Michigan Math. J. 60 (2011), 525–544

Lefschetz Fibration Structures on Knot Surgery 4-Manifolds

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1. Introduction

Since Seiberg–Witten theory was introduced in 1994, many techniques in 4-dimensional topology have been developed to show that a large class of simply connected smooth 4-manifolds admit infinitely many distinct smooth structures. Among them, a knot surgery technique introduced by R. Fintushel and R. Stern turned out to be one of the most powerful tools for changing the smooth structure on a given 4-manifold [4]. The knot surgery construction is as follows. Suppose that X is a simply connected smooth 4-manifold containing an embedded torus T of square 0. Then, for any knot $K \subset S^3$, one can construct a new 4-manifold, called *a knot surgery 4-manifold,*

$$X_K = X \sharp_{T=T_m} (S^1 \times M_K)$$

by taking a fiber sum along a torus T in X and $T_m = S^1 \times m$ in $S^1 \times M_K$, where M_K is the 3-manifold obtained by doing 0-framed surgery along K and m is the meridian of K. Then Fintushel and Stern proved that, under a mild condition on X and T, the knot surgery 4-manifold X_K is homeomorphic, but not diffeomorphic, to a given X [4]. Furthermore, if X is a simply connected elliptic surface $E(2)$, T is the elliptic fiber, and K is a fibered knot, then it is also known that the knot surgery 4-manifold $E(2)_K$ admits not only a symplectic structure but also a genus $2g(K) + 1$ Lefschetz fibration structure [6; 23]. Note that there are only two inequivalent genus 1 fibered knots, but there are infinitely many inequivalent genus g fibered knots for $g \geq 2$. So one may dig out some interesting properties of $E(2)_K$ by carefully investigating genus 2 fibered knots and related Lefschetz fibration structures.

On the one hand, Fintushel and Stern [5] conjectured that the set of all knot surgery 4-manifolds of the form $E(2)_K$ up to diffeomorphism is in one-to-one correspondence with the set of all knots in S^3 up to knot equivalence. Some progress related to the conjecture has been made by S. Akbulut [2] and M. Akaho [1]. However, a complete answer to the conjecture for prime knots up to mirror image is not known yet. Furthermore, Fintushel and Stern [6] also questioned whether or not any two in the following 4-manifolds,

Received December 4, 2009. Revision received October 26, 2010.

$$\{Y(2;\, K_1, K_2) := E(2)_{K_1} \sharp_{\mathrm{id}\colon \Sigma_{2g+1}\to\Sigma_{2g+1}} E(2)_{K_2} \mid K_1, K_2 \text{ are genus } g \text{ fibered knots}\},$$

are mutually diffeomorphic. The second author obtained a partial result related to this question under the constraint that one of K_i $(i = 1, 2)$ be fixed [23].

In this paper we investigate Lefschetz fibration structures on the knot surgery 4-manifold $E(2)_K$, where K ranges over a family of Kanenobu knots. Recall that Kanenobu [13; 14] found an interesting family of inequivalent genus 2 fibered prime knots

$$\{K_{p,q} \mid (p,q) \in \mathcal{R}\} \quad \text{for } \mathcal{R} = \{(p,q) \in \mathbb{Z}^2 \mid p \in \mathbb{Z}^+,\ -p \le q \le p\},$$

where no two of the knots are in mirror relation and all of them have the same Alexander polynomials. In Section 3 we consider the following family of simply connected symplectic 4-manifolds that have the same Seiberg–Witten invariants:

$$\{Y(2;\, K_{p,q}, K_{r,s}) := E(2)_{K_{p,q}} \sharp_{\mathrm{id}\colon \Sigma_5\to\Sigma_5} E(2)_{K_{r,s}} \mid (p,q), (r,s) \in \mathcal{R}\}.$$

By investigating the monodromy factorization expression corresponding to the Lefschetz fibration structure on $Y(2;\, K_{p,q}, K_{r,s})$, we answer the question raised in [6].

THEOREM 1.1. *Any two symplectic 4-manifolds in*

$$\{Y(2;\, K_{p,q}, K_{p+1,q}) \mid p, q \in \mathbb{Z}\}$$

are mutually diffeomorphic. Similarly, any two symplectic 4-manifolds in

$$\{Y(2;\, K_{p,q}, K_{p,q+1}) \mid p, q \in \mathbb{Z}\}$$

are mutually diffeomorphic.

In Section 4 we also study nonisomorphic Lefschetz fibration structures on simply connected symplectic 4-manifolds that share the same Seiberg–Witten invariants.

Let $\xi_{p,q}$ be the monodromy factorization of a genus 5 Lefschetz fibration structure on $E(2)_{K_{p,q}}$ corresponding to the fixed generic fiber (as in Theorem 2.8) and the specified monodromy $\Phi_{K_{p,q}}$ of the fibered knot $K_{p,q}$ (as in Section 3). Then, by investigating the monodromy group $G_F(\xi_{p,q})$ of $\xi_{p,q}$, we get the following theorem.

THEOREM 1.2. *$\xi_{p,q}$ is not equivalent to $\xi_{r,s}$ if $(p,q) \not\equiv (r,s) \pmod 2$.*

REMARK 1.3. For any $(p,q) \in \mathbb{Z}^2$, $K_{p,q}$ is equivalent to $K_{q,p}$ and therefore $E(2)_{K_{p,q}}$ is diffeomorphic to $E(2)_{K_{q,p}}$. Since $K_{p,q}$ and $K_{q,p}$ are equivalent fibered knots, their monodromy can be conjugated, which means that we can select a pair of isomorphic Lefschetz fibration structures from $E(2)_{K_{p,q}}$ and $E(2)_{K_{q,p}}$. But this does not imply that the Lefschetz fibration structure on $E(2)_{K_{p,q}} \approx E(2)_{K_{q,p}}$ is unique (see Remark 2.9 for details). In fact, Theorem 1.2 implies that, if $p \not\equiv q \pmod 2$, then we can select a pair of inequivalent special monodromy factorizations $\xi_{p,q}$ of $E(2)_{K_{p,q}}$ and $\xi_{q,p}$ of $E(2)_{K_{q,p}}$.

Acknowledgments. The authors thank the referee for invaluable comments and for correcting some errors in an earlier version of the paper. Jongil Park was supported by Basic Science Research Program through a National Research Foundation of Korea (NRF) Grant funded by the Korean government (no. 2009-0093866). Ki-Heon Yun was supported by a National Research Foundation of Korea (NRF) Grant funded by the Korean government (no. 2009-0066328).

2. Preliminaries

In this section we briefly review some well-known facts about Lefschetz fibrations on 4-manifolds and surface mapping class groups (refer to [8] for details).

Definition 2.1. Let X be a compact, oriented smooth 4-manifold. A *Lefschetz fibration* is a proper smooth map $\pi : X \to B$, where B is a compact connected oriented surface and $\pi^{-1}(\partial B) = \partial X$ such that:

(1) the set of critical points $C = \{p_1, p_2, \dots, p_n\}$ of π is nonempty and lies in $\operatorname{int}(X)$, and π is injective on C;
(2) for each p_i and $b_i := \pi(p_i)$, there are local complex coordinate charts agreeing with the orientations of X and B such that π can be expressed as $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Two Lefschetz fibrations $f_1 : X_1 \to B_1$ and $f_2 : X_2 \to B_2$ are called *isomorphic* if there are orientation-preserving diffeomorphisms $H : X_1 \to X_2$ and $h : B_1 \to B_2$ such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{H} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow{h} & B_2. \end{array} \tag{2.1}$$

Monodromy factorization of a Lefschetz fibration is an ordered sequence of right-handed Dehn twists along simple closed curves on the fixed generic fiber F of the Lefschetz fibration whose composition becomes the identity element in the mapping class group of F.

Two monodromy factorizations W_1 and W_2 are referred to as a *Hurwitz equivalence* if W_1 can be changed to W_2 in finitely many steps of the following two operations:

(1) *Hurwitz move:* $t_{c_n} \cdots t_{c_{i+1}} \cdot t_{c_i} \cdots t_{c_1} \sim t_{c_n} \cdots t_{c_{i+1}}(t_{c_i}) \cdot t_{c_{i+1}} \cdots t_{c_1}$;
(2) *inverse Hurwitz move:* $t_{c_n} \cdots t_{c_{i+1}} \cdot t_{c_i} \cdots t_{c_1} \sim t_{c_n} \cdots t_{c_i} \cdot t_{c_i}^{-1}(t_{c_{i+1}}) \cdots t_{c_1}$.

Here $t_a(t_b) = t_{t_a(b)}$, and $t_a(t_b) = t_a \circ t_b \circ t_a^{-1}$ as an element of mapping class group. This relation comes from the choice of Hurwitz system, a set of mutually disjoint arcs that connect b_0 to b_i but exclude the base point b_0.

