Absolute Chow–Künneth Decomposition for Rational Homogeneous Bundles and for Log Homogeneous Varieties

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1. Introduction

Suppose X is a nonsingular projective variety of dimension n defined over the complex numbers. Let $\mathrm{CH}^i(X) \otimes \mathbb{Q}$ be the Chow group of codimension-i algebraic cycles modulo rational equivalence, with rational coefficients. Murre [Mu2; Mu3] has made the following conjecture, which leads to a filtration on the rational Chow groups.

CONJECTURE. The motive $h(X) := (X, \Delta_X)$ of X has a Chow–Künneth decomposition:

$$\Delta_X = \sum_{i=0}^{2n} \pi_i \in \mathrm{CH}^n(X \times X) \otimes \mathbb{Q}$$

such that the π_i are orthogonal projectors (see Section 2.2).

In this paper, *absolute* Chow–Künneth decomposition (resp. projectors) is the same as Chow–Künneth decomposition (resp. projectors). We write "absolute" to emphasize the difference with the "relative" Chow–Künneth projectors that will appear in the paper.

Some examples where this conjecture is verified are: curves, surfaces, a product of a curve and surface [Mu1; Mu3], abelian varieties and abelian schemes [DenMu; Sh], uniruled threefolds [DM1], elliptic modular varieties [GHMu2; GMu], universal families over Picard modular surfaces [Mi+], and finite group quotients (which may be singular) of abelian varieties [AkJ], some varieties with nef tangent bundles [I], open moduli spaces of smooth curves [IM], and universal families over some Shimura surfaces [Mi].

In [I] we looked at varieties that have a nef tangent bundle. Given the structure theorems of Campana and Peternell [CP] and Demailly, Peternell, and Schneider [DePS], we know that such a variety X admits a finite étale surjective cover $X' \to X$ such that $X' \to A$ is a bundle of smooth Fano varieties over an abelian variety. Furthermore, any fibre that is a smooth Fano variety necessarily has a nef tangent bundle. It is an open question [CP, p. 170] whether such a Fano variety is

a rational homogeneous variety. They answered this question positively in dimension ≤ 3. We showed in [I] that if the étale cover is a relative cellular variety over A or if it admits a relative Chow–Künneth decomposition, then X' and X have a Chow–Künneth decomposition. In particular, this holds for varieties with a nef tangent bundle of dimension ≤ 3.

In this paper, we weaken the preceding hypothesis on the cover $X' \to A$ and obtain a Chow–Künneth decomposition whenever $X' \to A$ is a rational homogeneous bundle over an abelian variety A. This strengthens the results in [I] and, if the open question [C, p. 170] is answered positively in higher dimensions, then we obtain a Chow–Künneth decomposition for all varieties that have a nef tangent bundle. This question is answered positively in some higher-dimensional cases also; see [Hw, Sec. 4] and the references therein. Hence we also obtain a Chow–Künneth decomposition for new cases in higher dimensions.

We state the result and proofs in a more general situation.

THEOREM 1.1.   *Suppose S is a smooth projective variety over the complex numbers. Let G be a connected reductive algebraic group and let Z be a rational G homogeneous space over the variety S. Assume that S has a Chow–Künneth decomposition. Then the following statements hold*:

(a) *the motive of Z has an absolute Chow–Künneth decomposition*;
(b) *the motive of the bundle $Z \to S$ is expressed as a sum of tensor products of summands of the motive of S with the twisted Tate motive.*

A main observation in the proof is to note that a rational homogeneous bundle as just described is étale locally a relative cellular variety because the formal deformations of a rational homogeneous variety are trivial (see Lemma 3.2). Hence we can construct relative Chow–Künneth projectors (in the sense of [DenMu]) over étale morphisms of S. These projectors lie in the subspace generated by the relative algebraic cells. The corresponding relative cohomology classes patch up since they lie in the subspace generated by the relative analytic cells. Hence the relative orthogonal projectors can be patched up as algebraic cycles to obtain relative projectors in the rational Chow groups of the associated regular stack. In this case, we show that the relative Chow–Künneth projectors over the regular stack descend to relative Chow–Künneth projectors for $Z \to S$ (see Corollary 3.7). The criterion of Gordon, Hanamura, and Murre [GHMu2] for obtaining absolute Chow–Künneth projectors from relative Chow–Künneth projectors can be applied directly; see Proposition 3.8.

A similar proof also holds for a class of log homogeneous varieties studied by Brion [Br]. A log homogeneous variety consists of a pair (X, D), where X is a smooth projective variety and D is a normal crossing divisor on X, with the following property. The variety X is said to be *log homogeneous* with respect to D if the associated logarithmic tangent bundle $\mathcal{T}_X(-D)$ is generated by its global sections. It follows that X is almost homogeneous under the connected automorphism group $G := \mathrm{Aut}^0(X, D)$ with boundary D.

