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Two Extension Theorems of Hartogs–Chirka Type Involving Continuous Multifunctions

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1. Introduction and Statement of Results

This paper is motivated (i) by a version of Hartogs' lemma stating that, if Ω is some neighborhood of the union of $\partial\mathbb{D} \times \mathbb{D}$ and a complex analytic subvariety $\Sigma \subset \bar{\mathbb{D}} \times \mathbb{D}$ that is finitely sheeted over $\mathbb{D}$ (such that $\Omega \cap \mathbb{D}^2$ is connected) and if $f \in \mathcal{O}(\Omega)$, then f continues holomorphically to $\mathbb{D}^2$ and (ii) by the Hartogs-type extension theorem of Chirka. Chirka's theorem reads as follows.

Result 1.1 (Chirka). *Let $\phi\colon \bar{\mathbb{D}} \to \mathbb{C}$ be a continuous function having $\sup_{z\in\bar{\mathbb{D}}}|\phi(z)| < 1$ and let S be its graph. Let Ω be a connected open neighborhood of $S \cup (\partial\mathbb{D} \times \mathbb{D})$ contained in $\{(z,w) \in \mathbb{C}^2 : |w| < 1\}$. If $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^2$.*

(Here and in what follows, $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$.) This motivates the question of whether—given the "Weierstrass pseudopolynomial"

$$\mathcal{P}_a(z,w) := w^k + \sum_{j=0}^{k-1} a_j(z)w^j, \quad k \geq 2 \tag{1.1}$$

(where $a_0,\dots,a_{k-1} \in \mathcal{C}(\bar{\mathbb{D}})$ with $\mathcal{P}_a^{-1}\{0\} \subset \bar{\mathbb{D}} \times \mathbb{D}$) and given a neighborhood Ω of $\mathcal{P}_a^{-1}\{0\} \cup (\partial\mathbb{D} \times \bar{\mathbb{D}})$—the aforementioned results hold in this new setting.

One possible approach to this question is to investigate a version of Result 1.1 with one copy of $\mathbb{D}$ replaced by a bordered Riemann surface determined by $a := (a_0,\dots,a_{k-1})$, over which the graph of the multifunction is transformed to a graph. One is then reduced to solving a certain quasilinear $\bar\partial$-problem analogous to the one considered by Chirka in [4] (also see [5]). There is considerable literature on this subject (see, e.g., [7]). However, for this approach to work, one needs continuous dependence of solutions on the parameters as well as sup-norm estimates with small norm, neither of which seem to be known at this time. A second approach is suggested by the Kontinuitätssatz-based strategies of Bharali [2] and Barrett-Bharali [1], provided one is willing to allow $(a_0,\dots,a_{k-1})$ in (1.1) to belong to some strict subclass of $\mathcal{C}(\bar{\mathbb{D}}; \mathbb{C}^k)$. To motivate the origins of the two main theorems that follow, we state one of the results from [1] and [2].

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RESULT 1.2 (Bharali [2]). *Let Γ be the graph of the map $(\phi_1, \dots, \phi_k) \colon \bar{\mathbb{D}} \to \mathbb{C}^k$, where $\phi_j(z) := \psi_j(z, \bar{z})$ and*

$$\psi_j \in \left\{ \psi \in \mathcal{O}(\mathbb{D}^2) : \sup_{(z,\zeta)\in\mathbb{D}^2} |\psi(z,\zeta)| < 1 \text{ and } z \mapsto \psi(z,\bar{z}) \text{ is continuous on } \bar{\mathbb{D}} \right\} \tag{1.2}$$

for $j = 1, \dots, k$. If Ω is a connected neighborhood of $S := \Gamma \cup (\partial\mathbb{D} \times \mathbb{D}^k)$ contained in $\{(z, w) \in \mathbb{C} \times \mathbb{C}^k : w \in \mathbb{D}^k\}$ and if $f \in \mathcal{O}(\Omega)$, then f extends holomorphically to $\mathbb{D}^{k+1}$.

In the theorems in [1] and [2], the authors construct a continuous family of discs $\{\Phi_t \in \mathcal{C}(\bar{\mathbb{D}}; \mathbb{C}^k) : t \in [0,1]\}$ such that $\Phi_0 = (\phi_1, \dots, \phi_k)$ and each Φ_t is holomorphic on larger and larger subregions of $\mathbb{D}$ so that, eventually, $\Phi_1 \in \mathcal{O}(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$. This construction suggests the following strategy.

Step 1. Setting $(\phi_1, \dots, \phi_k) := (a_0, \dots, a_{k-1})$, we can try to construct a continuous family of discs $\{\Phi_t\}_{t\in[0,1]}$ with the properties mentioned before. We can then treat each $\Phi_t := (\Phi_{t,0}, \dots, \Phi_{t,k-1})$ as a k-tuple of the ordered coefficients of a Weierstrass pseudopolynomial and thereby obtain a continuous family of "pseudovarieties" $\left\{\Sigma_t := \left\{(z, w) \in \mathbb{D}\times\mathbb{C} : w^k + \sum_{j=0}^{k-1} \Phi_{t,j}(z) w^j = 0\right\}\right\}_{t\in[0,1]}$ such that $\Sigma_0 := \{(z, w) \in \bar{\mathbb{D}} \times \mathbb{C} : \mathcal{P}_a(z, w) = 0\}$, where each Σ_t is a finitely sheeted complex analytic subvariety fibered over larger and larger subregions of $\mathbb{D}$ and where Σ_1 is the graph of an analytic multifunction (i.e., a multigraph) over $\mathbb{D}$.