A choice of generic fiber also gives another equivalence relation. Two monodromy factorizations W_1 and W_2 are called a *simultaneous conjugation equivalence* if $W_2 = f(W_1)$ for some element f of the mapping class group of the chosen generic fiber of the Lefschetz fibration W_1.

It is well known that monodromy factorizations of two isomorphic Lefschetz fibrations are related by a finite sequence of Hurwitz equivalences and simultaneous conjugation equivalences [8; 15;18]. Therefore, in this paper we do not distinguish a monodromy factorization from the corresponding Lefschetz fibration up to isomorphism.

TERMINOLOGY. In order to emphasize that a chosen generic fiber is fixed, we sometimes use the term *marked* Lefschetz fibration to refer to a Lefschetz fibration whose chosen generic fiber is fixed. Two monodromy factorizations are also called *marked* equivalent if they are equivalent under a chosen fixed generic fiber.

NOTATION. We write $W_1 \cong W_2$ if two monodromy factorizations W_1 and W_2 give the isomorphic Lefschetz fibration. When two manifolds X_1 and X_2 are diffeomorphic, we write this as $X_1 \approx X_2$.

DEFINITION 2.2. Let $\pi : X \to S^2$ be a Lefschetz fibration and let F be a fixed generic fiber of the Lefschetz fibration. Let $W = w_n \cdots w_2 \cdot w_1$ be a monodromy factorization of the Lefschetz fibration corresponding to F. Then the *monodromy group* $G_F(W)$ is a subgroup of the mapping class group $\mathcal{M}_F = \pi_0(\mathrm{Diff}^+(F))$ generated by $w_1, w_2, \dots, w_n$. We will simply write $G(W)$ when the generic fiber F is clear from the context. The element $w_n \circ \cdots \circ w_2 \circ w_1$ in $\mathcal{M}_F$ is denoted by λ_W.

LEMMA 2.3. *If two monodromy factorizations W_1 and W_2 give isomorphic Lefschetz fibrations over S^2 with respect to chosen generic fibers F_1 and F_2 (respectively) that are homeomorphic to F, then the monodromy groups $G_{F_1}(W_1)$ and $G_{F_2}(W_2)$ are isomorphic as a subgroup of the mapping class group $\mathcal{M}_F$. Moreover, if a generic fiber $F = F_1 = F_2$ is fixed then $G_F(W_1) = G_F(W_2)$.*

REMARK 2.4. As mentioned previously, the role of simultaneous conjugation equivalence is in the choice of a generic fiber. If we use the same fixed generic fiber for W_1 and W_2 (i.e., if $F_1 = F = F_2$), then the global conjugation cannot occur. Therefore we get $G_F(W_1) = G_F(W_2)$.

A monodromy factorization of a Lefschetz fibration structure on $E(n)_K$ was studied by Fintushel and Stern [6]. We were able to find an explicit monodromy factorization of $E(n)_K$ [23] with the help of factorizations of the identity element in the mapping class group that were discovered by Y. Matsumoto [18], M. Korkmaz [17], and Y. Gurtas [9].

DEFINITION 2.5. Let $M(n, g)$ be the desingularization of the double cover of $\Sigma_g \times S^2$ branched over $2n(\{\text{point}\} \times S^2) \cup 2(\Sigma_g \times \{\text{point}\})$.

LEMMA 2.6 [17; 22]. *$M(2, g)$ has a monodromy factorization $\eta_{1,g}^2$, where*

$$\eta_{1,g} = t_{B_0} \cdot t_{B_1} \cdot t_{B_2} \cdots t_{B_{2g}} \cdot t_{B_{2g+1}} \cdot t_{b_{g+1}}^2 \cdot t_{b'_{g+1}}^2$$

and $\{B_j, b_{g+1}, b'_{g+1}\}$ are simple closed curves on Σ_{2g+1} as in Figure 1.

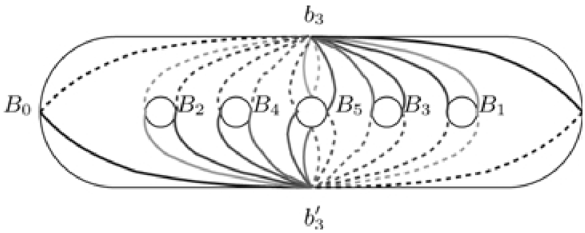

Figure 1 Vanishing cycles of $M(2,g)$ with $g=2$

REMARK 2.7. In this paper we assume that we have already fixed a reference generic fiber as in Figure 1 and read the monodromy factorization with respect to the chosen generic fiber. From now on we use the monodromy factorization $\eta_{1,g}^2$ in Lemma 2.6 for $M(2,g)$ as a genus $2g+1$ Lefschetz fibration with respect to the given fixed generic fiber.

THEOREM 2.8 [6; 23]. *Let $K \subset S^3$ be a fibered knot of genus g. Then $E(2)_K$, as a genus $2g+1$ Lefschetz fibration, has a monodromy factorization of the form*

$$\Phi_K(\eta_{1,g}) \cdot \Phi_K(\eta_{1,g}) \cdot \eta_{1,g} \cdot \eta_{1,g},$$

where $\eta_{1,g}^2$ is a monodromy factorization of $M(2,g)$ and

$$\Phi_K = \varphi_K \oplus \mathrm{id} \oplus \mathrm{id}\colon \Sigma_g \sharp \Sigma_1 \sharp \Sigma_g \to \Sigma_g \sharp \Sigma_1 \sharp \Sigma_g$$

is a diffeomorphism obtained by using a (geometric) monodromy φ_K of K defined by

$$S^3 \setminus \nu(K) = (I \times \Sigma_g^1)/((1,x) \sim (0, \varphi_K(x))),$$

where Σ_g^1 is an oriented surface of genus g with one boundary component.

REMARK 2.9. If two fibered knots K_1 and K_2 are equivalent with fiber surface Σ_g^1, then there is a homeomorphism $\phi\colon \Sigma_g^1 \to \Sigma_g^1$ such that

$$S^3 \setminus \nu(K_1) = (I \times \Sigma_g^1)/\sim_{\varphi_{K_1}} \approx (I \times \Sigma_g^1)/\sim_{\phi \circ \varphi_{K_1} \circ \phi^{-1}} = S^3 \setminus \nu(K_2).$$

So if we select a generic fiber $F' \approx \Sigma_{2g+1}$ of $M(2,g)$ such that $\Phi(\eta_{1,g}^2)$ is a monodromy factorization of $M(2,g)$ as a genus $2g+1$ Lefschetz fibration, then

$$\begin{aligned}\Phi(\eta_{1,g}^2) \cdot \Phi_{K_2}(\Phi(\eta_{1,g}^2)) &= \Phi(\eta_{1,g}^2) \cdot (\Phi \circ \Phi_{K_1} \circ \Phi^{-1})(\Phi(\eta_{1,g}^2)) \\ &= \Phi(\eta_{1,g}^2 \cdot \Phi_{K_1}(\eta_{1,g}^2)) \cong \eta_{1,g}^2 \cdot \Phi_{K_1}(\eta_{1,g}^2);\end{aligned}$$

this implies that we can select a pair of isomorphic Lefschetz fibration structures from $E(2)_{K_1}$ and $E(2)_{K_2}$.

On the other hand, for a given fibered knot K and its fiber surface Σ_K^1, we identify Σ_K^1 and $\Sigma_g^1 = \Sigma_g - \mathrm{int}(D^2) \subset \Sigma_g \sharp \Sigma_1 \sharp \Sigma_g$ by a fixed homeomorphism. Even though we fix a generic fiber Σ_{2g+1} of $M(2,g)$ and fix an identification between Σ_K^1 and Σ_g^1, there is still some ambiguity regarding the choice of monodromy factorization. For a given homeomorphism $\phi\colon \Sigma_g^1 \to \Sigma_g^1$ that fixes $\partial\Sigma_g^1$ pointwise, there is a fiber-preserving homeomorphism

$$(I \times \Sigma_g^1)/{\sim_{\varphi_K}} \to (I \times \Sigma_g^1)/{\sim_{\phi\circ\varphi_K\circ\phi^{-1}}} .$$

[3, 5.B]. Hence we do not change the fixed generic fiber and corresponding monodromy factorization $\eta_{1,g}^2$ of $M(2,g)$, but the gluing map is changed to $\Phi\circ\Phi_K\circ\Phi^{-1}$, where Φ is the extension of the homeomorphism ϕ to Σ_{2g+1}. We can interpret this phenomenon as a change of chosen generic fiber in $M(2,g)$ so that the monodromy factorization becomes $\Phi^{-1}(\eta_{1,g}^2)$. But it does not mean that $\Phi_K(\eta_{1,g}^2)\cdot\eta_{1,g}^2$ is isomorphic to $(\Phi\circ\Phi_K\circ\Phi^{-1})(\eta_{1,g}^2)\cdot\eta_{1,g}^2$ as a marked Lefschetz fibration. We will consider this phenomenon in Section 4.