THEOREM 1.2.   *Suppose X is log homogeneous with respect to a normal crossing divisor D. Then X has a Chow–Künneth decomposition. Moreover, the motive of*

X is expressed as a sum of tensor products of the summands of the motive of its Albanese reduction with the twisted Tate motive.

See Theorem 4.4.

The proof uses Brion's classification of log homogeneous varieties [Br]. The fibres of the Albanese morphism are smooth spherical varieties. In this case we check that étale local triviality of the Albanese fibration holds. The proof of Theorem 1.2 relies on the algebraicity of the cohomology of the spherical varieties, much as in Theorem 1.1, and follows by applying the criterion of [GHMu2].

2. Preliminaries

We work over the field of complex numbers. We begin by recalling the standard constructions of the category of motives. Since this is well covered in the literature, we give a brief account and refer to [Mu2; Sc] for details.

2.1. Category of Motives

The category of nonsingular projective varieties over $\mathbb{C}$ will be denoted by $\mathcal{V}$. For an object X of $\mathcal{V}$, let $\mathrm{CH}^i(X)_{\mathbb{Q}} = \mathrm{CH}^i(X) \otimes \mathbb{Q}$ denote the rational Chow group of codimension i algebraic cycles modulo rational equivalence. Suppose $X, Y \in \mathrm{Ob}(\mathcal{V})$ and let $X = \bigcup X_i$ be a decomposition into connected components X_i and $d_i = \dim X_i$. Then $\mathrm{Corr}^r(X, Y) = \bigoplus_i \mathrm{CH}^{d_i+r}(X_i \times Y)_{\mathbb{Q}}$ is the group of correspondences of degree r from X to Y.

We will use the standard framework of the category of Chow motives $\mathcal{M}_{\mathrm{rat}}$ in this paper and refer to [Mu2] for details. We denote the category of motives $\mathcal{M}_{\sim}$, where $\sim$ is any equivalence (e.g., homological or numerical equivalence). When S is a smooth variety, we also consider the category of relative Chow motives $\mathrm{CH}\mathcal{M}(S)$ introduced in [DenMu] and [GHMu1]. If $S = \mathrm{Spec}\,\mathbb{C}$ then the category $\mathrm{CH}\mathcal{M}(S) = \mathcal{M}_{\mathrm{rat}}$.

2.2. Chow–Künneth Decomposition for a Variety

Suppose X is a nonsingular projective variety over $\mathbb{C}$ of dimension n. Let $\Delta_X \subset X \times X$ be the diagonal. Consider the Künneth decomposition of Δ in the Betti cohomology:

$$\Delta_X = \bigoplus_{i=0}^{2n} \pi_i^{\mathrm{hom}}$$

where $\pi_i^{\mathrm{hom}} \in H^{2n-i}(X) \otimes H^i(X)$.

DEFINITION 2.1. The motive of X is said to have *Künneth decomposition* if each of the classes π_i^{hom} are algebraic and are projectors—that is, if π_i^{hom} is the image

of an algebraic cycle π_i under the cycle class map from the rational Chow groups to the Betti cohomology and satisfies $\pi_i \circ \pi_i = \pi_i$ and $\Delta_X = \bigoplus_{i=0}^{2n} \pi_i$ in the rational Chow ring of $X \times X$. The algebraic projectors π_i are known as the *algebraic Künneth projectors.*

DEFINITION 2.2.   The motive of X is said to have a *Chow–Künneth decomposition* if the algebraic Künneth projectors are orthogonal projectors—in other words, if $\pi_i \circ \pi_j = \delta_{i,j} \pi_i$ and $\Delta_X = \bigoplus_{i=0}^{2n} \pi_i$ in the rational Chow ring of $X \times X$.

3. Rational Homogeneous Bundles over a Variety

In this section, we first recall the motive of a rational homogeneous variety and later construct relative Chow–Künneth projectors for a bundle of homogeneous varieties. The criterion of [GHMu2] can then be applied to obtain absolute Chow–Künneth projectors on the total space of the bundle. For this purpose, we need to show that the bundle is étale locally trivial and to check patching conditions over the étale coverings. We begin by recalling the motive of a rational homogeneous variety.

3.1. The Motive of a Rational Homogeneous Variety

Suppose F is a rational homogeneous variety. Then F is identified as a quotient G/P for some reductive linear algebraic group G and P a parabolic subgroup of G. Notice that F is a cellular variety; that is, it has a cellular decomposition

$$\emptyset = F_{-1} \subset F_0 \subset \cdots \subset F_n = F$$

such that each $F_i \subset F$ is a closed subvariety and $F_i - F_{i-1}$ is an affine space. Then we have the following lemma.

LEMMA 3.1 [K, Thm., p. 363].   *The Chow motive $h(F) = (F, \Delta_F)$ of F decomposes as a direct sum of twisted Tate motives*

$$h(F) = \bigoplus_\omega \mathbb{L}^{\otimes \dim \omega}.$$

Here ω runs over the set of cells of F.

