
& \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{kj}^{e_{kj}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) \\
& \cdot K_{kj}^{-1} (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1} \\
& \cdot (K_{12}^{e_{12}} \cdots K_{ij}^{e_{ij}+1} \cdots K_{n(n-1)}^{e_{n(n-1)}}) K_{ij}^{-1} (K_{12}^{e_{12}} \cdots K_{n(n-1)}^{e_{n(n-1)}})^{-1}.
\end{aligned}$$

If $e_{pq} \neq 0$ for some $(k,j+1) \leq (p,q) \leq (n,n-1)$, then

$$\begin{aligned}
\tau(trt^{-1}) = {} & \gamma_{ik}(e_{12}, \ldots, e_{n(n-1)}) \gamma_{ij}(e_{12}, \ldots, e_{ik}+1, \ldots, e_{n(n-1)}) \\
& \cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{ik}+1, \ldots, e_{n(n-1)}) \\
& \cdot \gamma_{ik}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{kj}+1, \ldots, e_{n(n-1)})^{-1} \\
& \cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{n(n-1)})^{-1} \gamma_{ij}(e_{12}, \ldots, e_{n(n-1)})^{-1}.
\end{aligned}$$

If $\{(p,q) \in I \mid (k,j+1) \leq (p,q) \leq (n,n-1), e_{pq} \neq 0\} = \phi$, then the following statements hold.

(i) If $e_{pq} \neq 0$ for some $(i,k+1) \leq (p,q) \leq (k,j-1)$ or if $e_{kj} \neq 0, -1$, then

$$\begin{aligned}
\tau(trt^{-1}) = {} & \gamma_{ik}(e_{12}, \ldots, e_{kj}, 0, \ldots, 0) \gamma_{ij}(e_{12}, \ldots, e_{ik}+1, \ldots, e_{kj}, 0, \ldots, 0) \\
& \cdot \gamma_{ik}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{kj}+1, 0, \ldots, 0)^{-1} \\
& \cdot \gamma_{ij}(e_{12}, \ldots, e_{kj}, 0, \ldots, 0)^{-1}.
\end{aligned}$$

(ii) If $\{(p,q) \in I \mid (i,k+1) \leq (p,q) \leq (k,j-1), e_{pq} \neq 0\} = \phi$ and $e_{kj} = -1$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ik}&(e_{12},\dots,e_{ik},0,\dots,-1,\dots,0)\\ &\cdot \gamma_{ij}(e_{12},\dots,e_{ik}+1,0,\dots,-1,\dots,0)\\ &\cdot \gamma_{ij}(e_{12},\dots,e_{ik},0,\dots,-1,\dots,0)^{-1},\end{aligned}$$

where -1 appears in the (k,j) entries of the γ_{ik} and γ_{ij}.

(iii) If $\{(p,q)\in I \mid (i,k+1)\le (p,q)\le (k,j-1),\ e_{pq}\neq 0\}=\phi$ and $e_{kj}=0$, then

$$\begin{aligned}trt^{-1} &= (K_{12}^{e_{12}}\cdots K_{ik}^{e_{ik}+1})K_{ij}(K_{12}^{e_{12}}\cdots K_{ij}^{e_{ij}+1}\cdots K_{ik}^{e_{ik}+1})^{-1}\\ &\quad\cdot (K_{12}^{e_{12}}\cdots K_{ij}^{e_{ij}+1}\cdots K_{ik}^{e_{ik}+1}K_{kj})K_{ik}^{-1}(K_{12}^{e_{12}}\cdots K_{ij}^{e_{ij}+1}\cdots K_{ik}^{e_{ik}}K_{kj})^{-1}\\ &\quad\cdot (K_{12}^{e_{12}}\cdots K_{ij}^{e_{ij}+1}\cdots K_{ik}^{e_{ik}})K_{ij}^{-1}(K_{12}^{e_{12}}\cdots K_{ik}^{e_{ik}})^{-1}.\end{aligned}$$

Given (i)–(iii), we can make the following claims.