Step 2. In the construction just described, our hypotheses on $(a_0, \dots, a_{k-1})$ must also ensure that each Σ_t over $\mathbb{D}$, like the initial "pseudovariety", lies within the bidisc (i.e., $\Sigma_t \subset \bar{\mathbb{D}} \times \mathbb{D}$ for all $t \in [0,1]$) and that Σ_t is attached to $\partial\mathbb{D} \times \mathbb{D}$ along the border of Σ_t for all $t \in [0,1]$.

Step 3. Finally, we invoke a suitable version of the Kontinuitätssatz to achieve analytic continuation along the family so constructed in order to reduce the problem to the "finitely sheeted analytic variety" version of Hartogs' lemma mentioned in the beginning of this section.

It turns out that this second strategy is successful (with some refinement) if the coefficients $a_0, \dots, a_{k-1}$ are drawn from the subclasses studied in [1] and [2]. Now, one may ask why Step 1 cannot be attempted for $a_0, \dots, a_{k-1} \in \mathcal{C}(\bar{\mathbb{D}})$. This would amount to attempting to prove a vector-valued version of Chirka's main result (Result 1.1) in [4]. However, Rosay's counterexample in [9] establishes that Chirka's result cannot be generalized to higher dimensions in its entire generality—in other words, when $a_0, \dots, a_{k-1}$ $(k > 1)$ are merely continuous. The first theorem of this paper is stated for $a_0, \dots, a_{k-1}$ belonging to the subclass of $\mathcal{C}(\bar{\mathbb{D}})$ introduced by Barrett and Bharali in [1].

THEOREM 1.3. *Let $a_0, \dots, a_{k-1} \in \mathcal{C}(\bar{\mathbb{D}}; \mathbb{C})$ be such that the set*

$$\Sigma_a := \left\{(z, w) \in \bar{\mathbb{D}} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} a_j(z) w^j = 0\right\}$$

lies entirely in $\bar{\mathbb{D}} \times \mathbb{D}$. *For* $0 < r \leq 1$, *let* $A_\nu^j(r)$ *represent the* νth *Fourier coefficient of* $a_j(re^{i\bullet})$, $\nu \in \mathbb{Z}$. *Assume that* $A_\nu^j \equiv 0$ *for all* $\nu < 0$ *and* $j = 0, \dots, k-1$. *Let* Ω *be a connected neighborhood of* $S := \Sigma_a \cup (\partial\mathbb{D} \times \bar{\mathbb{D}})$ *such that* $\Omega \cap \mathbb{D}^2$ *is connected. Then, for every* $f \in \mathcal{O}(\Omega)$, *there exists an* $F \in \mathcal{O}(\mathbb{D}^2)$ *such that*

$$F|_{\Omega\cap\mathbb{D}^2} \equiv f|_{\Omega\cap\mathbb{D}^2}.$$

Our next theorem has its origins in Result 1.2, but see Remarks 1.5 and 1.6.

THEOREM 1.4. *Let* $a_j := \psi_j(z, \bar{z})$, *where*

$$\psi_j \in \Big\{\psi \in \mathcal{O}(\mathbb{D}^2) : \sup_{(\zeta,s)\in\mathbb{D}\times[0,1]} |\psi(\zeta, s\bar{\zeta})| < 1 \text{ and } z \mapsto \psi(z,\bar{z}) \text{ is continuous on } \bar{\mathbb{D}}\Big\} \tag{1.3}$$

for $j = 0, \dots, k-1$, *be such that the set*

$$\Sigma_a := \Big\{(z,w) \in \bar{\mathbb{D}} \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} a_j(z)w^j = 0\Big\}$$

lies entirely in $\bar{\mathbb{D}} \times D(0;2)$. *Let* Ω *be a connected neighborhood of* $S := \Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0;2)})$ *such that* $\Omega \cap (\mathbb{D} \times D(0;2))$ *is connected. Then, for every* $f \in \mathcal{O}(\Omega)$, *there exists an* $F \in \mathcal{O}(\mathbb{D} \times D(0;2))$ *such that*

$$F|_{\Omega\cap(\mathbb{D}\times D(0;2))} \equiv f|_{\Omega\cap(\mathbb{D}\times D(0;2))}.$$

REMARK 1.5. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be the classes of functions appearing in (1.2) and (1.3), respectively. Although Theorem 1.4 stems from Result 1.2, it must be admitted that the class $\mathfrak{F}_1$ is quite restrictive. However, when adapting the approach outlined previously, we were able to construct the deformation $\{\Phi_t : t \in [0,1]\}$ in a slightly different fashion from that suggested in [2], which allows us to work with $a_0, \dots, a_{k-1}$ belonging to a less restrictive class. Note that $\mathfrak{F}_2 \supsetneq \mathfrak{F}_1$; simply observe that if $\psi(z,w) := (M+\varepsilon)^{-1}\exp(z-w-2)$, where $M = \sup_{(\zeta,s)\in\bar{\mathbb{D}}\times[0,1]}|\exp(\zeta - s\bar{\zeta} - 2)|$, then $M < 1$ and for $\varepsilon \in (M, 1)$ we have $\psi \in \mathfrak{F}_2$ but $\psi \notin \mathfrak{F}_1$.

REMARK 1.6. Černe and Flores [3] have independently used the three-step method described previously in order to prove the following statement.

(∗) *Let* $a_0, \dots, a_{k-1}$ *be continuous functions on* $\bar{\mathbb{D}}$ *and let*

$$\Sigma_a := \{(z,w) \in \bar{\mathbb{D}} \times \mathbb{C} : w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z) = 0\}$$

be a continuous variety over $\bar{\mathbb{D}}$; *then every function holomorphic in a connected neighborhood of the set* $S = \Sigma_a \cup (\partial\mathbb{D} \times \mathbb{C})$ *extends holomorphically to a neighborhood of* $\bar{\mathbb{D}} \times \mathbb{C}$.