In particular, this lemma says that the Chow–Künneth decomposition holds for F. Next we consider bundles of homogeneous spaces $Z \to S$ over a smooth variety S. We want to describe the Chow motive of Z in terms of the Chow motive of S up to some Tate twists. For this we need to show étale local triviality of $Z \to S$, which we discuss next.

3.2. The Étale Local Triviality of a Rational Homogeneous Bundle

Suppose that $Z \to S$ is a smooth projective morphism and that the base variety S is smooth and projective.

   By *étale local triviality* we mean that there exist étale morphisms $p_\alpha : U_\alpha \to S$ such that the pullback bundle

$$Z_{U_\alpha} := Z \times_S U_\alpha \to U_\alpha$$

is a Zariski trivial fibration and the images of p_α cover S; that is, $\bigcup_\alpha p_\alpha(U_\alpha) = S$. Here α runs over some indexing set I. Consider a rational homogeneous bundle $f : Z \to S$; that is, π is a smooth projective morphism and any fibre $\pi^{-1}y$ is a rational homogeneous variety G/P. Here G is a reductive linear algebraic group and $P \subset G$ is a parabolic subgroup. Assume that S is a smooth complex projective variety.

We note that an étale cover $\{U_\alpha\}$ as just described exists for a rational homogeneous bundle $Z \to S$.

LEMMA 3.2.   *There are étale open sets* $p_\alpha : U_\alpha \to S$ $\left(\text{satisfying } \bigcup_\alpha p_\alpha(U_\alpha) = S\right)$ *such that the pullback bundle* $Z_{U_\alpha} \to U_\alpha$ *is a Zariski trivial fibration.*

Proof. We need to note that the formal deformations of a rational homogeneous variety are trivial. This is just a consequence of Bott's well-known vanishing theorem: $H^1(G/P, T) = 0$. The assertion on étale local triviality follows from [Se, Prop. 2.6.10]. □

Our aim is to obtain relative Chow–Künneth projectors for the bundle Z/S. For this purpose, we first construct relative projectors over the étale coverings of $Z \to S$ and check the patching conditions. This requires that we use the language of stacks, which enables us to descend the projectors down to $Z \to S$. Hence in Section 3.3 we recall some facts on regular stacks and the relationship of the rational Chow groups and cohomology of stacks to that of its coarse moduli space. These facts will be essentially applied to the simplest situation—the rational homogeneous bundle $Z \to S$. Also, the patching will be used for the étale open sets of Z that are of the type $Z_{U_\alpha} := Z \times_S U_\alpha$ for étale morphisms $U_\alpha \to S$. In this context, it is possible to avoid stacks because the regular stack associated to the étale coverings is again Z. But we use the stacks essentially to say that the algebraic cells that live in the fibres of $Z \to S$ patch together over the étale coverings. This will be needed in the proof of Lemma 3.5.

We remark that more general patching statements might also hold for other varieties using stacks. However, we do not yet know of concrete examples for which this can be checked.

3.3. Chow Groups of an Étale Site

Mumford [Mum] and Gillet [Gi] have defined Chow groups for Deligne–Mumford stacks and more generally for any algebraic stack $\mathcal{X}$. Furthermore, intersection products are defined whenever $\mathcal{X}$ is a regular stack. Let $\mathcal{X}$ be a regular stack. The coarse moduli space of $\mathcal{X}$ is denoted by X, and $p : \mathcal{X} \to X$ is the projection. So from [Gi, Thm. 6.8], the pullback maps p^* and pushforward maps p_* establish a ring isomorphism of rational Chow groups

$$\mathrm{CH}^*(\mathcal{X})_\mathbb{Q} \cong \mathrm{CH}^*(X)_\mathbb{Q}. \tag{1}$$

This can be applied to the product $p \times p : \mathcal{X} \times \mathcal{X} \to X \times X$ to get a ring isomorphism

$$\mathrm{CH}^*(\mathcal{X} \times \mathcal{X})_{\mathbb{Q}} \cong \mathrm{CH}^*(X \times X)_{\mathbb{Q}}. \tag{2}$$

Assume that X is a smooth projective variety. Then these isomorphisms also hold in the rational singular cohomology of $\mathcal{X}$ and $\mathcal{X} \times \mathcal{X}$ (see e.g. [Be]):

$$H^*(\mathcal{X}, \mathbb{Q}) \cong H^*(X, \mathbb{Q}); \tag{3}$$

$$H^*(\mathcal{X} \times \mathcal{X}, \mathbb{Q}) \cong H^*(X \times X, \mathbb{Q}). \tag{4}$$

Via these isomorphisms, we can pull back the Künneth decomposition of the diagonal class in $H^{2n}(X \times X, \mathbb{Q})$ to a decomposition of the diagonal class of $\mathcal{X}$ in $H^{2n}(\mathcal{X} \times \mathcal{X}, \mathbb{Q})$, whose components we refer to as the *Künneth components* of $\mathcal{X}$.

