Michigan Math. J. 59 (2010), 621–665

Lattice Zariski k-ples of Plane Sextic Curves and Z-Splitting Curves for Double Plane Sextics

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1. Introduction

By virtue of the theory of period mapping, lattice theory has become a strong computational tool in the study of complex $K3$ surfaces. In this paper, we apply this tool to the classification of complex projective plane curves of degree 6 with only simple singularities. In particular, we explain the phenomena of *Zariski pairs* from a lattice-theoretic point of view.

A *simple sextic* is a reduced (possibly reducible) complex projective plane curve of degree 6 with only simple singularities. For a simple sextic $B \subset \mathbb{P}^2$, we denote by μ_B the total Milnor number of B, by $\operatorname{Sing} B$ the singular locus of B, by R_B the ADE-type of the singular points of B, and by $\operatorname{degs} B = [d_1, \dots, d_m]$ the list of degrees $d_i = \deg B_i$ of the irreducible components $B_1, \dots, B_m$ of B.

We have the following equivalence relations among simple sextics.

Definition 1.1. Let B and B' be simple sextics.

(1) We write $B \sim_{\mathrm{eqs}} B'$ if B and B' are contained in the same connected component of an equisingular family of simple sextics.

(2) We say that B and B' are of the *same configuration type,* and write $B \sim_{\mathrm{cfg}} B'$, if there exist tubular neighborhoods $T \subset \mathbb{P}^2$ of B and $T' \subset \mathbb{P}^2$ of B' and a homeomorphism $\varphi\colon (T, B) \xrightarrow{\sim} (T', B')$ such that (a) $\deg \varphi(B_i) = \deg B_i$ holds for each irreducible component B_i of B, (b) φ induces a bijection $\operatorname{Sing} B \xrightarrow{\sim} \operatorname{Sing} B'$, and (c) φ is an analytic isomorphism of plane curve singularities locally around each $P \in \operatorname{Sing} B$. Note that R_B and $\operatorname{degs} B$ are invariants of the configuration type. (See [4, Rem. 3] for a combinatorial definition of $\sim_{\mathrm{cfg}}$.)

(3) We say that B and B' are of the *same embedding type,* and write $B \sim_{\mathrm{emb}} B'$, if there exists a homeomorphism $\psi\colon (\mathbb{P}^2, B) \xrightarrow{\sim} (\mathbb{P}^2, B')$ such that ψ induces a bijection $\operatorname{Sing} B \xrightarrow{\sim} \operatorname{Sing} B'$ and such that, locally around each $P \in \operatorname{Sing} B$, ψ is an analytic isomorphism of plane curve singularities.

It is obvious that

$$B \sim_{\mathrm{eqs}} B' \implies B \sim_{\mathrm{emb}} B' \implies B \sim_{\mathrm{cfg}} B',$$

although the converses do not necessarily hold.

Received March 19, 2009. Revision received June 5, 2009.

Partially supported by JSPS Grants-in-Aid for Scientific Research (20340002) and JSPS Core-to-Core Program (18005).

EXAMPLE 1.2. Zariski [34] showed that there exist irreducible simple sextics B_1 and B_2 with $R_{B_1} = R_{B_2} = 6A_2$ such that $\pi_1(\mathbb{P}^2 \setminus B_1) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ while $\pi_1(\mathbb{P}^2 \setminus B_2) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$, where $*$ denotes the free product of groups (see also Oka [18] and Shimada [22]). Therefore we have $B_1 \sim_{\mathrm{cfg}} B_2$, but $B_1 \not\sim_{\mathrm{emb}} B_2$ and hence $B_1 \not\sim_{\mathrm{eqs}} B_2$.

Artal Bartolo [3] revived the study of pairs of plane curves that are of the same configuration type but are not connected by equisingular deformation. Since then, many works have addressed the discrepancies between equisingular deformations and configuration types—not only for simple sextics but also for curves of higher degrees and with other types of singularities. (See the survey paper [4].) The main theme of these works is to find pairs of plane curves (called *Zariski pairs* or *Zariski couples*) that have the same configuration type but have different embedding topologies.

