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Geometries of Lines and Conics on the Quintic del Pezzo 3-fold and Its Application to Varieties of Power Sums

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1. Introduction

1.1. Varieties of Power Sums

The problem of representing a homogeneous form as a sum of powers of linear forms has been studied since the last decades of the 19th century. This is called the *Waring problem* for a homogeneous form. We are interested in the study of the global structure of a suitable compactification of the variety parameterizing all such representations of a homogeneous form. A precise definition of the claimed compactification is the following.

Definition 1.1.1. Let V be a $(v+1)$-dimensional vector space and let $F \in S^m \check{V}$ be a homogeneous form of degree m on V, where $\check{V}$ is the dual vector space of V. Let

$$\mathrm{VSP}(F,n)^o := \{(H_1,\dots,H_n) \mid H_1^m + \cdots + H_n^m = F\} \subset \mathrm{Hilb}^n(\mathbb{P}_*\check{V}).$$

The closed subset $\mathrm{VSP}(F,n) := \overline{\mathrm{VSP}(F,n)^o}$ is called the *varieties of power sums* of F.

Sometimes $\mathbb{P}_*\check{V}$ will be denoted by $\check{\mathbb{P}}^v$.

As far as we know, the first global descriptions of positive-dimensional VSPs were given by Mukai.

1.2. Mukai's Result

Let A_{22} be a smooth prime Fano 3-fold of genus 12—namely, a smooth projective 3-fold such that $-K_{A_{22}}$ is ample, the class of $-K_{A_{22}}$ generates $\mathrm{Pic}\,A_{22}$, and the genus $g(A_{22}) := (-K_{A_{22}})^3/2 + 1$ is equal to 12. The linear system $|-K_{A_{22}}|$ embeds A_{22} into $\mathbb{P}^{13}$.

Mukai discovered the following remarkable theorem [M1; M2].

Theorem 1.2.1. *Let $\{F_4 = 0\} \subset \mathbb{P}^2$ be a general plane quartic curve. Then*

(1) $\mathrm{VSP}(F_4,6) \subset \mathrm{Hilb}^6\check{\mathbb{P}}^2$ *is an A_{22}; and, conversely,*

(2) *every general A_{22} is of this form.*

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Mukai's motivation to discover this result was a characterization of a general A_{22}. For this purpose, he noticed that the Hilbert scheme of lines on a general $A_{22} \subset \mathbb{P}^{13}$ is isomorphic to a smooth plane quartic curve $\mathcal{H}_1 \subset \mathbb{P}^2$ (the notation $\mathbb{P}^2$ will be compatible with $\check{\mathbb{P}}^2$ in Theorem 1.2.1). He wanted to recover A_{22} by $\mathcal{H}_1$; for this, one more datum was necessary. In fact, he proved that the correspondence on $\mathcal{H}_1 \times \mathcal{H}_1$ defined by intersections of lines on A_{22} gives an ineffective theta characteristic θ on $\mathcal{H}_1$. We recall that an *ineffective theta characteristic* θ is a sheaf without global sections and such that the tensor product by itself gives the canonical sheaf of the curve. In Mukai's case, θ is constructed so that the following two sets in $\mathcal{H}_1 \times \mathcal{H}_1$ coincide:

$$\{(l,m) \mid l \cap m \neq \emptyset,\ l \neq m\} = \{(l,m) \mid h^0(\theta + l - m) > 0\}.$$

Now a deep and beautiful result of G. Scorza asserts that, associated to the pair $(\mathcal{H}_1, \theta)$, there exists another plane quartic curve $\{F_4 = 0\}$ in the same ambient plane as $\mathcal{H}_1$. (By way of saluting Scorza, $\{F_4 = 0\}$ is called the *Scorza quartic.*) Then, finally, Mukai proved that A_{22} is recovered as $\mathrm{VSP}(F_4, 6)$. This is the result (2) of Theorem 1.2.1. We recall also that, since the number of the moduli of A_{22} is equal to $\dim \mathcal{M}_4 = 6$, (1) follows from (2).

As it turns out, the geometries of lines on A_{22} is the main ingredient of Mukai's theorem—although this is not evident from the statement. Actually the geometry of conics is also deeply related. Indeed, Mukai observed that conics on A_{22} are parameterized by the plane $\mathcal{H}_2$ and that $\mathcal{H}_2$ is naturally considered as the plane $\check{\mathbb{P}}^2$ dual to $\mathbb{P}^2$ since, for a conic q on A_{22}, the lines intersecting q form a hyperplane section of $\mathcal{H}_1$. Further, he showed that the six points $H_1, \ldots, H_6$ such that $(H_1, \ldots, H_6) \in \mathrm{VSP}^o(F_4, 6)$ correspond to six conics through one point of A_{22}.

1.3. Generalization

To generalize Mukai's theorem, we study the relation between the concept of varieties of power sums and the geometries of lines and conics of other classes of 3-folds.

To explain our generalization naturally, we describe the famous double projection of A_{22} from a line (due to Iskovskih [Is2]) as follows:

$$\begin{tikzcd} & A' \arrow[dl, "f'"'] \arrow[r, dashed] & A \arrow[dr, "f"] & \\ A_{22} & & & B, \end{tikzcd}$$

where

- f' is the blow-up along a general line l,
- $A' \dashrightarrow A$ is a flop, and
- B is the smooth *quintic del Pezzo 3-fold*—namely, a smooth projective 3-fold such that $\mathrm{Pic}\, B \simeq \mathbb{Z}$ and $-K_B = 2H$ for H the ample generator of $\mathrm{Pic}\, B$ and with $H^3 = 5$. It is well known that the linear system $|H|$ embeds B into $\mathbb{P}^6$. Finally,

- f is the blow-up along a smooth rational curve C of degree 5, where the degree is measured by H.

Moreover, it is known that a general line on A_{22} is mapped to a general line on B intersecting C and that a general conic on A_{22} is mapped to a general conic on B intersecting C twice (these facts are easy to show because the exceptional divisor of f is the strict transform of the unique hyperplane section vanishing along l with multiplicity 3; see [Is1, Cor. 6.6, p. 513] for details). We can likewise switch from A_{22} to the pair (B, C).

The latter situation is generalizable by considering the following objects: (i) a general smooth rational curve C of degree d with $d \geq 5$ (mainly $d \geq 6$) on B; and (ii) the sets of the secant lines of C and of the multi-secant conics of C, respectively (see Section 2.2 for the construction of such a C). In this situation, we generalize Mukai's Theorem 1.2.1(2) as follows. Let $f\colon A \to B$ be the blow-up along C, and let $\rho\colon \tilde{A} \to A$ be the blow-up of A along the strict transforms β_i' of bi-secant lines β_i of C on B. Then there is a finite birational morphism from $\tilde{A}$ to $\mathrm{VSP}\big(F_4, \binom{d-1}{2}\big)$, where F_4 is a certain quartic homogeneous form whose $d-2$ variable is constructed from the geometries of multi-secant conics of C (see Theorem 1.5.1 for a more precise statement).

We also describe Mukai's Theorem 1.2.1(2) from our point of view in Appendix A.

1.4. Marked Lines and Marked Conics

Our generalization of Mukai's Theorem 1.2.1(2) is derived from geometries of the secant lines of C and of the multi-secant conics of C. It turns out that the latter are themselves interesting from the classical algebro-geometric point of view; we study them in detail in Section 4. To be precise, we introduce the following definition.

Definition 1.4.1. (1) A pair (l, t) of a line l on B and a point $t \in C \cap l$ is called a *marked line*.

(2) A pair consisting of a conic q on B and a 0-dimensional subscheme $\eta \subset C$ of length 2 contained in $q|_C$ is called a *marked conic.*

For marked lines we prove the following statement.

Proposition 1.4.2. *Marked lines are parameterized by a smooth trigonal canonical curve $\mathcal{H}_1$ of genus $d-2$.*

See the Section 4.1 for the proof, which we sketch here. A classically known geometric fact is that there are three lines (counted with multiplicities) through a point of B (see Section 2.1). This gives the triple cover $\mathcal{H}_1 \to C$ such that $(l, t) \mapsto t$. Moreover, points where "special lines" pass through form a divisor in $|2H|$, and the intersection of this divisor and C is nothing but the branch locus of this triple cover. We can show that all ramifications are simple. Thus, by the Riemann–Hurwitz formula

$$2g(\mathcal{H}_1) - 2 = 3(-2) + 2d,$$

we obtain

$$g(\mathcal{H}_1) = d - 2.$$

For marked conics we prove the following detailed structure theorem of their parameter space, which is one of the main results in this paper.

THEOREM 1.4.3. *If $d \geq 6$, then marked conics are parameterized by a so-called White surface $\mathcal{H}_2$ obtained by blowing up $S^2C \simeq \mathbb{P}^2$ at $\binom{d-2}{2}$ points. The surface $\mathcal{H}_2$ is embedded by the linear system $\left|(d-3)h - \sum_{i=1}^{s} e_i\right|$ into $\check{\mathbb{P}}^{d-3}$, where h is the pull-back of a line, e_i are the exceptional curves of $\mathcal{H}_2 \to \mathbb{P}^2$, and $s := \binom{d-2}{2}$.*

Here we use the notation $\check{\mathbb{P}}^{d-3}$ because the ambient projective spaces of $\mathcal{H}_1$ and $\mathcal{H}_2$ are reciprocally dual, as in Mukai's case. If $d = 6$ then $\mathcal{H}_2$ is a cubic surface. Gimigliano [Gi] showed that, in general, $\mathcal{H}_2$ is the intersection of cubics.

The proof of Theorem 1.4.3 is more involved than that of Proposition 1.4.2. See Corollary 4.2.10 and Theorem 4.2.11 for the proof, which again we sketch here. The morphism $\mathcal{H}_2 \to \mathbb{P}^2$ is just a natural one $\mathcal{H}_2 \to S^2C \simeq \mathbb{P}^2$ mapping $(q, \eta) \mapsto \eta$. Let β_i be a bi-secant line of C. We can show that there exist $s := \binom{d-2}{2}$ bi-secant lines of C (see Corollary 4.1.2). Then, for the length-2 subscheme $\beta_i|_C$, there exist infinitely many marked conics $(\beta_i \cup \alpha, \beta_i|_C)$, where the α are lines intersecting β_i and it is known that such α form a 1-dimensional family (see Proposition 2.1.3(5)). This explains why $\mathcal{H}_2 \to S^2C$ is the blow-up at s points, which are $\beta_i|_C \in S^2C$. Moreover, the birationality of $\mathcal{H}_2 \to \mathbb{P}^2$ follows because there exists a unique conic on B through two points t_1 and t_2 if there is no line on B through t_1 and t_2 (see Corollary 3.2.1).

Marked lines and marked conics are necessary yet are suitable for an intuitive understanding of $\mathcal{H}_1$ and $\mathcal{H}_2$ as just described. However, we switch to other objects (lines and conics on the blow-up A of B along C) because they are suitable for investigating their intersections. We consider the blow-up $f: A \to B$ of B along C. We say that a connected curve $l \subset A$ is a *line* on A if $-K_A \cdot l = 1$ and $E_C \cdot l = 1$, where E_C is the exceptional divisor of f. We say also that a connected and reduced curve $q \subset A$ is a *conic* on A if $-K_A \cdot q = 2$ and $E_C \cdot q = 2$.

Then we show that all the geometries of marked lines and of marked conics on B can be reinterpreted on A. In fact, we have as an independent result that for lines on A as described previously, the Hilbert scheme of lines on A is isomorphic to $\mathcal{H}_1$ (see Corollary 4.1.8). We also show that $\mathcal{H}_2$ is the normalization of the Hilbert scheme of conics on A and that the normalization morphism is injective. In particular, $\mathcal{H}_2$ parameterizes conics on A in a one-to-one way (see Corollary 4.2.10). This reinterpretation of $\mathcal{H}_1$ and $\mathcal{H}_2$ provides us with a flexible language that enables us to switch from B to A depending on the situation.

1.5. Construction of the Quartic Form F_4

As Mukai did, we can define an ineffective theta characteristic θ on $\mathcal{H}_1$ and construct the Scorza quartic hypersurface $\{F_4 = 0\}$ associated to this in the sense of [DK, Sec. 9]. This quartic hypersurface lives in the projective space $\mathbb{P}^{d-3} \supset \mathcal{H}_1$.

Yet because this construction is rather indirect in our context, we shall give a more direct construction of F_4 without introducing a theta characteristic on $\mathcal{H}_1$. In [TZ1] we show the quartic so constructed is actually Scorza.

For the construction of the quartic $\{F_4 = 0\}$, we make use of conics on A rather than lines on A. Indeed, assuming $d \geq 6$, consider the locus $D_l \subset \mathcal{H}_2$ parameterizing the conics on A that intersect a fixed line l on A. The locus D_l turns out to be a divisor that is linearly equivalent to $(d-3)h - \sum_{i=1}^{s} e_i$ on $\mathcal{H}_2$. Moreover, $|D_l|$ is very ample and embeds $\mathcal{H}_2$ in $\check{\mathbb{P}}^{d-3}$ (see Theorem 4.2.11(1)). Let

$$\mathcal{D}_2 := \{(q_1, q_2) \in \mathcal{H}_2 \times \mathcal{H}_2 \mid q_1 \cap q_2 \neq \emptyset\}$$

and denote by D_q the fiber of $\mathcal{D}_2 \to \mathcal{H}_2$ over a point q. It is easy to verify that $D_q \sim 2D_l = \mathcal{O}_{\mathcal{H}_2}(2)$. By the seesaw theorem, we have $\mathcal{D}_2 \sim p_1^* D_q + p_2^* D_q$. Since $\mathcal{H}_2$ is projectively Cohen–Macaulay and is not contained in a quadric (Theorem 4.2.11(4)), it follows that $H^0(\mathcal{H}_2 \times \mathcal{H}_2, \mathcal{D}_2) \simeq H^0(\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}, \mathcal{O}(2,2))$. Thus $\mathcal{D}_2$ is the restriction of a unique $(2,2)$-divisor $\mathcal{D}_2'$ on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$. Since $\mathcal{D}_2'$ is symmetric, we may assume that its equation $\tilde{\mathcal{D}}_2$ is also symmetric. By restricting $\tilde{\mathcal{D}}_2$ to the diagonal, we obtain a quartic hypersurface $\{\check{F}_4 = 0\}$ in $\check{\mathbb{P}}^{d-3}$. We can show that $\check{F}_4$ is nondegenerate in the sense of [D]; then there exists a unique quartic hypersurface $\{F_4 = 0\}$ in $\mathbb{P}^{d-3}$ that is dual to $\check{F}_4$ in the sense of [D] (see Appendix B).

Now we state our main result.

Theorem 1.5.1. *Let $f\colon A \to B$ be the blow-up along C, and let $\rho\colon \tilde{A} \to A$ be the blow-up of A along the strict transforms β_i' of $\binom{d-2}{2}$ bi-secant lines β_i of C on B. Then there is a finite birational morphism from $\tilde{A}$ to* VSP(F_4, n), *where $n := \binom{d-1}{2}$. Moreover, the image is uniquely determined by the incident variety $\mathcal{D}_2$ and is an irreducible component of*

$$\mathrm{VSP}(F_4, n; \mathcal{H}_2) := \overline{\{(H_1, \ldots, H_n) \mid H_i \in \mathcal{H}_2\}} \subset \mathrm{VSP}(F_4, n).$$

In particular, $\tilde{A}$ is reconstructed from VSP$(F_4, n; \mathcal{H}_2)$.

In Section 5 we prove this theorem in several steps. We remark that using conics on A is necessary not only to define the quartic F_4 but also to describe the subvariety of VSP(F_4, n).

Actually, the number n is equal to the number of multi-secant conics of C through a general point of B (see Proposition 3.2.6).

1.6. Projections of B

As in our explanation about Mukai's Theorem 1.2.1, the number of conics on V_{22} through one point is important. In our case, we need to count the number of conics on A through a general point of A or, equivalently, the number of multi-secant conics of C through a general point of B. For this, we use a special rational map from B called the *double projection from a point* of B, and Section 3 is devoted to applications of such a projection. Another application, Proposition 3.2.3, is essential; its refinement—Proposition 5.1.1 and Corollary 5.1.5—constitute rough forms of our main result (Theorem 1.5.1), as we explain next.

1.7. On Finiteness of the Number of Conics on A through a Point

The content of Mukai's Theorem 1.2.1(2) is (i) that there exists an injective morphism $V_{22} \to \mathrm{Hilb}^6 \check{\mathbb{P}}^2$ attaching to $v \in V_{22}$ the 0-dimensional subscheme of $\check{\mathbb{P}}^2$ of length 6 corresponding to the six conics through v and (ii) that it is an isomorphism onto its image. To prove Theorem 1.5.1, we would like to construct similarly a finite birational morphism $\Phi\colon \tilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$. The argument for this construction runs throughout the paper: it starts from Proposition 3.2.3; the result is refined in Proposition 4.2.13; and the final form is reflected in Proposition 5.1.1. Essentially, the meaning of Φ is to attach to a point $a \in A$ the 0-dimensional subscheme of $\check{\mathbb{P}}^{d-3}$ of length n corresponding to the n conics on A through a. However, this is impossible because there certainly exists a point of A through which infinitely many conics pass. To remedy that situation, we must take the blow-up $\rho\colon \tilde{A} \to A$ along the strict transforms of bi-secant lines of C. More precisely, Φ attaches to a point $\tilde{a} \in \tilde{A}$ the 0-dimensional subscheme of $\check{\mathbb{P}}^{d-3}$ of length n corresponding to canonically chosen n conics on A through $\rho(\tilde{a})$. We shall explain this in further detail.

First, we argue on B and determine points of B through which finitely many multi-secant conics of C pass: Proposition 3.2.3 (and its restatement, Corollary 3.2.4) shows that, for a point of B not contained in C or its bi-secant lines, there exist a finite number of multi-secant conics of C passing through it. Second, we argue on A and refine this finiteness: Proposition 4.2.13 shows that there exist a finite number of conics on A passing through a point $a \in A$ if a is outside of the strict transforms of the bi-secant lines of C.

It turns out that there are infinitely many conics on A passing through a point in the strict transforms of bi-secant lines of C, but we can remedy this situation by taking the blow-up $\rho\colon \tilde{A} \to A$ along the strict transforms of bi-secant lines of C. We show in Proposition 5.1.1 that, for each $\tilde{a} \in \tilde{A}$, it is possible to choose n conics on A through $\rho(\tilde{a})$.

We consider this aspect of the geometry of $\tilde{A}$ to be rather nonobvious, since the statement holds for *any* point of $\tilde{A}$ and not just for a general point of $\tilde{A}$. Hence we cannot avoid the argument becoming quite delicate.

1.8. Some Consequences of This Paper

We conclude this introduction by pointing out that unifying all the geometrical objects recalled here lays the foundation for a new geometry of the moduli space $S_g^{+\mathrm{tr}}$ of couples (Γ, θ), where Γ is a smooth trigonal curve of genus g and θ is an ineffective theta characteristic. Grounded on these foundations are: [TZ1], where we show that the above quartics F_4 are exactly the Scorza quartics associated to general pairs of trigonal curves and ineffective theta characteristics and that this implies the existence of the Scorza quartics for *any* general pairs of canonical curves and ineffective theta characteristics (this is an affirmative answer to the conjecture of Dolgachev and Kanev in [DK, Sec. 9]); and [TZ2], where we show that the moduli space S_4^+ is rational.

The trigonal curve $\mathcal{H}_1$ and the White surface $\mathcal{H}_2$ continue to play important roles. In this paper, $\mathcal{H}_2$ plays the leading role and $\mathcal{H}_1$ a supporting role; the converse is true in [TZ1; TZ2].

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We use quite a lot of notation, which we summarize next.

GLOSSARY OF NOTATION.