- If $e_{pq}\neq 0$ for some $(i,j+1)\le (p,q)\le (i,k-1)$ or if $e_{ik}\neq 0,-1$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ij}&(e_{12},\dots,e_{ik}+1,0,\dots,0)\\ &\cdot \gamma_{ik}(e_{12},\dots,e_{ij}+1,\dots,e_{ik},0,\dots,1,\dots,0)^{-1}\\ &\cdot \gamma_{ij}(e_{12},\dots,e_{ik},0,\dots,0)^{-1},\end{aligned}$$

  where 1 appears in the (k,j) entry of γ_{ik}.

- If $\{(p,q)\in I \mid (i,j+1)\le (p,q)\le (i,k-1),\ e_{pq}\neq 0\}=\phi$ and $e_{ik}=0$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ij}&(e_{12},\dots,e_{ij},0,\dots,1,\dots,0)\\ &\cdot \gamma_{ik}(e_{12},\dots,e_{ij}+1,0,\dots,1,\dots,0)^{-1},\end{aligned}$$

  where 1 appears in the (i,k) and (k,j) entries of γ_{ij} and γ_{ik}, respectively.

- If $\{(p,q)\in I \mid (i,j+1)\le (p,q)\le (i,k-1),\ e_{pq}\neq 0\}=\phi$ and $e_{ik}=-1$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ik}&(e_{12},\dots,e_{ij}+1,0,\dots,-1,\dots,1,\dots,0)^{-1}\\ &\cdot \gamma_{ij}(e_{12},\dots,e_{ij},0,\dots,-1,\dots,0)^{-1},\end{aligned}$$

  where -1 appears in the (i,k) entries of γ_{ik} and γ_{ij} while 1 appears in the (k,j) entry of γ_{ik}.

  We can derive the remaining three cases in a similar fashion.

Case 2: $i<k$ and $k<j$. Then $(i,k)<(i,j)<(k,j)$.

If $e_{pq}\neq 0$ for some $(k,j+1)\le (p,q)\le (n,n-1)$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ik}&(e_{12},\dots,e_{n(n-1)})\gamma_{ij}(e_{12},\dots,e_{ik}+1,\dots,e_{n(n-1)})\\ &\cdot \gamma_{kj}(e_{12},\dots,e_{ik}+1,\dots,e_{ij}+1,\dots,e_{n(n-1)})\\ &\cdot \gamma_{ik}(e_{12},\dots,e_{ij}+1,\dots,e_{kj}+1,\dots,e_{n(n-1)})^{-1}\\ &\cdot \gamma_{kj}(e_{12},\dots,e_{ij}+1,\dots,e_{n(n-1)})^{-1}\gamma_{ij}(e_{12},\dots,e_{n(n-1)})^{-1}.\end{aligned}$$

If $\{(p,q)\in I \mid (k,j+1)\le (p,q)\le (n,n-1),\ e_{pq}\neq 0\}=\phi$, then the following statements hold.

(i) If $e_{pq} \neq 0$ for some $(i, j+1) \leq (p,q) \leq (k, j-1)$ or if $e_{kj} \neq 0, -1$, then

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{ik}(e_{12}, \ldots, e_{kj}, 0, \ldots, 0)\gamma_{ij}(e_{12}, \ldots, e_{ik}+1, \ldots, e_{kj}, 0, \ldots, 0) \\ & \cdot \gamma_{ik}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{kj}+1, 0, \ldots, 0)^{-1} \\ & \cdot \gamma_{ij}(e_{12}, \ldots, e_{kj}, 0, \ldots, 0)^{-1}.\end{aligned}$$

(ii) If $\{(p,q) \in I \mid (i, j+1) \leq (p,q) \leq (k, j-1), e_{pq} \neq 0\} = \phi$ and $e_{kj} = -1$, then:

(a) if $e_{pq} \neq 0$ for some $(i, k+1) \leq (p,q) \leq (i, j-1)$ or if $e_{ij} \neq -1$, then

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{ik}(e_{12}, \ldots, e_{ij}, 0, \ldots, -1, \ldots, 0) \\ & \cdot \gamma_{ij}(e_{12}, \ldots, e_{ik}+1, \ldots, e_{ij}, 0, \ldots, -1, \ldots, 0) \\ & \cdot \gamma_{ik}(e_{12}, \ldots, e_{ij}+1, 0, \ldots, 0)^{-1} \\ & \cdot \gamma_{ij}(e_{12}, \ldots, e_{ij}, 0, \ldots, -1, \ldots, 0)^{-1},\end{aligned}$$

where -1 appears in the (k, j) entries of the γ_{ik} and γ_{ij};

(b) if $\{(p,q) \in I \mid (i, k+1) \leq (p,q) \leq (i, j-1), e_{pq} \neq 0\} = \phi$ and $e_{ij} = -1$, then

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{ik}(e_{12}, \ldots, e_{ik}, 0, \ldots, -1, \ldots, -1, \ldots, 0) \\ & \cdot \gamma_{ij}(e_{12}, \ldots, e_{ik}+1, 0, \ldots, -1, \ldots, -1, \ldots, 0) \\ & \cdot \gamma_{ij}(e_{12}, \ldots, e_{ik}, 0, \ldots, -1, \ldots, -1, \ldots, 0)^{-1},\end{aligned}$$

where -1 appears in the (i, j) and (k, j) entries of the γ_{ik} and γ_{ij}.

(iii) If $\{(p,q) \in I \mid (i, j+1) \leq (p,q) \leq (k, j-1), e_{pq} \neq 0\} = \phi$ and $e_{kj} = 0$, then:

(a) if $e_{pq} \neq 0$ for some $(i, k+1) \leq (p,q) \leq (i, j-1)$ or if $e_{ij} \neq 0, -1$, then

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{ik}(e_{12}, \ldots, e_{ij}, 0, \ldots, 0) \\ & \cdot \gamma_{ik}(e_{12}, \ldots, e_{ij}+1, 0, \ldots, 1, \ldots, 0)^{-1},\end{aligned}$$

where 1 appears in the (k, j) entry of γ_{ik};

(b) if $\{(p,q) \in I \mid (i, k+1) \leq (p,q) \leq (i, j-1), e_{pq} \neq 0\} = \phi$ and $e_{ij} = 0$, then

$$\tau(trt^{-1}) = \gamma_{ik}(e_{12}, \ldots, e_{ik}, 0, \ldots, 1, \ldots, 1, \ldots, 0)^{-1},$$

where 1 appears in the (i, j) and (k, j) entries;

(c) if $\{(p,q) \in I \mid (i, k+1) \leq (p,q) \leq (i, j-1), e_{pq} \neq 0\} = \phi$ and $e_{ij} = -1$, then

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{ik}(e_{12}, \ldots, e_{ik}, 0, \ldots, -1, \ldots, 0) \\ & \cdot \gamma_{ik}(e_{12}, \ldots, e_{ik}, 0, \ldots, 1, \ldots, 0)^{-1},\end{aligned}$$

where -1 appears in the (i, j) entry of γ_{ik} and 1 appears in the (k, j) entry of γ_{ik}.

Case 3: $k < i$ and $j < k$. Then $(k, j) < (i, j) < (i, k)$.

If $e_{pq} \neq 0$ for some $(i, k+1) \leq (p, q) \leq (n, n-1)$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ik}&(e_{12}, \ldots, e_{n(n-1)})\gamma_{ij}(e_{12}, \ldots, e_{ik}+1, \ldots, e_{n(n-1)}) \\ &\cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{ik}+1, \ldots, e_{n(n-1)}) \\ &\cdot \gamma_{ik}(e_{12}, \ldots, e_{kj}+1, \ldots, e_{ij}+1, \ldots, e_{n(n-1)})^{-1} \\ &\cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{n(n-1)})^{-1}\gamma_{ij}(e_{12}, \ldots, e_{n(n-1)})^{-1}.\end{aligned}$$

If $\{(p, q) \in I \mid (i, k+1) \leq (p, q) \leq (n, n-1), e_{pq} \neq 0\} = \phi$, then the following statements hold.