Note that $\mathcal{C}(\bar{\mathbb{D}}; \mathbb{D})$ is a subset of the uniform closure (on $\bar{\mathbb{D}}$) of the function space obtained if we drop the bound "$\sup_{(z,\zeta)\in\mathbb{D}^2}|\psi(z,\zeta)| < 1$" from $\mathfrak{F}_1$. It is this fact, coupled with their reliance on the three-step method outlined after Remark 1.5, that compels Černe and Flores [3] to work with the *unbounded cylinder* $\bar{\mathbb{D}} \times \mathbb{C}$.

Theorem 1.4 represents an alternative setting in which to exploit the same method with—in contrast to Černe and Flores—the following initial objectives.

- Use the ideas of Barrett and Bharali to demonstrate an analytic continuation theorem stated for a *compact* Hartogs figure ($S = \Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0;2)})$ in our case).
- Extend the applicability of Result 1.2 to a less restrictive class of graphs and coefficients—namely, $\mathfrak{F}_2$.

Because of considerations inherent to the three-step method that we intend to use (see Remark 2.3(i)) we—just like Černe and Flores—cannot work with the Hartogs configuration $\Sigma_a \cup (\partial\mathbb{D} \times \bar{\mathbb{D}})$ either. However, we *can* state a result involving $\Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0;2)})$.

Many of the mathematical details underlying Steps 2 and 3 are common to Theorems 1.3 and 1.4. These technicalities have been collected in Section 2. The actual proofs of Theorems 1.3 and 1.4 are presented in Sections 3 and 4, respectively.

2. Preliminary Lemmas

We will first isolate the technical elements of the two main proofs in the form of a few preliminary results. The following notation will be used.

- $D(a;r)$ will denote the open disc of radius r with center at a, and $\mathrm{Ann}(a;r,R)$ will denote the open annulus with center at $a \in \mathbb{C}$ and having inner radius r and outer radius R.
- $\mathcal{C}^\infty(\bar{\mathbb{D}};\mathbb{C})$ will denote the class of infinitely differentiable functions on the unit disc, all of whose derivatives extend to functions in $\mathcal{C}(\bar{\mathbb{D}})$.
- For $\alpha := (\alpha_0,\dots,\alpha_{k-1}) \in \mathcal{C}(\bar{G};\mathbb{C}^k)$, $k \in \mathbb{N}$, $G \subset \mathbb{C}$ a bounded domain, and $E \subset \bar{G}$, we set
$$\mathcal{P}_\alpha(z,w) := w^k + \sum_{j=0}^{k-1} \alpha_j(z) w^j,$$
$$\Sigma_{\alpha,E} := \left\{ (z,w) \in E \times \mathbb{C} : w^k + \sum_{j=0}^{k-1} \alpha_j(z) w^j = 0 \right\}.$$
For the sake of convenience, the subscript E will be omitted when $E = \bar{\mathbb{D}}$; for example, $\Sigma_{\alpha,\bar{\mathbb{D}}} =: \Sigma_\alpha$.

The first step of the three-step strategy outlined in Section 1 is not difficult, but the details involved are theorem-specific. This is due in part to the requirements described in Step 2. A crucial task is to determine sufficient yet not too strong conditions on the coefficient k-tuple $(a_0,\dots,a_{k-1})$ that will enable us to establish that each Σ_t, $t \in [0,1]$, is contained in the bidisc relevant to each theorem. The following lemma—a maximum principle for varieties—will prove useful.

LEMMA 2.1. *Let $G \subset \mathbb{C}$ be a bounded domain and let $a \in \mathcal{O}(G;\mathbb{C}^k) \cap \mathcal{C}(\bar{G};\mathbb{C}^k)$. Define*
$$M(z) := \max\{|w| : (z,w) \in \Sigma_{a,\bar{G}}\}.$$
If $M(z) \le K$ for all $z \in \partial G$, then $M(z) \le K$ for all $z \in \bar{G}$.

Proof. We would be done if we could obtain the conclusion of this lemma when $\Sigma_{a,G}$ is an irreducible subvariety. For $\Sigma_{a,G}$ irreducible, if we can show that M is subharmonic then the result would follow from the maximum principle.

Recall that the zeros of monic degree-k polynomials over $\mathbb{C}$, viewed as *unordered* k-tuples of zeros repeated according to multiplicity, vary continuously with the coefficients. Hence, since M is symmetric in the zeros of $\mathcal{P}_a$, it follows that $M \in \mathcal{C}(\bar{G})$.

Now, let

$$\mathfrak{R}(z) := \text{resultant of } \mathcal{P}_a(z,\cdot) \text{ and } \partial_w \mathcal{P}_a(z,\cdot), \quad z \in G.$$

By the irreducibility of $\Sigma_{a,G}$, we have $\mathfrak{R} \not\equiv 0$. Since $\mathfrak{R} \in \mathcal{O}(G)$, $\mathfrak{S} := \mathfrak{R}^{-1}\{0\}$ is a discrete set in G. Now, for any $z_0 \in G \setminus \mathfrak{S}$, $\Sigma_{a,\{z_0\}} = \{(z_0, w_{0,1}), \dots, (z_0, w_{0,k})\}$ with $w_{0,j} \neq w_{0,l}$ for $j \neq l$. Because $\partial_w \mathcal{P}_a(z_0, w_{0,j}) \neq 0$ for each $j = 1, \dots, k$, we may apply the implicit function theorem at each point of $\Sigma_{a,\{z_0\}}$ to obtain a common radius $r(z_0) > 0$ such that the k sheets of $\Sigma_{a,D(z_0;r(z_0))}$ are the graphs of functions $\phi_1^{z_0}, \dots, \phi_k^{z_0} \in \mathcal{O}(D(z_0; r(z_0)))$. Clearly,

$$M(z) = \max_{j \le k} |\phi_j^{z_0}(z)| \quad \forall z \in D(z_0; r(z_0)).$$

Thus, $M|_{D(z_0;r(z_0))}$ is subharmonic. Since z_0 was arbitrarily chosen from the open set $G \setminus \mathfrak{S}$, we infer that $M|_{G\setminus\mathfrak{S}}$ is a subharmonic function.