- C or C_d: a general smooth rational curve of degree d on B constructed as in Section 2.2
- β_i: bi-secant lines of C $\left(i = 1, \dots, \frac{(d-2)(d-3)}{2}\right)$
- $s := \frac{(d-2)(d-3)}{2}$
- $\{p_{i1}, p_{i2}\} := \beta_i \cap C$ $(i = 1, \dots, s)$
- $f: A \to B$: the blow-up of B along C
- E_C: the f-exceptional divisor
- $\zeta_{ij} := f^{-1}(p_{ij})$ $(i = 1, \dots, s,\ j = 1, 2)$
- $l_{ij} := \beta_i' \cup \zeta_{ij}$ $(i = 1, \dots, s,\ j = 1, 2)$
- $\mathcal{H}_1$: the smooth curve parameterizing marked lines or lines on A
- $\mathcal{U}_1 \subset A \times \mathcal{H}_1$: the universal family of lines on A
- $\mathcal{H}_2$: the smooth surface parameterizing marked conics or conics on A
- $\mathcal{U}_2 \subset A \times \mathcal{H}_2$: the universal family of conics on A
- $e_i \subset \mathcal{H}_2$: the locus in $\mathcal{H}_2$ parameterizing marked conics containing β_i as an irreducible component $(i = 1, \dots, s)$
- $L_b \subset \mathcal{H}_2$: the locus in $\mathcal{H}_2$ whose general point corresponds to a bi-secant conic through a point $b \in B$
- $D_l \subset \mathcal{H}_2$: the locus in $\mathcal{H}_2$ parameterizing conics on A that intersect the line l on A
- $D_q \subset \mathcal{H}_2$: the locus in $\mathcal{H}_2$ parameterizing conics on A that intersect the conic q on A
- $\pi_{1b}: B_b \to B$: the blow-up of B at a point b
- E_b: the π_{1b}-exceptional divisor
- $\rho: \tilde{A} \to A$: the blow-up of A along the strict transforms β_i' of β_i $(i = 1, \dots, s)$
- $n := \frac{(d-1)(d-2)}{2}$

2. Rational Curves on the Quintic del Pezzo 3-fold *B*

Let V be a vector space with $\dim_{\mathbb{C}} V = 5$. The Grassmannian $G(2, V)$ embeds into $\mathbb{P}^9$, and we denote the image by $G \subset \mathbb{P}^9$. It is well-known that the quintic del Pezzo 3-fold (i.e., the Fano 3-fold B of index 2 and degree 5) can be realized as $B = G \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to G (see [Fu; Is1, Proof of Thm. 4.2(iii), pp. 511–514].

First we collect basic known facts on lines and conics on B. Let $\mathcal{H}_1^B$ and $\mathcal{H}_2^B$ denote the Hilbert scheme of lines and conics (respectively) on B.

2.1. Lines on B

Let $\pi : \mathbb{P} \to \mathcal{H}_1^B$ be the universal family of lines on B and let $\varphi\colon \mathbb{P} \to B$ be the natural projection. By [FN1, Lemma 2.3 and Thm. I], $\mathcal{H}_1^B$ is isomorphic to $\mathbb{P}^2$ and φ is a finite morphism of degree 3. In particular, the number of lines passing through a point is 3 when counted with multiplicities. We shall recall some basic facts about π and φ that will be used in the sequel, but first we fix some notation.

NOTATION 2.1.1. For an irreducible curve C on B, denote by $M(C)$ the locus (contained in $\mathbb{P}^2$) of lines intersecting C; namely, $M(C) := \pi(\varphi^{-1}(C))$ with reduced structure. Since φ is flat, $\varphi^{-1}(C)$ is purely 1-dimensional. If $\deg C \geq 2$, then $\varphi^{-1}(C)$ does not contain a fiber of π and so $M(C)$ is a curve. See Proposition 2.1.3 for the description of $M(C)$ when C is a line.

DEFINITION 2.1.2. A line l on B is called a *special line* if $\mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

REMARK. If l is not a special line on B, then $\mathcal{N}_{l/B} = \mathcal{O}_l \oplus \mathcal{O}_l$.

PROPOSITION 2.1.3. (1) *For the branched locus B_φ of $\varphi\colon \mathbb{P} \to B$, we have*:

(1-1) $B_\varphi \in |-K_B|$;
(1-2) $\varphi^* B_\varphi = R_1 + 2R_2$, *where $R_1 \simeq R_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and where $\varphi\colon R_1 \to B_\varphi$ and $\varphi\colon R_2 \to B_\varphi$ are injective.*

(2) *R_2 is contracted to a conic Q_2 by $\pi : \mathbb{P} \to \mathcal{H}_1^B$. Moreover, Q_2 is the branched locus of the finite double cover $\pi|_{R_1}\colon R_1 \to \mathcal{H}_1^B$.*

(3) *Q_2 parameterizes special lines.*

(4) *If l is a special line, then $M(l)$ is the tangent line to Q_2 at l. If l is not a special line, then $\varphi^{-1}(l)$ is the disjoint union of the fiber of π corresponding to l and the smooth rational curve dominating a line on $\mathbb{P}^2$. In particular, $M(l)$ is the disjoint union of a line and the point l.*

By abuse of notation, we denote by $M(l)$ the 1-dimensional part of $M(l)$ for any line l. Conversely, any line in $\mathcal{H}_1^B$ is of the form $M(l)$ for some line l.

(5) *The locus swept by lines intersecting l is a hyperplane section T_l of B whose singular locus is l. For every point b of $T_l \setminus l$, there exists exactly one line that*

belongs to $M(l)$ and passes through b. Moreover, if l is not special, then the normalization of T_l is $\mathbb{F}_1$ and the inverse image of the singular locus is the negative section of $\mathbb{F}_1$; if l is special, then the normalization of T_l is $\mathbb{F}_3$ and the inverse image of the singular locus is the union of the negative section and a fiber.

Proof. See [FN1, Sec. 2] and [Ili, Sec. 1]. □

By the proof of [FN1], we see that B is stratified according to the ramification of $\varphi\colon \mathbb{P} \to B$ as follows:

$$B = (B \setminus B_\varphi) \cup (B_\varphi \setminus C_\varphi) \cup C_\varphi,$$

where C_φ is a smooth rational normal sextic. If $b \in B \setminus B_\varphi$ then exactly three distinct lines pass through this sextic; if $b \in (B_\varphi \setminus C_\varphi)$ then exactly two distinct lines pass through it and one of them is special. Finally, C_φ is the locus of $b \in B$ through which C_φ passes only one line, which is special.

2.2. *Construction of Rational Curves C_d of Degree d on B*

Definition 2.2.1. Let C and γ be smooth curves on B. We say that γ is a *secant curve* of C if $C \cap \gamma \neq \emptyset$. Moreover, we say that γ is a *k-secant curve* (resp., a *multi-secant curve*) if $\gamma|_C$ is a 0-dimensional subscheme of length k (resp., of length ≥ 2). For $k = 1, 2, \ldots$ we say uni-secant, bi-secant, … instead.

We construct smooth rational curves of degree d on B by smoothing the union of a smooth rational curve of degree $d-1$ and one of its uni-secant lines.

Proposition 2.2.2. *There exists a smooth rational curve C_d of degree d on B such that*

(a) *a general line on B intersecting C_d is uni-secant,*
(b) *C_d is obtained as a smoothing of the union of a smooth rational curve C_{d-1} of degree $d-1$ on B and a general uni-secant line of it on B, and*
(c) *$\mathcal{N}_{C_d/B} \simeq \mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)$; in particular, $h^1(\mathcal{N}_{C_d/B}) = 0$ and $h^0(\mathcal{N}_{C_d/B}) = 2d$.*

Note that the Hilbert scheme of smooth rational curves on B of degree d is smooth at C_d and is of dimension $2d$.

Proof. We argue by induction on d.

If $d = 1$, then we have the assertion since $\mathcal{N}_{C_1/B} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for a general line C_1.

Now assume that C_{d-1} is a smooth rational curve of degree $d-1$ on B constructed inductively. By induction, a general secant line l of C_{d-1} on B is uni-secant. Let $Z := C_{d-1} \cup l$ and $\mathcal{N}_{Z/B} := \mathcal{H}om_{\mathcal{O}_B}(\mathcal{I}_Z, \mathcal{O}_B)$. By induction, the normal bundle of C_{d-1} satisfies (c). Thus, by $\mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and [HHi, Thm. 4.1], $h^1(\mathcal{N}_{Z/B}) = 0$ and, moreover, $Z := C_{d-1} \cup l$ is strongly smoothable; namely, we can find a smoothing C_d of Z with the smooth total space. By the upper semi-continuity theorem, $h^1(\mathcal{N}_{C_d/B}) = 0$, and $h^0(\mathcal{N}_{C_d/B}) = 2d$ by the Riemann–Roch theorem.

We check the form of the normal bundle of C_d. Let $\mathcal{N}_{C_d/B} := \mathcal{O}_{\mathbb{P}^1}(a_d) \oplus \mathcal{O}_{\mathbb{P}^1}(b_d)$ ($a_d \geq b_d$) for the smoothing C_d of Z. We show that $a_d = b_d = d-1$. It suffices to prove $h^0(\mathcal{N}_{Z/B}(-d)) = 0$. In fact, then, by the upper semi-continuous theorem, we have $h^0(\mathcal{N}_{C_d/B}(-d)) = 0$ and $a_d, b_d \leq d-1$. Thus, since $a_d + b_d = 2d-2$, we have $a_d = b_d = d-1$. Given that $\mathcal{N}_{C_{d-1}/B} = \mathcal{O}_{\mathbb{P}^1}(d-2) \oplus \mathcal{O}_{\mathbb{P}^1}(d-2)$, the equality $h^0(\mathcal{N}_{Z/B}(-d)) = 0$ is an easy consequence of the following three exact sequences, where $t := C_{d-1} \cap l$:

$$0 \to \mathcal{N}_{Z/B} \to \mathcal{N}_{Z/B}|_{C_{d-1}} \oplus \mathcal{N}_{Z/B}|_l \to \mathcal{N}_{Z/B} \otimes_{\mathcal{O}_B} \mathcal{O}_t \to 0;$$

$$0 \to \mathcal{N}_{C_{d-1}/B} \to \mathcal{N}_{Z/B}|_{C_{d-1}} \to T_t^1 \to 0;$$

$$0 \to \mathcal{N}_{l/B} \to \mathcal{N}_{Z/B}|_l \to T_t^1 \to 0.$$

We can show by induction that a general line m intersecting C_{d-1} does not intersect l, so m is a uni-secant line of $C_{d-1} \cup l$. This implies part (a) of the proposition for C_d by a deformation-theoretic argument.

The last assertion follows from part (c). □

COROLLARY 2.2.3. *Let C_d be a general smooth rational curve constructed as in Proposition 2.2.2. If $d = 5$, then C_5 is a normal rational curve and is contained in a unique hyperplane section S, which is smooth. If $d \geq 6$, then C_d is not contained in a hyperplane section.*

Proof. If $d = 5$, then we can construct a general C_5 on a smooth quintic del Pezzo surface S that is a hyperplane section of B. Indeed, C_5 is a member of the linear system $|4h - 2e_1 - 2e_2 - 2e_3 - e_4|$ on S, where S is the blow-up of $\mathbb{P}^2$ at four distinct points, h is the strict transform of a general line of $\mathbb{P}^2$, and e_i ($1 \leq i \leq 4$) are the exceptional curves. Thus C_5 is contained in the unique hyperplane section S of B. We may assume that C_6 is obtained as a smoothing of the union of C_5 and a uni-secant line of C_5 that is not contained in S. Hence C_6 is not contained in a hyperplane section. We also have the assertion for the cases where $d \geq 7$ by smoothing constructions of C_d and the assertion for C_6. □

DEFINITION 2.2.4. We inductively define $\mathcal{H}_d^B$ as the union of the components of the Hilbert scheme whose general point parameterizes a smooth rational curve of degree d on B obtained as a smoothing of the union of a general smooth rational curve of degree $d-1$ belonging to $\mathcal{H}_{d-1}^B$ and its general uni-secant line.

2.3. *Relations of a General C_d with Lines and Conics*

We study multi-secant lines and conics of a general $C_d \in \mathcal{H}_d^B$.

For multi-secant lines, we have two statements: Propositions 2.3.1 and 2.3.3.

PROPOSITION 2.3.1. *A general C_d as in Proposition 2.2.2 satisfies the following conditions:*

(1) *there exist no k-secant lines of C_d on B with $k \geq 3$;*
(2) *there exist at most finitely many bi-secant lines of C_d on B, and any one of them intersects C_d simply;*

(3) *bi-secant lines of C_d on B are mutually disjoint*;
(4) *neither a bi-secant line nor a line through the intersection point between a bi-secant line and C_d is a special line*;
(5) *there exist at most finitely many points b outside C_d such that all the lines through b intersect C_d, and such points exist outside bi-secant lines of C_d.*

Proof. We can prove the assertions by simple dimension counts based upon Proposition 2.2.2. We only give a proof for (1). We assume that $d \geq 4$ since otherwise we can verify the assertion easily. Let $\mathcal{D}$ be the closure of the set

$$\{(C_d, l) \mid C_d \cap l \text{ consists of three points}\} \subset \mathcal{H}_d^B \times \mathcal{H}_1^B.$$

Let $\pi_d \colon \mathcal{D} \to \mathcal{H}_d^B$ and $\pi_1 \colon \mathcal{D} \to \mathcal{H}_1^B$ be the natural morphisms induced by the projections. Since $\dim \mathcal{H}_d^B = 2d$, the claim follows if we show that $\dim_{\mathbb{C}} \mathcal{D} \leq 2d - 1$.

We therefore estimate $\dim_{\mathbb{C}} \mathrm{Hom}^{2d}(\mathbb{P}^1, B;\ p_i \mapsto s_i, i = 1, 2, 3)$ at π, where p_i $(i = 1, 2, 3)$ are fixed distinct points of $\mathbb{P}^1$, s_i $(i = 1, 2, 3)$ are fixed distinct points of B, π is a general point, and the degree is measured by $-K_B$. By Proposition 2.2.2(c), since $d \geq 4$ it follows that $h^0(\mathbb{P}^1, \pi^* T_B(-p_1 - p_2 - p_3)) = 2d - 6$ and $h^1(\mathbb{P}^1, \pi^* T_B(-p_1 - p_2 - p_3)) = 0$. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}^{2d}(\mathbb{P}^1, B, p_i \mapsto s_i, i = 1, 2, 3)_\pi = h^0(\pi^* T_B(-p_1 - p_2 - p_3)) = 2d - 6.$$

This implies that $\dim_{\mathbb{C}} \pi_1^{-1}(l) \leq 2d - 6 + 3 = 2d - 3$, since the three points s_1, s_2, s_3 can be chosen arbitrarily on l. Then $\dim_{\mathbb{C}} \mathcal{D} \leq 2d - 1$ because $\dim_{\mathbb{C}} \mathcal{H}_1^B = 2$. $\square$

NOTATION 2.3.2. The bi-secant lines of C_d are denoted by β_i, where $i = 1, \ldots, s$.

In the following proposition, we describe some more relations of C_d with lines on B that can be translated into the geometry of $\mathcal{H}_1^B$. More explicitly, we prove that $M(C_d)$ is sufficiently general if C_d is general (recall the notation in Section 2.1).

PROPOSITION 2.3.3. *A general C_d as in Proposition 2.2.2 satisfies the following conditions*:

(1) *C_d intersects B_φ simply*;
(2) *$M_d := M(C_d)$ intersects Q_2 simply*;
(3) *M_d is an irreducible curve of degree d with only simple nodes* (*recall that, in Proposition 2.1.3(4), we abused notation by denoting the 1-dimensional part of $\pi(\varphi^{-1}(C_1))$ by $M(C_1)$*);
(4) *for a general line l intersecting C_d, $M_d \cup M(l)$ has only simple nodes as its singularities*;
(5) *$M_d \cup M(\beta_i)$ has only simple nodes as its singularities.*

Proof. We show the assertion inductively by using the smoothing construction of C_d from the union of C_{d-1} and a general uni-secant line l of C_{d-1}.

For $d = 1$, the assertion follows from Proposition 2.1.3 if we let C_1 be a general line. By induction on d, assume that we have a smooth C_{d-1} $(d \geq 2)$ satisfying

(1)–(5). We verify that $C_{d-1} \cup l$ satisfies the following (1′)–(5′), which are suitable modifications of (1)–(5):

(1′) $C_{d-1} \cup l$ intersects B_φ simply, by (1) for C_{d-1} and the generality of l;
(2′) $M_{d-1} \cup M(l)$ intersects Q_2 simply, by (2) for C_{d-1} and the generality of l;
(3′) $M_{d-1} \cup M(l)$ is not irreducible but is of degree d and has only simple nodes, by (4) for C_{d-1};
(4′) $M_{d-1} \cup M(l) \cup M(m)$ has only simple nodes as its singularities for a general line m intersecting C_{d-1}.

Indeed, since m is also general, $M_{d-1} \cup M(m)$ has only simple nodes by (4) for C_{d-1}. Thus we need only prove that $M_{d-1} \cap M(l) \cap M(m) = \emptyset$—in other words, that there is no secant line of C_{d-1} intersecting both l and m. Fix a general l and move m. If there are secant lines r_m of C_{d-1} intersecting both l and m for general m, then r_m moves; hence $M(l) \subset M_{d-1}$, a contradiction.

(5′) For a bi-secant line β of $C_{d-1} \cup l$ (except for the lines through $C_{d-1} \cap l$), the curve $M_{d-1} \cup M(l) \cup M(\beta)$ has only simple nodes as its singularities.

Indeed, if β is a bi-secant line of C_{d-1}, then the assertion follows from (5) for C_{d-1} in a way similar to the proof of (4′). Suppose that β is a uni-secant line of C_{d-1} intersecting l. We have only to prove that there is no secant line of C intersecting both l and β. If there is such a line r, then l, β, and r pass through one point. But by Proposition 2.3.1(5), this does not occur for general l and β.

Thus, by a deformation-theoretic argument, we see that C_d satisfies (1)–(5). □

The following proposition addresses multi-secant conics of C. Its proof is similar to the proof of Proposition 2.3.1, so we omit it.

PROPOSITION 2.3.4. *A general C_d as in Proposition 2.2.2 satisfies the following conditions*:

(1) *there exist no k-secant conics of C_d with $k \geq 5$;*
(2) *there exist at most finitely many quadri-secant conics of C_d on B, and no quadri-secant conic is tangent to C_d;*
(3) *$q|_{C_d}$ has no point of multiplicity greater than 2 for any multi-secant conic q.*

3. Double Projection of B from a Point

3.1. Basic Facts

DEFINITION 3.1.1. Let b be a point of B. We call the rational map from B defined by the linear system of hyperplane sections singular at b *the double projection from b*. We denote by π_b this rational map from B.

PROPOSITION 3.1.2. *Let b be a point of B.*

(1) *The double projection from b and the projection $B \dashrightarrow \bar{B}_b$ from b fit into the following diagram*:

$$
\begin{tikzcd}
 & B_b \arrow[dl, "\pi_{1b}"'] \arrow[dr] & & B'_b \arrow[dl] \arrow[dr, "\pi_{2b}"] & \\
B & & \bar{B}_b & & \mathbb{P}^2.
\end{tikzcd}
\tag{3.1}
$$

Here the bottom rational map $B \dashrightarrow \mathbb{P}^2$ *is the double projection* π_b *from* b, π_{1b} *is the blow-up of* B *at* b, $B_b \dashrightarrow B'_b$ *is the flop of the strict transforms of lines through* b, *and* $\pi_{2b}\colon B'_b \to \mathbb{P}^2$ *is a* (*unique*) $\mathbb{P}^1$*-bundle structure.*

(2) *We denote by* E_b *the* π_{1b}*-exceptional divisor and by* E'_b *the strict transform of* E_b *on* B'_b,

$$L = H - 2E'_b \quad \text{and} \quad -K_{B'_b} = H + L,$$

where H *is the strict transform of a general hyperplane section of* B *and where* L *is the pull-back of a line on* $\mathbb{P}^2$.

(3) *Case* (a). *If* $b \notin B_\varphi$, *then the strict transforms* l'_i *of three lines* l_i *through* b *on* B_b *have the normal bundle* $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. *The flop* $B_b \dashrightarrow B'_b$ *is the Atiyah flop. In particular,* $E'_b \to E_b$ *is the blow-up at the three points* $E_b \cap l'_i$.