3. Isomorphic Lefschetz Fibrations

In this section we construct examples of simply connected isomorphic symplectic Lefschetz fibrations with the same generic fiber but coming from a pair of inequivalent fibered knots. In [6], Fintushel and Stern constructed families of simply connected symplectic 4-manifolds with the same Seiberg–Witten invariants. Among them, they considered a set of the following symplectic 4-manifolds,

$$\{Y(2;\,K_1,K_2) := E(2)_{K_1}\sharp_{\mathrm{id}:\,\Sigma_{2g+1}\to\Sigma_{2g+1}}E(2)_{K_2} \mid K_1,K_2 \text{ are genus } g \text{ fibered knots}\},$$

and they showed that

$$\mathcal{SW}_{Y(2;\,K_1,K_2)} = t_L + t_L^{-1}$$

because the only basic classes of $Y(2;\,K_1,K_2)$ are $\pm L$, where L is the canonical class of $Y(2;\,K_1,K_2)$. In [23] we found examples such that $Y(2;\,K,K_1)$ and $Y(2;\,K,K_2)$ are diffeomorphic even though K_1 is not equivalent to K_2. In this section we will generalize such a construction. That is, we will construct infinitely many pairs (K,K') of inequivalent genus 2 fibered knots such that all the $Y(2;\,K,K')$ are mutually diffeomorphic.

A family of inequivalent knots with the same Alexander polynomials has been constructed by several authors. Among them, Kinoshita and Terasaka [16] constructed a nontrivial knot with the trivial Alexander polynomial by using a *knot union* operation. Thereafter, Kanenobu constructed infinitely many inequivalent knots $K_{p,q}$ $(p,q\in\mathbb{Z})$ with the same Alexander polynomials [13; 14]. These examples were constructed from the ribbon fibered knot $4_1\#(-4_1^*)$ by repeatedly applying the Stallings' twist [21] at two different locations where K^* is the mirror image of K.

The following lemma was cited by Kanenobu.

LEMMA 3.1 [13]. *Let $K_{p,q}$ be a Kanenobu knot as in Figure 2. Then*

(1) $K_{0,0} = 4_1\#(-4_1^*)$,
(2) *the Alexander matrix of* $K_{p,q}$ *is* $\begin{pmatrix} t^2-3t+1 & (p-q)t \\ 0 & t^2-3t+1 \end{pmatrix}$,
(3) $\Delta_{K_{p,q}}(t) \doteq (t-3+t^{-1})^2$,
(4) $K_{p,q}$ *is a fibered ribbon knot,*
(5) $K_{p,q}\sim K_{r,s}$ *if and only if* $(p,q)=(r,s)$ *or* (s,r),
(6) $K_{p,q}^*\sim K_{-q,-p}$, *and*
(7) $K_{p,q}$ *is a prime knot if* $(p,q)\neq(0,0)$.

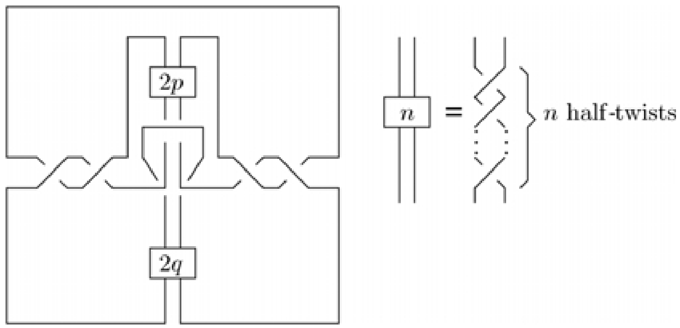

Figure 2 A Kanenobu knot $K_{p,q}$

It is not hard to see [10] that the monodromy map $\Phi_{K_{p,q}}$ of a Kanenobu knot $K_{p,q}$ is

$$\Phi_{K_{p,q}} = t_d^q \circ t_{c_2}^p \circ t_{a_2} \circ t_{b_2}^{-1} \circ t_{a_1}^{-1} \circ t_{b_1},$$

where $\{a_i, b_i, c_i, d\}$ are the simple closed curves shown in Figure 3. The reason is that we first perform Hopf plumbings of right-handed Hopf bands along the arc b_1 and of left-handed Hopf bands along b_2 and then perform Hopf plumbings of left-handed Hopf bands along arcs a_1 and of right-handed Hopf bands along a_2; see Figure 4. After that, we repeatedly perform Stallings' twists along simple closed curves c_2 and d as in Figure 4. The result is a monodromy of the fibered knot $K_{p,q}$ corresponding to the fiber surface, as in the right-hand side of Figure 4. We can naturally identify the simple closed curves a_1, b_1, a_2, b_2, c_2, and d in Figure 4 with the same lettered curves on the surface Σ_5 in Figure 3. We will read the monodromy factorization $\xi_{p,q}$ of $E(2)_{K_{p,q}}$ as a genus 5 Lefschetz fibration by using this identification.

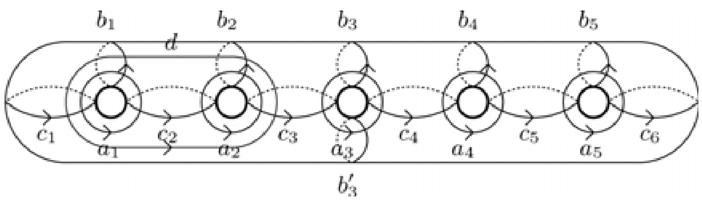

Figure 3 Standard simple closed curves

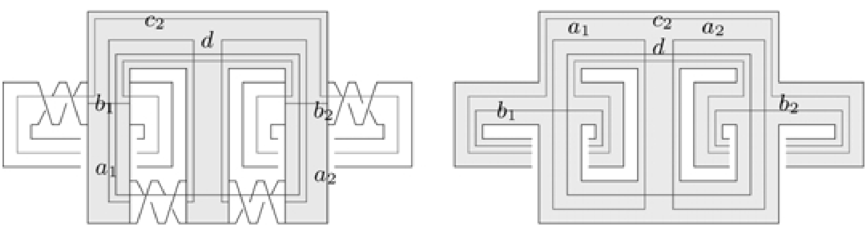

Figure 4 A fiber surface of $K_{p,q}$

Then we get that $Y(2; K_{p,q}, K_{r,s})$ has a monodromy factorization of the form

$$\Phi_{K_{r,s}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2 \cdot \Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2.$$

LEMMA 3.2. *For any $p, q \in \mathbb{Z}$ and $\Phi_{K_{p,q}} = t_d^q \circ t_{c_2}^p \circ t_{a_2} \circ t_{b_2}^{-1} \circ t_{a_1}^{-1} \circ t_{b_1}$, we have*

$$t_{c_2} \in G_F(\eta_{1,2}^2 \cdot t_{c_2}(\eta_{1,2}^2)), \qquad t_d \in G_F(\eta_{1,2}^2 \cdot t_d(\eta_{1,2}^2))$$

and

$$t_{c_2} \in G_F(\Phi_{K_{p+1,q}}(\eta_{1,2}^2) \cdot \Phi_{K_{p,q}}(\eta_{1,2}^2)), \qquad t_d \in G_F(\Phi_{K_{p,q+1}}(\eta_{1,2}^2) \cdot \Phi_{K_{p,q}}(\eta_{1,2}^2)).$$

Proof. Since B_2 and c_2 meet at one point on Σ_5, by the braid relation we get

$$t_{c_2} \circ t_{B_2} \circ t_{c_2} = t_{B_2} \circ t_{c_2} \circ t_{B_2}.$$

This implies that

$$t_{c_2} = t_{B_2} \circ t_{c_2} \circ t_{B_2} \circ t_{c_2}^{-1} \circ t_{B_2}^{-1} = t_{B_2} \circ t_{c_2}(t_{B_2}) \circ t_{B_2}^{-1}.$$