As for simple sextics, there have been two important works about $\sim_{\mathrm{eqs}}$ and $\sim_{\mathrm{cfg}}$. One is Yang [32], in which the configuration types of simple sextics are completely classified; the other is Degtyarev [11], which presents an algorithm to calculate the connected components of the equisingular family of simple sextics in a given configuration type. The main tool used in these two works is the theory of period mapping of complex $K3$ surfaces applied to double plane sextics.

In this paper, we introduce another equivalence relation $\sim_{\mathrm{lat}}$ by means of the structure of the Néron–Severi lattices of the $K3$ surfaces obtained as the double covers of $\mathbb{P}^2$ branching along the simple sextics. This relation is coarser than $\sim_{\mathrm{eqs}}$ but finer than $\sim_{\mathrm{cfg}}$, so it can play the same role as $\sim_{\mathrm{emb}}$. The definition of $\sim_{\mathrm{lat}}$ is, however, purely algebraic and therefore computationally easier to deal with than $\sim_{\mathrm{emb}}$. In fact, Yang's method [32] provides us with an algorithm to classify all the equivalence classes of the relation $\sim_{\mathrm{lat}}$, which are called the *lattice types* of simple sextics. Moreover we can sometimes conclude that $B \not\sim_{\mathrm{emb}} B'$ by looking at an invariant of the lattice types (Theorem 8.5).

We then define the notion of *Z-splitting curves* and investigate lattice types of simple sextics by means of this notion. The notion of *lattice Zariski couples* (or, more generally, *lattice Zariski k-ples*) is introduced for $\sim_{\mathrm{lat}}$ in the same way as the notion of classical Zariski couples was introduced for $\sim_{\mathrm{emb}}$ in [3]. The notion of Z-splitting curves provides us with a unifying tool to describe all lattice Zariski k-ples. In fact, the members of any lattice Zariski k-ple are distinguished by numbers of Z-splitting curves of degree ≤ 2 (Theorem 3.5).

Finally, we define *lattice types of Z-splitting curves* and classify all lineages via specialization of lattice types of Z-splitting curves of degree ≤ 3. It turns out that these lineages are completely indexed by the *class order* of the Z-splitting curves (Theorems 3.13, 3.19, and 3.23). These lineages yield many examples of simple sextics with interesting geometry. For example, the Z-splitting conics with class order 3 are the splitting conics of *torus sextics,* which have been studied intensively by Oka and others (see e.g. [19]).

The notion of Z-splitting curves is important also because, for a simple sextic B that is generic in an irreducible component of an equisingular family, the

Néron–Severi lattice of the corresponding $K3$ surface is generated by the reduced parts of the lifts of the irreducible components of B and the lifts of Z-splitting curves of degree ≤ 3 (Theorem 3.21).

The plan of this paper is as follows. In Section 2, we define various notions that are investigated in the paper. The relation $\sim_{\mathrm{lat}}$ is defined in Definition 2.8, and the notion of Z-splitting curves is defined in Definition 2.13. The main results are stated in Section 3. Most of these results are proved computationally with the assistance of a computer. We present lattice-theoretic algorithms to prove them in the following sections. In Section 4, we explain Yang's method of making the complete list of lattice types of simple sextics. In Section 5, we give an algorithm to determine the configuration type and the classes of lifts of smooth Z-splitting curves of degree ≤ 3 for a given lattice type of simple sextics. In Section 6, we present an algorithm about specialization of lattice types of Z-splitting curves; the results in this section are the main theoretical ingredients for our classification of the lineages of Z-splitting curves. In Section 7, we demonstrate the algorithms for a concrete example. We conclude this paper by presenting miscellaneous facts, examples, and remarks in Section 8.

As we were finishing the first version of this paper, a preprint by Yang and Xie [33] appeared on the e-print archive. In their paper, Yang and Xie also investigate the classical Zariski pairs of simple sextics by lattice theory and the result in [27; 28]. See also Theorem 8.5 of this paper.

Acknowledgments. Part of this work was done during the author's stay at National University of Singapore in September 2008. Thanks are due to Professor De-Qi Zhang for his warm hospitality. The author is also deeply grateful to the referee for many valuable comments on the first version of this paper.

2. Definitions

A *lattice* is a free $\mathbb{Z}$-module L of finite rank with a nondegenerate symmetric bilinear form $(\cdot,\cdot)\colon L \times L \to \mathbb{Z}$. We say that a lattice L is *even* if $x^2 \in 2\mathbb{Z}$ holds for any $x \in L$. We say that L is *negative definite* if $x^2 < 0$ holds for any nonzero $x \in L$.

