Cut Ideals of K_4-Minor Free Graphs
Are Generated by Quadrics

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1. Introduction

In this paper we prove a conjecture of Sturmfels and Sullivant [12] about toric ideals used in algebraic statistics. A new connection between commutative algebra and statistics was made by Diaconis and Sturmfels [5] when they introduced the fundamental notion of Markov basis. To explain the connection, we use the first example from the Oberwolfach lectures on algebraic statistics by Drton, Strumfels, and Sullivant [7].

EXAMPLE 1.1.   In a contingency table, both data and some marginals are reported. In Table 1, these marginals are the row and column sums. In order to test statistically the hypothesis that the verdicts are from a distribution independent of race, one must sample from a set of tables with the same marginals as Table 1. The usual way to sample is by a random walk on the set of tables with prescribed marginals, stopping when some test indicates that enough information has been collected. The nontrivial task is to find good steps (*Markov moves*) for the random walk, and here commutative algebra enters the picture.

Encode the numbers in Table 1 with monomials as in Table 2. The data entries in Table 2 are collected in the monomial $q_{11}^{19} q_{12}^{141} q_{21}^{17} q_{22}^{149} \in \mathbb{K}[q_{11}, q_{12}, q_{21}, q_{22}]$ and the marginal entries in the monomial $r_{1*}^{160} r_{2*}^{166} r_{*1}^{36} r_{*2}^{290} \in \mathbb{K}[r_{1*}, r_{2*}, r_{*1}, r_{*2}]$. The calculations that translate row and column sums into the algebraic setting are given by the ring homomorphism

Table 1   Data on death penalty verdicts
[1, 5.2.2]

Defendant's race	Yes	No	Total
White	19	141	160
Black	17	149	166
Total	36	290	326

Table 2 The commutative algebra
version of Table 1

q_{11}^{19}	q_{12}^{141}	r_{1*}^{160}
q_{21}^{17}	q_{22}^{149}	r_{2*}^{166}
r_{*1}^{36}	r_{*2}^{290}	

$$\phi \colon \mathbb{K}[q_{11}, q_{12}, q_{21}, q_{22}] \rightarrow \mathbb{K}[r_{1*}, r_{2*}, r_{*1}, r_{*2}]$$

defined by

$$\phi(q_{ab}) = r_{a*}r_{*b}.$$

The fibers of $r_{1*}^{160}r_{2*}^{166}r_{*1}^{36}r_{*2}^{290}$ are all monomials corresponding to tables with the same marginals as in Table 1, and that is the set of tables for which we need to find steps. The kernel of the map ϕ is a toric ideal, and a generating set of that ideal provides us with steps between the monomials in the fiber. In this easy example, the kernel is generated by $q_{11}q_{22} - q_{12}q_{21}$ and, for instance, provide a Markov move from $q_{11}^{19}q_{12}^{141}q_{21}^{17}q_{22}^{149}$ to $q_{11}^{20}q_{12}^{140}q_{21}^{16}q_{22}^{150}$, since their difference is a monomial multiplied by $q_{11}q_{22} - q_{12}q_{21}$. All monomials in the fiber can be reached by Markov moves using $q_{11}q_{22} - q_{12}q_{21}$, and the statisticians are able to sample from the set of tables with the same marginals as Table 1.

The benefit of translating problems from statistics to commutative algebra, as in Example 1.1, is a well-developed toolbox for finding generators of ideals—most prominently, using Gröbner basis.

Many statistical models are described by graphs, with a random variable for every vertex and marginals described by edges. If we flipped a coin for every vertex in a graph, then the vertex set would be partitioned into two parts: heads and tails. A partition of a graph into two parts is called a *cut,* and many questions in statistics, computer science, and optimization theory are naturally formulated (or easily transformed into) questions about cuts. There is also a rich geometric theory associated with cuts, as surveyed by Deza and Laurent [4].