(i) If $e_{pq} \neq 0$ for some $(i, j+1) \leq (p, q) \leq (i, k-1)$ or $e_{ik} \neq 0, -1$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ij}&(e_{12}, \ldots, e_{ik}+1, 0, \ldots, 0) \\ &\cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{ik}+1, 0, \ldots, 0) \\ &\cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, \ldots, e_{ik}, 0, \ldots, 0)^{-1} \\ &\cdot \gamma_{ij}(e_{12}, \ldots, e_{ik}, 0, \ldots, 0)^{-1}.\end{aligned}$$

(ii) If $\{(p, q) \in I \mid (i, j+1) \leq (p, q) \leq (i, k-1), e_{pq} \neq 0\} = \phi$ and $e_{ik} = -1$, then:

(a) if $e_{pq} \neq 0$ for some $(k, j+1) \leq (p, q) \leq (i, j-1)$ or if $e_{ij} \neq -1$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{kj}&(e_{12}, \ldots, e_{ij}+1, 0, \ldots, 0) \\ &\cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, 0, \ldots, -1, \ldots, 0)^{-1} \\ &\cdot \gamma_{ij}(e_{12}, \ldots, e_{ij}, 0, \ldots, -1, \ldots, 0)^{-1},\end{aligned}$$

where -1 appears in the (i, k) entries of γ_{kj} and γ_{ij};

(b) if $\{(p, q) \in I \mid (k, j+1) \leq (p, q) \leq (i, j-1), e_{pq} \neq 0\} = \phi$ and $e_{ij} = -1$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{kj}&(e_{12}, \ldots, e_{kj}, 0, \ldots, -1, \ldots, 0)^{-1} \\ &\cdot \gamma_{ij}(e_{12}, \ldots, e_{kj}, 0, \ldots, -1, \ldots, -1, \ldots, 0)^{-1},\end{aligned}$$

where -1 appears in the (i, j) and (i, k) entries of γ_{ij} and in the (i, k) entry of γ_{kj}.

(iii) If $\{(p, q) \in I \mid (i, j+1) \leq (p, q) \leq (i, k-1), e_{pq} \neq 0\} = \phi$ and $e_{ik} = 0$, then:

(a) if $e_{pq} \neq 0$ for some $(k, j+1) \leq (p, q) \leq (i, j-1)$ or if $e_{ij} \neq -1$, then

$$\begin{aligned}\tau(trt^{-1}) = \gamma_{ij}&(e_{12}, \ldots, e_{ij}, 0, \ldots, 1, \ldots, 0) \\ &\cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, 0, \ldots, 1, \ldots, 0) \\ &\cdot \gamma_{kj}(e_{12}, \ldots, e_{ij}+1, 0, \ldots, 0)^{-1},\end{aligned}$$

where 1 appears in the (i, k) entries of γ_{ij} and γ_{kj};

(b) if $\{(p, q) \in I \mid (k, j+1) \leq (p, q) \leq (i, j-1), e_{pq} \neq 0\} = \phi$ and $e_{ij} = -1$, then

$$\tau(trt^{-1}) = \gamma_{ij}(e_{12},\dots,e_{kj},0,\dots,-1,\dots,1,\dots,0) \\ \cdot \gamma_{kj}(e_{12},\dots,e_{kj},0,\dots,1,\dots,0),$$

where 1 appears in the (i,k) entries of γ_{ij} and γ_{kj} and -1 appears in the (i,j) entry of γ_{ij}.

Case 4: $k < i$ and $k < j$. Then $(k,j) < (i,k) < (i,j)$.