Given a smooth variety X, consider an atlas $\bigsqcup_{\alpha \in I} U_\alpha$ of X such that $p_\alpha \colon U_\alpha \to X$ is an étale morphism for each $\alpha \in I$ and such that the images of p_α cover X. Then one can associate a Q-variety [Mum] to this atlas. Furthermore, by [Gi, Prop. 9.2] there is a regular stack $\mathcal{X}$ associated to this data such that X is its coarse moduli space; in other words, there is a projection

$$p \colon \mathcal{X} \to X.$$

In this case we note that the regular stack $\mathcal{X}$ is the same as the variety X. Hence the isomorphisms in (1)–(4) hold trivially for the projection p. More precisely, we have

$$\mathrm{CH}^*(\mathcal{X})_{\mathbb{Q}} = \mathrm{CH}^*(X)$$

and

$$H^*(\mathcal{X}, \mathbb{Q}) = \mathrm{CH}^*(X, \mathbb{Q}).$$

3.4. The Motive of a Rational Homogeneous Bundle

Suppose $Z \to S$ is a rational homogeneous bundle over a smooth projective variety S. Let S^{et} be the étale site on S together with the natural morphism of the sites $f \colon S^{\mathrm{et}} \to S$. Here S is considered with the Zariski site. Consider the pullback bundle

$$Z^{\mathrm{et}} := Z \times_S S^{\mathrm{et}} \to S^{\mathrm{et}}$$

over S^{et}.

Since we are dealing with a rational homogeneous bundle, we can describe these covers explicitly as follows. By Lemma 3.2, the pullback bundles $Z_{U_\alpha} \to U_\alpha$ for $\alpha \in I$ are Zariski trivial. In other words, $Z_{U_\alpha} = F \times U_\alpha$, where F is a typical fibre of $Z \to S$. Hence $Z_{U_\alpha} \to U_\alpha$ is a relative cellular variety for each $\alpha \in I$.

The description of the rational Chow groups of relative cellular spaces $\pi \colon X \to T$ is given by Koeck [K] (see also [NZ, Thm. 5.9]), which is stated for the higher Chow groups. Suppose $X \to T$ is a relative cellular space. Then there is a sequence of closed embeddings

$$\emptyset = Z_{-1} \subset Z_0 \subset \cdots \subset Z_n = X \tag{5}$$

such that $\pi_k \colon Z_k \to T$ is a flat projective T-scheme. Furthermore, for any $k = 0, 1, \ldots, n$, the open complement $Z_k - Z_{k-1}$ is T-isomorphic to an affine space $\mathbb{A}_T^{m_k}$ of relative dimension m_k. Denote $i_k \colon Z_k \hookrightarrow X$.

THEOREM 3.3. *For any $a, b \in \mathbb{Z}$, the map*

$$\bigoplus_{k=0}^{n} H_{a-2m_k}(T, b - m_k) \rightarrow H_a(X, b),$$

$$(\alpha_0, \ldots, \alpha_n) \mapsto \sum_{k=0}^{n} (i_k)_* \pi_k^* \alpha_k$$

is an isomorphism. Here $H_a(T, b) = \mathrm{CH}_b(T, a - 2b)$ are the higher Chow groups of T.

Proof. See [K, Thm., p. 371]. □

Theorem 3.3 can equivalently be restated to express the rational Chow groups of X as

$$\mathrm{CH}^r(X)_{\mathbb{Q}} = \bigoplus_{k=0}^{r} \left(\bigoplus_{\gamma} \mathbb{Q}[\omega_k^{\gamma}] \right) . f^* \mathrm{CH}^k(T)_{\mathbb{Q}}. \tag{6}$$

Here the ω_k^{γ} are the $(r - k)$-codimensional relative cells and γ runs over the indexing set of $(r - k)$-codimensional relative cells in the T-scheme X.

We now apply this theorem to our situation: we have a homogeneous bundle $Z \rightarrow S$ and an étale atlas $S^{\mathrm{et}} := \bigsqcup_{\alpha} U_{\alpha} \rightarrow S$ such that $Z_{U_{\alpha}} \rightarrow U_{\alpha}$ is trivial.