Case (b). *If* $b \in B_\varphi \setminus C_\varphi$, *then* $E_b \dashrightarrow E'_b$ *can be described as follows. Let* l *and* m *be two lines through* b, *where* l *is special and* m *is not special. Let* l' *and* m' *be the strict transforms of* l *and* m *on* B_b. *First blow up* E_b *at two points* $t_1 := E_b \cap l'$ *and* $t_2 := E_b \cap m'$, *and then blow up at a point* t_3 *on the exceptional curve* e *over* t_1. *Finally, contract the strict transform of* e *to a point. Then we obtain* E'_b (*this is a degeneration of case* (a)).

(4) *A fiber of* π_{2b} *not contained in* E'_b *is either the strict transform of a conic through* b *or the strict transform of a line not containing* b *but that intersects a line through* b. *The description of the fibers of* π_{2b} *contained in* E'_b *is as follows.*

Case (a). *If* $b \notin B_\varphi$, *then* $\pi_{2b}|_{E'_b}\colon E'_b \to \mathbb{P}^2$ *is the blow-down of the strict transforms of three lines connecting two of* $E_b \cap l'_i$; *that is,* $E_b \dashrightarrow \mathbb{P}^2$ *is the Cremona transformation.*

Case (b). *Assume that* $b \in B_\varphi \setminus C_\varphi$. *Then* $\pi_{2b}|_{E'_b}\colon E'_b \to \mathbb{P}^2$ *is the blow-down of the strict transforms of two lines*: *one is the line connecting* t_1 *and* t_2; *the other is the line whose strict transform passes through* t_3. *In this case,* $E_b \dashrightarrow \mathbb{P}^2$ *is a degenerate Cremona transformation.*

Proof. This is a standard result in the birational geometry of Fano 3-folds. See [FN2] for a treatment of the most difficult case in which $b \in C_\varphi$ for statements (3) and (4) of the proposition. □

3.2. Applications

A first application of the preceding operations is the following result, which we will use often.

Corollary 3.2.1. *Let* b_1 *and* b_2 *be two* (*possibly infinitely near*) *points on* B *such that there exists no line on* B *through them. Then there is a unique conic on* B *through* b_1 *and* b_2.

Proof. We doubly project B from b_1 as in Proposition 3.1.2(1). Then the assertion follows by the description of fibers of π_{2b_1} not contained in E'_{b_1}, as in Proposition 3.1.2(4). □

NOTATION 3.2.2. Consider the double projection π_b from b (see Proposition 3.1.2). Throughout the paper, we denote by C'_b, C''_b, and C_b the strict transforms of $C := C_d$ on B_b, B'_b, and $\mathbb{P}^2$, respectively.

The following proposition is one of the key results for proving Theorem 1.5.1. Its importance and difficulty stem from its holding not only for a general $b \in B$ but also for every $b \in B$.

PROPOSITION 3.2.3. *Let C_d be a general smooth rational curve of degree d on B constructed as in Proposition 2.2.2. Assume that $d \geq 5$. Then, for any point $b \in B$, the restriction of the double projection π_b to C_d is birational.*

Proof. We prove this by induction based on the construction of C_d from $C_{d-1} \cup l$, where l is a general uni-secant line of C_{d-1} on B.

First we prove the assertion for $d = 5$. Assume by way of contradiction that $\pi_b|_{C_5}$ is not birational for a point b. Then, since $C \dashrightarrow C_b$ is a composite of linear projections, it follows that C_b is a line or conic in $\mathbb{P}^2$. Let S be the pull-back of C_b by π_{2b}. If C_b is a line, then C_5 is contained in a singular hyperplane section that is the strict transform of S on B (recall that $B \dashrightarrow \mathbb{P}^2$ is the double projection from b). This contradicts Corollary 2.2.3. Assume that C_b is a conic. The only possibility is that $L \cdot C''_b = 4$ and $C''_b \to C_b$ is a double cover, since $L \cdot C''_b = \deg C_b \cdot \deg(C''_b \to C_b) \leq 5$. Because the flop does not change the intersection numbers between the canonical divisor and curves, we have $-K_{B'_b} \cdot C''_b = -K_{B_b} \cdot C'_b$. If $b \in C$, then $-K_{B'_b} \cdot C''_b = 8$. Thus, by Proposition 3.1.2(2) and $L \cdot C''_b = 4$, we have $H \cdot C''_b = 4$. Since $L = H - 2E'_b$, it follows that $E'_b \cdot C''_b = 0$. However, this is a contradiction because $E'_b \cap C''_b \neq \emptyset$. Thus $b \notin C$ and, by Proposition 3.1.2(2), $H \cdot C''_b = 6$. Since $L = H - 2E'_b$, we have $E'_b \cdot C''_b = 1$. Next we compute $E'^2_b S$. Note that $-K_{B'_b} = 2H - 2E'_b = 2(L + 2E''_b) - 2E''_b = 2(L + E''_b)$. We have

$$E'^2_b L = \frac{1}{4}(-K_{B'_b} - 2L)^2 L = \frac{1}{4}(-K_L - L|_L)^2 = 1,$$

whence $E'^2_b S = 2E'^2_b L = 2$. The surface S is a Segre–del Pezzo scroll. Let C_0 be the negative section of S and l a fiber of $S \to C_b$, and let $e := -C_0^2$. We can write $E'_b|_S \sim C_0 + pl$ and $C''_b \sim 2C_0 + ql$ $(p, q \geq 0)$. From $E'_b \cdot C''_b = 1$ and $E'^2_b S = 2$ it follows that $q + 2p - 2e = 1$ and $2p - e = 2$; thus $e = 2p - 2$ and $q = 2p - 3$. Since C''_b is irreducible, we have $q \geq 2e$ and therefore $2p - 3 \geq 2(2p - 2)$; that is, $p = 0$ and $q = -3$, a contradiction.

Assume now that $d \geq 6$. Let $\mathcal{C} \to \Delta$ be the 1-parameter smoothing of $C_{d-1} \cup l$ such that $\mathcal{C}$ is smooth (the proof of Proposition 2.2.2 shows that this is possible). We consider the trivial family of the double projections $B \times \Delta \dashrightarrow \mathbb{P}^2 \times \Delta$ from $b \times \Delta$. Denote by $\mathcal{C}'_b$, $\mathcal{C}''_b$, and $\mathcal{C}_b$ the strict transforms of $\mathcal{C}$ on $B_b \times \Delta$, $B'_b \times \Delta$, and $\mathbb{P}^2 \times \Delta$, respectively. We also denote by $C'_{d-1,b}$, $C''_{d-1,b}$, and $C_{d-1,b}$ the strict transforms of C_{d-1} on B_b, B'_b, and $\mathbb{P}^2$, respectively. To prove the proposition it suffices to show that, for any b, there exists at least one point on $C_{d-1,b}$ over which $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is isomorphic. First, admitting this claim, we finish the proof of the proposition. Indeed, let

$$\mathcal{N} := \{(b,t) \in B \times \Delta \mid \mathcal{C} \dashrightarrow \mathcal{C}_b \text{ is not isomorphic over any point of } \mathcal{C}_{b,t}\}$$

and let $\Delta' \subset \Delta$ be the image of $\mathcal{N}$ by the projection to Δ. Here $\mathcal{N}$ is a closed subset, and so is Δ' since $B \times \Delta \to \Delta$ is proper. Thus Δ' consists of finitely many points because, by the claim, the origin is not contained in Δ'. So for a point $t \in \Delta$ sufficiently near the origin, $\mathcal{C}_t \dashrightarrow \mathcal{C}_{t,b}$ is birational for any b, which implies the proposition.

Now we show the preceding claim. By induction, we may assume that $C_{d-1} \dashrightarrow C_{d-1,b}$ is birational for any b. Note that $C_{d-1,b}$ is not a line, since otherwise C_{d-1} is contained in a singular hyperplane section (as in the foregoing case of C_5)—a contradiction. We investigate the image of l on $\mathbb{P}^2$. Recall the description of the fibers of π_{2b} outside E'_b (Proposition 3.1.2(4)). If $b \notin l$, then the image of l is a line or a point on $\mathbb{P}^2$; if $b \in l$, then the strict transform of l on B_b is a flopping curve. Thus $\mathcal{C}_b$ contains the image of the flopped curve, which is a line. We investigate the other possible irreducible components of the central fiber $\mathcal{C}_{b,0}$ of $\mathcal{C}_b \to \Delta$. If $b \notin C_{d-1} \cup l$, then the only possibility is that $\mathcal{C}_{b,0}$ contains the image of a flopped curve, which is a line on $\mathbb{P}^2$. Suppose $b \in C_{d-1} \cup l$. Let m'_b be the exceptional curve for $\mathcal{C}'_b \to \mathcal{C}$. Since $\mathcal{C}$ is a smooth surface, m'_b is a line on E_b. The curve $\mathcal{C}_{b,0}$ contains the strict transform m_b of m'_b. This is the only possibility for the other components of $\mathcal{C}_{b,0}$. Let l'_b be the strict transform of l on B_b. If $b \in l$ then, by the description of $E_b \dashrightarrow \mathbb{P}^2$, m_b is a line since l'_b is a flopping curve. Suppose that $b \in C_{d-1} \setminus l$. If m'_b intersects a flopping curve, then m_b is a line or a point; otherwise, m_b is a conic. If $b \notin \bigcup_i \beta_i$, then $\deg C_{d-1,b} = d-3$ by Proposition 3.1.2(2). Since $d \geq 6$, we know that $C_{d-1,b}$ is not a conic; hence $C_{d-1,b} \neq m_b$. Assume $b \in \beta_i$. Then $\deg C_{d-1,b} = d-4$. Thus, if $d \geq 7$ then $C_{d-1,b} \neq m_b$. We show that $C_{d-1,b} \neq m_b$ even if $d = 6$. By Proposition 2.3.1(4), the flop $B_b \dashrightarrow B'_b$ is of type (a) in Proposition 3.1.2(3). The strict transform m''_b of m'_b on B''_b intersects the three fibers of π_b contained in E'_b, which are the strict transforms of three lines on E_b. On the other hand, since $E'_b \cdot C''_{d-1,b} = 2$, the curve $C''_{d-1,b}$ intersects at most two fibers of π contained in E'_b. Therefore, $C_{d-1,b} \neq m_b$.

The above investigation shows that $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is isomorphic over a point of $C_{d-1,b}$. □

We restate the proposition in terms of the relation between C_d and multi-secant conics of C_d on B as follows.

Corollary 3.2.4 (Finiteness I). *Let b be a point of B not in any bi-secant line of C_d on B. If $d \geq 5$, then there exist finitely many k-secant conics of C_d on B through b with $k \geq 2$ if $b \notin C_d$ (or with $k \geq 3$ if $b \in C_d$).*

Proof. For a point $b \in B$ outside bi-secant lines of C_d on B, there is a finite number of singular multi-secant conics of C_d through b; the reason is that the number of lines through b is finite and, by Proposition 2.3.3(3), the number of lines intersecting both a line through b and C_d is also finite. Hence we need only consider smooth multi-secant conics q of C_d through b. By Proposition 3.1.2(4), the strict transform q' of such a conic q on B'_b is a fiber of π_{2b}. If $b \notin C_d$, then q' intersects C'_b twice or more counted with multiplicities; therefore, by Proposition 3.2.3, the finiteness of such a q follows. The assertion for $b \in C_d$ can be shown similarly, so we omit the proof. □

REMARK. We refine this statement in Lemma 4.2.13 and Proposition 5.1.1.

The number of multi-secant conics of C through a general point of B is rather important. We can obtain this number via the following lemma.

LEMMA 3.2.5. *Let l be a general uni-secant line of C, and let $l_b \subset \mathbb{P}^2$ be the image of l by the double projection π_b from a point b. Recall the notation in Notation 3.2.2. For a general point $b \notin C$, $\deg C_b = d$ and $C_b \cup l_b$ has only simple nodes. Assume that $d \geq 3$. For a general point b of C, $\deg C_b = d-2$ and $C_b \cup l_b$ has only simple nodes.*

Proof. The claims for $\deg C_b$ follow from Propositions 3.1.2(2) and 3.2.3. As for the singularity of $C_b \cup l_b$, the claim follows from a simple dimension count. For simplicity we prove only that, for a general point $b \notin C$, the curve C_b has only simple nodes. By Proposition 2.3.4, we may assume that any multi-secant conic through b is smooth and bi-secant and intersects C simply. Let q be a smooth bi-secant conic through b. We may assume that $\mathcal{N}_{q/B} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Let q' be the strict transform of q on B'_b. Let $\tilde{B}' \to B'_b$ be the blow-up along q', $E_{q'}$ the exceptional divisor, and $\tilde{C}''$ the strict transform of C''_b. Note that $E_{q'} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ since $\mathcal{N}_{q'/B'_b} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Then C_b has simple nodes at the image of q' if and only if the two points in $E_{q'} \cap \tilde{C}''$ do not belong to the same ruling with the opposite direction to a fiber of $E_{q'} \to q'$. Let $\tilde{B}_q \to B$ be the blow-up along q, E_q the exceptional divisor, and $\tilde{C}$ the strict transform of C. It is easy to see that a ruling of E_q with the opposite direction to a fiber of $E_q \to q$ corresponds to that of $E_{q'}$ with the opposite direction to a fiber of $E_{q'} \to q'$. Thus C_b has simple nodes at the image of q' if and only if the two points in $E_q \cap \tilde{C}$ do not belong to the same ruling with the opposite direction to a fiber of $E_q \to q$. We can show that this is the case for a general b by a simple dimension count. □

PROPOSITION 3.2.6. (1) *The number of multi-secant conics of C through a general point of B is $n := \frac{(d-1)(d-2)}{2}$.*

(2) *The number of k-secant conics of C with $k \geq 3$ through a general point of C is $\frac{(d-3)(d-4)}{2}$.*

(3) *Let l be a general uni-secant line of C. Then the number of multi-secant conics of C intersecting l and passing through a general point of C is $d-3$.*

Proof. We prove only (1) because the other statements can be shown similarly.

Let $b \notin C$ be a general point of B. Recall that, by Corollary 3.2.4, there exist only finitely many multi-secant conics of C through b. Moreover, since C_b is a nodal rational curve of degree d (by Lemma 3.2.5), the number of its nodes is exactly n—which is nothing but the number of multi-secant conics through b. □

4. Lines and Conics on A

We fix a general $C := C_d$ as in Section 2.2. Let $f\colon A \to B$ be the blow-up along C. We start the study of the geometry of A; in Sections 4.1 and 4.2 we study the families of curves on A of degree 1 or 2 with respect to the anticanonical sheaf of A (we call them, respectively, *lines* and *conics* on A). The curve $\mathcal{H}_1$ parameterizing lines on A and the surface $\mathcal{H}_2$ parameterizing conics on A are two of the main elements in this paper. See Corollary 4.1.1 and Theorem 4.2.11 for a quick view of their properties.

4.1. Curve $\mathcal{H}_1$ Parameterizing Marked Lines

4.1.1. *Construction of $\mathcal{H}_1$ and Marked Lines*
Let $\mathcal{H}_1 := \varphi^{-1}C \subset \mathbb{P}$ and $M := M_d$. We begin with a few corollaries to Proposition 2.3.3.

COROLLARY 4.1.1. *If $d \geq 2$, then $\mathcal{H}_1$ is a smooth curve of genus $d-2$ with the triple cover $\mathcal{H}_1 \to C$. In particular, if $d \geq 5$ then $\mathcal{H}_1$ is a smooth non-hyperelliptic trigonal curve of genus $d-2$.*

Proof. By Propositions 2.1.3(1) and 2.3.3(1), $\mathcal{H}_1$ is smooth and the ramification for $\mathcal{H}_1 \to C$ is simple. Since $B_\varphi \in |-K_B|$ and $d = \deg C$, we can compute $g(\mathcal{H}_1)$ by the Hurwitz formula:

$$2g(\mathcal{H}_1) - 2 = 3 \times (-2) + d \times 2;$$

equivalently,

$$g(\mathcal{H}_1) = d - 2.$$

□

The number of bi-secant lines of C is important in Theorem 4.2.11.

COROLLARY 4.1.2. *The number of nodes of M is $s := \frac{(d-2)(d-3)}{2}$, whence C has $\frac{(d-2)(d-3)}{2}$ bi-secant lines on B.*

Proof. By Proposition 2.3.3(3), $\pi|_{\mathcal{H}_1}\colon \mathcal{H}_1 \to M$ is birational and $p_a(M) = \frac{(d-1)(d-2)}{2}$. Then, since $g(\mathcal{H}_1) = d-2$, the number of nodes of M is

$$\frac{(d-1)(d-2)}{2} - (d-2) = \frac{(d-2)(d-3)}{2}.$$

The latter half follows because a bi-secant line of C corresponds to a node of M. □

Now we select some lines on B that will be used in the sequel. Note that

$$\mathcal{H}_1 = \{(l,t) \mid l \in M,\ t \in C \cap l\} \subset M \times C.$$

The elements of $\mathcal{H}_1$ deserve a name, as follows.

DEFINITION 4.1.3. A pair consisting of a secant line l of C on B and a point $t \in C \cap l$ is called a *marked line*.

Let (l, t) be a marked line. If $C \cap l$ is one point, then $\{t\} = C \cap l$ is uniquely determined. For a bi-secant line β_i of C, there are two choices of t. Thus $\mathcal{H}_1$ parameterizes marked lines.

4.1.2. *Lines on the Blow-up A of B along C_d*

We prove that each marked line corresponds to a curve of anticanonical degree 1 on the blow-up A of B along C. This gives us a suitable notion of line on A.

NOTATION 4.1.4. Let

(1) $f \colon A \to B$ be the blow-up along C,
(2) E_C be the f-exceptional divisor,
(3) $\{p_{i1}, p_{i2}\} = C \cap \beta_i \subset B$, and
(4) $\zeta_{ij} = f^{-1}(p_{ij}) \subset E_C \subset A$,

where $i = 1, \ldots, s$ and $j = 1, 2$.

DEFINITION 4.1.5. We shall say that a connected curve $l \subset A$ is a *line* on A if $-K_A \cdot l = 1$ and $E_C \cdot l = 1$.

We point out that, since $-K_A = f^*(-K_B) - E_C$ and $E_C \cdot l = 1$, it follows that $f(l)$ is a line on B *intersecting* C. More precisely, we have the following result.

PROPOSITION 4.1.6. *A line l on A is one of the following curves on A:*

(i) *the strict transform of a uni-secant line of C on B; or*
(ii) *the union $l_{ij} = \beta_i' \cup \zeta_{ij}$, where $i = 1, \ldots, s$ and $j = 1, 2$.*

In particular, l is reduced and $p_a(l) = 0$.

NOTATION 4.1.7. For a line l on A, we usually denote by $\bar{l}$ its image on B.

COROLLARY 4.1.8. *The curve $\mathcal{H}_1 \subset \mathbb{P}$ is the Hilbert scheme of the lines of A.*

Proof. Let $\mathcal{H}_1'$ be the Hilbert scheme of lines on A, which is a locally closed subset of the Hilbert scheme of A. By the obstruction calculation of the normal bundles of the components of lines on A, it is easy to see that $\mathcal{H}_1'$ is a smooth curve. Denote by $\mathcal{U}_1 \to \mathcal{H}_1'$ the universal family of the lines on A, and let $\bar{\mathcal{U}}_1$ be the image of $\mathcal{U}_1$ on $B \times \mathcal{H}_1'$ (with induced reduced structure).