DEFINITION 1.2.   For a graph G, the partition of $V(G)$ into A and B is the *cut* $A \mid B = B \mid A$. The edge set $\{ab \in E(G) : a \in A, b \in B\}$ induced by the cut $A \mid B$ is also denoted $A \mid B$ when no confusion can arise.

EXAMPLE 1.3.   We toss four coins 76 times and obtain the statistic on eight different cuts, as reported in Table 3. The marginals are encoded with the path graph $1-2-3-4$, and in Table 4 the cuts are tabulated together with how they cut the edges. Table 5 reports the marginals of Table 3—that is, how many times the different edges are cut. In algebraic statistics, the corresponding setup is two commutative rings,

$$\mathbb{K}\begin{bmatrix} q_{\{1,2,3,4\}|\emptyset}, & q_{\{1,2,3\}|\{4\}}, & q_{\{1,2,4\}|\{3\}}, & q_{\{1,2\}|\{3,4\}}, \\ q_{\{1,3,4\}|\{2\}}, & q_{\{1,3\}|\{2,4\}}, & q_{\{1,4\}|\{2,3\}}, & q_{\{1\}|\{2,3,4\}} \end{bmatrix}$$

and

$$\mathbb{K}[s_{12}, s_{23}, s_{34}, t_{12}, t_{23}, t_{34}],$$

Table 3   The number of cuts
of different types from
tossing four coins 76 times

Cut	Number of occurrences
$\{1,2,3,4\} \mid \emptyset$	8
$\{1,2,3\} \mid \{4\}$	13
$\{1,2,4\} \mid \{3\}$	12
$\{1,2\} \mid \{3,4\}$	6
$\{1,3,4\} \mid \{2\}$	9
$\{1,3\} \mid \{2,4\}$	8
$\{1,4\} \mid \{2,3\}$	11
$\{1\} \mid \{2,3,4\}$	9

Table 4   The cuts of the path graph $1-2-3-4$;
edges with vertices in different parts
(resp., the same part) are tabulated as 1 (resp., 0)

Cut	Edge 12	Edge 23	Edge 34
$\{1,2,3,4\} \mid \emptyset$	0	0	0
$\{1,2,3\} \mid \{4\}$	0	0	1
$\{1,2,4\} \mid \{3\}$	0	1	1
$\{1,2\} \mid \{3,4\}$	0	1	0
$\{1,3,4\} \mid \{2\}$	1	1	0
$\{1,3\} \mid \{2,4\}$	1	1	1
$\{1,4\} \mid \{2,3\}$	1	0	1
$\{1\} \mid \{2,3,4\}$	1	0	0

Table 5   The edge cuts of the path graph
$1-2-3-4$ given the cut statistic in Table 3

	Edge 12	Edge 23	Edge 34
# Cuts	37	35	44

together with a ring homomorphism $\phi\colon \mathbb{K}[q] \to \mathbb{K}[s,t]$ defined by

$$\phi(q_{\{1,2,3,4\}|\emptyset}) = t_{12}t_{23}t_{34}, \quad \phi(q_{\{1,2,3\}|\{4\}}) = t_{12}t_{23}s_{34},$$

$$\phi(q_{\{1,2,4\}|\{3\}}) = t_{12}s_{23}s_{34}, \quad \phi(q_{\{1,2\}|\{3,4\}}) = t_{12}s_{23}t_{34},$$

$$\phi(q_{\{1,3,4\}|\{2\}}) = s_{12}s_{23}t_{34}, \quad \phi(q_{\{1,3\}|\{2,4\}}) = s_{12}s_{23}s_{34},$$

$$\phi(q_{\{1,4\}|\{2,3\}}) = s_{12}t_{23}s_{34}, \quad \phi(q_{\{1\}|\{2,3,4\}}) = s_{12}t_{23}t_{34}$$

in accordance with Table 4, where s_{ij} denotes that the edge ij is separated and t_{ij} that it is kept together by the cut. The kernel of ϕ is a toric ideal generated by the binomials