Because $\mathfrak{S}$ is the zero set of a holomorphic function, it is a polar set. But $M|_{G\setminus\mathfrak{S}}$ is a bounded subharmonic function, and $M \in \mathcal{C}(\bar{G})$. Therefore, M must be subharmonic in G [8, p. 47]. □

REMARK 2.2. We paraphrase Lemma 2.1 as follows for use in our situation.

Let $G \subset \mathbb{C}$ be a bounded domain and let $a \in \mathcal{O}(G; \mathbb{C}^k) \cap \mathcal{C}(\bar{G}; \mathbb{C}^k)$. Then

$$\Sigma_{a,\partial G} \subset \partial G \times D(0; K) \implies \Sigma_{a,\bar{G}} \subset \bar{G} \times D(0; K).$$

REMARK 2.3. We will also need the following algebraic facts.

(i) If $\alpha_0, \dots, \alpha_{k-1} \in \mathbb{D}$ ($k \in \mathbb{N}$) and if $w_1, \dots, w_k$ are the zeros of the polynomial $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w^1 + \alpha_0$, then $w_j \in D(0; 2)$ for $j = 1, \dots, k$. For an easy proof of this fact, one can apply Rouché's theorem to $f(w) := w^k$ and $g(w) := w^k + \sum_{j=0}^{k-1} \alpha_j w^j$ on $\partial D(0; 2)$.

(ii) If $(\alpha_0, \dots, \alpha_{k-1}) \in \mathbb{C}^k$ and if $w_1, \dots, w_k$ are the zeros of the polynomial $w^k + \alpha_{k-1}w^{k-1} + \cdots + \alpha_1 w + \alpha_0$ then, for $\eta \in \mathbb{C}$, $w_1 + \eta, \dots, w_k + \eta$ are the zeros of the polynomial

$$w^k + \alpha_{k-1}^{(\eta)} w^{k-1} + \cdots + \alpha_1^{(\eta)} w + \alpha_0^{(\eta)};$$

here, for each j,

$$\alpha_j^{(\eta)} = \alpha_j + \sum_{l=j+1}^{k} (-1)^{l-j} \binom{l}{l-j} \alpha_l \eta^{l-j} \tag{2.1}$$

if we interpret $\alpha_k := 1$.

Theorems of a similar flavor as Theorems 1.3 and 1.4 have relied upon the Kontinuitätssatz. However, the earliest (and partially correct) works do not specify *which* form of the Kontinuitätssatz they rely upon. We wish to clarify here that the version that works for us is the one of Chirka and Stout [6]. However, merely using the Chirka–Stout Kontinuitätssatz yields a weaker conclusion than desired—that is, on the envelope of holomorphy of the domain in question. The next lemma follows the approach of Barrett and Bharali [1] to argue that it is, in fact, possible to obtain the strong conclusion of Chirka's extension theorem (i.e., Result 1.1) [4] in our situation.

LEMMA 2.4. *Let $a = (a_0, \dots, a_{k-1}) \in \mathcal{C}(\bar{\mathbb{D}}; \mathbb{C}^k)$ and let $\Sigma_a \subset \bar{\mathbb{D}} \times D(0; r)$, $r > 0$. Let Ω be a connected open neighborhood of $S := \Sigma_a \cup (\partial\mathbb{D} \times \overline{D(0; r)})$ and let $f \in \mathcal{O}(\Omega)$. Let $V := \mathrm{Ann}(0; 1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, be such that $V \times D(0; r) \subset \Omega$, and let $D \Subset \Omega$ be an open subset containing S. For any $\alpha \in \mathcal{C}(\bar{\mathbb{D}}; \mathbb{C}^k)$ and any $\eta \in \mathbb{C}$, let $\alpha^{(\eta)} \in \mathcal{C}(\bar{\mathbb{D}}; \mathbb{C}^k)$ denote the perturbation that is given by* (2.1) *so that $\Sigma_{\alpha^{(\eta)}} = \Sigma_\alpha + (0, \eta)$. Suppose there exist* (a) *a continuous function $A := (A_0, \dots, A_{k-1})$ on $\bar{\mathbb{D}} \times [0, 1]$ such that $A(\cdot, 0) = a(\cdot)$ and* (b) *a $\delta > 0$ so small that, defining $\Sigma_t^\eta := \Sigma_{A^{(\eta)}(\cdot, t)}$, we have*:

(1) *for each $\eta \in D(0; \delta)$, $\Sigma_t^\eta \subset \bar{\mathbb{D}} \times D(0; r)$ for all $t \in [0, 1]$; and*
(2) *for each $\eta \in D(0; \delta)$, $\Sigma_t^\eta \cap (\mathbb{D} \times D(0; r)) \setminus \bar{D}$ is a complex analytic subvariety of $\mathbb{D} \times D(0; r) \setminus \bar{D}$.*

Then there exist a connected neighborhood Ω_1 of $S_1 := \Sigma_1^0 \cup (\partial\mathbb{D} \times D(0; r))$ and an $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1|_{\Omega_1 \cap (V \times D(0; r))} \equiv f|_{\Omega_1 \cap (V \times D(0; r))}.$$

Proof. Let

$$\mathcal{T} := \bigcup_{\eta \in D(0; \delta)} \Sigma_1^0 + (0, \eta).$$

By the Chirka–Stout Kontinuitätssatz [6] we have $\mathcal{T} \subset \pi(\tilde{\Omega})$, where $(\tilde{\Omega}, \pi)$ denotes the envelope of holomorphy of Ω.