CLAIM 4.1.9. $\bar{\mathcal{U}} \to \mathcal{H}_1'$ *is a* $\mathbb{P}^1$*-bundle.*

Proof. Let $\mathcal{L}$ be the pull-back of the ample generator of Pic B by

$$\mathcal{U}_1 \hookrightarrow A \times \mathcal{H}_1' \to B \times \mathcal{H}_1' \to B.$$

Since $\varrho \colon \mathcal{U}_1 \to \mathcal{H}_1'$ is flat and since $h^0(l, \mathcal{L}|_l) = 2$ for a line l on B, it follows that $\mathcal{E} := \varrho_* \mathcal{L}$ is a locally free sheaf of rank 2. We remark that $\mathbb{P}(\mathcal{E})$ is nothing but the $\mathbb{P}^1$-bundle contained in $B \times \mathcal{H}_1'$ whose fiber is the image of a line on A. This implies that $\mathbb{P}(\mathcal{E}) = \bar{\mathcal{U}}$ as schemes and that $\bar{\mathcal{U}}$ is a $\mathbb{P}^1$-bundle. $\square$

By the claim, we have a natural morphism $\mathcal{H}_1' \to \mathbb{P}^2$ whose image is M. By Proposition 4.1.6, $\mathcal{H}_1' \to M$ is birational and surjective. Since $\mathcal{H}_1'$ and $\mathcal{H}_1$ are smooth, they are both normalizations of M; hence $\mathcal{H}_1' \simeq \mathcal{H}_1$, completing the proof of Corollary 4.1.8. $\square$

REMARK. For a bi-secant line β_i we have two choices of marking, p_{i1} or p_{i2}. We describe which line on A corresponds to (β_i, p_{ij}). Denote by $\mathcal{U}_1 \to \mathcal{H}_1$ the universal family of the lines on A and consider the following diagram:

$$\begin{array}{ccc} \mathcal{U}_1 & \subset & A \times \mathcal{H}_1 \\ \downarrow & & \downarrow \\ \bar{\mathcal{U}}_1 & \subset & B \times \mathcal{H}_1 . \end{array}$$

Then $\mathcal{U}_1 \to \bar{\mathcal{U}}_1$ is the blow-up along $(C \times \mathcal{H}_1) \cap \bar{\mathcal{U}}_1$, which is the union of a finite set of points $(p_{i,3-j}, (\beta_i, p_{ij}))$ and a section of $\bar{\mathcal{U}}_1 \to \mathcal{H}_1$ that consists of markings. Thus the marked line (β_i, p_{ij}) corresponds to the line $l_{i,3-j}$.

4.2. *Surface $\mathcal{H}_2$ Parameterizing Marked Conics*

Now we turn to define a notion of *conic* on A. We proceed as in the case of lines, first defining the notion of *marked conic.*

4.2.1. *Marked Conics and the Construction of $\mathcal{H}_2$*

DEFINITION 4.2.1. A pair consisting of a multi-secant conic q on B and a 0-dimensional subscheme $\eta \subset C$ of length 2 contained in $q|_C$ is called a *marked conic.*

From now on, we assume that $d \geq 3$.

By [Ili, Prop. 1.2.2], the Hilbert scheme of conics on B is isomorphic to $\mathbb{P}^4$. Marked conics are parameterized by

$$\mathcal{H}_2' := \{(q, \eta) \mid q \in \bar{\mathcal{H}}_2', \eta \subset q|_C\} \subset \bar{\mathcal{H}}_2' \times S^2C$$

with reduced structure, where $\bar{\mathcal{H}}_2' \subset \mathbb{P}^4$ is the locus of multi-secant conics of C on B. By Corollary 3.2.1 and $d \neq 1$, the natural projection of $\mathcal{H}_2' \to S^2C$ is one-to-one outside $\beta_i|_C$ and the diagonal of S^2C. Observe that $\mathcal{H}_2' \to \bar{\mathcal{H}}_2'$ is finite because, since $d \geq 3$, there are finitely many choices of markings of a multi-secant conic of C.

PROPOSITION 4.2.2. *$\mathcal{H}_2'$ is the union of the unique 2-dimensional component, which dominates S^2C, and possibly lower-dimensional components mapped into the diagonal of S^2C.*

Proof. We denote by e_i' the fiber of $\mathcal{H}_2' \to S^2C$ over a $\beta_i|_C$. Since B is the intersection of quadrics, no conic can properly intersect a line twice. Thus any conic containing $\beta_i|_C$ contains β_i. This implies that $e_i' \simeq \mathbb{P}^1$, and e_i' parameterizes marked conics of the form

$$\{(\beta_i \cup \alpha, \beta_i|_C) \mid \alpha \text{ is a line such that } \alpha \cap \beta_i \neq \emptyset\}.$$

Over the diagonal of S^2C, $\mathcal{H}'_2 \to S^2C$ is finite because, for $t \in C$, there is a finite number of reducible conics with t as a singular point or conics tangent to C at t. Hence $\mathcal{H}'_2$ is the union of the unique 2-dimensional component, which dominates S^2C, and possibly lower-dimensional components mapped into the diagonal of S^2C or e'_i.

We show that e'_i is contained in the unique 2-dimensional component of $\mathcal{H}'_2$. Indeed, we have only to prove that $\bar{\mathcal{H}}'_2$ is 2-dimensional near the generic point of the image of e'_i, given that $\mathcal{H}'_2 \to \bar{\mathcal{H}}'_2$ is one-to-one near the generic point of the image of e'_i. Let $\mathcal{V}_2 \to \mathcal{H}^B_2 \simeq \mathbb{P}^4$ be the universal family of conics on B, and let $\bar{\mathcal{H}}''_2$ be the inverse image of $C \times C$ by $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \to B \times B$. Since the morphism $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \to \mathcal{V}_2 \to \mathbb{P}^4$ is flat, $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2$ is purely 6-dimensional. Thus any component of $\bar{\mathcal{H}}''_2$ has dimension ≥ 2. Although the inverse image of the diagonal of $C \times C$ is 3-dimensional, any other component of $\bar{\mathcal{H}}''_2$ is at most 2-dimensional by a similar investigation of $\mathcal{H}'_2$. Thus $\bar{\mathcal{H}}'_2$ is 2-dimensional near the generic point of the image of e'_i since $\bar{\mathcal{H}}'_2$ is the image of the 2-dimensional part of $\bar{\mathcal{H}}''_2$ by $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \to \mathbb{P}^4$ near the generic point of the image of e'_i. □

NOTATION 4.2.3. Let $\mathcal{H}_2$ be the normalization of the unique 2-dimensional component of $\mathcal{H}'_2$, and let $\bar{\mathcal{H}}_2 \subset \bar{\mathcal{H}}'_2$ be the image of $\mathcal{H}_2$. Denote by η the natural morphism $\mathcal{H}_2 \to S^2C$. Let

$$c_i := \beta_i|_C \in S^2C \simeq \mathbb{P}^2 \quad \text{and} \quad e_i := \eta^{-1}(c_i),$$

where $i = 1, \dots, s$.

By the foregoing considerations, $\eta \colon \mathcal{H}_2 \to S^2C$ is isomorphic outside $\beta_i|_C$ (by the Zariski main theorem) and $\mathcal{H}_2 \to \bar{\mathcal{H}}_2$ is the normalization. Thus we see that $\mathcal{H}_2$ parameterizes marked conics in a one-to-one way outside the inverse image of c_i. We need to understand the inverse image by η of the diagonal.

CLAIM 4.2.4. *Assume that $(q, 2b) \in \mathcal{H}_2$ for $b \in C$ and a conic q. Then*:

(1) *q is reduced*;
(2) *if q is smooth at b, then q is tangent to C at b*; *and*
(3) *if q is singular at b, then the strict transform of q is connected on A. Moreover, $b \notin \beta_i$ and $b \notin B_\varphi$.*

Proof. We use the double projection from b. By Proposition 3.1.2(4) and a degeneration argument, q corresponds to the fiber of π_{2b} through the point t' in $C''_b \cap E'_b$ coming from $t := C'_b \cap E_b$.

(1) Assume by way of contradiction that q is nonreduced. By [Ili, Prop. 1.2.2], q is a multiple of a special line l. By Proposition 2.3.1(4), l is a uni-secant line of C. Let m be the other line through b (by generality of C, we have $l \neq m$). Let l' and m' be (respectively) the strict transforms of l and m on B_b. By Proposition 3.1.2(4), the fiber of π_{2b} through t' is the strict transform of the line in E_b joining $l' \cap E_b$ and $m' \cap E_b$. Then, by our assumption, the intersections $l' \cap E_b$, $m' \cap E_b$, and $C'_b \cap E_b$ are collinear. By a dimension count similar to that in the proof of Proposition 2.3.1, we can prove that a general C does not satisfy this condition.

(2) This part follows from the previous discussion.

(3) Set $q = l_1 \cup l_2$, where l_1 and l_2 are the irreducible components of q, and let l_i' be the strict transform of l_i on B_b. By (1) we have $l_1 \neq l_2$. Then the fiber of π_{2b} corresponding to q is the strict transform of the line on E_b through $E_b \cap l_1'$ and $E_b \cap l_2'$. Note that A is obtained from B_b by blowing up B_b along C_b' and then contracting the strict transform of E_b. Thus the former half of the assertion follows, and the latter half again follows by a simple dimension count. □

4.2.2. *Conics on A*

Definition 4.2.5. We say that a connected and reduced curve $q \subset A$ is a *conic* on A if $-K_A \cdot q = 2$ and $E_C \cdot q = 2$.

Using this definition, we can classify conics on A, similarly to Proposition 4.1.6, as follows.

Proposition 4.2.6. *Let q be a conic on A. Then $\bar{q} := f(q) \subset B$ is a multi-secant conic of C. Moreover, one of the following statements holds.*

(a) *$\bar{q}$ is smooth at $\bar{q} \cap C$; q is the union of the strict transform q' of $\bar{q}$ and $k-2$ distinct fibers $\zeta_1, \dots, \zeta_{k-2}$ of E_C such that $\zeta_i \cap q' \neq \emptyset$.*

(b) *$\bar{q}$ is the union of two uni-secant lines $\bar{l}$ and $\bar{m}$ such that $C \cap \bar{l} \cap \bar{m} \neq \emptyset$; q is the union of the strict transforms l and m of $\bar{l}$ and $\bar{m}$, respectively (we assume that $l \cap m \neq \emptyset$).*

(c) *$\bar{q}$ is the union of β_i and a line $\bar{r}$ through a p_{ij}; q is the union of the fiber ζ_{ij} over p_{ij} and the strict transforms β_i' and r' of β_i and $\bar{r}$, respectively.*

Notation 4.2.7. We usually denote by $\bar{q} \subset B$ the image of a conic q on A.

Let $\mathcal{H}_2^A$ be the normalization of the 2-dimensional part of the Hilbert scheme of conics on A, which is a locally closed subset of the Hilbert scheme of A. Let $\mu\colon \mathcal{U}_2 \to \mathcal{H}_2^A$ be the pull-back of the universal family of conics on A.

The proof of the following lemma is similar to that of Claim 4.1.9, so we omit it.

Lemma 4.2.8. *Let $\bar{\mathcal{U}}_2$ be the image of $\mathcal{U}_2$ on $B \times \mathcal{H}_2^A$ (with induced reduced structure). Then $\bar{\mathcal{U}}_2 \to \mathcal{H}_2^A$ is a conic bundle.*

Proposition 4.2.9. *There exists a natural bijection between the set of marked conics belonging to $\mathcal{H}_2$ and the set of conics on A. Moreover, the two surfaces $\mathcal{H}_2^A$ and $\mathcal{H}_2$ are isomorphic.*

Proof. The first assertion follows from Claims 4.2.4(1) and (3) and Proposition 4.2.6.

By Lemma 4.2.8, there exists a natural morphism $\bar{\nu}\colon \mathcal{H}_2^A \to \bar{\mathcal{H}}_2'$. By Proposition 4.2.6, $\bar{\nu}$ is finite and birational; hence $\bar{\nu}$ lifts to the morphism $\nu\colon \mathcal{H}_2^A \to \mathcal{H}_2$ because $\mathcal{H}_2 \to \bar{\mathcal{H}}_2$ is the normalization. By the Zariski main theorem, ν is an inclusion. By Claims 4.2.4(1) and (3) and Proposition 4.2.6, ν is also surjective. □

By Proposition 4.2.9, we can pass freely from conics on A—that is, from elements of $\mathcal{H}_2^A$ to marked conics (and vice versa) according to the kind of argument we will need. In particular, when we speak of the universal family $\mu\colon \mathcal{U}_2 \to \mathcal{H}_2$ of marked conics, we mean $\mathcal{U}_2 := \mathcal{U}_2^A$ and $\mathcal{H}_2^A$ identified with $\mathcal{H}_2$ via ν.

COROLLARY 4.2.10. *The Hilbert scheme of conics on A is an irreducible surface* (*and $\mathcal{H}_2$ is the normalization*). *The normalization is injective; namely, $\mathcal{H}_2$ parameterizes conics on A in a one-to-one way.*

Proof. By Proposition 4.2.6, the image of $\mathcal{H}_2$ in the Hilbert scheme of A parameterizes all the conics on A; hence the first part follows.

For the second part, we have already seen that $\mathcal{H}_2$ parameterizes marked conics belonging to the unique 2-dimensional component of $\mathcal{H}_2'$ in a one-to-one way outside $\bigcup_i e_i$. Thus, by Proposition 4.2.9, $\mathcal{H}_2$ parameterizes conics on A in a one-to-one way outside $\bigcup_i e_i$. Let α be a general line intersecting β_i, and let α' be the strict transform of α on A. By an easy obstruction calculation, we see that the Hilbert scheme of conics on A is smooth at $\beta_i' \cup \alpha'$. Thus general points of e_i also parameterize conics on A in a one-to-one way. But since $e_i' \simeq \mathbb{P}^1$, where e_i' is the inverse image of $\beta_i|_C$ by $\mathcal{H}_2' \to S^2C$, it follows that $e_i \simeq e_i' \simeq \mathbb{P}^1$ ($\mathcal{H}_2 \to S^2C$ has only connected fibers). This implies the assertion. $\square$

In Section 4.2.5 we develop the following complete description of $\mathcal{H}_2$. See Notation 4.2.16 for the definition of D_l in the statement.

THEOREM 4.2.11. (1) *The morphism $\eta\colon \mathcal{H}_2 \to \mathbb{P}^2$ is the blow-up at $c_1, \dots, c_s$, and the e_i are η-exceptional curves. We have*

$$D_l \sim (d-3)h - \sum_{i=1}^{s} e_i,$$

where h is the strict transform of a general line on $\mathbb{P}^2$.

(2) *We have*

$$h^1\left(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}\left((d-4)h - \sum_{i=1}^{s} e_i\right)\right) = 0.$$

(3) $|D_l|$ *is base point free. When $d = 5$, the image of $\Phi_{|D_l|}$ is $\check{\mathbb{P}}^2$; when $d \geq 6$, D_l is very ample and $|D_l|$ embeds $\mathcal{H}_2$ into $\check{\mathbb{P}}^{d-3}$.* (*Here we use the dual notation $\check{\mathbb{P}}^{d-3}$ for later convenience.*)

(4) *If $d \geq 6$, then $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$ is projectively Cohen–Macaulay. Equivalently,*

$$h^i(\check{\mathbb{P}}^{d-3}, \mathcal{I}_{\mathcal{H}_2}(j)) = 0 \quad \textit{for } i = 1, 2 \textit{ and } j \in \mathbb{Z},$$

where $\mathcal{I}_{\mathcal{H}_2}$ is the ideal sheaf of $\mathcal{H}_2$ in $\check{\mathbb{P}}^{d-3}$. Moreover, $\mathcal{H}_2$ is the intersection of cubics.

REMARK. If $d \geq 6$, then $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$ is so called the *White surface* (see [Gi; W]).

In Sections 4.2.3 and 4.2.4 we give some results that are preliminary to proving this theorem.

4.2.3. *Quasi-finiteness of* $\psi : \mathcal{U}_2 \to A$

NOTATION 4.2.12. For a point $b \in C$, let

$$L_b := \overline{\{q \in \mathcal{H}_2 \mid \exists b' \neq b,\ f(q) \cap C = \{b, b'\}\}}.$$

By Corollary 3.2.1, $\eta(L_b)$ is a line in $S^2C \simeq \mathbb{P}^2$.

Let $\psi : \mathcal{U}_2 \to A$ be the morphism obtained via the universal family $\mu : \mathcal{U}_2 \to \mathcal{H}_2$. The following result refines Proposition 3.2.3. We need this result here to investigate the intersection of lines and conics on A in Section 4.2.4, but it is important also for the proof of the main result and is refined again in Section 5.1 (Proposition 5.1.1).

From now on in this paper, unless otherwise noted we assume that $d \geq 5$.

PROPOSITION 4.2.13 (Finiteness II). *The morphism ψ is finite of degree $n = \frac{(d-1)(d-2)}{2}$ and is flat outside $\bigcup_{i=1}^{s} \beta_i'$.*

Proof. Let $a \in A \setminus \bigcup_{i=1}^{s} \beta_i'$ and let $b := f(a)$. If $b \notin C$, then the finiteness of ψ over a follows from Corollary 3.2.1. Moreover, by Proposition 3.2.6, the number of conics through a general a is n. Thus $\deg \psi = n$. We will prove that ψ is finite over $a \in E_C \setminus \bigcup_{i=1}^{s} \beta_i'$. Once we prove this, the assertion follows. Indeed, $\mathcal{U}_2$ is Cohen–Macaulay because $\mathcal{H}_2$ is smooth and any fiber of $\mathcal{U}_2 \to \mathcal{H}_2$ is reduced; hence ψ is flat.

Let $a \in E_C \setminus \bigcup_{i=1}^{s} \beta_i'$. The assertion is equivalent to stating that only finitely many conics belonging to L_b pass through a. If $b \notin \bigcup_{i=1}^{s} \beta_i$, then L_b is irreducible. If $b \in \bigcup_{i=1}^{s} \beta_i$, then $L_b = L_b' \cup e_i$, where L_b' is the strict transform of $\eta(L_b)$ and so is irreducible. Note that almost none of the conics belonging to e_i pass through $a \notin \bigcup_{i=1}^{s} \beta_i'$. Let $S_b \subset A$ be the locus swept by the conics of the family L_b if $b \notin \bigcup_{i=1}^{s} \beta_i$ or the locus swept by the conics of the family L_b' if $b \in \bigcup_{i=1}^{s} \beta_i$. Then S_b is irreducible. Let $\bar{S}_b := f(S_b)$, where $\bar{S}_b'$ and $\bar{S}_b''$ are the strict transforms of $\bar{S}_b$ on B_b and B_b', respectively. Then $\bar{S}_b'' = \pi_{2b}^* C_b$. Let $d_b := \deg C_b$. By Proposition 3.1.2(2), $d_b = d - 2$ if $b \notin \bigcup_{i=1}^{s} \beta_i$ or $d_b = d - 3$ if $b \in \bigcup_{i=1}^{s} \beta_i$. Since $\bar{S}_b'' \sim d_b L$ and since $L = H - 2E_b'$, it follows that $\bar{S}_b'|_{E_b}$ is a curve of degree $2d_b$ in $E_b \simeq \mathbb{P}^2$.

Because A is obtained from B_b by blowing up C_b' and then contracting the strict transform of E_b, a point a over b corresponds to a line l_a in E_b through $t := E_b \cap C_b'$. The image on B_b of the strict transform of a conic on A through a intersects E_b at a point of $l_a \cap \bar{S}_b'$. If C_b'' does not intersect fibers of π_{2b} contained in E_b', then $\bar{S}_b''|_{E_b'}$ is irreducible. Thus no l_a is contained in $\bar{S}_b'|_{E_b}$ and we are done. So assume that C_b'' intersects a fiber l' of π_{2b} contained in E_b'. This situation is covered by Claim 4.2.4(3); hence $b \notin B_\varphi$ and $b \notin \bigcup_{i=1}^{s} \beta_i$ for a general C. Since L_b is irreducible by $b \notin \bigcup_{i=1}^{s} \beta_i$, it suffices to prove the finiteness and nonemptyness of the set of conics through a general point a over b. Equivalently, we need only show that a general l_a intersects $\bar{S}_b'|_{E_b}$ outside t. Because l' intersects C_b'' simply at one point, C_b is smooth at the image t' of l' on $\mathbb{P}^2$. Thus $\bar{S}_b'|_{E_b} = C_b''' + l$, where C_b''' and l are (respectively) the strict transforms of C_b and l'. Note that C_b''' is smooth at t and that $\deg C_b''' = 2d_b - 1 = 2d - 5 \geq 5$ since $d \geq 5$. Thus a general l_a intersects C_b''' outside t. □

EXAMPLE 4.2.14. Here we describe the fiber of ψ over a general point $a \in E_C \setminus \bigcup_{i=1}^{s} \beta_i'$. We use the description in the proof of Corollary 5.1.5.