Since $t_{B_2}, t_{c_2}(t_{B_2}) \in G_F(\eta_{1,2}^2 \cdot t_{c_2}(\eta_{1,2}^2))$, we get

$$t_{c_2} \in G_F(\eta_{1,2}^2 \cdot t_{c_2}(\eta_{1,2}^2)).$$

Each of B_1, B_2, B_3, B_4 meets at one point with the simple closed curve d. So by the braid relation we get

$$t_d \circ t_{B_i} \circ t_d = t_{B_i} \circ t_d \circ t_{B_i}, \quad i = 1, 2, 3, 4,$$

which implies

$$t_d = t_{B_i} \circ t_d(t_{B_i}) \circ t_{B_i}^{-1}, \quad i = 1, 2, 3, 4.$$

Since $t_{B_i}, t_d(t_{B_i}) \in G_F(\eta_{1,2}^2 \cdot t_d(\eta_{1,2}^2))$, we get

$$t_d \in G_F(\eta_{1,2}^2 \cdot t_d(\eta_{1,2}^2)).$$

Observe that $\Phi_{K_{0,0}}(B_3)$ meets with c_2 at one point and $\Phi_{K_{0,0}}(B_4)$ meets with d at one point. Therefore,

$$t_{\Phi_{K_{0,0}}(B_3)} \circ t_{c_2} \circ t_{\Phi_{K_{0,0}}(B_3)} = t_{c_2} \circ t_{\Phi_{K_{0,0}}(B_3)} \circ t_{c_2},$$
$$t_{\Phi_{K_{0,0}}(B_4)} \circ t_d \circ t_{\Phi_{K_{0,0}}(B_4)} = t_d \circ t_{\Phi_{K_{0,0}}(B_4)} \circ t_d.$$

This implies that

$$\begin{aligned}
t_{c_2} &= t_d^q \circ t_{c_2}^p \circ t_{c_2} \circ t_{c_2}^{-p} \circ t_d^{-q} \\
&= t_d^q \circ t_{c_2}^p \circ (t_{\Phi_{K_{0,0}}(B_3)} \circ t_{c_2} \circ t_{\Phi_{K_{0,0}}(B_3)} \circ t_{c_2}^{-1} \circ t_{\Phi_{K_{0,0}}(B_3)}^{-1}) \circ t_{c_2}^{-p} \circ t_d^{-q} \\
&= t_d^q \circ t_{c_2}^p \circ (\Phi_{K_{0,0}} \circ t_{B_3} \circ \Phi_{K_{0,0}}^{-1}) \circ t_{c_2} \circ (\Phi_{K_{0,0}} \circ t_{B_3} \circ \Phi_{K_{0,0}}^{-1}) \\
&\quad \circ t_{c_2}^{-1} \circ (\Phi_{K_{0,0}} \circ t_{B_3}^{-1} \circ \Phi_{K_{0,0}}^{-1}) \circ t_{c_2}^{-p} \circ t_d^{-q} \\
&= t_d^q \circ t_{c_2}^p \circ (\Phi_{K_{0,0}} \circ t_{B_3} \circ \Phi_{K_{0,0}}^{-1}) \circ (t_{c_2}^{-p} \circ t_d^{-q} \circ t_{c_2}^{p+1} \circ t_d^q) \circ (\Phi_{K_{0,0}} \circ t_{B_3} \circ \Phi_{K_{0,0}}^{-1}) \\
&\quad \circ (t_{c_2}^{-p-1} \circ t_d^{-q} \circ t_{c_2}^p \circ t_d^q) \circ (\Phi_{K_{0,0}} \circ t_{B_3}^{-1} \circ \Phi_{K_{0,0}}^{-1}) \circ t_{c_2}^{-p} \circ t_d^{-q} \\
&= (t_d^q \circ t_{c_2}^p \circ \Phi_{K_{0,0}}) \circ t_{B_3} \circ (\Phi_{K_{0,0}}^{-1} \circ t_{c_2}^{-p} \circ t_d^{-q}) \circ (t_{c_2}^{p+1} \circ t_d^q \circ \Phi_{K_{0,0}}) \circ t_{B_3} \\
&\quad \circ (\Phi_{K_{0,0}}^{-1} \circ t_{c_2}^{-p-1} \circ t_d^{-q}) \circ (t_{c_2}^p \circ t_d^q \circ \Phi_{K_{0,0}}) \circ t_{B_3}^{-1} \circ (\Phi_{K_{0,0}}^{-1} \circ t_{c_2}^{-p} \circ t_d^{-q}) \\
&= t_{\Phi_{K_{p,q}}(B_3)} \circ t_{\Phi_{K_{p+1,q}}(B_3)} \circ t_{\Phi_{K_{p,q}}(B_3)}^{-1}.
\end{aligned}$$

By the same method we also get

$$t_d = t_{\Phi_{K_{p,q}}(B_4)} \circ t_{\Phi_{K_{p,q+1}}(B_4)} \circ t^{-1}_{\Phi_{K_{p,q}}(B_4)}.$$

Since

$$\Phi_{K_{p,q}}(t_{B_3}), \Phi_{K_{p+1,q}}(t_{B_3}) \in G_F(\Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}))$$

and

$$\Phi_{K_{p,q}}(t_{B_4}), \Phi_{K_{p,q+1}}(t_{B_4}) \in G_F(\Phi_{K_{p,q+1}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2})),$$

we obtain the conclusion

$$t_{c_2} \in G_F(\Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2})), \qquad t_d \in G_F(\Phi_{K_{p,q+1}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2})). \quad \square$$

LEMMA 3.3 [23]. *Let $W_i = w_{i,n_i} \cdots w_{i,2} \cdot w_{i,1}$ be a sequence of right-handed Dehn twists along a simple closed curves on Σ_g such that $\lambda_{W_i} := w_{i,n_i} \circ \cdots \circ w_{i,1} =$ id in $\mathcal{M}_F$ for $i = 1, 2$. Then*

$$W_1 \cdot W_2 \sim W_2 \cdot W_1.$$

Suppose $f \in G(W_2)$; then

$$f(W_1) \cdot W_2 \sim W_1 \cdot W_2.$$

THEOREM 3.4. *For each pair $p, q \in \mathbb{Z}$, we get diffeomorphisms*

$$Y(2;\, K_{p,q}, K_{p+1,q}) \approx Y(2;\, K_{p+1,q}, K_{p+2,q})$$

and

$$Y(2;\, K_{p,q}, K_{p,q+1}) \approx Y(2;\, K_{p,q+1}, K_{p,q+2}).$$

Proof. $Y(2;\, K_{p,q}, K_{p+1,q})$ has a monodromy factorization of the form

$$\Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2},$$

where $\Phi_{K_{p,q}} = t^q_d \circ t^p_{c_2} \circ t_{a_2} \circ t^{-1}_{b_2} \circ t^{-1}_{a_1} \circ t_{b_1}$.

By Lemma 3.2, we have

$$t_{c_2} \in G_F(\Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2})),$$

$$t_{c_2} \in G_F(t_{c_2}(\eta^2_{1,2}) \cdot \eta^2_{1,2}).$$

Therefore,

$$\Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \tag{3.1}$$

$$\sim \eta^2_{1,2} \cdot \Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \tag{3.2}$$

$$\sim t_{c_2}(\eta^2_{1,2}) \cdot \Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \tag{3.3}$$

$$\sim \Phi_{K_{p+1,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot t_{c_2}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \tag{3.4}$$

$$\sim \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p-1,q}}(\eta^2_{1,2}) \cdot t_{c_2}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \tag{3.5}$$

$$\sim t_{c_2}(\eta^2_{1,2}) \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p-1,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \tag{3.6}$$

$$\sim \eta^2_{1,2} \cdot \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \Phi_{K_{p-1,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \tag{3.7}$$

$$\sim \Phi_{K_{p,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2} \cdot \Phi_{K_{p-1,q}}(\eta^2_{1,2}) \cdot \eta^2_{1,2}. \tag{3.8}$$

In particular:

- since $\lambda_{\eta_{1,2}^2} = \mathrm{id}$, we get (3.1) to (3.2), (3.3) to (3.4), (3.5) to (3.6), and (3.7) to (3.8);
- Lemma 3.2 and Lemma 3.3 together imply (3.2) to (3.3), (3.4) to (3.5), and (3.6) to (3.7).

This implies that, for each fixed q, $Y(2; K_{p,q}, K_{p+1,q})$ and $Y(2; K_{p-1,q}, K_{p,q})$ have isomorphic Lefschetz fibration structure; hence they are diffeomorphic.

Similarly, by using

$$t_d \in G_F(\Phi_{K_{p,q+1}}(\eta_{1,2}^2) \cdot \Phi_{K_{p,q}}(\eta_{1,2}^2)),$$
$$t_d \in G_F(t_d(\eta_{1,2}^2) \cdot \eta_{1,2}^2)$$

in Lemma 3.2 we obtain

$$Y(2; K_{p,q}, K_{p,q+1}) \approx Y(2; K_{p,q+1}, K_{p,q+2}). \qquad \square$$

4. Nonisomorphic Lefschetz Fibrations

In this section we investigate some algebraic and graph-theoretic properties of $\xi_{p,q} = \Phi_{K_{p,q}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2$ and its monodromy group $G_{\Sigma_5}(\xi_{p,q})$ corresponding to the fixed generic fiber Σ_5. In [11], Humphries showed that the minimal number of Dehn twist generators of the mapping class group $\mathcal{M}_g$ or $\mathcal{M}_g^1$ is $2g+1$; he did this by using symplectic transvection and the Euler number (mod 2) of a graph.