There is a canonical holomorphic imbedding of Ω into $\tilde{\Omega}$. We denote this imbedding by $j : \Omega \hookrightarrow \tilde{\Omega}$. Corresponding to each $f \in \mathcal{O}(\Omega)$ there is a holomorphic function $\mathcal{E}(f) \in \mathcal{O}(\tilde{\Omega})$ such that $\mathcal{E}(f) \circ j = f$. By [6] (and analogously to the situation in [1]), there exists a holomorphic mapping (note that Σ_1^η varies analytically in η) $H : \mathcal{T} \to \tilde{\Omega}$ such that

$$\pi \circ H(\Sigma_1^\eta \cap (\{z\} \times \mathbb{C}_w)) = \Sigma_1^\eta \cap (\{z\} \times \mathbb{C}_w) \quad \text{for all } \eta \in D(0; \delta) \text{ and } z \in \bar{\mathbb{D}}.$$

Now, for each $p := (z_1, w_1) \in \mathcal{T} \cap (V \times D(0; r))$ there exist

- an $\eta_0 \in D(0; \delta)$ and
- a point $q \in \Sigma_0^{\eta_0} \cap \{z_1\} \times \mathbb{C}_w$

such that the continuous family $\{\Sigma_t^{\eta_0}\}_{t \in [0,1]}$ determines a path $\gamma_{qp} : [0, 1] \to \{z_1\} \times \mathbb{C}_w$ with $\gamma_{qp}(0) = q$ and $\gamma_{qp}(1) = p$. Let $\mathfrak{S}_\Omega :=$ the sheaf of $\mathcal{O}(\Omega)$-germs over $\mathbb{C}^2$ (refer to [8, Chap. 6] for the definition of an $\mathcal{O}(\Omega)$-germ), and let

$$\widetilde{\gamma_{qp}} := \text{the lift of } \gamma_{qp} \text{ to } \mathfrak{S}_\Omega \text{ starting at the germ } [g : g \in \mathcal{O}(\Omega)]_q.$$

Examining the Kontinuitätssatz, we find $H(p) = \widetilde{\gamma_{qp}}(1)$.

We know that if $[s_g : g \in \mathcal{O}(\Omega)]_z$ is an $\mathcal{O}(\Omega)$-germ in $\tilde{\Omega}$ then

$$\mathcal{E}(f)([s_g : g \in \mathcal{O}(\Omega)]_z) = s_f(z).$$

By the monodromy theorem, $\widetilde{\gamma_{qp}}(1) = [g : g \in \mathcal{O}(\Omega)]_p$ and so

$$\mathcal{E}(f) \circ H(p) = \mathcal{E}(f)(\widetilde{\gamma_{qp}}(1)) = f(p).$$

Since this expression holds for any arbitrary $p \in \mathcal{T} \cap (V \times D(0; r))$, it follows that

$$\mathcal{E}(f) \circ H = f \quad \text{on } \mathcal{T} \cap (V \times D(0; r)).$$

Finally, let $\Omega_1 := \mathcal{T} \cup (V \times D(0; r))$ and

$$f_1(z, w) := \begin{cases} \mathcal{E}(f) \circ H(z, w) & \text{if } (z, w) \in \mathcal{T}, \\ f(z, w) & \text{if } (z, w) \in V \times D(0; r). \end{cases}$$

Then $f_1 \in \mathcal{O}(\Omega_1)$ and

$$f_1|_{\Omega_1 \cap (V \times D(0;r))} \equiv f|_{\Omega_1 \cap (V \times D(0;r))}. \qquad \square$$

3. Proof of Theorem 1.3

By [1, Lemma 5] and the continuous dependence of the zeros of a polynomial on its coefficients, we know that it is enough to prove Theorem 1.3 for $a_0, \dots, a_{k-1} \in \mathfrak{G}_1$, where $\mathfrak{G}_1 \subsetneq \mathcal{C}(\bar{\mathbb{D}}; \mathbb{C})$ is the following set:

$$\Big\{ g \in \mathcal{C}^\infty(\bar{\mathbb{D}}; \mathbb{C}) : \exists N \in \mathbb{N},\ G_n \in \mathcal{C}^\infty([0, 1]; \mathbb{C}) \text{ such that } g(re^{i\theta}) = \sum_{n=0}^{N} G_n(r) e^{in\theta},\ r \in (0, 1] \Big\}.$$

Thus, we replace $a = (a_0, \dots, a_{k-1})$ in Theorem 1.3 by $b := (b_0, \dots, b_{k-1}) \in \mathfrak{G}_1^k$. This is because we can find a Σ_b so close to Σ_a that Σ_b is a subset of Ω and is attached to $\partial\mathbb{D} \times \bar{\mathbb{D}}$.

Fix a $j \in \{0, \dots, k-1\}$. Let

$$b_j(re^{i\theta}) = \sum_{n=0}^{n(j)} B_n^j(r) e^{in\theta}, \quad \theta \in [0, 2\pi),$$

where $n(j) \in \mathbb{N}$ and $B_n^j \in \mathcal{C}^\infty([0, 1]; \mathbb{C})$. Using Lemma 3 from [1], where Barrett and Bharali constructed an explicit family of analytic discs in $\bar{\mathbb{D}} \times \mathbb{C}$ with boundaries in $\{(z, b_0(z), \dots, b_{k-1}(z)) : z \in \mathbb{D}\}$, we define a family of continuous discs $\{\mathfrak{B}_t = (\mathfrak{B}_{t,0}, \dots, \mathfrak{B}_{t,k-1})\}_{t \in [0,1]}$ as follows:

$$\mathfrak{B}_{t,j}(\zeta) := \begin{cases} \sum_{n=0}^{n(j)} B_n^j(t) \left(\frac{\zeta}{t}\right)^n & \text{if } \zeta \in D(0; t), \\ b_j(\zeta) & \text{if } \zeta \in \overline{\mathrm{Ann}(0; t, 1)}. \end{cases} \tag{3.1}$$

Note that $\mathfrak{B}_0 = b$. Also, by [1, Lemma 4], $\{\mathfrak{B}_t\}_{t\in[0,1]}$ is a continuous family and $\mathfrak{B}_1 \in \mathcal{O}(\mathbb{D}; \mathbb{C}^k) \cap \mathcal{C}(\bar{\mathbb{D}}; \mathbb{C}^k)$.