$$q_{\{1,3,4\}|\{2\}}q_{\{1,2,3\}|\{4\}} - q_{\{1,4\}|\{2,3\}}q_{\{1,2\}|\{3,4\}},$$

$$q_{\{1,3\}|\{2,4\}}q_{\{1,2,3,4\}|\emptyset} - q_{\{1\}|\{2,3,4\}}q_{\{1,2,4\}|\{3\}},$$

$$q_{\{1,2,4\}|\{3\}}q_{\{1\}|\{2,3,4\}} - q_{\{1,4\}|\{2,3\}}q_{\{1,2\}|\{3,4\}},$$

$$q_{\{1,3\}|\{2,4\}}q_{\{1,2,3,4\}|\emptyset} - q_{\{1,2,3\}|\{4\}}q_{\{1,3,4\}|\{1\}}.$$

The toric ideal in Example 1.3 is the cut ideal of a path on four vertices. The theory of cut ideals was initiated by Sturmfels and Sullivant [12].

DEFINITION 1.4.   The *cut ideal of the graph G*, I_G, is the kernel of the ring homomorphism $\phi_G \colon \mathbb{K}[q] \to \mathbb{K}[s,t]$ defined by

$$q_{A|B} \mapsto \prod_{ij \text{ is in } A|B} s_{ij} \prod_{ij \text{ is not in } A|B} t_{ij},$$

where
$$\mathbb{K}[q] = \mathbb{K}[q_{A|B} : \text{there is a cut } A \mid B \text{ of } G],$$

$$\mathbb{K}[s,t] = \mathbb{K}[s_{ij}, t_{ij} : ij \text{ is an edge of } G].$$

The potential uses of cut ideals in statistics and the applied sciences are not apparent from Example 1.3 because it's too small. As described in [12], there are applications in biology [11] that use, for example, the Jukes–Cantor model.

Based on theorems about similar constructions as well as computer calculations, it is reasonable to believe that topological properties of G should be reflected in algebraic properties of I_G.

THEOREM (conjectured by Sturmfels and Sullivant [12]).   *The cut ideal is generated by quadrics if and only if G is free of K_4 minors.*

Several partial results have been proved; for instance, Brennan and Chen [2] showed this for subdivisions of books and outerplanar graphs. A ring graph is, more or less, a bunch of disjoint cycles that are connected by a tree that touches any cycle in at most one vertex. For ring graphs, the conjecture was proved by Nagel and Petrović [10].

The conjecture follows as a corollary of Theorem 2.6, which is a fiber product–type theorem. In the same way as the fiber product theorems in [3] and [12] could be generalized in [13], we will present a more general form of Theorem 2.6 in [8]. Methods from this paper were used on ideals of graph homomorphisms in Engström and Norén's paper [9].

BASIC NOTIONS OF CUT IDEALS.   The largest degree of a minimal generator of I_G is $\mu(G)$. By [12, Cor. 3.3], the contraction of an edge or deletion of a vertex cannot increase μ. In [12, Thm. 2.1] it is proved that if G is glued together from two graphs G_1 and G_2 over a complete graph with zero, one, or two vertices, then the cut ideal I_G is generated by (i) lifts of generators of I_{G_1} and I_{G_2} and (ii) quadratic

binomials for sorting cuts. The main theorem of this paper is a variation of [12, Thm. 2.1] for the case of gluing over an edge.

2. Decompositions of Graphs and Ideals

The induced subgraph of G on S is denoted $G[S]$.

DEFINITION 2.1. Let u, v be two vertices of G, and let $A_1 \mid B_1, A_2 \mid B_2, \ldots,$ $A_n \mid B_n$ be a list of cuts. The *height $h_{u,v}(q)$* of

$$q = q_{A_1|B_1} q_{A_2|B_2} \cdots q_{A_n|B_n}$$

with respect to u and v is the number of cuts in the list that put u and v in different parts.