DEFINITION 4.1 [11]. Let $\{\gamma_1, \gamma_2, \dots, \gamma_{2g}\}$ be a set of simple closed curves on Σ_g that generate $H_1(\Sigma_g; \mathbb{Z}_2)$. Let $\Gamma(\gamma_1, \gamma_2, \dots, \gamma_{2g})$ be a graph defined by:

- a vertex for each homology class $[\gamma_i]$ of simple closed curves $\gamma_i, i = 1, 2, \dots, 2g$;
- an edge between γ_i and γ_j if $i_2(\gamma_i, \gamma_j) = 1$, where $i_2(\gamma_i, \gamma_j)$ is the modulo 2 algebraic intersection between $[\gamma_i]$ and $[\gamma_j]$; and
- no intersections between any two edges.

Let γ be a simple closed curve on Σ_g such that $[\gamma] = \sum_{i=1}^{2g} \varepsilon_i [\gamma_i]$ ($\varepsilon_i = 0$ or 1) as an element of $H_1(\Sigma_g; \mathbb{Z}_2)$. We define $\bar{\gamma} := \bigcup_{\varepsilon_i = 1} \overline{\gamma_i}$, where $\overline{\gamma_i}$ is the union of all closures of half-edges with one end vertex γ_i. We define $\chi_\Gamma(\gamma) := \chi(\bar{\gamma}) \pmod 2$, where $\chi(\bar{\gamma})$ is the Euler number of the graph $\bar{\gamma}$.

LEMMA 4.2 [11]. *Let $\Gamma(\gamma_1, \dots, \gamma_{2g})$ be the graph of simple closed curves $\{\gamma_1, \dots, \gamma_{2g}\}$ that generate the $\mathbb{Z}_2$ vector space $H_1(\Sigma_g; \mathbb{Z}_2)$. Let $G_{\Gamma,g}$ be the subgroup of $\mathcal{M}_g$ that is generated by*

$$\{t_\alpha \mid \alpha \text{ is a nonseparating simple closed curve on } \Sigma_g \text{ such that } \chi_\Gamma(\alpha) = 1\}.$$

Then $G_{\Gamma,g}$ is a nontrivial proper subgroup of $\mathcal{M}_g$. Moreover, if β is a nonseparating simple closed curve on Σ_g with $\chi_\Gamma(\beta) = 0$, then $t_\beta \notin G_{\Gamma,g}$.

Proof. Let us prove that $G_{\Gamma,g}$ is a nontrivial proper subgroup of $\mathcal{M}_g$.

The mapping class group $\mathcal{M}_g$ acts transitively on $H_1(\Sigma_g; \mathbb{Z}_2) \setminus \{0\}$. The action is defined by

$$t_c\colon H_1(\Sigma_g;\mathbb{Z}_2)\to H_1(\Sigma_g;\mathbb{Z}_2),\quad t_c(x)=i_2(c,x)[c]+x,$$

where c is a simple closed curve on Σ_g, $x\in H_1(\Sigma_g;\mathbb{Z}_2)$, and $i_2(c,x)$ is the modulo 2 algebraic intersection number between $[c]$ and x.

If c is a nonseparating simple closed curve on Σ_g such that $\chi_\Gamma(c)=1$, then in $H_1(\Sigma_g;\mathbb{Z}_2)$ we have

$$t_c([\gamma])=\begin{cases}[\gamma] & \text{if } i_2(c,\gamma)=0,\\ [c]+[\gamma] & \text{if } i_2(c,\gamma)=1.\end{cases}$$

For the $i_2(c,\gamma)=0$ case, it is clear that $\chi_\Gamma(t_c(\gamma))=\chi_\Gamma(\gamma)$. For the $i_2(c,\gamma)=1$ case, if $[c]=\sum_{i=1}^{2g}\varepsilon_{c,i}[\gamma_i]$ and $[\gamma]=\sum_{i=1}^{2g}\varepsilon_{\gamma,i}[\gamma_i]$ in $H_1(\Sigma_g;\mathbb{Z}_2)$, then

$$\overline{t_c(\gamma)}=\bigcup_{\varepsilon_{c,i}+\varepsilon_{\gamma,i}=1}\overline{\gamma}_i.$$

Let

$$A=\sum_{\varepsilon_{c,i}=1,\,\varepsilon_{\gamma,i}=1}[\gamma_i],$$

$$B=\sum_{\varepsilon_{c,i}=1,\,\varepsilon_{\gamma,i}=0}[\gamma_i],$$

$$C=\sum_{\varepsilon_{c,i}=0,\,\varepsilon_{\gamma,i}=1}[\gamma_i].$$

Then

$$\chi(\bar{c})=\chi(\bar{A}\cup\bar{B})=\chi(\bar{A})+\chi(\bar{B})+i_2(A,B)\ (\mathrm{mod}\,2),$$

$$\chi(\bar{\gamma})=\chi(\bar{A}\cup\bar{C})=\chi(\bar{A})+\chi(\bar{C})+i_2(A,C)\ (\mathrm{mod}\,2),$$

$$\chi(\overline{t_c(\gamma)})=\chi(\bar{B}\cup\bar{C})=\chi(\bar{B})+\chi(\bar{C})+i_2(B,C)\ (\mathrm{mod}\,2),$$

and $i_2(c,\gamma)=i_2(A+B,A+C)=i_2(A,A)+i_2(A,B)+i_2(A,C)+i_2(B,C)=i_2(A,B)+i_2(A,C)+i_2(B,C)$ (mod 2) because $i_2(A,A)=0$. Therefore,

$$\chi_\Gamma(t_c(\gamma))=\chi(\overline{t_c(\gamma)})=\chi(\bar{c})+\chi(\bar{\gamma})+i_2(c,\gamma)=\chi(\bar{\gamma})=\chi_\Gamma(\gamma)\ (\mathrm{mod}\,2).$$

For any $f\in G_{\Gamma,g}$, f is of the form $t_{c_k}^{\varepsilon_k}\circ t_{c_{k-1}}^{\varepsilon_{k-1}}\circ\cdots\circ t_{c_2}^{\varepsilon_2}\circ t_{c_1}^{\varepsilon_1}$, where each c_i is a nonseparating simple closed curve with $\chi_\Gamma(c_i)=1$ and $\varepsilon_i\in\{\pm1\}$. This implies that $\chi_\Gamma(f(\gamma))\equiv\chi_\Gamma(\gamma)$ (mod 2). Therefore, if $G_{\Gamma,g}=\mathcal{M}_g$ then, for any nonseparating simple closed curves γ on Σ_g, we must have $\chi_\Gamma(\gamma)=1$—which is clearly impossible. Hence $G_{\Gamma,g}$ is a nontrivial proper subgroup of $\mathcal{M}_g$.

Let β be a nonseparating simple closed curve with $\chi_\Gamma(\beta)=0$. Then, for a simple closed curve γ on Σ_g with $i_2(\beta,\gamma)=1$, we have $\chi_\Gamma(t_\beta(\gamma))\not\equiv\chi_\Gamma(\gamma)$ (mod 2). Therefore, $t_\beta\notin G_{\Gamma,g}$. □

REMARK 4.3. By Lemma 4.2, we know that:

- if $\chi_\Gamma(c)=1$ then, for any γ,

$$\chi_\Gamma(t_c(\gamma))=\chi_\Gamma(\gamma);$$

- if $\chi_\Gamma(c)=0$ then, for any γ,

$$\chi_\Gamma(t_c(\gamma))=\chi_\Gamma(\gamma)+i_2(c,\gamma).$$

LEMMA 4.4. *For each pair of integers* (p, q) *there is a basis* $\mathcal{B}_i$ *for* $H_1(\Sigma_5; \mathbb{Z}_2)$ ($depending only on$ (p, q) *modulo* 2) *with the property that*

$$G_F(\xi_{p,q}) \leq G_{\Gamma_i,5}$$

but with $\chi_{\Gamma_i}(c_2) = \chi_{\Gamma_i}(d) = 0$, *where* Γ_i *is the corresponding graph to a basis* $\mathcal{B}_i$.

Proof. We will prove this in four cases.

Case 1: p and q are even integers. Let us consider a basis

$$\mathcal{B}_1 = \{c_1, a_1, a_2, b_2, a_3, b_3, a_4, a_5, B_2, B_4\}$$

of $H_1(\Sigma_5; \mathbb{Z}_2)$, where $\{a_i, b_i, c_i, d_i, B_i\}$ are simple closed curves on Σ_5 as in Figure 1 and Figure 3. Then the graph of $\mathcal{B}_1$,

$$\Gamma_1 = \Gamma(\{c_1, a_1, a_2, b_2, a_3, b_3, a_4, a_5, B_2, B_4\}),$$

is as given in Figure 5.