We fix several conventions about lattices. Let L be a lattice, and let S be a subset of L. We use $\langle S\rangle$ to denote the sublattice of L generated by S and use $\langle S\rangle^+$ to denote the *monoid* of vectors $\sum a_v v$ ($v \in S$) with $a_v \in \mathbb{Z}_{\geq 0}$. When $S = \{v\}$, we write $\langle v\rangle$ for $\langle\{v\}\rangle$. We denote by $S^\perp$ or $(S \subset L)^\perp$ the orthogonal complement of $\langle S\rangle$ in L.

Let L' be another lattice. An *embedding* of L into L' is a homomorphism of $\mathbb{Z}$-modules $\phi\colon L \to L'$ that satisfies $(x, y) = (\phi(x), \phi(y))$ for any $x, y \in L$. Note that such a homomorphism is necessarily injective. An embedding ϕ is said to be *primitive* if the cokernel of ϕ is torsion free. For an embedding ϕ, we use the same letter ϕ to denote the induced linear homomorphism $L \otimes \mathbb{C} \to L' \otimes \mathbb{C}$.

Definition 2.1. Let L be an even negative-definite lattice. A vector $d \in L$ is called a *root* if $d^2 = -2$. Let D_L be the set of roots in L. A subset F of D_L is called a *fundamental system of roots in* L if F is a basis of $\langle D_L \rangle$ and every $d \in D_L$ can be written as a linear combination of elements of F with coefficients all nonpositive or all nonnegative. An even negative-definite lattice L is called a *root lattice* if $\langle D_L \rangle = L$ holds.

A fundamental system of roots exists for any even negative-definite lattice. The isomorphism classes of fundamental systems of roots (and hence root lattices) are classified by means of *Dynkin diagrams.* See for example Ebeling [14, Sec. 1.4] or Bourbaki [7] for the proof of these facts.

We denote by $L^\vee$ the *dual lattice* $\{v \in L \otimes \mathbb{Q} \mid (x, v) \in \mathbb{Z}$ for any $x \in L\}$ of L, which is a submodule of $L \otimes \mathbb{Q}$ with finite rank containing L.

DEFINITION 2.2. Let L be a lattice. A submodule L' of $L^\vee$ is called an *overlattice* of L if L' contains L and the $\mathbb{Q}$-valued symmetric bilinear form on $L^\vee$ extending the symmetric bilinear form on L takes values in $\mathbb{Z}$ on L'.

DEFINITION 2.3. *Lattice data* is a triple $[\mathcal{E}, h, \Lambda]$, where $\mathcal{E}$ is a fundamental system of roots in the negative-definite root lattice $\langle \mathcal{E} \rangle$ generated by $\mathcal{E}$, h is a vector with $h^2 = 2$ that generates a positive-definite lattice $\langle h \rangle$ of rank 1, and Λ is an even overlattice of the orthogonal direct sum $\langle h \rangle \oplus \langle \mathcal{E} \rangle$.

Extended lattice data is a quartet $[\mathcal{E}, h, \Lambda, S]$, where $[\mathcal{E}, h, \Lambda]$ is lattice data and S is a subset of Λ with cardinality 2.

REMARK 2.4. In the geometric application, S is the placeholder for the classes of the lifts of a Z-splitting curve (see Definition 2.26).

DEFINITION 2.5. An *isomorphism* from lattice data $[\mathcal{E}, h, \Lambda]$ to lattice data $[\mathcal{E}', h', \Lambda']$ is an isomorphism of lattices $\phi\colon \Lambda \xrightarrow{\sim} \Lambda'$ that satisfies $\phi(\mathcal{E}) = \mathcal{E}'$ and $\phi(h) = h'$. If $\phi\colon \Lambda \xrightarrow{\sim} \Lambda'$ is an isomorphism of lattice data, then ϕ induces an isomorphism of fundamental systems of roots between $\mathcal{E}$ and $\mathcal{E}'$.

DEFINITION 2.6. An *isomorphism* from extended lattice data $[\mathcal{E}, h, \Lambda, S]$ to extended lattice data $[\mathcal{E}', h', \Lambda', S']$ is an isomorphism $\phi\colon \Lambda \xrightarrow{\sim} \Lambda'$ of lattice data from $[\mathcal{E}, h, \Lambda]$ to $[\mathcal{E}', h', \Lambda']$ that induces a bijection from S to S'.