If $e_{pq} \neq 0$ for some $(i,j+1) \leq (p,q) \leq (n,n-1)$, then

$$\tau(trt^{-1}) = \gamma_{ik}(e_{12},\dots,e_{n(n-1)})\gamma_{ij}(e_{12},\dots,e_{ik}+1,\dots,e_{n(n-1)}) \\ \cdot \gamma_{kj}(e_{12},\dots,e_{ik}+1,\dots,e_{ij}+1,\dots,e_{n(n-1)}) \\ \cdot \gamma_{ik}(e_{12},\dots,e_{kj}+1,\dots,e_{ij}+1,\dots,e_{n(n-1)})^{-1} \\ \cdot \gamma_{kj}(e_{12},\dots,e_{ij}+1,\dots,e_{n(n-1)})^{-1}\gamma_{ij}(e_{12},\dots,e_{n(n-1)})^{-1}.$$

If $\{(p,q) \in I \mid (i,j+1) \leq (p,q) \leq (n,n-1), e_{pq} \neq 0\} = \phi$, then the following statements hold.

(i) If $e_{pq} \neq 0$ for some $(i,k+1) \leq (p,q) \leq (i,j-1)$ or if $e_{ij} \neq 0,-1$, then

$$\tau(trt^{-1}) = \gamma_{ik}(e_{12},\dots,e_{ij},0,\dots,0)\gamma_{kj}(e_{12},\dots,e_{ik}+1,\dots,e_{ij}+1,0,\dots,0) \\ \cdot \gamma_{ik}(e_{12},\dots,e_{kj}+1,\dots,e_{ij}+1,0,\dots,0)^{-1} \\ \cdot \gamma_{kj}(e_{12},\dots,e_{ij}+1,0,\dots,0)^{-1}.$$

(ii) If $\{(p,q) \in I \mid (i,k+1) \leq (p,q) \leq (i,j-1), e_{pq} \neq 0\} = \phi$ and $e_{ij} = -1$, then:

(a) if $e_{pq} \neq 0$ for some $(k,j+1) \leq (p,q) \leq (i,k-1)$ or if $e_{ik} \neq 0,-1$, then

$$\tau(trt^{-1}) = \gamma_{ik}(e_{12},\dots,e_{ik},0,\dots,-1,\dots,0) \\ \cdot \gamma_{kj}(e_{12},\dots,e_{ik}+1,0,\dots,0) \\ \cdot \gamma_{kj}(e_{12},\dots,e_{ik},0,\dots,0)^{-1},$$

where -1 appears in the (i,j) entry of γ_{ik};

(b) if $\{(p,q) \in I \mid (k,j+1) \leq (p,q) \leq (i,k-1), e_{pq} \neq 0\} = \phi$ and $e_{ik} = -1$, then

$$\tau(trt^{-1}) = \gamma_{ik}(e_{12},\dots,e_{kj},0,\dots,-1,\dots,-1,\dots,0) \\ \cdot \gamma_{kj}(e_{12},\dots,e_{kj},0,\dots,-1,\dots,0)^{-1},$$

where -1 appears in the (i,k) and (i,j) entries of γ_{ik} and in the (i,k) entry of γ_{kj};

(c) if $\{(p,q) \in I \mid (k,j+1) \leq (p,q) \leq (i,k-1), e_{pq} \neq 0\} = \phi$ and $e_{ik} = 0$, then

$$\tau(trt^{-1}) = \gamma_{ik}(e_{12},\dots,e_{kj},0,\dots,-1,\dots,0) \\ \cdot \gamma_{kj}(e_{12},\dots,e_{kj},0,\dots,1,\dots,0),$$

where -1 appears in the (i,j) entry of γ_{ik} and 1 appears in the (i,k) entry of γ_{kj}.