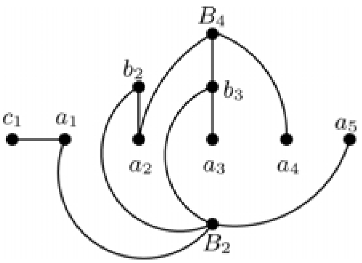

Figure 5 Graph Γ_1

We can easily obtain the following relations in $H_1(\Sigma_5; \mathbb{Z}_2)$:

$$\begin{aligned}
B_0 &= a_1 + a_2 + a_3 + a_4 + a_5,\\
B_1 &= B_2 + a_1 + a_5,\\
B_3 &= B_4 + a_2 + a_4,\\
B_5 &= a_3 = \Phi_{K_{0,0}}(B_5);\\
\Phi_{K_{0,0}}(B_4) &= B_4 + a_2,\\
\Phi_{K_{0,0}}(B_3) &= B_4 + a_2 + a_4 + b_2,\\
\Phi_{K_{0,0}}(B_2) &= B_2 + a_1 + b_2 + a_2,\\
\Phi_{K_{0,0}}(B_1) &= B_2 + a_1 + a_2 + a_5 + c_1 + b_2,\\
\Phi_{K_{0,0}}(B_0) &= a_3 + a_4 + a_5 + c_1 + b_2.
\end{aligned}$$

Hence the graph yields

$$\chi_{\Gamma_1}(a_i) = \chi_{\Gamma_1}(B_i) = \chi_{\Gamma_1}(\Phi_{K_{0,0}}(B_i)) = 1 \quad \text{for } i = 0, 1, 2, 3, 4, 5$$

and $\chi_{\Gamma_1}(c_1) = \chi_{\Gamma_1}(c_6) = 1$. So we have

$$\{t_{B_i}, \Phi_{K_{0,0}}(t_{B_i}), t_{a_j}, t_{b_3}, t_{b_3'}, t_{c_1}, t_{c_6} \mid i = 0, 1, 2, 3, 4, 5,\ j = 1, 2, 3, 4, 5\} \subset G_{\Gamma_1,5},$$

and each generator of the group $G_F(\Phi_{K_{0,0}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)$ is an element of $G_{\Gamma_1,5}$. This implies that $G_F(\Phi_{K_{0,0}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)\le G_{\Gamma_1,5}$.

But we have

$$\chi_{\Gamma_1}(c_j)=\chi_{\Gamma_1}(d)=0$$

for $j=2,3,4,5$ and therefore

$$t_{c_2},t_{c_3},t_{c_4},t_{c_5},t_d\notin G_{\Gamma_1,5}.$$

This implies that $t_{c_2},t_d\notin G_F(\Phi_{K_{0,0}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)$.

Since the $\mathbb{Z}_2$-homology class of $\Phi_{K_{2p,2q}}(B_i)$ and $\Phi_{K_{0,0}}(B_i)$ are the same for any $p,q\in\mathbb{Z}$, we get

$$\chi_{\Gamma_1}(\Phi_{K_{2p,2q}}(B_i))=\chi_{\Gamma_1}(\Phi_{K_{0,0}}(B_i))$$

for $i=0,1,2,3,4,5$. This implies that $G_F(\Phi_{K_{2p,2q}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)\le G_{\Gamma_1,5}$, so we have $t_{c_2},t_d\notin G_F(\Phi_{K_{2p,2q}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)$.

Case 2: p is an odd integer and q is an even integer. Let us consider a basis $\mathcal{B}_2=\{a_3,b_3,B_1,B_2,B_3,B_4,d_1,d_2,d_3,d_4\}$ of $\mathbb{Z}_2$-vector space $H_1(\Sigma_5;\mathbb{Z}_2)$ and its graph

$$\Gamma_2=\Gamma(\{a_3,b_3,B_1,B_2,B_3,B_4,d_1,d_2,d_3,d_4\});$$

here $\{a_i,b_i,c_i,d_i,B_i\}$ are simple closed curves on Σ_5 as in Figure 1, Figure 3, and Figure 6. Then the graph Γ_2 is as in Figure 7.

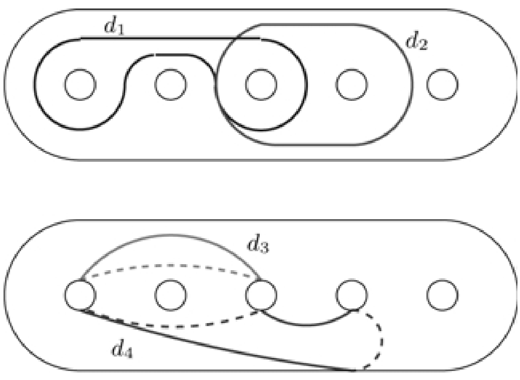

Figure 6 Simple closed curves d_i

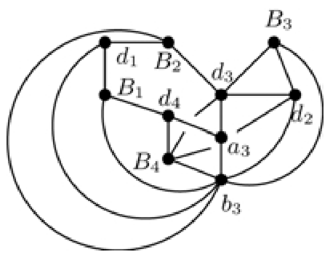

Figure 7 Graph Γ_2

Since $\Phi_{K_{1,0}} = t_{c_2}\circ t_{a_2}\circ t_{b_2}^{-1}\circ t_{a_1}^{-1}\circ t_{b_1}$, we get the following relations in $H_1(\Sigma_5;\mathbb{Z}_2)$:

$$\begin{aligned}
B_0 &= B_1 + B_2 + B_3 + B_4 + a_3;\\
\Phi_{K_{1,0}}(B_0) &= B_1 + B_2 + B_4 + b_3 + d_1 + d_2 + d_4,\\
\Phi_{K_{1,0}}(B_1) &= B_1 + B_3 + B_4 + a_3 + d_2,\\
\Phi_{K_{1,0}}(B_2) &= B_2 + B_3 + B_4 + b_3 + d_1 + d_2 + d_3,\\
\Phi_{K_{1,0}}(B_3) &= B_3 + b_3 + d_3,\\
\Phi_{K_{1,0}}(B_4) &= B_3 + B_4 + b_3 + d_2 + d_4,\\
\Phi_{K_{1,0}}(B_5) &= B_5 = a_3;\\
c_2 &= a_3 + b_3 + d_4 + B_4,\\
d &= B_3 + B_4 + d_1 + d_2.
\end{aligned}$$

A computation of χ_{Γ_2} shows that

$$\chi_{\Gamma_2}(B_i) = \chi_{\Gamma_2}(\Phi_{K_{1,0}}(B_i)) = \chi_{\Gamma_2}(b_3) = \chi_{\Gamma_2}(b_3') = \chi_{\Gamma_2}(a_3) = 1 \tag{4.1}$$

for each $i = 0, 1, 2, 3, 4, 5$ and that

$$\chi_{\Gamma_2}(c_1) = \chi_{\Gamma_2}(c_2) = \chi_{\Gamma_2}(a_1) = \chi_{\Gamma_2}(a_2) = \chi_{\Gamma_2}(b_2) = \chi_{\Gamma_2}(d) = 0. \tag{4.2}$$

Hence $G_F(\Phi_{K_{1,0}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2) \le G_{\Gamma_2,5}$ and, since $t_{c_2}, t_d \notin G_{\Gamma_2,5}$, we get

$$t_{c_2}, t_d \notin G_F(\Phi_{K_{1,0}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2).$$

Furthermore, since $\Phi_{K_{2p+1,2q}}(B_i)$ and $\Phi_{K_{1,0}}(B_i)$ represent the same element in $H_1(\Sigma_2;\mathbb{Z}_2)$, we get $\chi_{\Gamma_2}(\Phi_{K_{2p+1,2q}}(B_i)) = \chi_{\Gamma_2}(\Phi_{K_{1,0}}(B_i)) = 1$; this implies that

$$t_{c_2}, t_d \notin G_F(\Phi_{K_{2p+1,2q}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)$$

for any $p, q \in \mathbb{Z}$ because $G_F(\Phi_{K_{2p+1,2q}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2) \le G_{\Gamma_2,5}$.

Case 3: p is an even integer and q is an odd integer. We want to find a graph

$$\Gamma_3 = \Gamma(\{\gamma_1, \gamma_2, \dots, \gamma_{10}\})$$

satisfying

$$\chi_{\Gamma_3}(B_i) = \chi_{\Gamma_3}(\Phi_{K_{0,1}}(B_i)) = \chi_{\Gamma_3}(b_3) = \chi_{\Gamma_3}(b_3') = \chi_{\Gamma_3}(a_3) = 1 \tag{4.3}$$

for $i = 0, 1, 2, 3, 4, 5$ and

$$\chi_{\Gamma_3}(c_2) = \chi_{\Gamma_3}(d) = 0. \tag{4.4}$$

Note that we observe the following relations in $H_1(\Sigma_5;\mathbb{Z}_2)$.