Let $B \subset \mathbb{P}^2$ be a simple sextic. Consider the double covering $\pi_B\colon Y_B \to \mathbb{P}^2$ branching exactly along B. Then Y_B has only rational double points of type R_B as its singularities, and the minimal resolution $\rho_B\colon X_B \to Y_B$ of Y_B yields a $K3$ surface X_B. Let $\tilde{\rho}_B\colon X_B \to \mathbb{P}^2$ denote the composite of ρ_B and π_B.

We denote by $\mathrm{NS}(X_B) \subset H^2(X_B, \mathbb{Z})$ the Néron–Severi lattice of X_B. Let $\mathcal{E}_B$ be the set of (-2)-curves on X_B that are contracted by $\tilde{\rho}_B\colon X_B \to \mathbb{P}^2$. We regard $\mathcal{E}_B$ as a subset of $\mathrm{NS}(X_B)$ by $E \mapsto [E]$, where $[E] \in \mathrm{NS}(X_B)$ denotes the class of the curve $E \in \mathcal{E}_B$. We consider the sublattice

$$\Sigma_B := \langle h_B \rangle \oplus \langle \mathcal{E}_B \rangle \subset \mathrm{NS}(X_B)$$

of $\mathrm{NS}(X_B)$ generated by the polarization class

$$h_B := [\tilde{\rho}_B^*(\mathcal{O}_{\mathbb{P}^2}(1))]$$

and $\mathcal{E}_B \subset \mathrm{NS}(X_B)$. Observe that $\mathcal{E}_B$ is a fundamental system of roots in the root lattice $\langle \mathcal{E}_B \rangle$ of type R_B. We then denote by

$$\Lambda_B := (\Sigma_B \otimes \mathbb{Q}) \cap H^2(X_B, \mathbb{Z})$$

the primitive closure of Σ_B in $H^2(X_B, \mathbb{Z})$, which is an even overlattice of Σ_B. Since $\mathrm{NS}(X_B)$ is primitive in $H^2(X_B, \mathbb{Z})$, Λ_B is the primitive closure of Σ_B in $\mathrm{NS}(X_B)$. Finally, we define the finite abelian group G_B by

$$G_B := \Lambda_B / \Sigma_B.$$

DEFINITION 2.7. We denote by $\ell(B)$ the lattice data $[\mathcal{E}_B, h_B, \Lambda_B]$ and call it the *lattice data of B*.

DEFINITION 2.8. Let B and B' be simple sextics. We write $B \sim_{\mathrm{lat}} B'$ if there exists an isomorphism between the lattice data $\ell(B)$ and $\ell(B')$. An equivalence class of the relation $\sim_{\mathrm{lat}}$ is called a *lattice type* of simple sextics. The lattice type containing a simple sextic B is denoted by $\lambda(B)$.

By definition, an isomorphism of lattice data from $\ell(B)$ to $\ell(B')$ is an isomorphism of lattices $\Lambda_B \xrightarrow{\sim} \Lambda_{B'}$ that preserves the polarization class and the set of classes of the exceptional (-2)-curves.

It is obvious that the isomorphism class of the finite abelian group G_B is an invariant of the lattice type $\lambda(B)$.

Let $B_1, \dots, B_m$ be the irreducible components of B. We denote by $\tilde{B}_i \subset X_B$ the reduced part of the strict transform of B_i and put

$$\Theta_B := \Sigma_B + \langle [\tilde{B}_1], \dots, [\tilde{B}_m] \rangle \subset \mathrm{NS}(X_B).$$

Then we have

$$\Sigma_B \subset \Theta_B \subset \Lambda_B \subset \mathrm{NS}(X_B).$$

We see that the implications

$$B \sim_{\mathrm{eqs}} B' \implies B \sim_{\mathrm{lat}} B' \implies B \sim_{\mathrm{cfg}} B'$$

hold, where the second implication was proved by Yang [32] (see also Corollary 5.26). Hence the isomorphism class of the finite abelian group

$$F_B := \Lambda_B / \Theta_B$$

is also an invariant of the lattice type $\lambda(B)$.