If there is an edge between u and v in G, then we will use $h_{u,v}(q)$ to denote the *degree* of s_{uv} in $\phi_G(q)$. Another way to define the height of q with respect to u and v is as the degree of s_{uv} in $\phi_{G+uv}(q)$, and that is a good way to think of it.

DEFINITION 2.2. A set of generators

$$q_{A_{i,1}|B_{i,1}} q_{A_{i,2}|B_{i,2}} \cdots q_{A_{i,n_i}|B_{i,n_i}} - q_{A'_{i,1}|B'_{i,1}} q_{A'_{i,2}|B'_{i,2}} \cdots q_{A'_{i,n_i}|B'_{i,n_i}}$$

of I_G is *slow-varying* with respect to the vertices u and v of G if

$$\left| h_{u,v}(q_{A_{i,1}|B_{i,1}} \cdots q_{A_{i,n_i}|B_{i,,n_i}}) - h_{u,v}(q_{A'_{i,1}|B'_{i,1}} \cdots q_{A'_{i,n_i}|B'_{i,n_i}}) \right| \leq 2$$

for all i.

LEMMA 2.3. *If $w_1 - w_2 - \cdots - w_k$ is a path in G, then*

$$h_{w_1, w_k}(q_{A|B}) \equiv \sum_{i=1}^{k-1} (s_{w_i w_{i+1}} - \text{degree of } \phi_G(q_{A|B}))$$

modulo 2.

Proof. A walk on the path from w_1 to w_k crosses the cut an odd number of times if and only if w_1 and w_k are in different parts. $\qquad \square$

LEMMA 2.4. *If there is a path in G from u to v and if $\phi_G(q) = \phi_G(q')$, then $h_{u,v}(q) \equiv h_{u,v}(q')$ modulo 2.*

Proof. Use Lemma 2.3. $\qquad \square$

PROPOSITION 2.5. *Any set of generators of I_G validating $\mu(G) \leq 2$ is slow-varying with respect to any vertex pair.*

Proof. The statement's truth is self-evident. $\qquad \square$

THEOREM 2.6. *Let G be a graph with two special nonadjacent vertices u and v. Assume that G can almost be decomposed into a left and right part: there*

are $L, R \subseteq V(G)$ such that $L \cup R = V(G)$, $L \cap R = \{u, v\}$, and $E(G) = E(G[L]) \cup E(G[R])$.

If there is a path from u to v in both $G[L]$ and $G[R]$, and if there exist slow-varying generators of both $I_{G[L]}$ and $I_{G[R]}$ with respect to u and v, then

$$\mu(G) \le \max\{2\mu(G[L]) - 2, 2\mu(G[R]) - 2, \mu(G[L] + uv), \mu(G[R] + uv)\}.$$

The cut ideal of G is generated by a union of

  (i) *lifts of generators of $I_{G[L]+uv}$,*
 (ii) *lifts of generators of $I_{G[R]+uv}$,*
(iii) *joins of generators $q_1 - q_2$ of $I_{G[L]}$ and $q_3 - q_4$ of $I_{G[R]}$ such that*
      $|h_{u,v}(q_1) - h_{u,v}(q_2)| = |h_{u,v}(q_3) - h_{u,v}(q_4)| = 2$, and
 (iv) *quadratic binomials with which to reorder.*

Proof. The basic part of this proof, which involves only $G[L]$ and $G[R]$, is in the spirit of the proof of Theorem 2.1 in [12].