Let $b := f(a)$. As in the proof of Proposition 4.2.13, a point a over b corresponds to a line l_a in E_b passing through $E_b \cap C_b'$. By Lemma 3.2.5, $\deg C_b = d-2$ and C_b has $\frac{(d-3)(d-4)}{2}$ simple nodes for a general $b \in C$. This means that $\frac{(d-3)(d-4)}{2}$ tri-secant conics pass through b. By Proposition 4.2.6, corresponding to a tri-secant conic $\bar{q}$ there is a unique conic q on A containing the fiber of E_C over b, and such a conic on A contains a. Thus we obtain $\frac{(d-3)(d-4)}{2}$ conics through a. By definition of L_b, these conics do not belong to L_b.

We need additional $n - \frac{(d-3)(d-4)}{2} = 2d-5$ conics through a. We show that there exist $2(d-2)-1$ conics through a on A coming from the family parameterized by L_b. We use the notation of the proof of Proposition 4.2.13. For a general $b \in C$, note that C_b'' does not intersect fibers of π_{2b} contained in E_b'. Thus $\bar{S}_b'|_{E_b}$ is an irreducible curve of degree $2(d-2)$ on E_b. Hence there are $2(d-2)$ intersection points of $\bar{S}_b'|_{E_b}$ and l_a. Among these, the intersection point $C_b' \cap E_b$ does not correspond to a conic on A through a because it comes from the tangent of C. Thus we have $2(d-2)-1$ conics as desired.

4.2.4. *Intersection of Lines and Conics on A*

To describe $\mathcal{H}_2$, we use the divisor on $\mathcal{H}_2$ parameterized by lines on A, which now we define. Let $\hat{\mathcal{U}}_1 \subset \mathcal{U}_2 \times \mathcal{H}_1$ be the pull-back of $\mathcal{U}_1$ via the following diagram:

$$\begin{tikzcd} \hat{\mathcal{U}}_1 \subset \mathcal{U}_2 \times \mathcal{H}_1 \arrow[r] \arrow[d] & A \times \mathcal{H}_1 \supset \mathcal{U}_1 \arrow[d] \\ \hat{\mathcal{D}}_1 \subset \mathcal{H}_2 \times \mathcal{H}_1 \arrow[r] & \mathcal{H}_1, \end{tikzcd} \tag{4.1}$$

where $\hat{\mathcal{D}}_1$ is the image of $\hat{\mathcal{U}}_1$ on $\mathcal{H}_2 \times \mathcal{H}_1$. By definition,

$$\hat{\mathcal{D}}_1 = \{(q,l) \mid q \cap l \neq \emptyset\} \subset \mathcal{H}_2 \times \mathcal{H}_1.$$

First we need to know which component of $\hat{\mathcal{D}}_1$ is divisorial or dominates $\mathcal{H}_1$. For this purpose, we study the mutual intersection of a conic and a line in special cases. Let $\mathcal{F} \subset \mathcal{H}_2 \times \mathcal{H}_1$ be the image in $\mathcal{H}_2 \times \mathcal{H}_1$ of the inverse image of $\left(\left(\bigcup \beta_i'\right) \times \mathcal{H}_1\right) \cap \mathcal{U}_1$; that is,

$$\mathcal{F} := \{(q,l) \mid q \cap \beta_i' \cap l \neq \emptyset\}.$$

A point (q,l) is in $\mathcal{F}$ if and only if (i) $l = l_{ij}$ $(:= \beta_i' \cup \zeta_{ij})$ and $q \cap \beta_i' \neq \emptyset$ *or* (ii) $l \neq l_{ij}$ and $q \cap \beta_i' \cap l \neq \emptyset$. For every $i = 1, \dots, s$ and $j = 1, 2$, the family of those (q,l) that satisfy (i) or (ii) has dimension 1 and clearly does not dominate $\mathcal{H}_1$.

PROPOSITION 4.2.15. *Any component of $\hat{\mathcal{D}}_1$ that is not contained in $\mathcal{F}$ dominates $\mathcal{H}_1$. Moreover, any nondivisorial component of $\hat{\mathcal{D}}_1$ outside $\mathcal{F}$ (if it exists) is a*

1-dimensional component whose generic point parameterizes reducible conics—namely, a 1-dimensional component of

$$\{(q,l) \mid l \subset q\}.$$

REMARK. At this juncture, it is still possible that a 1-dimensional component whose generic point parameterizes reducible conics is contained in a divisorial component of $\hat{\mathcal{D}}_1$. However, we prove in Corollary 4.2.20 that this is not the case. Hence, in the end, the fiber of $\hat{\mathcal{D}}_1 \to \mathcal{H}_1$ over a general $l \in \mathcal{H}_1$ parameterizes conics that properly intersect l.

Proof of Proposition 4.2.15. By Proposition 4.2.13, $\mathcal{U}_2 \to A$ is finite and flat outside $\bigcup \beta_i'$. Therefore, $\mathcal{U}_2 \times \mathcal{H}_1 \to A \times \mathcal{H}_1$ is flat outside $\left(\bigcup \beta_i'\right) \times \mathcal{H}_1$. By base change, $\hat{\mathcal{U}}_1 \to \mathcal{U}_1$ is flat and finite outside $\left(\left(\bigcup \beta_i'\right) \times \mathcal{H}_1\right) \cap \mathcal{U}_1$. Thus every irreducible component of $\hat{\mathcal{U}}_1$ that is not mapped to $\left(\left(\bigcup \beta_i'\right) \times \mathcal{H}_1\right) \cap \mathcal{U}_1$ is 2-dimensional and dominates $\mathcal{U}_1$ and so dominates $\mathcal{H}_1$. Hence any component of $\hat{\mathcal{D}}_1$ that is not contained in $\mathcal{F}$ dominates $\mathcal{H}_1$.

We next find a possible nondivisorial component of $\hat{\mathcal{D}}_1$ outside $\mathcal{F}$. Let $\gamma \subset \hat{\mathcal{U}}_1$ be a curve mapped to a point—say, (q,l) on $\mathcal{H}_2 \times \mathcal{H}_1$. The image of γ on A is an irreducible component of q—say, q_1. The image of γ on $\mathcal{U}_1$ is $q_1 \times l$ and thus q_1 is also an irreducible component of l. We have the following three possibilities.

(1) l is irreducible; hence $q_1 = l$ and $q = l \cup m$, where m is another line. Such (q,l) form the 1-dimensional family of reducible conics.
(2) $l = l_{ij}$ and $\beta_i' \subset q$. Namely, $q \in e_i$ or $q = \beta_i' \cup \alpha \cup \zeta_{ik}$, where α is the strict transform of a line on B intersecting β_i and C outside $\beta_i \cap C$.
(3) $l = l_{ij}$ and $\zeta_{ij} \subset q$, and $f(q)$ is a tri- or quadri-secant conic of C such that $p_{ij} \in f(q)$.

Thus we have the second assertion. □

NOTATION 4.2.16. Let $\mathcal{D}_1 \subset \mathcal{H}_2 \times \mathcal{H}_1$ be the divisorial part of $\hat{\mathcal{D}}_1$. Since $\mathcal{H}_1$ is a smooth curve, it follows that $\mathcal{D}_1 \to \mathcal{H}_1$ is flat. Let D_l be the fiber of $\mathcal{D}_1 \to \mathcal{H}_1$ over $l \in \mathcal{H}_1$. Clearly we can write $D_l \hookrightarrow \mathcal{H}_2$.

4.2.5. *Description of $\mathcal{H}_2$*

Now we prove Theorem 4.2.11. We need the following two lemmas, which are applications of the projection of B from a line. We refer to [Fu] for the facts on the projection of B from a line. Here we recall that the target of the projection is the smooth quadric 3-fold, which we denote by Q.

Let $C := C_d$ be a general rational curve of degree d constructed as in Proposition 2.2.2, and let l_1 and l_2 be two general secant lines of C such that $l_1 \cap l_2 = \emptyset$. We need to count the number of multi-secant conics of C intersecting l_1 and l_2 in the proof of Theorem 4.2.11.

LEMMA 4.2.17. *Assume that $d \geq 3$. Let $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ be the successive linear projections from l_1 and then the strict transform of l_2 on Q. Let l be another general secant line of C, and let C' and $l' \subset \mathbb{P}^2$ be the images of C and l, respectively. Then $C \cup l \dashrightarrow C' \cup l'$ is generically one-to-one and* $\deg C' \cup l' = d - 1$. *Moreover, $C' \cup l'$ has only simple nodes as its singularities.*

In particular (*since* $\deg C' = d-2$ *and* C' *is rational*), C' *has* $\frac{(d-3)(d-4)}{2}$ *simple nodes*; *equivalently, there exist* $\frac{(d-3)(d-4)}{2}$ *bi-secant conics of* C *intersecting both* l_1 *and* l_2.

REMARK. The line l is needed for the inductive proof that follows.

Proof of Lemma 4.2.17. We show the assertion using the inductive construction of $C = C_d$. The assertion follows for $d = 3$ directly. Consider a smoothing from $C_{d-1} \cup m$ to C_d. Let m_1 and m_2 be two general secant lines of C_{d-1} such that $m_1 \cap m_2 = \emptyset$. Let $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ be the successive linear projections from m_1 and then from the strict transform of m_2 on Q. Let r be another general secant line of C_{d-1}, and let C'_{d-1}, m', and $r' \subset \mathbb{P}^2$ be the respective images of C_{d-1}, m, and r. Then we have to show that: $C_{d-1} \cup m \cup r \dashrightarrow C'_{d-1} \cup m' \cup r'$ is generically one-to-one; $\deg C'_{d-1} \cup m' \cup r' = d-1$ and $C'_{d-1} \cup m' \cup r'$ has only simple nodes as its singularities (assuming $C_{d-1} \cup r \dashrightarrow C'_{d-1} \cup r'$ is generically one-to-one); and $\deg C'_{d-1} \cup r' = d-2$ and $C'_{d-1} \cup r'$ has only simple nodes as its singularities.

Since m is also general, it follows that $C_{d-1} \cup m \dashrightarrow C'_{d-1} \cup m'$ is generically one-to-one, that $\deg C'_{d-1} \cup m' = d-2$, and that $C'_{d-1} \cup m'$ has only simple nodes as its singularities. Thus $C_{d-1} \cup m \cup r \dashrightarrow C'_{d-1} \cup m' \cup r'$ is generically one-to-one and $\deg C'_{d-1} \cup m' \cup r' = d-1$. To show $C'_{d-1} \cup m' \cup r'$ has only simple nodes as its singularities, it suffices to prove that there are no secant conics of C_{d-1} intersecting all the m_1, m_2, m, and r. This follows because a secant conic q of C_{d-1} intersects finitely many secant lines of C_{d-1} since $M(q) \not\subset M(C_{d-1})$.

The last statement of the lemma follows because, by the generality of l_1 and l_2, any multi-secant conic of C intersecting l_1 and l_2 is bi-secant. $\square$

The following is a variant of Lemma 4.2.17. The proof is similar to that of Lemma 4.2.17, so we omit it.

LEMMA 4.2.18. *Assume that* $d \geq 4$, *and let* l_0 *be a general uni-secant line of* C. *Let* $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ *be the successive linear projections from* l_0 *and then the strict transform of a bi-secant line* β_i *on* Q. *Let* l *be another general uni-secant line of* C, *and let* C' *and* $l' \subset \mathbb{P}^2$ *be the images of* C *and* l, *respectively. Then* $C \cup l \dashrightarrow C' \cup l'$ *is generically one-to-one,* $\deg C' \cup l' = d-2$, *and* $C' \cup l'$ *has only simple nodes as its singularities.*

In particular (*since* $\deg C' = d-3$ *and* C' *is rational*), C' *has* $\frac{(d-4)(d-5)}{2}$ *simple nodes*; *equivalently, there exist* $\frac{(d-4)(d-5)}{2}$ *bi-secant conics of* C *intersecting* β_i *and* l_0 *except conics containing* β_i.

Proof of Theorem 4.2.11. (1) We first compute the intersection number $D_l \cdot L_b$ for general l and b (this intersection number will be well-defined because the intersection points of D_l and L_b are contained in the smooth locus of $\mathcal{H}_2$). We prove that D_l and L_b intersect simply. Indeed, let $\pi_C \colon C \times C \to S^2C$ be the natural projection and let L'_b be a ruling of $C \times C \to C$ in one fixed direction such that $\pi_C(L'_b) = \eta(L_b)$. Applying the Bertini theorem to $|L'_b|$ shows that $\pi_C^*\eta(D_l)$ and L'_b intersect simply for a general $b \in C$, whence $\eta(D_l)$ intersects $\eta(L_b)$ simply since π_C is étale at $\pi_C^*\eta(D_l) \cap L'_b$. Then D_l intersects L_b simply since η is isomorphic at $D_l \cap L_b$. Thus we need only count the number of points in $D_l \cap L_b$, which is $d-3$ by Proposition 3.2.6(3). Now we see that $D_l \cdot L_b = d-3$, so $\eta(D_l)$ is a curve of degree $d-3$.

Second, we compute the intersection number $D_{l_1} \cdot D_{l_2}$ for two general lines l_1 and l_2 on A. Let the images $\bar{l}_1 := f(l_1)$ and $\bar{l}_2 := f(l_2)$ be two general secant lines of C such that $\bar{l}_1 \cap \bar{l}_2 = \emptyset$. By Lemma 4.2.17, $\#(D_{l_1} \cap D_{l_2}) = \frac{(d-3)(d-4)}{2}$. This immediately gives $D_{l_1} \cdot D_{l_2} \geq \frac{(d-3)(d-4)}{2}$ as the intersection product. Unfortunately, we cannot show that the intersection is simple a priori, so we need some argument. We have $D_l \cap e_i \neq \emptyset$ for a general l because $D_l \cap e_i$ contains the point corresponding to a marked conic $(\beta_i \cup \alpha, \beta_i|_C)$, where α is the unique line intersecting β_i and l. Moreover, for two general l_1 and l_2 we have that $D_{l_1} \cap D_{l_2} \cap e_i = \emptyset$ and that $D_{l_1} \cap e_i$ and $D_{l_2} \cap e_i$ are contained in the smooth locus of $\mathcal{H}_2$. Thus, taking the minimal resolution of $\mathcal{H}_2$ near e_i if necessary, we can see that $D_{l_1} \cdot D_{l_2} \leq (d-3)^2 - s = \frac{(d-3)(d-4)}{2}$ and hence $D_{l_1} \cdot D_{l_2} = \frac{(d-3)(d-4)}{2}$. Moreover, $e_i^2 = -1$ and since $e_i \cap e_j = \emptyset$ we obtain that $\eta\colon \mathcal{H}_2 \to \mathbb{P}^2$ is the blow-up at $c_1, \dots, c_s$. Thus $D_l \sim (d-3)h - \sum_{i=1}^{s} e_i$ for a general $l \in \mathcal{H}_1$ and, by the flatness of $\mathcal{D}_1 \to \mathcal{H}_1$, this expression holds for any $l \in \mathcal{H}_1$.

(2) Let $L'_{p_{ij}} = L_{p_{ij}} - e_i$ (note that $e_i \subset L_{p_{ij}}$). We see that $L'_{p_{ij}} \subset D_{l_{ij}}$ and $D_{l_{i1}} - L'_{p_{i1}} = D_{l_{i2}} - L'_{p_{i2}}$, which we denote by D_{β_i}. Now

$$D_{\beta_i} \sim (d-4)h - \sum_{k \neq i} e_k.$$

It is easy to see that the D_{β_i} have the following properties:

$$D_{\beta_i} \cap e_i = \emptyset; \tag{4.2}$$

$$D_{\beta_i} \cap D_{\beta_j} \cap D_{\beta_k} = \emptyset. \tag{4.3}$$

We only prove (4.2). Since $D_{\beta_i} \cap e_i \neq \emptyset$ would imply that e_i is a component of D_{β_i}, it suffices to prove that, for a general l, $D_{\beta_i} \cap D_l$ does not contain a point of e_i. By Lemma 4.2.18, $D_{\beta_i} \cap D_l$ contains $\frac{(d-4)(d-5)}{2}$ points corresponding to bi-secant conics intersecting β_i and l except conics containing β_i. That being said, we have $D_l \cdot D_{\beta_i} = \frac{(d-4)(d-5)}{2}$ and so the conics we count in Lemma 4.2.18 correspond to all the intersection points of $D_{\beta_i} \cap D_l$. Consequently, $D_{\beta_i} \cap D_l$ does not contain a point of e_i.

From (4.2) and the trivial equality

$$(d-4)h - \sum_{i \geq k+1} e_i = D_{\beta_k} + e_1 + \cdots + e_{k-1}$$

we obtain $e_k \not\subset \mathrm{Bs}\left|(d-4)h - \sum_{i \geq k+1} e_i\right|$. Since

$$\mathcal{O}_{\mathcal{H}_2}\left((d-4)h - \sum_{i \geq k+1} e_i\right) \otimes_{\mathcal{O}_{\mathcal{H}_2}} \mathcal{O}_{e_k} \simeq \mathcal{O}_{e_k},$$

it follows that

$$H^0\left(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}\left((d-4)h - \sum_{i \geq k+1} e_i\right)\right) \to H^0(\mathcal{H}_2, \mathcal{O}_{e_k})$$

is surjective. Therefore, by the exact sequence

$$0 \to \mathcal{O}_{\mathcal{H}_2}\Big((d-4)h - \sum_{i \geq k} e_i\Big) \to \mathcal{O}_{\mathcal{H}_2}\Big((d-4)h - \sum_{i \geq k+1} e_i\Big) \to \mathcal{O}_{e_k} \to 0$$

we have $H^1\big(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}\big((d-4)h - \sum_{i=1}^{s} e_i\big)\big) \simeq H^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(d-4)h)$. It is now easy to see that $h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(d-4)h) = 0$, which shows (2).

(3) Since no conic on A intersects all the lines on A, it follows that $|D_l|$ has no base point. When $d = 5$, the image of $\Phi_{|D_l|}$ is $\mathbb{P}^2$ by $(D_l)^2 = 1$.

For $d \geq 6$, we prove that D_l is very ample. By (2) and [DaG, Thm. 3.1], it suffices to prove that

$$h^0\Big(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}\Big(h - \sum_{j=1}^{d-3} e_{i_j}\Big)\Big) = 0$$

for any set of $d-3$ exceptional curves $e_{i_1}, \dots, e_{i_{d-3}}$. Assume by way of contradiction that there exists an effective divisor $L \in \big|h - \sum_{j=1}^{d-3} e_{i_j}\big|$ for a set of $d-3$ exceptional curves $e_{i_1}, \dots, e_{i_{d-3}}$. Since $\frac{(d-2)(d-3)}{2} - (d-3) \geq 3$, we can find at least three e_i such that $i \notin \{j_1, \dots, j_{d-3}\}$. For $i \notin \{j_1, \dots, j_{d-3}\}$, noting that $D_l \sim D_{\beta_i} + h - e_i$, $D_l \cdot L = 0$, and $L \cdot (h - e_i) > 0$ yields $L \subset D_{\beta_i}$. This contradicts (4.3), since the number of i such that $i \notin \{j_1, \dots, j_{d-3}\}$ is at least 3.

Next we show that $h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = d-2$. By the Riemann–Roch theorem, $\chi(\mathcal{O}_{\mathcal{H}_2}(D_l)) = d-2$. Since $h^2(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(-D_l + K_{\mathcal{H}_2})) = 0$, we see that $h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = d-2$ is equivalent to $h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = 0$; since $|D_l|$ has no base point, $\big|(d-3)h - \sum_{i \geq k+1} e_i\big|$ is likewise equivalent. Thus the proof that $h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = 0$ is much the same as our preceding proof of (2), so we omit it.

(4) This part of the theorem follows from [Gi, Prop. 1.1]. □

REMARK. When $d = 5$, the morphism defined by $|D_l|$ contracts three curves D_{e_i} $(i = 1, 2, 3)$, which are nothing but the strict transforms of three lines passing through two of the c_j. In other words, the composite $S^2C \leftarrow \mathcal{H}_2 \to \check{\mathbb{P}}^2$ is the Cremona transformation.