LEMMA 3.4. *Given a Zariski trivial homogeneous bundle $p_{\alpha} \colon Z_{U_{\alpha}} \rightarrow U_{\alpha}$, the rational Chow groups are described as follows:*

$$\mathrm{CH}^r(Z_{U_{\alpha}})_{\mathbb{Q}} = \bigoplus_{k=0}^{r} \left(\bigoplus_{\gamma} \mathbb{Q}[\omega_k^{\gamma}] \right) . p_{\alpha}^* \mathrm{CH}^k(U_{\alpha})_{\mathbb{Q}}.$$

Proof. Since the homogeneous bundle $p_{\alpha} \colon Z_{U_{\alpha}} \rightarrow U_{\alpha}$ is a Zariski trivial bundle, it is a relative cellular variety. Hence Theorem 3.3 can be applied to give a natural isomorphism

$$\mathrm{CH}^r(Z_{U_{\alpha}})_{\mathbb{Q}} = \bigoplus_{k=0}^{r} \left(\bigoplus_{\gamma} \mathbb{Q}[\omega_k^{\gamma}] \right) . f_{\alpha}^* \mathrm{CH}^k(U_{\alpha})_{\mathbb{Q}}.$$

Equivalently, since $Z_{U_{\alpha}} = F \times U_{\alpha}$, we have the equality (see [FMSS, Thm. 2])

$$\mathrm{CH}^r(Z_{U_{\alpha}})_{\mathbb{Q}} = \mathrm{CH}^r(F \times U_{\alpha})_{\mathbb{Q}} = \sum_{p,q, p+q=r} \mathrm{CH}^p(F)_{\mathbb{Q}} . \mathrm{CH}^q(U_{\alpha})_{\mathbb{Q}}. \tag{7}$$

Here F is a typical fibre of $Z \rightarrow S$ that is a cellular variety. This gives the assertion. □

For our applications it suffices to consider the piece $k = 0$, which consists of only the relative algebraic cells of codimension r:

$$\mathrm{RCH}^r(Z_{U_{\alpha}})_{\mathbb{Q}} := \bigoplus_{\gamma} \mathbb{Q}[\omega_0^{\gamma}].$$

In other words, we look only at the subgroup consisting of the direct summand

$$\mathrm{CH}^r(F) \subset \mathrm{CH}^r(F \times U_\alpha)$$

in (7).

A similar equality as in (7) holds in the rational singular cohomology of $Z_{U_\alpha} \to U_\alpha$. So we can also define the piece

$$RH^{2r}(Z_{U_\alpha})_{\mathbb{Q}} := \bigoplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

in the rational singular cohomology of Z_{U_α} and the piece

$$RH^{2r}(Z)_{\mathbb{Q}} := \bigoplus_\gamma \mathbb{Q}[\omega_0^\gamma]$$

as a subspace of the rational Betti cohomology $H^{2r}(Z, \mathbb{Q})$ generated by the relative analytic cells ω_0^γ. Here, we use that $Z \to S$ is locally trivial in the analytic topology and that there is an analytic cellular decomposition similar to (5).

LEMMA 3.5.   *The cycles ω_0^γ in $\mathrm{RCH}^*(Z_{U_\alpha})_{\mathbb{Q}}$ patch together in the étale site to determine a subspace $\mathrm{RCH}^*(Z)_{\mathbb{Q}}$ of $\mathrm{CH}^*(Z)_{\mathbb{Q}}$, generated by the patched cycles and that maps isomorphically onto the subspace $RH^{2r}(Z)_{\mathbb{Q}} \subset H^{2r}(Z, \mathbb{Q})$, under the cycle class map*

$$\mathrm{CH}^*(Z)_{\mathbb{Q}} \to H^{2*}(Z, \mathbb{Q}).$$

Proof. Note that the cycles $\omega_0^\gamma \in \mathrm{RCH}^*(Z_{U_\alpha})_{\mathbb{Q}}$ patch together as analytic cycles in the étale site and determine a subspace $RH^{2r}(Z)_{\mathbb{Q}} \subset H^{2r}(Z, \mathbb{Q})$.

Since the fibre F is a cellular variety, there is a natural isomorphism

$$\mathrm{RCH}^*(Z_{U_\alpha})_{\mathbb{Q}} \xrightarrow{\simeq} RH^{2*}(Z_{U_\alpha})_{\mathbb{Q}} \tag{8}$$

between the 0th piece of the rational Chow group and the relative Betti cohomology for each α.

Via the isomorphism in (8), the patching conditions required over the étale site to define the piece $\mathrm{RCH}^{2r}(Z)_{\mathbb{Q}}$ are the same as those for $RH^{2r}(Z)_{\mathbb{Q}}$. More precisely, the patching conditions are given in [Gi, Sec. 4]. The identification in (8) together with the fact that the patching conditions are fulfilled for the singular cohomology of the étale site means that the cycles ω_0^γ patch together to give a class in $RH^{2r}(Z)_{\mathbb{Q}}$, and hence they also patch together to give a class in $\mathrm{RCH}^{2r}(Z)_{\mathbb{Q}}$. These patched classes generate the $\mathbb{Q}$-subspace $\mathrm{RCH}^{2r}(Z)_{\mathbb{Q}} \subset \mathrm{CH}^*(Z)_{\mathbb{Q}}$, which maps isomorphically onto the subspace $RH^{2r}(Z)_{\mathbb{Q}} \subset H^{2r}(Z, \mathbb{Q})$ under the cycle class map.   □

COROLLARY 3.6.   *There is a canonical isomorphism*

$$\mathrm{RCH}^r(Z)_{\mathbb{Q}} \simeq RH^{2r}(Z)_{\mathbb{Q}}$$

between the rational Chow groups and the rational cohomology generated by the relative cells.