In fact, Yang [32] gave an algorithm to classify all lattice types and configuration types of simple sextics using the idea of Urabe [30; 31]. The numbers of these types are given in Table 2.1. (Yang did not present the complete table in his paper, so in [26] we reproduced the classification table along with the complete list of configurations of rational double points on normal $K3$ surfaces.) Table 2.1 shows that, for $\mu_B > 11$, there exist many lattice Zariski k-ples ($k > 1$) defined as follows.

Table 2.1 Numbers of Configuration Types and Lattice Types

μ_B	0	1	2	3	4	5	6	7	8	9	10	11
$\sim_{\mathrm{cfg}}$	1	1	2	3	6	10	18	30	53	89	148	246
$\sim_{\mathrm{lat}}$	1	1	2	3	6	10	18	30	53	89	148	246

μ_B	12	13	14	15	16	17	18	19	Total
$\sim_{\mathrm{cfg}}$	415	684	1090	1623	2139	2283	1695	623	11159
$\sim_{\mathrm{lat}}$	416	686	1096	1639	2171	2330	1734	629	11308

DEFINITION 2.9. A configuration type γ of simple sextics is called a *lattice Zariski k-ple* if γ contains exactly k lattice types.

EXAMPLE 2.10. The configuration type of irreducible simple sextics B with $R_B = 6A_2$ is a lattice Zariski couple with $\mu_B = 12$. Indeed, for B_1 and B_2 in Example 1.2, we have $G_{B_1} = 0$ and $G_{B_2} \cong \mathbb{Z}/3\mathbb{Z}$.

REMARK 2.11. See Section 7 for an example of lattice Zariski triples. Looking at the classification table, we see that there exist no lattice Zariski k-ples with $k > 3$.

Next we define the notion of *Z-splitting curves,* where Z stands for *Zariski.* Let B be a simple sextic. We denote by

$$\iota_B : X_B \xrightarrow{\sim} X_B$$

the involution of X_B over $\mathbb{P}^2$, and we use the same letter ι_B to denote the induced involution on the lattice $H^2(X_B, \mathbb{Z})$. Note that ι_B preserves the sublattices Σ_B, Λ_B, Θ_B, and $\mathrm{NS}(X_B)$.

DEFINITION 2.12. A reduced irreducible projective plane curve $\Gamma \subset \mathbb{P}^2$ is said to be *splitting for B* if the strict transform of Γ by $\tilde{\rho}_B : X_B \to \mathbb{P}^2$ splits into two (possibly equal) irreducible components $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^- = \iota_B(\tilde{\Gamma}^+)$. We call $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ the *lifts* of the splitting curve Γ.

We have $\tilde{\Gamma}^+ = \tilde{\Gamma}^-$ if and only if Γ is an irreducible component of B.

DEFINITION 2.13. A splitting curve Γ is said to be *pre-Z-splitting* if the class $[\tilde{\Gamma}^+]$ of a lift $\tilde{\Gamma}^+ \subset X_B$ of Γ is contained in Λ_B. (Note that $[\tilde{\Gamma}^+] \in \Lambda_B$ if and only if $[\tilde{\Gamma}^-] \in \Lambda_B$ because we have $[\tilde{\Gamma}^+] + [\tilde{\Gamma}^-] \in \Sigma_B$.)

DEFINITION 2.14. A pre-Z-splitting curve Γ is said to be *Z-splitting* if the classes $[\tilde{\Gamma}^+]$ and $[\tilde{\Gamma}^-] = \iota_B([\tilde{\Gamma}^+])$ are distinct.

REMARK 2.15. Since ι_B acts on the orthogonal complement of Λ_B in $H^2(X_B, \mathbb{Z})$ as the multiplication by -1, it follows that, if a splitting curve Γ is not pre-Z-splitting, then we have $[\tilde{\Gamma}^+] \neq [\tilde{\Gamma}^-]$. See Table 2.2.

Table 2.2 Three Notions of Splittingness

	$[\tilde{\Gamma}^+] \in \Lambda_B$	$[\tilde{\Gamma}^+] \notin \Lambda_B$	
			Splitting : $I + II + III$
$[\tilde{\Gamma}^+] = [\tilde{\Gamma}^-]$	I	$\emptyset$	Pre-Z-splitting : $I + II$
$[\tilde{\Gamma}^+] \neq [\tilde{\Gamma}^-]$	II	III	Z-splitting : II

We have an easy numerical criterion of pre-Z-splittingness (see Proposition 8.2). We also have the following result.