COROLLARY 4.2.19. *$H^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(i)) \simeq H^0(\check{\mathbb{P}}^{d-3}, \mathcal{O}_{\check{\mathbb{P}}^{d-3}}(i))$ for $i = 1, 2$.*

Proof. The assertion follows from Theorem 4.2.11(4). □

The following corollary contains the nontrivial result that, for a general $l \in \mathcal{H}_1$, D_l parameterizes conics that properly intersect l.

COROLLARY 4.2.20. *For a general $l \in \mathcal{H}_1$, the locus D_l does not contain any point corresponding to the line pairs $l \cup m$ with $m \in \mathcal{H}_1$. Hence D_l parameterizes all conics that properly intersect l.*

Proof. Fix $m \in \mathcal{H}_1$ such that $l \cup m$ is a line pair. If $(\bar{m}, b)$ is the marked line given by m, then we have $d-2$ line pairs $l \cup m, l_1 \cup m, \dots, l_{d-3} \cup m$. Since $L_b \sim h$, it follows that $h \cdot D_l = d-3$ and so $l_1 \cup m, \dots, l_{d-3} \cup m \in D_l$. Thus $l \cup m \notin D_l$. □

5. Varieties of Power Sums for Special Quartics F_4

From now on we assume $d \geq 6$. In this section we prove our main result (Theorem 1.5.1). The proof consists of several steps, which we summarize as follows.

In the first step (Section 5.1), we construct a finite birational morphism $\Phi\colon \tilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$ (see Corollary 5.1.5); this is a part of the statement of Theorem 1.5.1. For that purpose we modify the morphism $\psi\colon \mathcal{U}_2 \to A$ (as in the Section 4.2.3) to obtain a finite one; see Proposition 5.1.1, which is a refinement of Proposition 4.2.13.

In the second step (Section 5.2), we describe the image of Φ by constructing the special quartic hypersurfaces F_4 that live in the projective space dual to the ambient of $\mathcal{H}_2$. Finally, we complete the proof of Theorem 1.5.1 in Section 5.3.

5.1. Construction of the Finite Birational Morphism Φ

Let $\rho\colon \tilde{A} \to A$ be the blow-up along $\bigcup_{i=1}^s \beta_i'$. Let $\tilde{\mathcal{U}}_2 := \mathcal{U}_2 \times_A \tilde{A}$; in other words, $\tilde{\mathcal{U}}_2$ is the blow-up of $\mathcal{U}_2$ along $\Gamma := \mathcal{U}_2 \cap \left(\bigcup_{i=1}^s \beta_i' \times \mathcal{H}_2\right)$. Note that $\tilde{\mathcal{U}}_2$ is naturally contained in $\tilde{A} \times \mathcal{H}_2$ because $\mathcal{U}_2$ is contained in $A \times \mathcal{H}_2$. The next proposition contains the final finiteness result we need.

Proposition 5.1.1 (Finiteness III). *$\tilde{\mathcal{U}}_2$ is Cohen–Macaulay, and the natural morphism $\tilde{\psi}\colon \tilde{\mathcal{U}}_2 \to \tilde{A}$ is finite $\left(\text{of degree } n := \frac{(d-1)(d-2)}{2}\right)$. In particular, $\tilde{\psi}$ is flat.*

Before proving Proposition 5.1.1, we construct a morphism $\Phi\colon \tilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$ and show that Φ is finite and birational if Proposition 5.1.1 is admitted.

We may take $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$ since we assume that $d \geq 6$. Consider the following diagram:

$$\begin{tikzcd} & \tilde{\mathcal{U}}_2 \arrow[dl, "\tilde{\psi}"'] \arrow[dr, "\tilde{\mu}"] & \\ \tilde{A} & & \mathcal{H}_2, \end{tikzcd} \tag{5.1}$$

where $\tilde{\mathcal{U}}_2 \subset \tilde{A} \times \mathcal{H}_2$.

Definition 5.1.2. Let $\tilde{a}$ be a point of $\tilde{A}$. We say that $\tilde{\psi}^{-1}(\tilde{a}) \in \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$ is *the cluster of conics attached to $\tilde{a}$* and denote it by $\mathcal{Z}_{\tilde{a}}$. A conic q such that $q \in \operatorname{Supp} \mathcal{Z}_{\tilde{a}}$ is called *a conic attached to $\tilde{a}$.*

We add the following pieces of notation.

Notation 5.1.3. (1) $E_i := \rho^{-1}(\beta_i')$ for $i = 1, \ldots, s$.

(2) By Proposition 2.3.3(5), there exist $d-4$ lines $\alpha_{i1}, \ldots, \alpha_{id-4}$ that are distinct from β_i and intersect both C and β_i outside $C \cap \beta_i$. Let $t_{ik} := \alpha_{ik} \cap C$. Corresponding to α_{ik} are two marked conics $(\alpha_{ik} \cup \beta_i;\, p_{i1}, t_{ik})$ and $(\alpha_{ik} \cup \beta_i;\, p_{i2}, t_{ik})$. We denote by ξ_{ijk} the conics on A corresponding to $(\alpha_{ik} \cup \beta_i;\, p_{ij}, t_{ik})$, where $i = 1, \ldots, s$, $j = 1, 2$, and $k = 1, \ldots, d-4$.

EXAMPLE 5.1.4. We describe the fiber of $\tilde{\psi}$ over a general point $\tilde{a} \in E_i$ for some i; that is, we exhibit n conics attached to $\tilde{a}$. We need the description in the proof of Corollary 5.1.5.

Let $a := \rho(\tilde{a}) \in A$ and $b := f(a) \in \beta_i$. We use the notation of Proposition 4.2.13. Since $\deg C_b = d - 2$, the number of bi-secant conics through b not belonging to the family e_i is given by the number of double points of C_b, which is $\frac{(d-3)(d-4)}{2}$. Moreover, there are $2(d-4)$ conics ξ_{ijk} through a.

The number of remaining conics is $3 = n - \frac{(d-3)(d-4)}{2} - 2(d-4)$. Such conics will belong to e_i. We look for three such conics. By Lemma 5.1.6 to follow, $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let $\sigma_i \colon E_i \to \mathbb{P}^1$ be a projection that differs from $E_i \to \beta_i'$, and let S_i be the strict transform on $\tilde{A}$ of the locus of lines intersecting β_i. Then it is easy to see that $S_i|_{E_i}$ does not contain any fiber γ_i of σ_i. Moreover, $S_i|_{E_i} \sim 2\gamma_i + 3f_i$, where f_i is a fiber of $E_i \to \beta_i'$. Let γ_i' be the fiber of σ_i through $\tilde{a}$; then γ_i' intersects S_i at three points. Corresponding to these three points are three lines on B intersecting β_i. Denote by $l_1, l_2, l_3 \subset A$ the strict transforms of these three lines. Then $\beta_i' \cup l_j$ $(j = 1, 2, 3)$ are the conics on A that we seek.

COROLLARY 5.1.5. *There exists a finite birational morphism $\Phi \colon \tilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$ that attaches to $\tilde{a}$ the class of the cluster $\mathcal{Z}_{\tilde{a}}$.*

To show Corollary 5.1.5 and also Proposition 5.1.1, we need the following small technicality. Let $f \colon A \to B$ be the blow-up of B along a general smooth rational curve C_d. The following lemma can be regarded as asserting the generality of C_d.

LEMMA 5.1.6. *Let $\beta_i' \subset A$ be the strict transform of a bi-secant line β_i of C_d. Then*

$$\mathcal{N}_{\beta_i'/A} = \mathcal{O}_{\beta_i'}(-1) \oplus \mathcal{O}_{\beta_i'}(-1).$$

Proof. We prove this lemma by using the inductive construction of C_d. The assertion is clear for $d = 1$ because C_1 has no bi-secant line.

Suppose the assertion holds for C_{d-1}. Choose a general uni-secant line $l \subset B$ of C_{d-1}. Let $m_1, \dots, m_{d-2}$ be the lines on B intersecting both C_{d-1} and l outside $C_{d-1} \cap l$. By the generality of C_{d-1}, we can assume that all of $m_1, \dots, m_{d-2}$ are uni-secant lines of C_{d-1}.

Let $A' \to B$ be the blow-up along $C_{d-1} \cup l$. Observe that the smoothing of $C_{d-1} \cup l$ to C_d induces the smoothing of A' to A. Let $\tilde{m}_i$ be the strict transform of m_i on A'. By the smoothing construction of C_d from $C_{d-1} \cup l$ and our assumption on induction, we have only to prove $\mathcal{N}_{\tilde{m}_i/A'} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let $A_1' \to B$ be the blow-up along l and $A_2' \to A_1'$ the blow-up along the strict transform of C_{d-1}. Denote by m_i' and m_i'' the respective strict transforms of m_i on A_1' and A_2'. Then $\mathcal{N}_{\tilde{m}_i/A'} = \mathcal{N}_{m_i''/A_2'}$. We consider the projection of B from the line l. Since m_i' is a fiber of $A_1' \to Q$, we have $\mathcal{N}_{m_i'/A_1'} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let F be the exceptional divisor of $A_1' \to Q$ and let F' be the strict transform of F on A_2'. We may suppose that F and C_{d-1}' intersect transversely, in which case $F' \to F$ is the blow-up at $d-2$ points $m_i' \cap C_{d-1}'$ ($i = 1, \ldots, d-2$). Thus $F' \cdot m_i'' = -1$ and $\mathcal{N}_{m_i''/F'} = \mathcal{O}_{\mathbb{P}^1}(-1)$, which implies the assertion. $\square$

Proof of Corollary 5.1.5. By Proposition 5.1.1 and the universal property of Hilbert schemes, we obtain a naturally defined map $\Phi\colon \tilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$.

For a general point $\tilde{a}$ of $\tilde{A}$, there exist n distinct attached conics; by Corollary 3.2.1, the only intersection point of all these conics is $\tilde{a}$. Hence the cluster $\mathcal{Z}_{\tilde{a}}$ determines such an $\tilde{a}$ and so Φ is birational.

We prove Φ to be finite. First we show that no curve in E_i is contracted by Φ. Since $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$ by Lemma 5.1.6, it suffices to show that no general fibers of $\mathbb{P}^1 \times \mathbb{P}^1$ in two directions are contracted by Φ. This follows from Example 5.1.4 and the definition of Φ. Similarly, we can use Example 4.2.14 to show that no curve in the strict transform of E_C is contracted by Φ. Thus, a possible positive-dimensional and irreducible component of a fiber of Φ is the strict transform of an irreducible component of a multi-secant conic of C. We show that the strict transform of an irreducible component of a multi-secant conic $\bar{q}$ of C cannot be contracted by Φ, treating several cases separately.

Case 1: $\bar{q}$ intersects a β_i. Let $\bar{q}_0$ be an irreducible component of $\bar{q}$ other than β_i. By the preceding considerations, we need only show that the strict transform of $\bar{q}_0$ is not contracted by Φ. Let $\tilde{a}$ be the intersection point of E_i and the strict transform of $\bar{q}_0$. By Example 5.1.4, we can choose a conic ξ_{ijk} attached to $\tilde{a}$ such that ξ_{ijk} is not attached to any other point on the strict transform of $\bar{q}_0$. Thus the strict transform of $\bar{q}_0$ cannot be contracted by Φ.

Case 2: $\bar{q}$ is a bi-secant conic of C and intersects none of the β_i. Let t be any one of two points in $C \cap \bar{q}$. By the assumptions of this case, $t \notin \bigcup_{i=1}^{s} \beta_i$. By Proposition 3.2.6(2), there exists a k-secant conic $\bar{q}'$ ($k \geq 3$) through t. (In Proposition 3.2.6(2) we assumed generality of a point of C, but the same proof works for any point of C by counting the number of conics with multiplicities.) By Proposition 4.2.6(a), there exists a conic q' on A such that (i) its image on B is $\bar{q}'$ and (ii) q' contains the fiber of $E_C \to C$ over t. By the assumptions of this case, we see that $\bar{q}$ and $\bar{q}'$ do not have a common irreducible component. Let $\tilde{a} \in \tilde{A}$ be the intersection point of the strict transform of $\bar{q}$ and an irreducible component of q'. Then q' is a conic attached to $\tilde{a}$ (by Example 4.2.14) but is not attached to any other points on the strict transform of $\bar{q}$. Thus the strict transform of $\bar{q}$ cannot be contracted by Φ.

Both in Cases 1 and 2, we found a special point $\tilde{a}$ on the strict transform of $\bar{q}$ such that there are at least two conics attached to $\tilde{a}$. Hence there are at least two multi-secant conics of C through a general point of such a $\bar{q}$.

Case 3: $\bar{q}$ is a k-secant conic ($k \geq 3$) *and intersects none of the β_i.* In this case we use induction on the degree d of C to show that there are at least two multi-secant conics of C through a general point of $\bar{q}$. This suffices for the assertion because $\bar{q}$ is smooth by the assumption of Case 3.

Assume that $d = 6$. By the two-ray game starting from the blow-up of B along C (see [TZ2, Prop. 3.11(1)]), we see that there are no quadri-secant conics (for otherwise its strict transform on A is a flopping curve, in contradiction to [TZ2, Prop. 3.11(1)]]). Let $A \dashrightarrow A'$ be the flop of the strict transforms of the β_i. Then there exists a birational morphism $f'\colon A' \to B$. By [TZ2, Prop. 3.11(2)], the strict transform $\hat{q}$ of $\bar{q}$ on A' is a nontrivial fiber of f'. Let $s := f'(\hat{q})$ and take a general bi-secant conic of C' through s. Then, coming back, its strict transform $\bar{q}'$ on B is a bi-secant conic of C intersecting $\bar{q}$. Hence there are at least two multi-secant conics of C through a general point of $\bar{q}$.

Now assume that $d \geq 7$; then $C = C_d$ is obtained by smoothing $C_{d-1} \cup l$, where l is a uni-secant line of C_{d-1}. It follows that $\bar{q}$ is the deformation of a k-secant conic $\bar{q}_0$ of $C_{d-1} \cup l$, so $\bar{q}_0$ is at least a $(k-1)$-secant conic of C_{d-1}. By the conclusion of Cases 1 and 2 and by the assumption of the induction, there are at least two multi-secant conics of C_{d-1} through a general point of $\bar{q}_0$. Thus there exist at least two multi-secant conics of C through a general point of $\bar{q}$. □

Proof of Proposition 5.1.1. The proof consists mostly of a local analysis of the morphism $\mathcal{U}_2 \to A$ in the neighborhood of Γ.

It is easy to describe Γ set-theoretically. Recall Notation 5.1.3, and let D_{β_i} be as in the proof of Theorem 4.2.11. Then Γ is set-theoretically the union of $\beta_i' \times e_i$,

$$\Gamma_i := \{(x, q) \mid q \in D_{\beta_i},\ x = q \cap \beta_i'\}$$

(which is a section of μ over D_{β_i}), and

$$\Gamma_{ijk} := \beta_i' \times \xi_{ijk} \quad (i = 1, \ldots, s,\ j = 1, 2,\ k = 1, \ldots, d-4).$$

The conic ξ_{ijk} does not belong to e_i by our choice of marking. Moreover, we have the following statement.

CLAIM 5.1.7. *The conic ξ_{ijk} does not belong to D_{β_i}.*

Proof. We consider the projection of B from a bi-secant line β_i. Let $C' \subset Q$ be the image of C by this projection and let p_{ij}' be the point of C' corresponding to p_{ij}, where p_{ij} is one of the two points of $C \cap \beta_i$. By this projection, the line α_{ik} maps to a point, which we denote by s_{ik}. Let F be the exceptional divisor of the blow-up along β_i and let F' be the image of F on Q. We say that a ruling of $F' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is *horizontal* if it does not come from a fiber of $F \to \beta_i$. Note that the image $q' \subset Q$ of a general conic q belonging to D_{β_i} is a bi-secant line of C'. So if $\xi_{ijk} \in D_{\beta_i}$ then ξ_{ijk} would also correspond to a bi-secant line of C', which must be the horizontal ruling of F' through p_{ij}' and s_{ik}. By inductive construction of C, however, we can prove that p_{ij}' and s_{ik} do not lie on a horizontal ruling (cf. the proof of Lemma 5.1.6). Thus we have the claim. □

Therefore, all of $\beta_i' \times e_i$, Γ_i, and Γ_{ijk} are disjoint ($i = 1, \ldots, s$, $j = 1, 2$, $k = 1, \ldots, d-4$).

CLAIM 5.1.8. *Γ is a reduced scheme, and $\mathcal{U}_2$ is smooth along Γ.*

Let us finish the proof of Proposition 5.1.1 while first admitting this claim, by which the morphism $\tilde{\mathcal{U}}_2 \to \mathcal{U}_2$ is the blow-up along the reduced subscheme Γ

contained in the smooth locus of $\mathcal{U}_2$. The subscheme $\beta_i' \times e_i$ is a Cartier divisor of $\mathcal{U}_2$, so $\tilde{\mathcal{U}}_2 \to \mathcal{U}_2$ is isomorphic over $\beta_i' \times e_i$. The curve Γ_{ijk} is smooth. Moreover, the curve Γ_i has only planar singularities because so does D_{β_i}. Thus $\tilde{\mathcal{U}}_2$ is Cohen–Macaulay since $\mathcal{U}_2$ is.

We have only to prove that $\tilde{\psi}$ is finite. By Proposition 4.2.13, $\tilde{\psi}$ is finite outside $\bigcup_i E_i$. Note that $\tilde{\psi}^{-1}(E_i)$ consists of nothing but the inverse images of $\beta_i' \times e_i$, Γ_i, and Γ_{ijk} by $\tilde{\mathcal{U}}_2 \to \mathcal{U}_2$, all of which are $\mathbb{P}^1$-bundles over curves and are mapped to E_i finitely. Hence we are done.

Proof of Claim 5.1.8. We study $\mathcal{U}_2$ locally along Γ.

Let q be a conic on A belonging to D_{β_i}. Then—by Proposition 2.3.1(5), Claim 5.1.7, and $D_{\beta_i} \cap e_i = \emptyset$ (see (4.2) in the proof of Theorem 4.2.11)—we see that q is smooth near β_i' and intersects β_i' transversely. This implies that $\tilde{\mathcal{U}}_2$ is smooth along Γ_i. Observe that, near Γ_i, the morphism $\psi : \mathcal{U}_2 \to A$ is finite and hence flat. Since Γ is the pull-back of β_i' near Γ_i and since Γ_i is not contained in the ramification locus of ψ, it follows that Γ is reduced along Γ_i.

Let q be the fiber of $\mathcal{U}_2 \to \mathcal{H}_2$ over ξ_{ijk} or a point of e_i. Note that q is a conic on A and has only nodes as its singularities. We show that $h^1(\mathcal{N}_{q/A}) = 0$ and that the natural map $H^0(\mathcal{N}_{q/A}) \to H^0(T_p^1) \simeq \mathbb{C}$ is surjective, where p is any node of q and T_p^1 is the local deformation space of p. As in the proof of [HHi, Prop. 1.1], this implies that $\mathcal{H}_2$ coincides with the Hilbert scheme of conics on A at ξ_{ijk} or a point of e_i and that $\mathcal{U}_2$ is smooth near q.

We first treat the case where $q = \xi_{ijk} = \alpha_{ik}' \cup \beta_i' \cup \zeta_{i,3-j}$. Note that $\mathcal{N}_{\alpha_{ik}'/A} \simeq \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$, $\mathcal{N}_{\beta_i'/A} \simeq \mathcal{O}_{\mathbb{P}_1}(-1)^{\oplus 2}$, and $\mathcal{N}_{\zeta_{i,3-j/A}} \simeq \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$. We apply [HHi, Thm. 4.1] after setting $X = \xi_{ijk}$, $C = \beta_i'$, and $D = \alpha_{ik}' \cup \zeta_{i,3-j}$. Checking the conditions (a) and (b) of [HHi, Thm. 4.1], we see that condition (a) clearly holds. That condition (b) holds can be shown as follows.