Let $n := \dim(Z/S)$.

COROLLARY 3.7. *The bundle $Z \to S$ has a relative Chow–Künneth decomposition in the sense of* [GHMu1].

Proof. This follows from Lemma 3.5 applied to the relative product $Z \times_S Z \to S$. We notice that the relative orthogonal Künneth projectors in $H^{2n}(Z \times_S Z, \mathbb{Q})$ lift to relative orthogonal projectors in $H^{2n}(Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha}, \mathbb{Q})$, which add to the relative diagonal cycle. Now we note that the relative diagonal $\Delta_{Z/S}$ and its orthogonal Künneth components actually lie in the piece $RH^{2n}(Z \times_S Z)_{\mathbb{Q}}$ (generated by the relative algebraic cells) and, under the isomorphisms in (3) and (4), lift to an orthogonal decomposition

$$\Delta_{Z/S} = \sum_{i=0}^{2n} \Pi_i \in RH^{2n}(Z \times_S Z)_{\mathbb{Q}}$$

over the étale site—that is, over $Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha}$ for each $\alpha \in I$. Now apply Corollary 3.6 to the product space $Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha} \to U_\alpha$ to lift the relative orthogonal projectors just described to orthogonal algebraic projectors in $\mathrm{RCH}^n(Z_{U_\alpha} \times_{U_\alpha} Z_{U_\alpha})_{\mathbb{Q}}$, and these patch to give relative Chow–Künneth projectors and a relative Chow–Künneth decomposition:

$$\Delta_{Z/S} = \sum_{i=0}^{2n} \Pi_i \in \mathrm{CH}^n(Z \times_S Z)_{\mathbb{Q}}. \qquad \square$$

PROPOSITION 3.8. *Suppose $Z \to S$ is a rational homogeneous bundle over a smooth variety S. Then the motive of the bundle $Z \to S$ is expressed as a sum of tensor products of summands of the motive of S with the twisted Tate motive. More precisely, the motive of Z can be written as*

$$h(Z) = \bigoplus_i h^i(Z),$$

where $h^i(Z) = \bigoplus_{j+k} r_{\omega_\alpha} . \mathbb{L}^j \otimes h^k(S)$. Here r_{ω_α} is the number of j-codimensional cells on a fibre $\mathbb{F}$.

In particular, if S has a Chow–Künneth decomposition then Z also admits an absolute Chow–Künneth decomposition.

Proof. By Corollary 3.7, we know that the bundle Z/S has a relative Chow–Künneth decomposition. Since the map $Z \to S$ is a smooth morphism and since the fibres of $Z \to S$ have only algebraic cohomology, we can directly apply the criterion in [GHMu2, Main Theorem 1.3] to obtain absolute Chow–Künneth projectors for Z and the decomposition stated previously (see e.g. [I, Lemma 3.2, Cor. 3.3]). $\qquad \square$

REMARK 3.9. Suppose X is a smooth projective variety with a nef tangent bundle. Then, by [CP; DePS], we know that there exists an étale cover $X' \to X$ of X such that $X' \to A$ is a smooth morphism over an abelian variety A whose fibres are smooth Fano varieties with a nef tangent bundle. It is an open question [CP, p. 170]

whether such a Fano variety is a rational homogeneous variety. A positive answer
to this question, together with Proposition 3.8, will give absolute Chow–Künneth
projectors for all varieties with a nef tangent bundle. See also [Hw, Sec. 4] for a
discussion on new cases where this question is answered positively.

4. Chow–Künneth Decomposition for Log Homogeneous Varieties

Log homogeneous varieties were introduced by Brion [Br]. Suppose X is a smooth
projective variety and $D \subset X$ is a normal crossing divisor. Then X is said to be
log homogeneous with respect to D if the logarithmic tangent bundle $\mathcal{T}_X(-D)$ is
generated by its global sections. Then X is almost homogeneous under the con-
nected automorphism group $G := \mathrm{Aut}^0(X, D)$ with boundary D. The G-orbits in
X are exactly the strata defined by D; in particular, their number is finite.

   A classification of log homogeneous varieties is given by Brion as follows.

THEOREM 4.1.   *Any log homogeneous variety X can be written uniquely as $G \times^I Y$,
where*:

(1) *G is a connected algebraic group;*
(2) *$I \subset G$ is a closed subgroup containing G_{aff} as a subgroup of finite index;*
(3) *for any Levi subgroup $L \subset G_{\mathrm{aff}}$, Y is a complete smooth I-variety contain-
     ing an open L-stable subset Y_L such that the L-variety Y is spherical and the
     projection*

$$X \to G/I =: A \qquad (9)$$

   is the Albanese morphism.