(iii) If $\{(p,q) \in I \mid (i,k+1) \leq (p,q) \leq (i,j-1),\, e_{pq} \neq 0\} = \phi$ and $e_{ij} = 0$, then:

(a) if $e_{pq} \neq 0$ for some $(k,j+1) \leq (p,q) \leq (i,k-1)$ or if $e_{ik} \neq -1$, then

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{kj}(e_{12},\ldots,e_{ik}+1,0,\ldots,1,\ldots,0)\\ & \cdot \gamma_{ik}(e_{12},\ldots,e_{kj}+1,\ldots,e_{ik},0,\ldots,1,\ldots,0)^{-1}\\ & \cdot \gamma_{kj}(e_{12},\ldots,e_{ik},0,\ldots,1,\ldots,0)^{-1},\end{aligned}$$

where 1 appears in the (i,j) entries of the γ_{ik} and γ_{kj};

(b) if $\{(p,q) \in I \mid (k,j+1) \leq (p,q) \leq (i,k-1),\, e_{pq} \neq 0\} = \phi$ and $e_{ik} = -1$, then

$$\begin{aligned}\tau(trt^{-1}) = {} & \gamma_{kj}(e_{12},\ldots,e_{kj},0,\ldots,1,\ldots,0)\\ & \cdot \gamma_{ik}(e_{12},\ldots,e_{kj}+1,0,\ldots,-1,\ldots,1,\ldots,0)^{-1}\\ & \cdot \gamma_{kj}(e_{12},\ldots,e_{kj},0,\ldots,-1,\ldots,1,\ldots,0)^{-1},\end{aligned}$$

where -1 appears in the (i,k) entries of γ_{ik} and γ_{kj} and 1 appears in the (i,j) entries of γ_{ik} and γ_{kj}.

3.2. Infinitely Many Linearly Independent Elements in $(\mathrm{P}\Sigma_n^{(2)})^{\mathrm{ab}}$

In this section we detect infinitely many linearly independent elements in $(\mathrm{P}\Sigma_n^{(2)})^{\mathrm{ab}}$ using the presentation for $\mathrm{P}\Sigma_n^{(2)}$ obtained in Section 3.1. Then we show that $(\mathrm{P}\Sigma_n^{(k)})^{\mathrm{ab}}$ for $k \geq 3$ also contains infinitely many linearly independent elements. Finally, we confirm that for $k \geq 2$, the abelianization of each of the subgroups $\mathcal{P}_n(k)$ of the Johnson filtration of $\mathrm{P}\Sigma_n$ has the same property.

To begin, let

$$A := \mathrm{Span}_{\mathbf{Z}}\{b_{ij}(e,e') \mid 1 \leq i < j \leq n,\ e \in \mathbf{Z},\ e' \in \mathbf{Z} \setminus \{0\}\}.$$

Clearly, A is a free abelian group of infinite rank. Because $N = \varphi^{-1}(\mathrm{P}\Sigma_n^{(2)})$ is a free group with basis $\mathfrak{E}$, we can define a surjective homomorphism $\Psi'\colon N \to A$ by

$$\Psi'(\gamma_{ij}(e_{12},\ldots,e_{n(n-1)})) = \begin{cases} b_{ij}(e_{ij},e_{ji}) & \text{if } i < j \text{ and } e_{ji} \neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Then, by the results obtained in Section 3.1, it is easily checked that $\Psi'(\tau(trt^{-1})) = 0$ for any $t \in T$ and $r =$ (R1), (R2), (R3). This shows that Ψ' induces a surjective homomorphism

$$\Psi\colon \mathcal{P}_n(2) \to A.$$

Since the target of Ψ is abelian, we know that Ψ factors through the abelianization $\mathcal{P}_n(2)^{\mathrm{ab}}$ of $\mathcal{P}_n(2)$. Hence we obtain the following result.

THEOREM 3.1. *For any $n \geq 3$, $(\mathrm{P}\Sigma_n^{(2)})^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

As a corollary, we have this statement.