(i) Let F be the exceptional divisor of the blow-up of B along α_{ik}. Observe that $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We call a fiber of $F \to \mathbb{P}^1$ in the other direction to $F \to \alpha_{ik}$ a horizontal fiber. Then the intersection points of the strict transform of C and F and of the strict transform of β_i and F do not lie on a common horizontal fiber. This can be proved by the inductive construction of $C = C_d$ in a similar fashion to the proof of Lemma 5.1.6—or by a straightforward dimensional computation as in the proof of Proposition 2.3.1(2).

(ii) Let G be the exceptional divisor of the blow-up of A along $\zeta_{i,3-j}$, and note that $G \simeq \mathbb{F}_1$. Then the intersection points of the strict transform of β_i' and G do not lie on the negative section of G. Indeed, since $E_C \cdot \zeta_{i,3-j} = -1$, the intersection of G and the strict transform of E_C is the negative section of G. However, the strict transforms of E_C and β_i' are disjoint.

Thus, by [HHi, Thm 4.1], ξ_{ijk} satisfies the desired properties.

Second, we treat the case when q is a fiber over a point of e_i. Note that $\bar{q} = \beta_i \cup \alpha$, where α is a line intersecting β_i. Denote by α' the strict transform of α. We present the discussion in terms of four cases:

(a) $\alpha \cap C = \emptyset$ and $\mathcal{N}_{\alpha/B} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$;
(b) $\alpha \cap C = \emptyset$ and $\mathcal{N}_{\alpha/B} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$;

(c) $\alpha = \alpha_{ik}$ for some k;
(d) α passes through a point of $\beta_i \cap C$.

In case (a) or (b), it is easy to see that the proof of [HHi, Thm 4.1] works as before by setting $X = q$, $C = \beta_i'$, and $D = \alpha'$; in case (c) or (d), we must modify that proof. Here we treat only case (c) because case (d) can be handled similarly. Note that $q = \beta_i' \cup \alpha_{ik}' \cup \gamma_{ik}$, where γ_{ik} is the fiber of E_C over t_{ik}, and observe that C is smooth. By [HHi, Cor. 3.2] and a simple dimension count, we can describe the restrictions of the normal bundle $\mathcal{N}_{q/A}$ to the components of q as follows:

$$\mathcal{N}_{q/A}|_{\beta_i'} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1), \quad \mathcal{N}_{q/A}|_{\alpha_{ik}'} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1), \quad \mathcal{N}_{q/A}|_{\gamma ik} = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}.$$

Set $C = \beta_i' \cup \gamma_{ik}$ and $D = \alpha_{ik}'$. As in [HHi, Thm. 4.1], let $S := C \cap D$. From our description of $\mathcal{N}_{q/A}|_{\beta_i'}$, $\mathcal{N}_{q/A}|_{\alpha_{ik}'}$, and $\mathcal{N}_{q/A}|_{\gamma ik}$, it follows that $H^1(\mathcal{N}_{q/A}|_C) = H^1(\mathcal{N}_{q/A}|_D) = \{0\}$. Moreover, by considering the tautological linear systems of $\mathbb{P}(\mathcal{N}_{q/A}|_{\beta_i'})$, $\mathbb{P}(\mathcal{N}_{q/A}|_{\alpha_{ik}'})$, $\mathbb{P}(\mathcal{N}_{q/A}|_{\gamma ik})$, and $\mathbb{P}(\mathcal{N}_{q/A})$, we see that $H^0(\mathcal{N}_{q/A}|_C) \oplus H^0(\mathcal{N}_{q/A}|_D) \to H^0(\mathcal{N}_{q/A}|_S)$ is surjective. Thus $h^1(\mathcal{N}_{q/A}) = 0$ holds. By [HHi, Cor. 3.2] again, we have the following exact sequences (cf. [HHi, (3) in the proof of Thm. 4.1]):

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{N}_{q/A}|_{\beta_i'} \to \mathcal{N}_{q/A}|_S \to 0;$$
$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{N}_{q/A}|_{\alpha_{ik}'} \to \mathcal{N}_{q/A}|_S \to 0;$$
$$0 \to \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(-1) \to \mathcal{N}_{q/A}|_{\gamma ik} \to \mathcal{N}_{q/A}|_S \to 0.$$

We can therefore view $H^0(\mathcal{N}_{q/A}|_C)$ and $H^0(\mathcal{N}_{q/A}|_D)$ as subspaces of $H^0(\mathcal{N}_{q/A}|_S)$. By [HHi, (2) in the proof of Thm 4.1], $H^0(\mathcal{N}_{q/A}|_C) \to H^0(T_p^1)$ and $H^0(\mathcal{N}_{q/A}|_D) \to H^0(T_p^1)$ are both surjective. Moreover, considering the tautological linear systems of $\mathbb{P}(\mathcal{N}_{q/A}|_{\beta_i'})$, $\mathbb{P}(\mathcal{N}_{q/A}|_{\alpha_{ik}'})$, $\mathbb{P}(\mathcal{N}_{q/A}|_{\gamma ik})$, and $\mathbb{P}(\mathcal{N}_{q/A})$, we see that the kernels of $H^0(\mathcal{N}_{q/A}|_C) \to H^0(T_p^1)$ and $H^0(\mathcal{N}_{q/A}|_D) \to H^0(T_p^1)$ do not coincide for any $p \in S$. Thus any nonzero element of $H^0(T_p^1) \simeq \mathbb{C}$ comes from an element of $H^0(\mathcal{N}_{q/A}|_C) \cap H^0(\mathcal{N}_{q/A}|_D)$, as in the end of the proof of [HHi, Thm. 4.1]. This implies that the natural map $H^0(\mathcal{N}_{q/A}) \to H^0(T_p^1)$ is surjective for any $p \in S$.

Note that, near e_i, the family $\mathcal{U}_2 \to \mathcal{H}_2$ is locally a deformation of a node with smooth discriminant locus e_i. Hence a local computation shows that Γ is reduced along $\beta_i' \times e_i$.

Next we prove that Γ is reduced along Γ_{ijk}. We need only show that $\mathcal{U}_2 \to A$ is unramified along Γ_{ijk}, since then Γ is the étale pull-back of β_i' near Γ_{ijk} and so is reduced. Recall that $S := (\alpha_k' \cap \beta_i') \cup (\zeta_{i,3-j} \cap \beta_i')$. Using a simple dimension count and the exact sequence

$$0 \to \mathcal{N}_{\beta_i'/A} \to \mathcal{N}_{\xi_{ijk}/A}|_{\beta_i'} \to T_S^1 \to 0,$$

we can prove that $\mathcal{N}_{\xi_{ijk}/A}|_{\beta_i'} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Thus $H^0(\mathcal{N}_{\xi_{ijk}/A}) \otimes \mathcal{O}_{\xi_{ijk}} \to \mathcal{N}_{\xi_{ijk}/A}$ is surjective at a point of Γ_{ijk} because it factors through the surjection $H^0(\mathcal{N}_{\xi_{ijk}/A}|_{\beta_i'}) \otimes \mathcal{O}_{\beta_i'} \to \mathcal{N}_{\xi_{ijk}/A}|_{\beta_i'}$. Therefore, $\mathcal{U}_2 \to A$ is unramified along Γ_{ijk}. $\square$

This completes the proof of Proposition 5.1.1. $\square$

5.2. Construction of the Special Quartics

To construct the special quartic hypersurface, we need the incidence variety in $\mathcal{H}_2 \times \mathcal{H}_2$ defined by the intersections of conics.

Much as with (4.1), we consider the diagram

$$\begin{array}{ccc} \hat{\mathcal{U}}_2 \subset \mathcal{U}_2 \times \mathcal{H}_2 & \xrightarrow{(\psi,\mathrm{id})} & A \times \mathcal{H}_2 \supset \mathcal{U}_2 \\ \downarrow & & \downarrow \\ \hat{\mathcal{D}}_2 \subset \mathcal{H}_2 \times \mathcal{H}_2 & \longrightarrow & \mathcal{H}_2, \end{array} \tag{5.2}$$

where $\hat{\mathcal{U}}_2 \subset \mathcal{U}_2 \times \mathcal{H}_2$ is the base change of $\mathcal{U}_2$ and $\hat{\mathcal{D}}_2$ is the image of $\hat{\mathcal{U}}_2$ on $\mathcal{H}_2 \times \mathcal{H}_2$. Similarly to our investigation of (4.1), we see here that the image $\mathcal{F}'$ in $\mathcal{H}_2 \times \mathcal{H}_2$ $\big($of the inverse image of $\bigcup_{i=1}^{n} \beta_i' \times \mathcal{H}_2\big)$ is not divisorial and does not dominate $\mathcal{H}_2$. Moreover, any component of $\hat{\mathcal{D}}_2$ outside $\mathcal{F}'$ dominates $\mathcal{H}_2$ and is divisorial or possibly the diagonal of $\mathcal{H}_2 \times \mathcal{H}_2$. Observe that, unlike the diagram (4.1), in this case there is no other nondivisorial component (cf. our proof of Proposition 4.2.15). Here we leave the possibility that the diagonal of $\mathcal{H}_2 \times \mathcal{H}_2$ is contained in the divisorial component of $\hat{\mathcal{D}}_2$, but in Lemma 5.2.2 we prove that this is not the case.

Let $\mathcal{D}_2 \subset \mathcal{H}_2 \times \mathcal{H}_2$ be the union of the divisorial components of $\hat{\mathcal{D}}_2$ with reduced structure. We have that $\mathcal{D}_2$ is Cartier since $\mathcal{H}_2 \times \mathcal{H}_2$ is smooth and that $\mathcal{D}_2 \to \mathcal{H}_2$ is flat since $\mathcal{D}_2$ is Cohen–Macaulay; also, $\mathcal{H}_2$ is smooth and $\mathcal{D}_2 \to \mathcal{H}_2$ is equidimensional. Let D_q be the fiber of $\mathcal{D}_2 \to \mathcal{H}_2$ over $q \in \mathcal{H}_2$ via the projection to the second factor.

LEMMA 5.2.1. *$D_q \sim 2(d-3)h - 2\sum_{i=1}^{e} e_i$ for a conic q, and D_q is a quadric section of $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$.*

Proof. The proof of the first statement is almost identical to the proof of Theorem 4.2.11(1). The second statement follows from Corollary 4.2.19. □

Now we proceed with construction of the quartic hypersurface, which occupies the balance of Section 5.2.

From now on we write $\mathbb{P}^{d-3} = \mathbb{P}_* V$, where V is the $(d-2)$-dimensional vector space. The crucial point in the following considerations is the equality

$$n = \dim S^2 V. \tag{5.3}$$

By the seesaw theorem, we have $\mathcal{D}_2 \sim p_1^* D_q + p_2^* D_q$. Consider the morphism $\mathcal{H}_2 \times \mathcal{H}_2$ into $\check{\mathbb{P}}^{d-2} \times \check{\mathbb{P}}^{d-3}$ defined by $|p_1^* D_l + p_2^* D_l|$, which is an embedding since $d \geq 6$. By Corollary 4.2.19,

$$H^0(\mathcal{H}_2 \times \mathcal{H}_2, \mathcal{D}_2) \simeq H^0(\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}, \mathcal{O}(2,2)).$$

Hence $\mathcal{D}_2$ is the restriction of a unique $(2,2)$-divisor on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$ that we denote by $\{\tilde{\mathcal{D}}_2 = 0\}$. Since $\{\tilde{\mathcal{D}}_2 = 0\}$ is symmetric, we may assume the equation $\tilde{\mathcal{D}}_2$ to be symmetric also. Actually, the desired quartic is obtained by restricting $\tilde{\mathcal{D}}_2$ to the diagonal and taking the dual in the sense of Dolgachev (see Appendix B), but we need more argument to obtain the generalization of Mukai's theorem.

For $q \in \mathcal{H}_2$, we denote by $\tilde{D}_q$ the restriction of $\tilde{\mathcal{D}}_2$ to the fiber over q. Note that $\tilde{\mathcal{D}}_2 \in S^2V \otimes S^2V$, so $\tilde{\mathcal{D}}_2$ defines a linear map $\lambda\colon S^2\check{V} \simeq (S^2V)^{\check{}} \to S^2V$. Let H_q be a linear form on $\check{V}$ corresponding to q. Then $\lambda(H_q^2) = \tilde{D}_q$ up to scalar, so we may choose H_q such that $\lambda(H_q^2) = \tilde{D}_q$ holds. We prove that λ is an isomorphism.

LEMMA 5.2.2. *$\mathcal{D}_2$ does not contain the diagonal of $\mathcal{H}_2 \times \mathcal{H}_2$. In particular, we have the following statement: Let $\tilde{a}$ be a general point of $\tilde{A}$, and let $q_1, q_2, \dots, q_n \in \mathcal{H}_2$ be the conics attached to $\tilde{a}$; then*

$$\tilde{D}_{q_i}(q_i) \neq 0$$

for $1 \leq i \leq n$.

Proof. Here we assume $d \geq 3$. It suffices to prove that $\tilde{D}_q(q) \neq 0$ for a general $q \in \mathcal{H}_2$. This is equivalent to showing that the image $D_q^\flat$ on $\bar{\mathcal{H}}_2$ of D_q does not contain $\bar{q}$, where $D_q^\flat$ is the closure of the locus of multi-secant conics of C that properly intersect $\bar{q}$. Now the assertion follows from the inductive construction of C_d from $C_{d-1} \cup \bar{l}$. From now on, we use $D_{q,d}^\flat$ to denote $D_q^\flat$ for C_d. If $d = 3$, then $D_q \sim 0$ and so the assertion is trivially true. If $D_{q',d-1}^\flat(\bar{q}') \neq 0$ for a general multi-secant conic $\bar{q}'$ of C_{d-1}, then $D_{q,d}^\flat(\bar{q}) \neq 0$ for a general multi-secant conic $\bar{q}$ of C_d. $\square$

Let $\tilde{a}$ be a general point of $\tilde{A}$, and let $q_1, \dots, q_n$ be the conics attached to $\tilde{a}$. By the definition of $\tilde{D}_{q_i}$ and the generality of $\tilde{a}$,

$$\tilde{D}_{q_j}(q_i) = 0 \ \ (j \neq i) \quad \text{and} \quad \tilde{D}_{q_i}(q_i) \neq 0. \tag{5.4}$$

The implications of (5.4) are that the $\tilde{D}_{q_1}, \dots, \tilde{D}_{q_n}$ are linearly independent and that, by (5.3), they span the vector space S^2V. Thus λ is an isomorphism.

The inverse $\lambda^{-1}\colon S^2V \to S^2\check{V}$ defines an element $\check{\mathcal{D}}_2 \in S^2\check{V} \otimes S^2\check{V}$. We consider the polarization map $\mathrm{pl}_2\colon S^2\check{V} \to \mathrm{Sym}_2 V$ (see Appendix B). We show that $\tilde{U} := \mathrm{pl}_2 \otimes \mathrm{pl}_2(\check{\mathcal{D}}_2) \in \mathrm{Sym}_2 V \otimes \mathrm{Sym}_2 V \subset \check{V}^{\otimes 4}$ is contained in $\mathrm{Sym}_4 V$, which implies that $\mathrm{pl}_2 \otimes \mathrm{pl}_2(\tilde{\mathcal{D}}_2)$ is the image of a quartic form in $S^4\check{V}$ by pl_4.

The following argument is almost identical to the proof of [DK, Thm. 9.3.1] (the identification becomes clearer when we construct the theta characteristic on $\mathcal{H}_1$ in [TZ1]). Let l be a general line on A, and let $l_1, \dots, l_{d-2}$ be the lines intersecting l. Note that $l_1, \dots, l_{d-2}$ correspond to lines on B intersecting both C and the image $\bar{l}$ of l on B except those through $C \cap \bar{l}$. Hence the number of such lines is $d-2$. Because l is general, so are $l_1, \dots, l_{d-2}$. We have $d-2$ reducible conics $r_1 := l \cup l_1, \dots, r_{d-2} := l \cup l_{d-2}$, and $D_{r_i} = D_l + D_{l_i}$. By Corollary 4.2.19, D_l and D_{l_i} are defined by the linear forms L and L_i. We may assume that $\lambda(H_{r_i}^2) = \tilde{D}_{r_i} = L_i L$. By Corollary 4.2.20, $L_i(r_i) \neq 0$ and $L_i(r_j) = 0$ for $i \neq j$. In other words, $\langle L_i, H_{r_i}\rangle \neq 0$ and $\langle L_i, H_{r_j}\rangle = 0$ for $i \neq j$, where $\langle \cdot, \cdot \rangle$ is the natural dual pairing. Thus $L_1, \dots, L_{d-2}$ and $H_{r_1}, \dots, H_{r_{d-2}}$ span $\check{V}$ and V, respectively, since $\dim \check{V} = d-2$. Moreover, $\{H_{r_i}\}$ and $\{L_i\}$ are dual to each other. Choose coordinates of V and $\check{V}$ such that H_{r_i} and L_i are the coordinate hyperplanes $\{x_i = 0\}$ and $\{u_i = 0\}$, respectively. Set $L = \sum a_i u_i$. For any $y = (y_1, \dots, y_{d-2}) \in V$, we have $\lambda\big(\sum y_i x_i^2\big) = \big(\sum a_i u_i\big)\big(\sum y_i u_i\big)$ since $\lambda(H_{r_i}^2) = L_i L$. Given $\tilde{U} \in \check{V}^{\otimes 4}$, this implies that $\tilde{U}(L, y, x, x) = \sum y_i x_i^2 = P_y\big(\sum x_i^3\big)$, where $x = (x_1, x_2, \dots, x_{d-2})$ and P_y is the polar with respect to y (see Appendix B). Thus we have $\tilde{U}(L, y, x, z) = \sum y_i x_i z_i$ for $z = (z_1, z_2, \dots, z_{d-2})$, whence $\tilde{U}(L, y, x, z)$ is symmetric for y, x, and z. Since $\tilde{U} \in \mathrm{Sym}_2 \check{V} \otimes \mathrm{Sym}_2 \check{V}$ and $\tilde{\mathcal{D}}_2$ is symmetric, we have shown that $\tilde{U} \in \mathrm{Sym}_4 \check{V}$.

Let F_4 be the quartic form associated to $\tilde{U}$; namely, $F_4 := \tilde{U}(x, x, x, x)$. From the construction, we obtain the following rather important property of F_4.

PROPOSITION 5.2.3. *The following equality holds*:

$$P_{\tilde{D}_q}(F_4) = H_q^2. \tag{5.5}$$

By the theory of polarity (see Appendix B), what we have done can be interpreted as $\lambda^{-1} = \mathrm{ap}_{F_4}^2$. Since λ^{-1} is an isomorphism, F_4 is nondegenerate.

5.3. *Description of the Image of* Φ

As we saw in Proposition 3.2.6(1), a general point of B gives n multi-secant conics of C through it. Conversely, we ask whether or not mutually intersecting n multi-secant conics of C do pass through one point. The next lemma partially answers this question, and it is sufficient for our purpose in the proof of Theorem 1.5.1. We remark that the case $d = 5$ is treated in [D, 4.3].

LEMMA 5.3.1. *Let $q_1, \dots, q_n$ be mutually intersecting n distinct multi-secant conics of C such that*:

(1) *all q_i are smooth*;
(2) *no two of q_i intersect at a point of $C \cup \bigcup_i \beta_i$; and*
(3) *if three of q_i pass through a point b, then no other q_i intersects a line through b outside b.*

Then all q_i pass through one point.

REMARK. The set of n conics through a general point satisfies the conditions of the lemma.

Proof of Lemma 5.3.1. The proof is combinatorial and proceeds in three steps as follows.

Step 1. Let $b \in B$ be a point such that five of q_i (say, $q_1, \dots, q_5$) pass through b. Then all the q_i pass through b.