Let $\delta > 0$ be so small that $\eta \in D(0; \delta)$ implies $\Sigma_b + (0, \eta) \subset \Omega \cap (\bar{\mathbb{D}} \times \mathbb{D})$. Let $b^{(\eta)} = (b_1^{(\eta)}, \dots, b_{k-1}^{(\eta)})$ be defined pointwise by (2.1). By Remark 2.3(ii), each $b_j^{(\eta)}$, as a linear combination of $b_j, \dots, b_{k-1}$, is in $\mathfrak{G}_1$. Thus, we can define continuous discs $\{\mathfrak{B}_t^{(\eta)} = (\mathfrak{B}_{t,0}^{(\eta)}, \dots, \mathfrak{B}_{t,k-1}^{(\eta)})\}_{t\in[0,1]}$ using the Fourier coefficients of $b_j^{(\eta)}(re^{i\bullet})$, $r \in (0, 1]$, just as in (3.1). It is a simple observation that the same discs can be obtained by defining, on $\bar{\mathbb{D}}$,

$$\mathfrak{B}_{t,j}^{(\eta)} := \mathfrak{B}_{t,j} + \sum_{l=j+1}^{k-1} (-1)^{l-j} \binom{l}{l-j} \mathfrak{B}_{t,l} \eta^{l-j} + (-1)^{k-j} \binom{k}{k-j} \eta^{k-j}. \quad (3.2)$$

It is important to note that $\mathfrak{B}_t^{(0)} \equiv \mathfrak{B}_t$ for all $t \in [0, 1]$.

Fix a domain $D \Subset \Omega$ such that $S \subset D$. We claim that the continuous family $\{\mathfrak{B}_t^{(\eta)}\}_{t\in[0,1]}$ satisfies the following properties.

(a) $\mathfrak{B}_0^{(\eta)} = b^{(\eta)}$ for all $\eta \in D(0; \delta)$;
(b) for a fixed t, $\mathfrak{B}_t^{(\eta)}$ depends analytically on η;
(c) for each $\mathfrak{B}_t^{(\eta)}$, $\Sigma_{\mathfrak{B}_t^{(\eta)}} \setminus \bar{D}$ is an analytic subvariety of $\bar{\mathbb{D}} \times \mathbb{C} \setminus \bar{D}$; and
(d) for each t, $\Sigma_{\mathfrak{B}_t^{(\eta)}} \subset \bar{\mathbb{D}} \times \mathbb{D}$ for all $\eta \in D(0; \delta)$.

Properties (a) and (b) follow from construction. For (c), it is enough to observe that

$$\Sigma_{\mathfrak{B}_t^{(\eta)}} = (\Sigma_{b^{(\eta)}, \overline{\mathrm{Ann}(0;t,1)}}) \cup (\Sigma_{\mathfrak{B}_t^{(\eta)}, D(0;t)})$$

and that $\mathfrak{B}_t^{(\eta)}|_{D(0;t)} \in \mathcal{O}(D(0; t); \mathbb{C}^k)$. For property (d), it is enough to show that

$$\Sigma_{\mathfrak{B}_t^{(\eta)}, D(0;t)} \subset \bar{\mathbb{D}} \times \mathbb{D}.$$

But this follows from Lemma 2.1 applied to $\Sigma_{\mathfrak{B}_t^{(\eta)}, D(0;t)}$, with $D(0; t)$ acting as G, since

$$\Sigma_{\mathfrak{B}_t^{(\eta)}, \partial D(0;t)} \equiv \Sigma_{b^{(\eta)}, \partial D(0;t)} \subset \partial D(0; t) \times \mathbb{D}.$$

From this expression we can conclude that the mapping $A: \bar{\mathbb{D}} \times [0, 1] \to \mathbb{C}^k$ with $A(z, t) := \mathfrak{B}_t(z)$ satisfies the hypotheses of Lemma 2.4. Hence there exist a connected open neighborhood Ω_1 of $S_1 := \Sigma_{\mathfrak{B}_1^{(0)}} \cup (\partial\mathbb{D} \times \mathbb{D})$ and an $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1|_{\Omega_1 \cap (V \times \mathbb{D})} \equiv f|_{\Omega_1 \cap (V \times \mathbb{D})},$$

where $V := \mathrm{Ann}(0; 1 - \varepsilon, 1 + \varepsilon)$, $\varepsilon > 0$, chosen so that $V \times \mathbb{D} \subset \Omega$.

However, $\mathfrak{B}_1^{(0)}$ is holomorphic by construction. So in view of the ideas presented in Remark 2.3, we can argue as follows.

1. Let $\{\mathfrak{U}_s\}_{s\in[0,1]}$ be defined as

$$\mathfrak{U}_s = (\mathfrak{U}_{s,0}, \dots, \mathfrak{U}_{s,k-1}) := (s^k \mathfrak{B}_{1,0}^{(0)}, \dots, s^{k-j} \mathfrak{B}_{1,j}^{(0)}, \dots, s \mathfrak{B}_{1,k-1}^{(0)}).$$

Then, for each $s \in [0, 1]$, $\mathfrak{U}_s$ is analytic and $\Sigma_{\mathfrak{U}_s} = \{(z, sw) : (z\, w) \in \Sigma_{\mathfrak{B}_1^{(0)}}\}$.