COROLLARY 3.2. *For any $n \geq 3$, $\mathrm{P}\Sigma_n^{(2)}$ is not finitely generated.*

Next, let us consider $\mathrm{P\Sigma}_n^{(k)}$ for $k \geq 3$. For $1 \leq i < j \leq n$, $e \in \mathbf{Z}$, and $e' \in \mathbf{Z}_{\geq 1}$, let

$$\alpha_{ij}(e,e') := K_{ij}^{e}[K_{ji},[K_{ji},\ldots,[K_{ji},K_{ij}]]\cdots]K_{ij}^{-e} \quad \text{and}$$

$$\beta_{ij}(e,e') := \gamma_{ij}(0,\ldots,e,\ldots,e',\ldots,0) = K_{ij}^{e}K_{ji}^{e'}K_{ij}K_{ji}^{-e'}K_{ij}^{-(e+1)},$$

where K_{ji} appears e' times in the definition of $\alpha_{ij}(e,e')$.

Here we construct one lemma. In general, for any group G and $x, y \in G$, let

$$\theta_{e'}(x,y) := [x,[x,\ldots,[x,y]]\cdots] \in G$$

for $e' \geq 1$, where x appears e' times in the commutator. Then we have the following lemma.

LEMMA 3.3. *With notation as before, $[x^{e'},y] \in G$ may be written as*

$$\theta_{e'}(x,y)\theta_{e'_1}(x,y)\cdots\theta_{e'_p}(x,y)$$

in G for some $1 \leq e'_1,\ldots,e'_p \leq e'-1$.

Proof. We proceed by induction on e'. If $e' = 1$ then it is obvious that $[x,y] = \theta_1(x,y)$, so assume that $e' \geq 2$. Using the commutator formula

$$[ab,c] = [a,[b,c]][b,c][a,c],$$

we see that

$$[x^{e'},y] = [x^{e'-1},[x,y]][x,y][x^{e'-1},y].$$

Hence, by the inductive hypothesis, we have

$$\begin{aligned}[x^{e'},y] = {} & \theta_{e'-1}(x,[x,y])\theta_{e'_1}(x,[x,y])\cdots\theta_{e'_p}(x,[x,y])\theta_1(x,y) \\ & \cdot\theta_{e'-1}(x,y)\theta_{e''_1}(x,y)\cdots\theta_{e''_q}(x,y)\end{aligned}$$

for some $1 \leq e'_j, e''_j \leq e'-2$. On the other hand, since $\theta_{e-1}(x,[x,y]) = \theta_e(x,y)$ for any $e \geq 2$, we obtain the required result. This completes the proof. □

Now we consider a relation between $\alpha_{ij}(e,e')$ and $\beta_{ij}(e,e')$.

LEMMA 3.4. *For any $1 \leq i < j \leq n$, $e \in \mathbf{Z}$, and $e' \in \mathbf{Z}_{\geq 1}$, there exist some $1 \leq e'_1,\ldots,e'_p \leq e'-1$ such that*

$$\alpha_{ij}(e,e') = \beta_{ij}(e,e')\beta_{ij}(e,e'_1)^{d_1}\cdots\beta_{ij}(e,e'_p)^{d_p} \quad \text{for } d_j = \pm 1.$$

Proof. We prove this lemma by induction on e'. If $e' = 1$ then clearly $\alpha_{ij}(e,1) = \beta_{ij}(e,1)$, so assume $e' \geq 2$. By Lemma 3.3, we have

$$\begin{aligned}\beta_{ij}(e,e') &= K_{ij}^{e}[K_{ji}^{e'},K_{ij}]K_{ij}^{-e} \\ &= K_{ij}^{e}\theta_{e'}(K_{ji},K_{ij})\theta_{e'_1}(K_{ji},K_{ij})\cdots\theta_{e'_p}(K_{ji},K_{ij})K_{ij}^{-e} \\ &= \alpha_{ij}(e,e')\alpha_{ij}(e,e'_1)\cdots\alpha_{ij}(e,e'_p)\end{aligned}$$

for some $1 \leq e'_j \leq e'-1$. Using the inductive hypothesis then yields the required result, completing the proof. □

Next we prove our main theorem.