PROPOSITION 2.16. *Let Γ be a pre-Z-splitting curve for a simple sextic B. Let B' be a general member of the connected component $\mathcal{F}$ of the equisingular family containing B, and let $\phi\colon H^2(X_B,\mathbb{Z}) \xrightarrow{\sim} H^2(X_{B'},\mathbb{Z})$ be an isomorphism of lattices induced by an equisingular deformation from B to B'. Then there exists a pre-Z-splitting curve Γ' for B' such that the class of a lift of Γ' is equal to $\phi([\tilde{\Gamma}^+])$. If Γ is Z-splitting, then so is Γ'.*

Proof. Since ϕ is induced by an equisingular deformation, we see that ϕ induces an isomorphism $\Lambda_B \xrightarrow{\sim} \Lambda_{B'}$. The second assertion follows from the first assertion because ϕ commutes with the involutions ι_B and $\iota_{B'}$. Since Γ is irreducible by definition, the lift $\tilde{\Gamma}^+$ is also irreducible and hence we have $H^1(X_B, \mathcal{O}(\tilde{\Gamma}^+)) = 0$ by [21, Lemma 3.5]. Since B' is general in $\mathcal{F}$, we see that $\tilde{\Gamma}^+$ is deformed to an effective divisor $\tilde{\Gamma}'^+$ on $X_{B'}$ (see Lemmas 6.8 and 6.9) and that $\tilde{\Gamma}'^+$ is irreducible and mapped birationally to a curve Γ' on $\mathbb{P}^2$. Hence $\phi([\tilde{\Gamma}^+])$ is the class of a lift $\tilde{\Gamma}'^+$ of a splitting curve Γ' for B'. Since $\phi([\tilde{\Gamma}'^+]) \in \Lambda_{B'}$, Γ' is pre-Z-splitting. □

EXAMPLE 2.17. Let $f(x_0,x_1,x_2)$ and $g(x_0,x_1,x_2)$ be general homogeneous polynomials of degree 5 and 3, respectively. Then $B = \{x_0 f + g^2 = 0\}$ is smooth, and the triple tangent line $\Gamma = \{x_0 = 0\}$ is splitting for B but not pre-Z-splitting because a general sextic has no triple tangents.

EXAMPLE 2.18. Every irreducible component of B is pre-Z-splitting but not Z-splitting.

EXAMPLE 2.19. Suppose that B is a union of cubic curves E_0 and E_∞. Then the general member E_t of the pencil in $|\mathcal{O}_{\mathbb{P}^2}(3)|$ spanned by E_0 and E_∞ is pre-Z-splitting. The lifts $\tilde{E}_t^+$ and $\tilde{E}_t^-$ of E_t are, however, contained in the same elliptic pencil on X_B, so E_t is not Z-splitting.

If a pre-Z-splitting curve Γ is of degree ≤ 2 and not contained in B, then its lifts $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ are distinct (-2)-curves on X_B and hence Γ is Z-splitting.

EXAMPLE 2.20. Let $f(x_0,x_1,x_2)$ and $g(x_0,x_1,x_2)$ be general homogeneous polynomials of degree 2 and 3, respectively. Then the *torus sextic* $B_{\mathrm{trs}} := \{f^3 + g^2 = 0\}$ is a simple sextic, with $R_{B_{\mathrm{trs}}} = 6A_2$, and the conic $\Gamma = \{f = 0\}$ is

Z-splitting, as can be seen by the numerical criterion Proposition 8.2 (see Example 8.3). In fact, this torus sextic B_{trs} is the simple sextic B_2 in Examples 1.2 and 2.10, and the class $[\tilde{\Gamma}^+]$ generates the cyclic group $G_{B_2} = G_{B_{\mathrm{trs}}}$ of order 3.

DEFINITION 2.21. A simple sextic B is said to be *lattice-generic* if $\Lambda_B = \mathrm{NS}(X_B)$ holds or, equivalently, if the Picard number of X_B is equal to $\mu_B + 1$.

REMARK 2.22. It is easy to see that lattice-generic simple sextics are dense in any equisingular family (see Corollary 4.14). In particular, every lattice type contains a lattice-generic member.