2. Let $\tilde{\delta} > 0$ be so small that $\eta \in D(0;\tilde{\delta})$ implies $\Sigma_{\mathfrak{U}_1} + (0,\eta) \subset \Omega_1 \cap (\bar{\mathbb{D}} \times \mathbb{D})$. Note that, by Remark 2.3(ii), each $\mathfrak{U}_s^{(\eta)}$ depends analytically on η.
3. We are now in a position to apply Lemma 2.4 to the continuous mapping $U\colon \bar{\mathbb{D}} \times [0,1] \to \mathbb{C}^k$ with $U(z,s) := \mathfrak{U}_{1-s}(z)$. Thus, there exist a connected open neighborhood Ω_2 of $S_2 := \Sigma_{\mathfrak{U}_0} \cup (\partial\mathbb{D} \times \mathbb{D}) = (\bar{\mathbb{D}} \times \{0\}) \cup (\partial\mathbb{D} \times \mathbb{D})$ and an $f_2 \in \mathcal{O}(\Omega_2)$ such that

$$f_2|_{\Omega_2 \cap (V \times \mathbb{D})} \equiv f_1|_{\Omega_2 \cap (V \times \mathbb{D})},$$

where $V := \mathrm{Ann}(0; 1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, chosen so that $V \times \mathbb{D} \subset \Omega$.

By the classical theorem of Hartogs, there exists an $F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F|_{\Omega_2 \cap \mathbb{D}^2} \equiv f_2|_{\Omega_2 \cap \mathbb{D}^2}.$$

Therefore, F and f must coincide in $\Omega_2 \cap \Omega_1 \cap (V \times \mathbb{D}) \cap \mathbb{D}^2$. Since the latter is an open subset of the connected set $\Omega \cap \mathbb{D}^2$, we conclude that

$$F|_{\Omega \cap \mathbb{D}^2} \equiv f|_{\Omega \cap \mathbb{D}^2}.$$

4. Proof of Theorem 1.4

The proof of this theorem is similar to that of Theorem 1.3. The main difference lies in the specific method of constructing—starting from the given multigraph—a continuous family of multigraphs along which we can achieve analytic continuation by invoking the Kontinuitätssatz. Recall that, in Section 3, the form of each coefficient function a_j facilitated the construction of functions that were holomorphic on increasing concentric discs in $\mathbb{D}$. In the present case, in order to perturb the coefficients we will construct analytic annuli attached to the graphs of a_j along their inner boundaries and attached to $\partial\mathbb{D} \times \mathbb{D}$ along their outer boundaries. In view of Remark 2.3(i), we are compelled to work with a polydisc longer than $\mathbb{D}^2$.

Let $a(z) = \psi(z,\bar{z}) := (\psi_0(z,\bar{z}), \dots, \psi_{k-1}(z,\bar{z}))$. Set $\mathcal{R} := \bar{\mathbb{D}} \times [0,1]$. Note that, by hypothesis, we can find an $\varepsilon > 0$ such that $\mathrm{Ann}(0; 1-\varepsilon, 1+\varepsilon) \times \mathbb{D}(0;2) \subset \Omega$. Hence, if we keep in mind the closing arguments in Section 3, it suffices to work with $\Sigma_{a,\overline{D(0;1-\varepsilon/2)}}$ and the Hartogs configuration

$$S_\varepsilon := \Sigma_{a,\overline{D(0;1-\varepsilon/2)}} \cup (\partial D(0; 1-\varepsilon/2) \times \overline{D(0;2)}).$$

This affords us the following useful property:

$$(\zeta, s) \mapsto \psi_j(\zeta, s\bar{\zeta}) \text{ is continuous on } \overline{D(0;1-\varepsilon/2)} \times [0,1] \quad \forall j = 0, \dots, k-1.$$

Hence it actually suffices to prove Theorem 1.4 under the assumption that $\psi_0, \dots, \psi_{k-1} \in \mathfrak{G}_2$, where

$$\mathfrak{G}_2 := \Big\{\psi \in \mathcal{O}(\mathbb{D}^2) \cap \mathcal{C}(\bar{\mathbb{D}}^2) : \sup_{(\zeta,s)\in\mathcal{R}} |\psi(\zeta, s\bar{\zeta})| < 1\Big\}.$$

In order to avoid messy subscripted notation such as $\Sigma_{a,\overline{D(0;1-\varepsilon/2)}}$ and messy normalizations, we shall hereafter assume that $\psi_j \in \mathfrak{G}_2$ for $j = 0, \dots, k-1$.

We define a family of continuous discs $\{\Psi_t = (\Psi_{t,0}, \dots, \Psi_{t,k-1})\}_{t\in[0,1)}$ as follows:

$$\Psi_t(\zeta) := \begin{cases} a(\zeta) = \psi(\zeta, \bar{\zeta}) & \text{if } \zeta \in D(0; 1-t), \\ \psi\big(\zeta, \frac{(1-t)^2}{\zeta}\big) & \text{if } \zeta \in \overline{\mathrm{Ann}(0; 1-t, 1)}. \end{cases} \tag{4.1}$$

Therefore, $\Psi_0 = a$. We observe that $\{\Psi_t\}_{t\in[0,1)}$ is a continuous family in the sense that, for a fixed $\zeta_0 \in \bar{\mathbb{D}}$, $t \mapsto \Psi_t(\zeta_0)$ is continuous in the interval $[0, 1)$. Furthermore, we may define

$$\Psi_1(\zeta) := \lim_{t\to 1^-} \Psi_t(\zeta) = \psi(\zeta, 0). \tag{4.2}$$

Thus, $\Psi_1 \in \mathcal{O}(\mathbb{D}; \mathbb{C}^k)$. We also note that, for each $t \in [0, 1]$,

$$\sup_{\zeta\in\partial\mathbb{D}} |\Psi_{t,j}(\zeta)| = \sup_{\zeta\in\partial\mathbb{D}} |\psi_j(\zeta, (1-t)^2\bar{\zeta})| < 1, \quad j = 0, \dots, k-1. \tag{4.3}$$