Proof. See [Br, Thm. 3.2.1].   □

Recall that a smooth spherical variety Y is a G-variety such that the Borel sub-
group B of G has an open dense orbit in Y. It is known that Y contains a finite
number of B-orbits. Since we are looking at varieties defined over $\mathbb{C}$, it follows
that a spherical variety is a linear variety (in the sense of [T, Addendum, p. 5]).
In particular, we have the following statement.

LEMMA 4.2.   *Suppose Y is a smooth complete spherical variety. Then there is an
isomorphism*

$$\mathrm{CH}^i(Y) \xrightarrow{\cong} H^{2i}(Y, \mathbb{Z})$$

for each i.

Proof. See [FMSS, Cor. to Thm. 2].   □

LEMMA 4.3.   *Suppose Y is a smooth complete spherical variety. Then Y has a
Chow–Künneth decomposition.*

Proof. This follows from Lemma 4.2 and the construction of orthogonal projec-
tors given in [IM, Lemma 5.2].   □

We will show that X has a Chow–Künneth decomposition under the following assumption.

THEOREM 4.4.   *Suppose X is a log homogeneous variety. Then the variety X has a Chow–Künneth decomposition. Moreover, the motive of X is expressed as a sum of tensor products of the summands of the motive of its Albanese reduction with the twisted Tate motive.*

Proof. With notation as in Theorem 4.1, suppose the spherical variety Y is a Fano variety. Then, by [BiB, Prop. 4.2(i)], we have the vanishing $H^1(Y, T_Y) = 0$. In particular, this implies that the formal deformations of Y are trivial. Hence, by [Se, Prop. 2.6.10], the Albanese fibration in (9) is étale locally trivial. In general, consider the Albanese fibration

$$X = G \times^I Y \to G/I = A,$$

which is easily seen to be étale locally trivial. The following explanation is due to Totaro. Notice that all the fibres of this morphism are isomorphic to Y. In more detail, this morphism is étale locally trivial because the morphism $G \to G/I$ is étale locally trivial, which is a standard fact about the quotient of an algebraic group by a smooth closed subgroup. See the discussion of homogeneous spaces in [Bo, 6.14].

Hence we can apply the methods from the previous section. By Lemma 4.3, relative Chow–Künneth projectors can be constructed for Zariski trivializations of (9) over étale covers $U_\alpha \to A$. Hence the proof of Proposition 3.8 applies to this situation. Indeed, Lemma 3.4 holds for a relative spherical variety over U_α. This can be applied to the Albanese fibration in (9) over étale morphisms, where it is Zariski trivial. In this case, the following piece of the rational Chow ring $\mathrm{RCH}^*(U_\alpha \times Y)_{\mathbb{Q}}$ is identified with the Chow ring $\mathrm{CH}^*(Y)_{\mathbb{Q}}$. A formula similar to (6) holds for the Chow groups of $U_\alpha \times Y$, since Y is cellular; see [FMSS, Thm. 2]. Hence, by Lemma 4.2, $\mathrm{CH}^*(U_\alpha \times Y)_{\mathbb{Q}} \simeq H^{2*}(U_\alpha \times Y)_{\mathbb{Q}}$. Similarly, Lemma 3.5 and Corollary 3.6 hold for (9) over étale morphisms. The rest of the arguments are the same as given for a rational homogeneous bundle. □

References

[AkJ] R. Akhtar and R. Joshua, *Künneth decomposition for quotient varieties,* Indag. Math. (N.S.) 17 (2006), 319–344.

[ADh] D. Arapura and A. Dhillon, *The motive of the moduli stack of G-bundles over the universal curve,* Proc. Indian Acad. Sci. Math. Sci. 118 (2008), 389–411.

[Be] K. Behrend, *On the de Rham cohomology of differential and algebraic stacks,* Adv. Math. 198 (2005), 583–622.

[BiB] F. Bien and M. Brion, *Automorphisms and local rigidity of regular varieties,* Compositio Math. 104 (1996), 1–26.

[Bo] A. Borel, *Linear algebraic groups,* Grad. Texts in Math., 126, Springer-Verlag, New York, 1991.

[B]   F. Brion, *Log homogeneous varieties,* Proceedings of the VI Coloquio
         Latinoamericano de Algebra (Colonia, 2005), pp. 1–39, Bibl. Rev. Mat.
         Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2007.

[CP]   F. Campana and T. Peternell, *Projective manifolds whose tangent bundles are
         numerically effective,* Math. Ann. 289 (1991), 169–187.

[DM1]   P. del Angel and S. Müller-Stach, *Motives of uniruled 3-folds,* Compositio
         Math. 112 (1998), 1–16.

[DM2]   ———, *On Chow motives of 3-folds,* Trans. Amer. Math. Soc. 352 (2000),
         1623–1633.