COROLLARY 2.23. *A splitting curve Γ for a simple sextic B is pre-Z-splitting if and only if Γ is stable under general equisingular deformation of B.*

Proof. The "only if" part follows from Proposition 2.16. The "if" part follows from Remark 2.22. □

The requirement that B' be a general member of $\mathcal{F}$ in Proposition 2.16 is indispensable, as the next example shows.

EXAMPLE 2.24. Let f_1, f_2, and g be general homogeneous polynomials with $\deg f_1 = \deg f_2 = 1$ and $\deg g = 3$. We put $B_0 := \{f_1^3 f_2^3 + g^2 = 0\}$. Then we have $B_{\mathrm{trs}} \sim_{\mathrm{eqs}} B_0$. The Z-splitting conic $\Gamma = \{f = 0\}$ for B_{trs} degenerates into the union of two lines $\{f_1 = 0\}$ and $\{f_2 = 0\}$. Both of them are splitting but not pre-Z-splitting for B_0. Note that B_{trs} is lattice-generic whereas B_0 is not lattice-generic.

DEFINITION 2.25. We call a pair (B, Γ) of a simple sextic B and a Z-splitting curve Γ for B a *Z-splitting pair.* If B is lattice-generic, we say that (B, Γ) is *lattice-generic.*

DEFINITION 2.26. The *lattice data $\ell^P(B, \Gamma)$ of a Z-splitting pair* (B, Γ) is the extended lattice data

$$\ell^P(B, \Gamma) := [\mathcal{E}_B, h_B, \Lambda_B, \{[\tilde{\Gamma}^+], [\tilde{\Gamma}^-]\}].$$

We write $(B, \Gamma) \sim_{\mathrm{lat}} (B', \Gamma')$ if there exists an isomorphism of extended lattice data between $\ell^P(B, \Gamma)$ and $\ell^P(B', \Gamma')$. The equivalence class of $\sim_{\mathrm{lat}}$ is called a *lattice type,* and the lattice type containing a Z-splitting pair (B, Γ) is denoted by $\lambda^P(B, \Gamma)$.

By definition, an isomorphism of lattice data from $\ell^P(B, \Gamma)$ to $\ell^P(B', \Gamma')$ is an isomorphism of lattices $\Lambda_B \xrightarrow{\sim} \Lambda_{B'}$ that preserves the polarization class, the set of exceptional (-2)-curves, and maps the classes of the lifts $\tilde{\Gamma}^{\pm}$ of Γ to the classes of the lifts $\tilde{\Gamma}'^{\pm}$ of Γ'.

REMARK 2.27. By Proposition 2.16 and Remark 2.22, every lattice type λ^P of Z-splitting pairs contains a lattice-generic member.

3. Main Results

3.1. Classes of Lifts of Z-Splitting Curves

Let B be a simple sextic. For $n = 1, 2, 3$, we define

$$\mathcal{Z}_n(B)\{[\tilde{\Gamma}^+], [\tilde{\Gamma}^-] \mid \Gamma \text{ is a smooth } Z\text{-splitting curve of degree } n\} \subset \Lambda_B.$$

REMARK 3.1. In this definition, the condition that Γ should be smooth is of course redundant when $n < 3$. For $n = 3$, there may be a Z-splitting nodal cubic curve Γ such that $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ are (-2)-curves on X_B, but we do not consider such Z-splitting curves.

The main reason why we treat only smooth Z-splitting curves of degree ≤ 3 will be revealed in Theorem 3.21.

Our first main result is as follows.

THEOREM 3.2. *Let B and B' be lattice-generic simple sextics such that $B \sim_{\mathrm{lat}} B'$. If $\phi\colon \Lambda_B \xrightarrow{\sim} \Lambda_{B'}$ is an isomorphism of lattice data from $\ell(B)$ to $\ell(B')$, then ϕ induces a bijection between $\mathcal{Z}_n(B)$ and $\mathcal{Z}_n(B')$ for $n = 1, 2, 3$.*

More precisely, we will give in Section 5 an algorithm to calculate the sets $\mathcal{Z}_1(B)$, $\mathcal{Z}_2(B)$, and $\mathcal{Z}_3(B)$ for a lattice-generic simple sextic B from the lattice data $\ell(B)$.