Let $\delta > 0$ be so small that

- $\eta \in D(0; \delta)$ implies $\Sigma_a + (0, \eta) \subset \Omega \cap (\bar{\mathbb{D}} \times \mathbb{D})$ and
- for all $\eta \in D(0; \delta)$ and $j = 0, \dots, k-1$,

$$\sup_{(\zeta,s)\in\mathcal{R}} |\psi_j(\zeta, s\bar{\zeta})| + \sum_{l=j+1}^{k-1} \binom{l}{l-j} \sup_{(\zeta,s)\in\mathcal{R}} |\psi_l(\zeta, s\bar{\zeta})||\eta|^{l-j} + \binom{k}{k-j}|\eta|^{k-j} < 1. \tag{4.4}$$

Let $\psi^{(\eta)} = (\psi_1^{(\eta)}, \dots, \psi_{k-1}^{(\eta)}) \in \mathcal{O}(\mathbb{D}^2; \mathbb{C}^k)$ be defined pointwise on $\bar{\mathbb{D}}^2$ by (2.1). By (4.4), we have

$$\sup_{(\zeta,s)\in\mathcal{R}} |\psi_j^{(\eta)}(\zeta, s\bar{\zeta})| < 1 \quad \text{for all } \eta \in D(0; \delta) \text{ and } j = 0, \dots, k-1.$$

Thus, each $\psi_j^{(\eta)} \in \mathfrak{G}_2$.

Now, just as in the proof of Theorem 1.3, we use $\{\Psi_t\}_{t\in[0,1]}$ to construct continuous families of continuous discs $\{\Psi_t^{(\eta)} = (\Psi_{t,0}^{(\eta)}, \dots, \Psi_{t,k-1}^{(\eta)})\}_{t\in[0,1]}$ on $\bar{\mathbb{D}}$ as follows:

$$\Psi_{t,j}^{(\eta)} := \Psi_{t,j} + \sum_{l=j+1}^{k-1} (-1)^{l-j}\binom{l}{l-j}\Psi_{t,l}\eta^{l-j} + (-1)^{k-j}\binom{k}{k-j}\eta^{k-j}. \tag{4.5}$$

Observe that $\Psi_t^{(0)} = \Psi_t$ and, by construction,

$$\sup_{\zeta\in\partial\mathbb{D}} |\Psi_t^{(\eta)}(\zeta)| = \sup_{\zeta\in\partial\mathbb{D}} |\psi_t^{(\eta)}(\zeta, (1-t)^2\bar{\zeta})| < 1. \tag{4.6}$$

As before, we fix a domain $D \Subset \Omega$ such that $S \subset D$ and claim that the following properties are satisfied:

(a*) $\Psi_0^{(\eta)} = a^{(\eta)}$ for all $\eta \in D(0; \delta)$;
(b*) for a fixed t, $\Psi_t^{(\eta)}$ depends analytically on η;
(c*) for each $\Psi_t^{(\eta)}$, $\Sigma_{\Psi_t^{(\eta)}} \setminus \bar{D}$ is an analytic subvariety of $\bar{\mathbb{D}} \times \mathbb{C} \setminus \bar{D}$; and
(d*) for each t, $\Sigma_{\Psi_t^{(\eta)}} \subset \bar{\mathbb{D}} \times D(0; 2)$ for all $\eta \in D(0; \delta)$.

Properties (a*) and (b*) pose no problem, and (c*) can be argued in exactly the same way as in Section 3. For (d*) we write, in the notation established in Section 2,

$$\Sigma_{\Psi_t^{(\eta)}, \partial\,\mathrm{Ann}(0;1-t,1)} = \Sigma_{\Psi_t^{(\eta)}, \partial D(0;1-t)} \cup \Sigma_{\Psi_t^{(\eta)}, \partial\mathbb{D}}. \tag{4.7}$$

Note that $\Sigma_{\Psi_t^{(\eta)}, \partial D(0;1-t)} \subset \partial D(0; 1 - t) \times D(0; 2)$ whereas, by (4.6) and Remark 2.3(i), we have $\Sigma_{\Psi_t^{(\eta)}, \partial\mathbb{D}} \subset \partial\mathbb{D} \times D(0; 2)$. Thus, applying Lemma 2.1 (specifically, its paraphrasing in Remark 2.2) to $\Sigma_{\Psi_t^{(\eta)}, \mathrm{Ann}(0;1-t,1)}$ shows that (d*) holds.

We can now conclude that the mapping $A : \bar{\mathbb{D}} \times [0, 1] \to \mathbb{C}^k$ defined as $A(z, t) := \Psi_t(z)$ satisfies the hypotheses of Lemma 2.4. Hence there exist a connected open neighborhood Ω_1 of $S_1 := \Sigma_{\Psi_1^{(0)}} \cup (\partial\mathbb{D} \times D(0; 2))$ and an $f_1 \in \mathcal{O}(\Omega_1)$ such that

$$f_1|_{\Omega_1 \cap (V \times D(0;2))} \equiv f|_{\Omega_1 \cap (V \times D(0;2))},$$

where $V := \mathrm{Ann}(0; 1-\varepsilon, 1+\varepsilon)$, $\varepsilon > 0$, chosen so that $V \times D(0; 2) \subset \Omega$. Because $\Psi_1^{(0)}$ is holomorphic by construction, we can repeat the argument presented in the proof of Theorem 1.3 to conclude that there exists an $F \in \mathcal{O}(\mathbb{D}^2)$ such that

$$F|_{\Omega \cap \mathbb{D}^2} \equiv f|_{\Omega \cap \mathbb{D}^2}.$$

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