	$\Phi_{K_{0,0}}(B_i)$	$\Phi_{K_{0,1}}(B_i)$
B_0	$B_0 + a_1 + b_1 + a_2 + b_2$	$B_0 + a_1 + b_1 + a_2 + b_2$
B_1	$B_1 + b_1 + b_2 + a_2$	$B_1 + b_1 + a_2 + b_2 + d$
B_2	$B_2 + a_1 + b_2 + a_2$	$B_2 + a_1 + b_2 + a_2$
B_3	$B_3 + b_2$	$B_3 + b_2$
B_4	$B_4 + a_2$	$B_4 + a_2 + d$
B_5	B_5	B_5

(4.5)

From equation (4.3), we may assume that B_i $(i = 1, 2, 3, 4)$, b_3, and a_3 are in the generating set, which we will extend to a basis of $H_1(\Sigma_5; \mathbb{Z}_2)$. For each $i = 0, 1, 2, 3, 4, 5$, B_i and $\Phi_{K_{0,1}}(B_i)$ are elements of $G_{\Gamma_{3,5}}$ at the same time. Since $i_2(\Phi_{K_{0,0}}(B_0), d) = 0$, we get $\chi_{\Gamma_3}(\Phi_{K_{0,1}}(B_0)) = \chi_{\Gamma_3}(\Phi_{K_{0,0}}(B_0))$. We also know that

$$\begin{aligned} i_2(B_0, b_1) &= i_2(t_{b_1}(B_0), a_1) = i_2(t_{a_1}^{-1}(t_{b_1}(B_0)), b_2) \\ &= i_2(t_{b_2}^{-1}(t_{a_1}^{-1}(t_{b_1}(B_0))), a_2) = 1. \end{aligned}$$

So by Lemma 4.2 and Remark 4.3 it follows that

$$\chi_{\Gamma_3}(\Phi_{K_{0,0}}(B_0)) = \chi_{\Gamma_3}(B_0) + |\{a_1, b_1, a_2, b_2\} - G_{\Gamma_{3,5}}| = \chi_{\Gamma_3}(B_0).$$

Therefore, if B_0 and $\Phi_{K_{0,1}}(B_0)$ are elements of $G_{\Gamma_{3,5}}$ at the same time, then an even number of elements in $\{a_1, b_1, a_2, b_2\}$ must have $\chi_{\Gamma_3} = 0$. By the same method, we derive the following statements:

- an even number of elements in $\{b_1, b_2, a_2, d\}$ must have $\chi_{\Gamma_3} = 0$ because $\chi_{\Gamma_3}(\Phi_{K_{0,1}}(B_1)) = \chi_{\Gamma_3}(B_1)$;
- an even number of elements in $\{a_1, b_2, a_2\}$ must have $\chi_{\Gamma_3} = 0$ because $\chi_{\Gamma_3}(\Phi_{K_{0,1}}(B_2)) = \chi_{\Gamma_3}(B_2)$;
- an even number of elements in $\{b_2\}$ must have $\chi_{\Gamma_3} = 0$ because $\chi_{\Gamma_3}(\Phi_{K_{0,1}}(B_3)) = \chi_{\Gamma_3}(B_3)$;
- an even number of elements in $\{a_2, d\}$ must have $\chi_{\Gamma_3} = 0$ because $\chi_{\Gamma_3}(\Phi_{K_{0,1}}(B_4)) = \chi_{\Gamma_3}(B_4)$.

When combined with these constraints, equation (4.4) yields

$$\chi_{\Gamma_3}(a_1) = \chi_{\Gamma_3}(a_2) = 0,$$
$$\chi_{\Gamma_3}(b_1) = \chi_{\Gamma_3}(b_2) = 1.$$

Hence $\{B_1, B_2, B_3, B_4, b_1, b_2, b_3, a_3\}$ might be a subset of $G_{\Gamma_{3,5}}$, and we will extend it to a basis of $H_1(\Sigma_5; \mathbb{Z}_2)$ by adding two simple closed curves d_1, d_2 as in Figure 6. Let

$$\Gamma_3 = \Gamma(\{B_1, B_2, B_3, B_4, b_1, b_2, b_3, a_3, d_1, d_2\});$$

then Γ_3 is the graph in Figure 8 and satisfies equations (4.3) and (4.4).

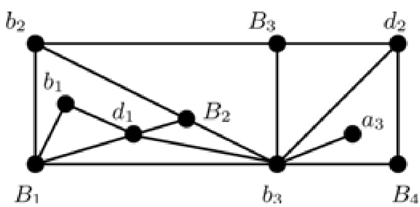

Figure 8 Graph Γ_3

Therefore, $G_F(\Phi_{K_{0,1}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)\le G_{\Gamma_{3},5}$ and, since $t_{c_2}, t_d\notin G_{\Gamma_3,5}$, we get

$$t_{c_2}, t_d \notin G_F(\Phi_{K_{0,1}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)$$

and

$$t_{c_2}, t_d \notin G_F(\Phi_{K_{2p,2q+1}}(\eta_{1,2}^2)\cdot\eta_{1,2}^2)$$

for any $p, q \in \mathbb{Z}$.

Case 4: p and q are odd integers. We want to find a graph

$$\Gamma_4 = \Gamma(\{\gamma_1,\gamma_2,\dots,\gamma_{10}\})$$

satisfying

$$\chi_{\Gamma_4}(B_i) = \chi_{\Gamma_4}(\Phi_{K_{1,1}}(B_i)) = \chi_{\Gamma_4}(b_3) = \chi_{\Gamma_4}(b_3') = \chi_{\Gamma_4}(a_3) = 1 \tag{4.6}$$

for $i = 0,1,2,3,4,5$ and

$$\chi_{\Gamma_4}(c_2) = \chi_{\Gamma_4}(d) = 0. \tag{4.7}$$

We may assume that each element of $\{B_1, B_2, B_3, B_4, a_3, b_3\}$ is in the generating set, and we will extend it to a basis of $H_1(\Sigma_5;\mathbb{Z}_2)$.

Note that we observe the following relations in $H_1(\Sigma_5;\mathbb{Z}_2)$.

	$\Phi_{K_{0,0}}(B_i)$	$\Phi_{K_{1,1}}(B_i)$
B_0	$B_0+a_1+b_1+a_2+b_2$	$B_0+a_1+b_1+a_2+b_2$
B_1	$B_1+b_1+b_2+a_2$	$B_1+b_1+a_2+b_2+c_2+d$
B_2	$B_2+a_1+b_2+a_2$	$B_2+a_1+b_2+a_2+c_2$
B_3	B_3+b_2	$B_3+b_2+c_2$
B_4	B_4+a_2	$B_4+a_2+c_2+d$
B_5	B_5	B_5

(4.8)

Hence, by Lemma 4.2 and (4.6)–(4.8), we have the following statements:

- an even number of elements in $\{a_1, b_1, a_2, b_2\}$ must have $\chi_{\Gamma_4}=0$ because $\chi_{\Gamma_3}(\Phi_{K_{1,1}}(B_0)) = \chi_{\Gamma_3}(B_0)$;
- an even number of elements in $\{a_2, b_1, b_2, c_2, d\}$ must have $\chi_{\Gamma_4}=0$ because $\chi_{\Gamma_3}(\Phi_{K_{1,1}}(B_1)) = \chi_{\Gamma_3}(B_1)$;
- an even number of elements in $\{a_1, a_2, b_2, c_2\}$ must have $\chi_{\Gamma_4}=0$ because $\chi_{\Gamma_3}(\Phi_{K_{1,1}}(B_2)) = \chi_{\Gamma_3}(B_2)$;
- an even number of elements in $\{b_2, c_2\}$ must have $\chi_{\Gamma_4}=0$ because $\chi_{\Gamma_3}(\Phi_{K_{1,1}}(B_3)) = \chi_{\Gamma_3}(B_3)$;
- an even number of elements in $\{a_2, c_2, d\}$ must have $\chi_{\Gamma_4}=0$ because $\chi_{\Gamma_3}(\Phi_{K_{1,1}}(B_4)) = \chi_{\Gamma_3}(B_4)$.