By the double projection from b, the conics $q_1, \dots, q_5$ are mapped to points $p_1, \dots, p_5$ on $\mathbb{P}^2$. Suppose by way of contradiction that a smooth conic q_j does not pass through b. Let q_j', q_j'', and $\tilde{q}_j$ be the strict transforms of q_j on B_b, B_b', and $\mathbb{P}^2$ (respectively), and let $S := \pi_{2b}^* \tilde{q}_j$. By condition (3), q_j does not intersect a line through b; hence $\tilde{q}_j$ is a smooth conic through $p_1, \dots, p_5$. The conic $\tilde{q}_j$ is unique because a conic through five points is unique. It follows that $-K_{B'_b} \cdot q''_j = 4$ and $S \cdot q''_j = 4$, so $S \simeq \mathbb{F}_2$ and q''_j is the negative section. This implies that q_j is also unique. By reordering, we may assume that $j = n$ to derive a configuration such that all the conics pass through b except q_n. Denote by p_i the image of q_i $(i \neq n)$. Then $\tilde{q}_n$ and C_b intersect at p_i. Since $d \geq 6$ we have $\deg C_b \geq 3$, so $\tilde{q}_n \neq C_b$. By condition (2), $b \notin C$. Therefore, $\tilde{q}_n$ and C_b intersect at $n-1$ singular points of C_b. Since $\deg C_b \leq d$, it follows that $2(n-1) \leq 2d$—a contradiction.

Step 2. If four conics $q_1, \ldots, q_4$ pass through one point b, then all the conics pass through b.

By contradiction and Step 1, we assume that only the conics $q_1, \ldots, q_4$ pass through b. Pick any two conics (say, q_5 and q_6) not passing through b, and consider the double projection from b as in Step 1. Denote by $\tilde{q}_j$ $(j \geq 5)$ the image of q_j on $\mathbb{P}^2$. By condition (3), q_5 and q_6 do not intersect a line through b; hence $\tilde{q}_5$ and $\tilde{q}_6$ are conics on $\mathbb{P}^2$. Therefore, $q_5 \cap q_6$ must lie on one of $q_1, \ldots, q_4$ because otherwise $\tilde{q}_5$ and $\tilde{q}_6$ would intersect at five points, a contradiction (as in Step 1). Thus any two conics intersect on $q_1, \ldots, q_4$. Let p_i be the intersection $q_i \cap q_5$ for $i = 1, \ldots, 4$. Then the q_j $(j \geq 5)$ pass through one of p_i, so one of the p_i there (say, p_1) passes pass through at least $\lceil \frac{(n-5)}{4} \rceil$ conics. By Step 1, $\lceil \frac{(n-5)}{4} \rceil \leq 2$ (already q_1 and q_5 pass through p_1), which implies that $d = 6$. We exclude this case in Step 3. Note that if $d = 6$, then the four conics q_1, q_2, q_5, and q_6 mutually intersect and all the intersection points are different. By reordering conics, we assume that q_i $(1 \leq i \leq 4)$ satisfy this property in Step 3.

Step 3. We complete the proof.

Assume by way of contradiction that $q_1, \ldots, q_n$ do not pass through one point on B. If $d \geq 7$ then, by Steps 1 and 2,

$$\text{at most three of the } q_i \text{ pass through any intersection point.} \tag{5.6}$$

Let m be the number of conics in a maximal tree T of the q_i such that two conics in T pass through any intersection point. Observe that T is connected since the q_i mutually intersect. The number of intersection points of the q_i contained in T is $\frac{m(m-1)}{2}$.

By the maximality of T, a conic not belonging to T passes through one of the intersection points of conics in T. By (5.6), no two conics not belonging to T pass through one of the intersection points of conics in T. Hence $\frac{m(m-1)}{2} + m \geq n$, which implies that $m \geq d-2$ because $n = \frac{(d-1)(d-2)}{2}$. By reordering, we assume that $q_1, \ldots, q_m$ belong to T. If $d = 6$, then we take $q_1, \ldots, q_4$ as in the last part of Step 2. Consider the projection $B \dashrightarrow \mathbb{P}^3$ from the conic q_1. (For facts on the projection of B from a smooth conic, we refer to no. 22 (resp. no. 26) of [MoM1] for condition (2) (resp. (1)); see also [MoM2, p. 533] for a discussion.) Then $q_2, \ldots, q_m$ are mapped to lines $l_2, \ldots, l_m$ intersecting mutually on $\mathbb{P}^3$ and all the intersection points are different. Thus $l_2, \ldots, l_m$ span a plane, which shows that $q_1, \ldots, q_m$ span a hyperplane section H on B. Since C intersects q_i at two points or more, it follows from condition (2) that C intersects H at $2m$ points or more. But $2m \geq 2(d-2) > d$ and so C must be contained in H, in contradiction with Corollary 2.2.3. □

We can now complete the proof of Theorem 1.5.1.

Proof of Theorem 1.5.1 (*conclusion*). First we show that $\operatorname{Im}\Phi$ is an irreducible component of

$$\mathrm{VSP}(F_4, n; \mathcal{H}_2) := \overline{\{(H_1, \ldots, H_n) \mid H_i \in \mathcal{H}_2\}} \subset \mathrm{VSP}(F_4, n).$$

Toward this end, we let

$$Z := \{(H_1, \ldots, H_n) \in \operatorname{Hilb}^n \check{\mathbb{P}}^{d-3} \mid H_1^4 + \cdots + H_n^4 = F_4,\ H_i \in \mathcal{H}_2\}.$$

For a general point $\tilde{a}$ and conics $q_1, \ldots, q_n$ attached to $\tilde{a}$ in the sense of Definition 5.1.2, we have (5.4). Conversely, n conics q_i satisfying (5.4) and conditions (1)–(3) of Lemma 5.3.1 determine a point of $\tilde{A}$. Note that (1)–(3) of Lemma 5.3.1 are *open* conditions, so we need only prove that (5.4) is equivalent to

$$\alpha_1 H_{q_1}^4 + \cdots + \alpha_n H_{q_n}^4 = F_4 \quad \text{with some nonzero } \alpha_i \in \mathbb{C}. \tag{5.7}$$

We see that (5.7) is equivalent to the following statement:

$$\text{if } \{G = 0\} \subset \check{\mathbb{P}}^{d-3} \text{ is any quartic through } q_1, \ldots, q_n, \text{ then } P_{F_4}(G) = 0. \tag{5.8}$$

Indeed, by the apolarity pairing, $\langle G, H_{q_i}^4 \rangle = 0$ if and only if $G(q_i) = 0$; hence the assumption on G is equivalent to $G \in \langle H_{q_1}^4, \ldots, H_{q_n}^4 \rangle^{\perp}$. Therefore, (5.7) is equivalent to $\langle H_{q_1}^4, \ldots, H_{q_n}^4 \rangle^{\perp} \subset \langle F_4 \rangle^{\perp}$. Since F_4 is nondegenerate, this is equivalent to (5.7).

We now show that (5.4) implies (5.8). If (5.4) holds then $\tilde{D}_{q_i}$ $(i \neq 1)$ generate the space of quadric forms passing through $[q_1]$, so we may write $G = Q_2 \tilde{D}_{q_2} + \cdots + Q_n \tilde{D}_{q_n}$ for Q_i the quadratic forms on $\check{\mathbb{P}}^{d-3}$. Since $G(q_i) = 0$ for $i \neq 1$, we have $Q_i(q_i)\tilde{D}_{q_i}(q_i) = 0$. Now $\tilde{D}_{q_i}(q_i) \neq 0$ implies that $Q_i(q_i) = 0$, so Q_i is a linear combination of $\tilde{D}_{q_j}$ $(j \neq i)$. As a result, G is a linear combination of $\tilde{D}_{q_i}\tilde{D}_{q_j}$ $(1 \leq i < j \leq n)$. Thus $P_{F_4}(G) = 0$ is a consequence of

$$P_{F_4}(\tilde{D}_{q_i}\tilde{D}_{q_j}) = P_{H_{q_i}}(\tilde{D}_{q_j}) = \tilde{D}_{q_j}(q_i) = 0.$$

Finally, we show that (5.7) implies (5.4). By (5.7),

$$H_{q_i}^2 = P_{\tilde{D}_{q_i}}(F_4) = \sum \alpha_j \langle \tilde{D}_{q_i}, H_{q_j}^4 \rangle H_{q_j}^2.$$

Since the $\tilde{D}_{q_i}$ are linearly independent, so are the $H_{q_j}^2$. Thus (5.4) holds.

Next we show that $\operatorname{Im}\Phi$ is uniquely identified from the incident variety $\mathcal{D}_2$. We prove a more precise statement as follows.

Claim 5.3.2. *Let* $(\mathcal{H}_2^k)^o$ *and* $(\operatorname{Hilb}^k \check{\mathbb{P}}^{d-3})^o$ $(k \in \mathbb{N})$ *be the complements of all the small diagonals of* $\mathcal{H}_2^k$ (k *times product of* $\mathcal{H}_2$) *and* $\operatorname{Hilb}^k \check{\mathbb{P}}^{d-3}$, *respectively. Let*

$$\mathrm{VSP}^o(F_4, n; \mathcal{H}_2) := \{(H_1, \ldots, H_n) \mid H_i \in \mathcal{H}_2,\ H_1^m + \cdots + H_n^m = F_4\}.$$

Let V^o be the inverse image of $\mathrm{VSP}^o(F_4, n; \mathcal{H}_2)$ *by the natural projection* $(\mathcal{H}_2^n)^o \to (\mathrm{Hilb}^n \check{\mathbb{P}}^{d-3})^o$*, and let* $(\mathcal{H}_2^n)^o \to (\mathcal{H}_2^2)^o$ *be the projection to any of two factors. Then a component of* V^o *dominating* $\mathcal{D}_2$ *dominates* $\mathrm{Im}\,\Phi$*. In particular,* $\mathrm{Im}\,\Phi$ *is uniquely identified from* $\mathcal{D}_2$*.*

Proof. Let $(q_1, q_2) \in \mathcal{D}_2 \cap (\mathcal{H}_2^2)^o$ be a general point, and let $\{q_i\}$ $(i = 1, \dots, n)$ be any set of mutually conjugate n conics that include q_1 and q_2. Since q_1 and q_2 are general, we may assume that all the q_i are general. By Lemma 5.3.1, it suffices to prove that $q_1, \dots, q_n$ satisfies conditions (1)–(3) of Lemma 5.3.1 because $\mathrm{Im}\,\Phi$ is an irreducible component of $\mathrm{VSP}(F_4, n; \mathcal{H}_2)$.

(1) Let $\bar{r}_1$ and $\bar{r}_2$ be mutually intersecting smooth conics on B, and let $\bar{r}_3$ be a line pair on B that intersects both $\bar{r}_1$ and $\bar{r}_2$. Since the Hilbert scheme of conics on B is 4-dimensional, the pair of $\bar{r}_1$ and $\bar{r}_2$ depends on seven parameters; if we fix $\bar{r}_1$ and $\bar{r}_2$, then $\bar{r}_3$ depends on one parameter. Thus the configuration $\bar{r}_1, \bar{r}_2, \bar{r}_3$ depends on eight parameters. Fix $\bar{r}_1$, $\bar{r}_2$, and $\bar{r}_3$. We count the number of parameters of C_d such that C_d intersects each of $\bar{r}_i$ $(i = 1, 2, 3)$ twice. The number of these parameters is $h^0((\mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) + 6 = 2d - 12 + 6 = 2d - 6$, where $+6$ means the sum of the numbers of parameters of two points on $\bar{r}_i$ $(i = 1, 2, 3)$. Since $2d - 6 + 8 = 2d + 2$, a general C_d has 2-dimensional pairs of mutually intersecting bi-secant conics that intersect at least one bi-secant line pair of C_d. Therefore, general pairs of mutually intersecting bi-secant conics of C_d that form a 3-dimensional family do not intersect a bi-secant line pair of C_d.

(2) Assume by way of contradiction that $\bar{q}_i$, $\bar{q}_j$, and $\bar{q}_k$ pass through a point b and that $\bar{q}_l$ does not pass through b but does intersect a line through b. Then, by the double projection from b, $\bar{q}_l$ is mapped to a line through the three singular points of the image of C_b corresponding to $\bar{q}_i$, $\bar{q}_j$, and $\bar{q}_k$. Thus we have only to prove that, for a general point of b on B, three double points of the image of C_b do not lie on a line.

Fix a general point $b \in B$. Let $\bar{r}_1, \bar{r}_2, \bar{r}_3$ be three conics on B through b such that, by the double projection from b, they are mapped to three collinear points on $\mathbb{P}^2$. The number of parameters of the C_d that intersect each of $\bar{r}_i$ twice is $h^0((\mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) = 2d - 12$, since $h^1((\mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-6)) = 0$. Observe that the number of parameters of $\bar{r}_1, \bar{r}_2, \bar{r}_3$ is 5 because the number of parameters of lines in $\mathbb{P}^2$ is 2 and that of three points on a line is 3. Hence the number of parameters of the C_d such that the image of C_d by the double projection from b has three collinear double points is at most $2d - 1$. Therefore, a general C_d does not satisfy this property.

(3) Let r_1 and r_2 be a general pair of mutually conjugate conics on A such that $\bar{r}_1$ and $\bar{r}_2$ are smooth and intersect at a point on $C \cup \bigcup_i \beta_i$. Such general pairs of conics r_1 and r_2 form a 2-dimensional family because $\dim\bigl(C \cup \bigcup_i \beta_i\bigr) = 1$; if one point t of $C \cup \bigcup_i \beta_i$ is fixed, then the pairs of conics such that $t \in \bar{r}_1 \cap \bar{r}_2$ form a 1-dimensional family. For a general pair r_1 and r_2, the number of the sets of n mutually conjugate conics that include both r_1 and r_2 is finite because D_{r_1}

and D_{r_2} have no common component. Thus $\{q_i\}$ contains no such pair (by generality), whence $\{q_i\}$ satisfies (3). $\square$

We have thus finished the proof of Theorem 1.5.1. $\square$

Appendix A: Relation to Mukai's Result

Here we sketch how the argument goes on if $d = 5$ and explain how the results are related to Theorem 1.2.1

Assume that $d = 5$. Associated to the birational morphism $\mathcal{H}_2 \to \check{\mathbb{P}}^2$ is a non-finite birational morphism,

$$\Phi\colon \tilde{A} \to A_{22} := \mathrm{VSP}(F_4, 6) \subset \mathrm{Hilb}^6 \check{\mathbb{P}}^2,$$

that fits into the following diagram:

$$\begin{tikzcd}
& & \tilde{A} \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] \arrow[ddrr, bend left, "\Phi"] & & \\
& A \arrow[rr, dashed] \arrow[dl, "f"'] & & A' \arrow[dr, "f'"] & \\
B & & & & A_{22}\,.
\end{tikzcd}$$

Here:

- A_{22} is a smooth prime Fano 3-fold of genus 12;
- ρ' is the blow-down of the three ρ-exceptional divisors E_i $(i = 1, 2, 3)$ over the strict transforms β_i' in the other direction (i.e., $A \dashrightarrow A'$ is the flops of β_1', β_2', and β_3'; cf. Lemma 5.1.6); and
- the morphism f' contracts the strict transform of the unique hyperplane section S containing C (see Corollary 2.2.3) to a general line on A_{22}.

The rational map $A_{22} \dashrightarrow B$ is the famous double projection of A_{22} from a general line m that was first discovered by Iskovskih (see [Is2]).

We now explain how f' and ρ' are interpreted in our context. As remarked after the proof of Theorem 4.2.11, the morphism $\mathcal{H}_2 \to \check{\mathbb{P}}^2$ defined by $|D_l|$ contracts three curves D_{e_i} that parameterize conics intersecting β_i'. Given that S is covered by the images of such conics, this contraction corresponds to the morphism f' contracting the strict transform of S.

We can see that any conic on A (except one belonging to D_{e_i}) corresponds to that conic on A_{22} in the usual sense, and the component of a Hilbert scheme of A_{22} that parameterizes conics is naturally isomorphic to $\check{\mathbb{P}}^2$. The three conics on A_{22} corresponding to the images of D_{e_i} are $\beta_i'' \cup m$, where the β_i'' are images of the flopped curve corresponding to β_i'.

Let $a \in E_i$. Then the six conics on A that are attached to a are ξ_{ij1} $(j = 1, 2)$, a conic q_a from D_{e_i}, and three conics from e_i (see the Remark at the end of Section 5.1). Moreover, if a moves in a fiber γ of the other projection $E_i \to \mathbb{P}^1$, then only the conic q_a from D_{e_i} varies. By the contraction $\mathcal{H}_2 \to \check{\mathbb{P}}^2$, there is no difference among points on γ. This is the meaning of the contraction ρ' of E_i in the other direction.

Finally, we remark that $\mathcal{H}_1$ is also naturally isomorphic to the component of a Hilbert scheme of A_{22} that parameterizes lines.

Appendix B: Theory of Polarity

We give a quick review of basic facts in the theory of polarity. The main references are [DK, Secs. 1, 2] and [D, Sec. 2].

(1) Denote by $\mathrm{Sym}_m V$ the image of the linear map

$$\check{V}^{\otimes m} \to \check{V}^{\otimes m},$$
$$t \mapsto \sum_{\sigma \in S_m} \sigma(t).$$

The map $\check{V}^{\otimes m} \to \mathrm{Sym}_m V$ is decomposed as $\check{V}^{\otimes m} \xrightarrow{s_m} S^m \check{V} \xrightarrow{p_m} \mathrm{Sym}_m V$, where s_m is the natural quotient map. Denote by $\mathrm{pl}_m \colon S^m \check{V} \to \mathrm{Sym}_m V$ the map that is equal to $\frac{1}{m!} p_m$. This is called the *polarization map.* Let $r_m \colon \mathrm{Sym}_m V \hookrightarrow \check{V}^{\otimes m} \xrightarrow{s_m} S^m \check{V}$ be the natural map. Then $\mathrm{pl}_m \circ r_m = r_m \circ \mathrm{pl}_m = \mathrm{id}$.

(2) For $F \in S^m \check{V}$, let $\tilde{F} := \mathrm{pl}_m(F)$. Then $F(x) = \tilde{F}(x, x, \ldots, x)$ for $x \in V$.

(3) For $F \in S^m \check{V}$ and $a \in V$, let $P_a(F)(x) := \tilde{F}(a, x, \ldots, x)$. It is easy to verify that

$$P_a(F) = \frac{1}{m} \sum_i a_i \frac{\partial F}{\partial x_i},$$

where a_i are coordinates of a and x_i are coordinates of V. Similarly, letting

$$P_{a,b,\ldots,c}(F) := \tilde{F}(a, b, \ldots, c, x, \ldots, x),$$

where the number of $a, b, \ldots, c$ is k, yields

$$P_{a,b,\ldots,c,x,\ldots,x}(F) = \frac{(m-k)!}{m!} \sum_{i_1,\ldots,i_k} a_{i_1} b_{i_2} \cdots c_{i_k} \frac{\partial^k F}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

This is called *the mixed polar of F with respect to $a, b, \ldots, c$.*

It is possible to regard this expression as the pairing between $F \in S^m \check{V}$ and $ab \cdots c \in S^k V$. By extending this pairing, we have

$$S^k V \times S^m \check{V} \to S^{m-k} \check{V},$$
$$(G, F) \mapsto P_G(F).$$

Furthermore, fixing F allows us to write

$$\mathrm{ap}_F^k \colon S^k V \to S^{m-k} \check{V},$$
$$G \mapsto P_G(F).$$

This is called the *apolarity map.*

When $m = k$, this pairing is sometimes denoted by $\langle G, F\rangle$ and is called the *apolarity pairing.*

(4) A basic property of the apolarity pairing is that

$$\langle F, ab\cdots c\rangle = \tilde{F}(a, b, \ldots, c),$$

where the number of $a, b, \ldots, c$ is m. In particular,

$$\langle F, a^m\rangle = \tilde{F}(a, a, \ldots, a) = F(a).$$

(5) If $m = 2k$, then F is said to be *nondegenerate* if

$$\mathrm{ap}_F^k\colon S^kV \to S^k\check{V}$$

is an isomorphism. In this case, there is an $\check{F} \in S^kV$ such that

$$(\mathrm{ap}_F^k)^{-1} = \mathrm{ap}_{\check{F}}^k;$$

here $\check{F}$ is called the *form dual to F.*

(6) Apolarity maps are usually considered in the projective setting; in other words, we typically consider $a \in \mathbb{P}_*V$ rather than $a \in V, \ldots$. In this situation, we denote by $H_a \in V$ an element corresponding to $a \in \mathbb{P}_*V$ that is unique up to scalar. By abuse of notation, we sometimes continue to write $P_a(F)$ rather than $P_{H_a}(F)$.

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