THEOREM 3.5. *For $n \geq 3$ and $k \geq 3$, $\mathrm{P\Sigma}_n^{(k)\mathrm{ab}}$ contains infinitely many linearly independent elements.*

Proof. For any $d \geq k$, consider $d-k+2$ elements

$$\alpha_{ij}(e,k-1), \alpha_{ij}(e,k), \ldots, \alpha_{ij}(e,d)$$

in $\mathrm{P}\Sigma_n^{(k)}$ for any $e \in \mathbf{Z}$. We denote by

$$c_{ij}(e,k-1), c_{ij}(e,k), \ldots, c_{ij}(e,d)$$

the respective images of $\alpha_{ij}(e,k-1), \alpha_{ij}(e,k), \ldots, \alpha_{ij}(e,d)$ by the composition map $\Psi_k' \colon \mathrm{P}\Sigma_n^{(k)} \hookrightarrow \mathrm{P}\Sigma_n^{(2)} \xrightarrow{\Psi} A$.

We will show that $c_{ij}(e,k-1), \ldots, c_{ij}(e,d)$ are linearly independent in A. Assume that

$$a_{k-1}c_{ij}(e,k-1) + \cdots + a_d c_{ij}(e,d) = 0$$

for $a_{k-1}, \ldots, a_d \in \mathbf{Z}$. Then, by Lemma 3.4, it follows that the coefficient of $b_{ij}(e,d)$ is exactly a_d and hence $a_d = 0$. This implies

$$a_{k-1}c_{ij}(e,k-1) + \cdots + a_{d-1}c_{ij}(e,d-1) = 0.$$

By the same argument as before, we can show that $a_{d-1} = a_{d-2} = \cdots = a_{k-1} = 0$ recursively. Therefore, $c_{ij}(e,k-1), \ldots, c_{ij}(e,d)$ are linearly independent in A.

Since Ψ_k' factors through the abelianization $(\mathrm{P}\Sigma_n^{(k)})^{\mathrm{ab}}$ of $\mathrm{P}\Sigma_n^{(k)}$, we see that $(\mathrm{P}\Sigma_n^{(k)})^{\mathrm{ab}}$ contains a free abelian group of rank $d-k+2$. Yet because we can take $d \geq k$ arbitrarily, $(\mathrm{P}\Sigma_n^{(k)})^{\mathrm{ab}}$ must contain a free abelian group of infinite rank. This completes the proof. $\square$

We also have the following statement.

COROLLARY 3.6. *For $n \geq 3$ and $k \geq 3$, $\mathrm{P}\Sigma_n^{(k)}$ is not finitely generated.*

Finally, we consider the Johnson filtration of $\mathrm{P}\Sigma_n$.

COROLLARY 3.7. *For $n \geq 3$ and $k \geq 3$, $\mathcal{P}_n(k)^{\mathrm{ab}}$ contains infinitely many linearly independent elements.*

Proof. Consider a homomorphism $\Psi : \mathcal{P}_n(k) \hookrightarrow \mathrm{P}\Sigma_n^{(2)} \xrightarrow{\Psi} A$ that factors through the abelianization $\mathcal{P}_n(k)^{\mathrm{ab}}$ of $\mathcal{P}_n(k)$. In light of the proof of Theorem 3.5, we see that for any $d \geq k$ the images of $\alpha_{ij}(e,k-1), \ldots, \alpha_{ij}(e,d) \in \mathrm{P}\Sigma_n^{(k)} \subset \mathcal{P}_n(k)$ by Ψ_k are linearly independent in A. Hence we obtain the required result, completing the proof. $\square$

Consequently, we obtain our final result as follows.

COROLLARY 3.8. *For $n \geq 3$ and $k \geq 3$, $\mathcal{P}_n(k)$ is not finitely generated.*

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