[DePS]   J. P. Demailly, T. Peternell, and M. Schneider, *Compact complex manifolds with
         numerically effective tangent bundles,* J. Algebraic Geom. 3 (1994), 295–345.

[DenMu] Ch. Deninger and J. Murre, *Motivic decomposition of abelian schemes and the
         Fourier transform,* J. Reine Angew. Math. 422 (1991), 201–219.

[FMSS]   W. Fulton, R. MacPherson, F. Sottile, and B. Sturmfels, *Intersection theory on
         spherical varieties,* J. Algebraic Geom. 4 (1995), 181–193.

[Gi]   H. Gillet, *Intersection theory on algebraic stacks and Q-varieties,* Proceedings
         of the Luminy conference on algebraic *K*-theory (Luminy, 1983), J. Pure Appl.
         Algebra 34 (1984), 193–240.

[GHMu1] B. B. Gordon, M. Hanamura, and J. P. Murre, *Relative Chow–Künneth
         projectors for modular varieties,* J. Reine Angew. Math. 558 (2003), 1–14.

[GHMu2] ———, *Absolute Chow–Künneth projectors for modular varieties,* J. Reine
         Angew. Math. 580 (2005), 139–155.

[GMu]   B. Gordon and J. Murre, *Chow motives of elliptic modular threefolds,* J. Reine
         Angew. Math. 514 (1999), 145–164.

[GuPe]   V. Guletskiĭ and C. Pedrini, *Finite-dimensional motives and the conjectures of
         Beilinson and Murre,* K-Theory 30 (2003), 243–263.

[Hw]   J.-K. Hwang, *Rigidity of rational homogeneous spaces,* International Congress
         of Mathematicians, vol. II, pp. 613–626, Eur. Math. Soc., Zürich, 2006.

[I]   J. N. Iyer, *Murre's conjectures and explicit Chow Künneth projectors for
         varieties with a nef tangent bundle,* Trans. Amer. Math. Soc. 361 (2009),
         1667–1681.

[IM]   J. N. Iyer and D. Müller-Stach, *Chow–Künneth decomposition for some moduli
         spaces,* Doc. Math. 14 (2009), 1–18.

[Ja]   U. Jannsen, *Motivic sheaves and filtrations on Chow groups,* Motives (Seattle,
         1991), Proc. Sympos. Pure Math., 55, pp. 245–302, Amer. Math. Soc.,
         Providence, RI, 1994.

[K]   B. Köck, *Chow motif and higher Chow theory of G/P*, Manuscripta Math. 70
         (1991), 363–372.

[Ma]   Yu. Manin, *Correspondences, motifs and monoidal transformations,* Mat. Sb.
         (N.S.) 77 (119) (1968), 475–507.

[Mi]   A. Miller, *Chow motives of universal families over some Shimura surfaces,*
         preprint, 2010, arXiv math.AG/0710.4209.

[Mi+]   A. Miller, S. Müller-Stach, S. Wortmann, Y. H. Yang, and K. Zuo,
         *Chow–Künneth decomposition for universal families over Picard modular
         surfaces,* Algebraic cycles and motives, vol. 2, pp. 241–276, London Math.
         Soc. Lecture Note Ser., 344, Cambridge Univ. Press, Cambridge, 2007.

[Mum]   D. Mumford, *Towards an enumerative geometry of the moduli space of curves,*
         Arithmetic and geometry, vol. II, pp. 271–328, Progr. Math., 36, Birkhäuser,
         Boston, 1983.

[Mu1] J. P. Murre, *On the motive of an algebraic surface,* J. Reine Angew. Math. 409 (1990), 190–204.

[Mu2] ———, *On a conjectural filtration on the Chow groups of an algebraic variety, I. The general conjectures and some examples,* Indag. Math. (N.S.) 4 (1993), 177–188.

[Mu3] ———, *On a conjectural filtration on the Chow groups of an algebraic variety, II. Verification of the conjectures for threefolds which are the product on a surface and a curve,* Indag. Math. (N.S.) 4 (1993), 189–201.

[NZ] A. Nenashev and K. Zainoulline, *Oriented cohomology and motivic decompositions of relative cellular spaces,* J. Pure Appl. Algebra 205 (2006), 323–340.

[Sa] M. Saito, *Chow–Künneth decomposition for varieties with low cohomological level,* preprint, 2004, arXiv math.AG/0604254.

[Sc] A. J. Scholl, *Classical motives,* Motives (Seattle, 1991), Proc. Sympos. Pure Math., 55, pp. 163–187, Amer. Math. Soc., Providence, RI, 1994.

[Se] E. Sernesi, *Deformations of algebraic schemes,* Grundlehren Math. Wiss., 334, Springer-Verlag, Berlin, 2006.

[Sh] A. M. Shermenev, *The motive of an abelian variety,* J. Funct. Anal. 8 (1974), 55–61.

[T] B. Totaro, *Chow groups, Chow cohomology, and linear varieties,* J. Algebraic Geom. (to appear).

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