We will prove Theorem 2.6 by an explicit construction of generators for I_G. Let

$$q = \prod_{i=1}^{n} q_{A_i | B_i} \quad \text{and} \quad q' = \prod_{i=1}^{n} q_{A'_i | B'_i}$$

be two elements of $\mathbb{K}[q_{A|B} : A \sqcup B = V(G)]$ with $\phi_G(q) = \phi_G(q')$. If, for any such q and q', we can construct a sequence of moves from q to q', then we can generate I_G. A move from q_1 to q_2 is a composition of a q_3 with a binomial generator $q_4 - q_5$ such that

$$q_1 - q_2 = q_3(q_4 - q_5).$$

We can assume that $h_{u,v}(q) \ge h_{u,v}(q')$.

Main idea. To construct the sequence from q to q', we use sequences from q_L to q'_L and from q_R to q'_R. (Note that q_L is q induced on L, and likewise for q_R.) If we simply took a sequence from q_L to q'_L given by $I_{G[L]}$ and a corresponding one on R and tried to glue them together, it would sometimes fail on the vertex pair u and v. What goes wrong is that the number of cuts with u and v in different parts might not be the same. That is, the height $h_{u,v}$ could be different on the left and the right side. Yet we know that the height is the same for q_L and q_R at the start of the sequence and also for q'_L and q'_R at the end of the sequence.

In the sequence $q_L, \ldots, q'_L$, the number of cuts with u and v in different parts can look like the fat gray line in Figure 1. If it changes, it does so by an even number (per Lemma 2.3). It never changes by more than 2 because $I_{G[L]}$ is slow-varying. Since the height of the sequence $q_R, \ldots, q'_R$ need not have the same shape as the gray line, we must normalize the sequences.

How to normalize the sequence $q_L, \ldots, q'_L$. We do this as described in Figure 1. Let $q'_{L,h}$ be the last element in the sequence with height h for $h = h_{u,v}(q_L)$, $h_{u,v}(q_L) - 2, \ldots, h_{u,v}(q'_L) + 2, h_{u,v}(q'_L)$; let $q_{L,h}$ be the element after $q'_{L,h+2}$ in the sequence for $h = h_{u,v}(q_L) - 2, \ldots, h_{u,v}(q'_L) + 2, h_{u,v}(q'_L)$; and let $q_{L,h_{u,v}(q_L)} = q_L$. In our normalized sequence we still go from $q'_{L,h}$ to $q_{L,h-2}$ via a generator
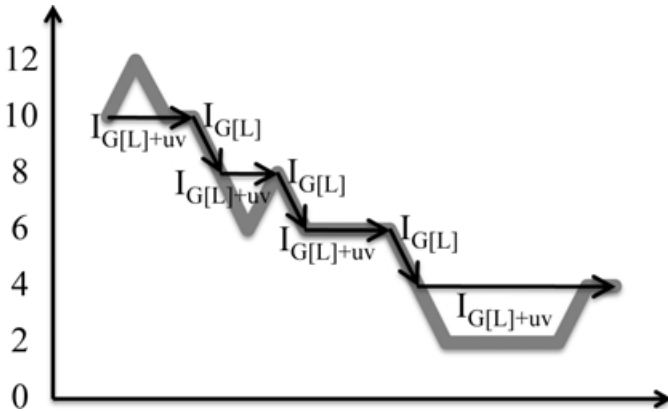
Figure 1 Vertical axis: height with respect to u, v

of $I_{G[L]}$. But from $q_{L,h}$ to $q'_{L,h}$ we build up the sequence by using generators of $I_{G[L]+uv}$; this is possible because the heights of $q_{L,h}$ and $q'_{L,h}$ are the same. For our normalized sequence, the height is never increasing.

Normalize $q_R, \ldots, q'_R$ in the same way. The plots of the heights for the normalized sequences on L and R now look the same, so we can put the sequences together without any conflicts regarding u and v.

Thus we need four kinds of moves:

($\mathbf{F}_1$) all from $I_{G[L]+uv}$;
($\mathbf{F}_2$) all from $I_{G[R]+uv}$;
($\mathbf{F}_3$) those from $I_{G[L]}$ and $I_{G[R]}$ that change height by 2; and
($\mathbf{F}_4$) reorderings to match cuts.