This implies that

$$\chi_{\Gamma_3}(a_1) = \chi_{\Gamma_3}(a_2) = 1,$$
$$\chi_{\Gamma_3}(b_1) = \chi_{\Gamma_3}(b_2) = 0,$$

so $\{B_1, B_2, B_3, B_4, a_1, a_2, b_3, a_3\}$ might be a subset of $G_{\Gamma_4,5}$. We will extend this subset to a basis of $H_1(\Sigma_5; \mathbb{Z}_2)$ by adding two simple closed curves d_3, d_4 as in Figure 6. Let

$$\Gamma_4 = \Gamma(\{B_1, B_2, B_3, B_4, a_1, a_2, a_3, b_3, d_3, d_4\});$$

then Γ_4 is graphed as in Figure 9 and satisfies equations (4.6) and (4.7).

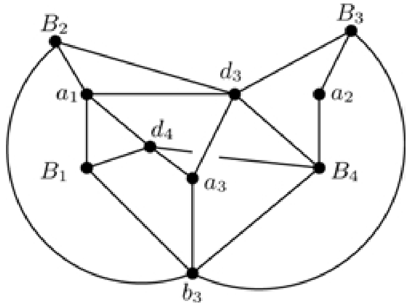

Figure 9 Graph Γ_4

Therefore, $G_F(\Phi_{K_{1,1}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2) \leq G_{\Gamma_4,5}$ and, since $t_{c_2}, t_d \notin G_{\Gamma_4,5}$, we get

$$t_{c_2}, t_d \notin G_F(\Phi_{K_{1,1}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2)$$

and

$$t_{c_2}, t_d \notin G_F(\Phi_{K_{2p+1,2q+1}}(\eta_{1,2}^2) \cdot \eta_{1,2}^2)$$

for any $p, q \in \mathbb{Z}$. □

Remark 4.5. We can double-check the preceding statements by using the representation of a mapping class group in a symplectic group (this approach was suggested by S. Humphries [12]). There is a natural map

$$\psi_n \colon \mathcal{M}_5 \xrightarrow{\psi} \mathrm{Sp}(10, \mathbb{Z}) \xrightarrow{q_n} \mathrm{Sp}(10, \mathbb{Z}/n\mathbb{Z})$$

where, for each $t_\gamma \in \mathcal{M}_5$,

$$\psi(t_\gamma) \colon H_1(\Sigma_5, \mathbb{Z}) \to H_1(\Sigma_5, \mathbb{Z})$$

is an integral matrix representation of the mapping class group action on the integral first homology group. We then reduce the coefficient of the symplectic group to $\mathbb{Z}/n\mathbb{Z}$ by taking a quotient map q_n. It is easy to check that

$$\psi_2(t_{c_2}^2) = \mathrm{Id}_{10\times 10} = \psi_2(t_d^2),$$

which implies that

$$\psi_2(G_F(\xi_{p,q})) = \psi_2(G_F(\xi_{r,s})) \quad \text{if } (p,q) \equiv (r,s) \pmod 2.$$

An explicit group order computation (using a computer algebra system such as *GAP* [7] or *Sagemath* [20]) shows that

$$\mathrm{Order}(\psi_2(G_F(\xi_{p,q}))) = 50030759116800,$$
$$\mathrm{Order}(\langle\psi_2(G_F(\xi_{p,q}) \cup \{t_{c_2}\})\rangle) = 24815256521932800,$$
$$\mathrm{Order}(\langle\psi_2(G_F(\xi_{p,q}) \cup \{t_d\})\rangle) = 24815256521932800,$$
$$\mathrm{Order}(\psi_2(\mathcal{M}_5)) = 24815256521932800,$$

and this implies that

$$t_{c_2}, t_d \notin G_F(\xi_{p,q}) \quad \text{for any} \quad p, q \in \mathbb{Z}.$$

THEOREM 4.6. *$\xi_{p,q}$ is not marked equivalent to $\xi_{r,s}$ if $(p,q) \not\equiv (r,s) \pmod 2$.*

Proof. Let us consider the Γ_1 case, in which

$$i_2(\Phi_{K_{0,0}}(B_i), c_2) = \begin{cases} 1, & i = 1,2,3,4, \\ 0, & i = 0; \end{cases} \qquad i_2(\Phi_{K_{0,0}}(B_i), d) = \begin{cases} 1, & i = 1,4, \\ 0, & i = 0,2,3. \end{cases}$$

Then, by Lemma 4.2, it follows that $\chi_{\Gamma_1}(\Phi_{K_{1,0}}(B_i)) = 0$ for $i = 1,2,3,4$, $\chi_{\Gamma_1}(\Phi_{K_{0,1}}(B_i)) = 0$ for $i = 1,4$, and $\chi_{\Gamma_1}(\Phi_{K_{1,1}}(B_i)) = 0$ for $i = 2,3$; this gives the result for Γ_1. Other rows are obtained by the same method.

	$G_{\Gamma_i,5}$ does not contain
Γ_1	$t_{\Phi_{K_{1,0}}(B_j)}$ $(j = 1,2,3,4)$, $t_{\Phi_{K_{0,1}}(B_1)}$, $t_{\Phi_{K_{0,1}}(B_4)}$, $t_{\Phi_{K_{1,1}}(B_2)}$, $t_{\Phi_{K_{1,1}}(B_3)}$
Γ_2	$t_{\Phi_{K_{0,0}}(B_j)}$ $(j = 1,2,3,4)$, $t_{\Phi_{K_{0,1}}(B_2)}$, $t_{\Phi_{K_{0,1}}(B_3)}$, $t_{\Phi_{K_{1,1}}(B_1)}$, $t_{\Phi_{K_{1,1}}(B_4)}$
Γ_3	$t_{\Phi_{K_{0,0}}(B_1)}$, $t_{\Phi_{K_{0,0}}(B_4)}$, $t_{\Phi_{K_{1,0}}(B_2)}$, $t_{\Phi_{K_{1,0}}(B_3)}$, $t_{\Phi_{K_{1,1}}(B_j)}$ $(j = 1,2,3,4)$
Γ_4	$t_{\Phi_{K_{0,0}}(B_2)}$, $t_{\Phi_{K_{0,0}}(B_3)}$, $t_{\Phi_{K_{1,0}}(B_1)}$, $t_{\Phi_{K_{1,0}}(B_4)}$, $t_{\Phi_{K_{0,1}}(B_j)}$ $(j = 1,2,3,4)$

It is clear that $t_{\Phi_{K_{p,q}}(B_j)}$ is contained in $G_{\Gamma_i,5}$ if and only if $t_{\Phi_{K_{\varepsilon_p,\varepsilon_q}}(B_j)}$ is contained in $G_{\Gamma_i,5}$, where $\varepsilon_p, \varepsilon_q \in \{0,1\}$ such that $p \equiv \varepsilon_p$ and $q \equiv \varepsilon_q$ modulo 2. The reason is that $\chi_{\Gamma_i}(\Phi_{K_{p,q}}(B_j)) = \chi_{\Gamma_i}(\Phi_{K_{\varepsilon_p,\varepsilon_q}}(B_j))$, which implies that

$$\xi_{p,q} \not\cong \xi_{r,s} \quad \text{if} \quad (p,q) \not\equiv (r,s) \pmod 2.$$

For example, if $(p,q) \equiv (0,0)$ and $(r,s) \equiv (1,0)$ modulo 2, then

$$t_{\Phi_{K_{p,q}}(B_j)} \notin G_{\Gamma_2,5} \quad (j = 1,2,3,4)$$

and $G_F(\xi_{r,s}) \leq G_{\Gamma_2,5}$. Hence $t_{\Phi_{K_{p,q}}(B_j)} \in G_F(\xi_{p,q})$, but $t_{\Phi_{K_{p,q}}(B_j)} \notin G_F(\xi_{r,s})$ for $j = 1,2,3,4$. This implies that $G_F(\xi_{p,q}) \neq G_F(\xi_{r,s})$ and $\xi_{p,q} \not\cong \xi_{r,s}$. □

COROLLARY 4.7. *If $p \not\equiv q$ modulo 2, then the knot surgery 4-manifold $E(2)_{K_{p,q}}$ has at least two nonisomorphic genus 5 Lefschetz fibration structures.*

Proof. This follows from Lemma 3.1. Since $K_{p,q}$ is equivalent to $K_{q,p}$, we get a diffeomorphism $E(n)_{K_{p,q}} \approx E(n)_{K_{q,p}}$. However, by Theorem 4.6 we know that $\xi_{p,q} \not\cong \xi_{q,p}$. □

REMARK 4.8. We are interested in the question of whether the knot surgery 4-manifold $E(2)_K$ admits infinitely many nonisomorphic Lefschetz fibrations over S^2 with the same generic fiber. In Theorem 3.4 we constructed a family of simply connected genus 5 Lefschetz fibrations over S^2, all of whose underlying spaces are diffeomorphic, from a pair of inequivalent prime fibered knots. We expect that these knots are strong candidates for admitting infinitely many nonisomorphic Lefschetz fibrations. We leave this problem for a future research project.

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