Let $\mathbf{F}_L, \mathbf{F}_{L+uv}, \mathbf{F}_R$, and $\mathbf{F}_{R+uv}$ be the respective binomial generating sets of $I_{G[L]}$, $I_{G[L]+uv}$, $I_{G[R]}$, and $I_{G[R]+uv}$. If the maximal degree of a binomial in $\mathbf{F}_L$ or $\mathbf{F}_R$ is M, then extend $\mathbf{F}_L$ to

$$\tilde{\mathbf{F}}_L = \{q_1(q_2 - q_3) : \text{degree of } q_1 q_2 \leq 2M - 2 \text{ and } q_2 - q_3 \in \mathbf{F}_L\}$$

and $\mathbf{F}_R$ to

$$\tilde{\mathbf{F}}_R = \{q_1(q_2 - q_3) : \text{degree of } q_1 q_2 \leq 2M - 2 \text{ and } q_2 - q_3 \in \mathbf{F}_R\}.$$

The extension is needed to allow binomial generators of different degree from the left and right side to be joined when the height decreases by 2. In the definitions of $\mathbf{F}_1, \mathbf{F}_2$, and $\mathbf{F}_3$, any product of the type

$$\prod_{i=1}^{m} q_{C_i|D_i}$$

is assumed to have an order such that

$$h_{u,v}(q_{C_1|D_1}) \geq \cdots \geq h_{u,v}(q_{C_m|D_m}).$$

Let

$$\mathbf{F}_1 = \left\{ \prod_{i=1}^{m} q_{C_i|D_i} - \prod_{i=1}^{m} q_{C_i'|D_i'} \in \mathbb{K}[q_G] : \right.$$

$$\prod_{i=1}^{m} q_{C_i \cap L|D_i \cap L} - \prod_{i=1}^{m} q_{C_i' \cap L|D_i' \cap L} \in \mathbf{F}_{L+uv},$$

$$\left. C_i \cap R = C_i' \cap R \text{ for } i = 1, \ldots, m \right\};$$

$$\mathbf{F}_2 = \left\{ \prod_{i=1}^{m} q_{C_i|D_i} - \prod_{i=1}^{m} q_{C_i'|D_i'} \in \mathbb{K}[q_G] : \right.$$

$$\prod_{i=1}^{m} q_{C_i \cap R|D_i \cap R} - \prod_{i=1}^{m} q_{C_i' \cap R|D_i' \cap R} \in \mathbf{F}_{R+uv},$$

$$\left. C_i \cap L = C_i' \cap L \text{ for } i = 1, \ldots, m \right\};$$

$$\mathbf{F}_3 = \left\{ \prod_{i=1}^{m} q_{C_i|D_i} - \prod_{i=1}^{m} q_{C_i'|D_i'} \in \mathbb{K}[q_G] : \right.$$

$$\prod_{i=1}^{m} q_{C_i \cap L|D_i \cap L} - \prod_{i=1}^{m} q_{C_i' \cap L|D_i' \cap L} \in \tilde{\mathbf{F}}_L,$$

$$\prod_{i=1}^{m} q_{C_i \cap R|D_i \cap R} - \prod_{i=1}^{m} q_{C_i' \cap R|D_i' \cap R} \in \tilde{\mathbf{F}}_R,$$

$$\left. h_{u,v}\left(\prod_{i=1}^{m} q_{C_i|D_i} \right) \neq h_{u,v}\left(\prod_{i=1}^{m} q_{C_i|D_i} \right) \right\};$$

$$\mathbf{F}_4 = \left\{ \prod_{i=1}^{2} q_{C_i|D_i} - \prod_{i=1}^{2} q_{C_i'|D_i'} \in \mathbb{K}[q_G] : \right.$$

$$C_1 \cap L = C_1' \cap L, \ C_2 \cap L = C_2' \cap L,$$

$$\left. C_1 \cap R = C_2' \cap R, \ C_2 \cap R = C_1' \cap R \right\}.$$

We have that $\mathbf{F} = \mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3 \cup \mathbf{F}_4$ is a generating set of I_G. From that we get

$$\mu(G) \leq \max\{2, 2\mu(G[L]) - 2, 2\mu(G[R]) - 2, \mu(G[L] + uv), \mu(G[R] + uv)\}.$$

In $G[L]$ there is an induced path from u to v with more than one edge. For the path with two edges we have $\mu = 2$ and thus, by contraction, $\mu \geq 2$ for any path; this shows that $\mu(G[L]) \geq 2$. The 2 can be removed to yield

$$\mu(G) \leq \max\{2\mu(G[L]) - 2, 2\mu(G[R]) - 2, \mu(G[L] + uv), \mu(G[R] + uv)\}. \quad \square$$

COROLLARY 2.7. *Let H_1 and H_2 be two graphs on different vertex sets that satisfy the following conditions*:

- *u_1, v_1 are two distinct nonadjacent vertices of H_1;*
- *u_2, v_2 are two distinct nonadjacent vertices of H_2;*
- *H_1 and H_2 are connected: and*
- *$\mu(H_1), \mu(H_2), \mu(H_1 + u_1 v_1), \mu(H_2 + u_2 v_2) \le 2$.*

Then $\mu \le 2$ for the graph obtained by gluing $u_1 = u_2$ and $v_1 = v_2$ in $H_1 \cup H_2$.

Proof. Insert Proposition 2.5 into Theorem 2.6. ☐

The graphs without K_4-minors are also called *series-parallel* graphs. Starting with the complete graphs on less than four vertices, the connected series-parallel graphs can be constructed by the gluing of two smaller ones: either in series over one vertex or in parallel over two vertices (that may or may not be connected) [6].

COROLLARY 2.8 [12, Conj. 3.5]. *The cut ideal is generated by quadrics if and only if G is free of K_4-minors.*

Proof. We prove that if G is series-parallel then $\mu(G) \le 2$; the other direction was proved in [12]. We need only prove the statement for connected series-parallel graphs.

The proof is by induction on the number of vertices of G. If there are fewer than four vertices, then $\mu(G) \le 2$ by explicit calculations in [12]. So assume that G has at least four vertices. If G is constructed by two graphs H_1 and H_2 put in series and glued at one vertex, then $\mu(G) = \max\{\mu(H_1), \mu(H_2)\} \le 2$ by the fiber construction in [12]. However, if G is constructed by two graphs H_1 and H_2 glued parallel together in two vertices, then there are two cases.

Case 1: No matter how the subgraphs H_1 and H_2 are chosen to be glued together in parallel to create G, one of them will be only an edge. Assume that H_2 is only the edge uv and that uv is not in H_1. If H_1 came from a parallel gluing of H_1' and H_1'' at u and v, then G could be parallel constructed from H_1' and $H_1'' + uv$; since none of these is only an edge, we have a contradiction. So H_1 is from a series gluing at some vertex $w \notin \{u, v\}$. Both graphs glued together to get H_1 cannot be only edges, for then G would be a triangle yet we assumed that G had more than three vertices. Hence we can assume that the part of H_1 between v and w has more than two vertices. But then G can be formed as a parallel construction glued at v and w when none of the parts is only an edge. That is the situation in Case 2.

Case 2: The graph G can be created by a parallel construction at u, v of two graphs H_1 and H_2, and both of them have more than two vertices. If uv is an edge of G, then $\mu(G) = \max\{\mu(H_1 + uv), \mu(H_2 + uv)\} \le 2$ because H_1 and H_2 are series-parallel. If there is no edge between u and v in G, then we use that H_1, H_2, $H_1 + uv$, and $H_2 + uv$ are series-parallel, together with Corollary 2.7, to obtain $\mu(G) \le 2$. ☐

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