When $n < 3$, each element of $\mathcal{Z}_n(B)$ is the class of a unique (-2)-curve, which is a lift of a Z-splitting curve of degree n. Hence the cardinality of $\mathcal{Z}_1(B)$ (resp. $\mathcal{Z}_2(B)$) is twice the number of Z-splitting lines (resp. Z-splitting conics). By Theorem 3.2, we can make the following definition.

DEFINITION 3.3. For a lattice type $\lambda = \lambda(B)$ of simple sextics, we define $z_1(\lambda)$ and $z_2(\lambda)$ to be the numbers of Z-splitting lines and of Z-splitting conics, respectively, for a lattice-generic member B of λ.

In Definition 3.3, the condition that B should be lattice-generic is indispensable.

EXAMPLE 3.4. The non–lattice-generic member B_0 of the lattice type $\lambda(B_{\mathrm{trs}}) = \lambda(B_0)$ in Example 2.24 has no Z-splitting conics, whereas $z_2(\lambda(B_{\mathrm{trs}})) = 1$.

The usefulness of the notion of Z-splitting curves in the study of lattice Zariski k-ples comes from the following theorem.

THEOREM 3.5. *Let λ and λ' be lattice types of simple sextics in the same configuration type. If $z_1(\lambda) = z_1(\lambda')$ and $z_2(\lambda) = z_2(\lambda')$, then $\lambda = \lambda'$. Namely, the lattice types in any lattice Zariski k-ple are distinguished by the numbers $z_1(\lambda)$ and $z_2(\lambda)$.*

The set $\mathcal{Z}_3(B)$ is in two-to-one correspondence with a set of 1-dimensional families of Z-splitting cubic curves.

Table 3.1 Numbers of Lattice Types with Z-Splitting Lines or Conics

μ_B	12	13	14	15	16	17	18	19	Total
Lines	0	0	0	1	2	7	13	18	41
Conics	1	2	7	18	47	86	108	55	324

PROPOSITION 3.6. *Let $\tilde{E}$ be an effective divisor on X_B. We have $[\tilde{E}] \in \mathcal{Z}_3(B)$ if and only if $|\tilde{E}|$ is an elliptic pencil on X_B whose general member is a lift of a Z-splitting cubic curve.*

Proof. Let $\tilde{E}$ be a lift of a smooth Z-splitting cubic curve E. Then $\tilde{E}$ is smooth of genus 1 and hence $|\tilde{E}|$ is an elliptic pencil. Conversely, if $|\tilde{E}|$ is an elliptic pencil on X_B whose general member $\tilde{E}$ is a lift of a Z-splitting cubic curve E, then E must be smooth because E is birational to $\tilde{E}$ and hence of genus 1. $\square$

3.2. *Classification of Z-Splitting Curves of Degree ≤ 2*

Next we give a classification of lattice types of Z-splitting pairs (B, Γ) with $\deg \Gamma \leq 2$. The numbers of lattice types λ of simple sextics with $z_1(\lambda) > 0$ or $z_2(\lambda) > 0$ are given in Table 3.1. If $\mu_B < 12$, then B has no Z-splitting curves of degree ≤ 2. (Note that there are lattice types λ for which both $z_1(\lambda) > 0$ and $z_2(\lambda) > 0$ hold; such lattice types are counted twice in Table 3.1.)

The entire classification table is too huge to be presented in a paper. In order to state our classification in a concise way, we introduce the notion of *specialization* of lattice types.

DEFINITION 3.7. Let λ_0 and λ be lattice types of simple sextics. We say that λ_0 is a *specialization* of λ if there exists an analytic family $f : \mathcal{B} \to \Delta$ of simple sextics $f^{-1}(t) = B_t$ parameterized by a unit disc $\Delta \subset \mathbb{C}$, where $\mathcal{B}$ is a surface in $\mathbb{P}^2 \times \Delta$ and f is a projection, such that the central fiber B_0 is a member of λ_0 and the other fibers B_t $(t \neq 0)$ are members of λ.

DEFINITION 3.8. Let λ_0^P and λ^P be lattice types of Z-splitting pairs. We say that λ_0^P is a specialization of λ^P if there exists an analytic family $f : \mathcal{P} \to \Delta$ of Z-splitting pairs $f^{-1}(t) = (B_t, \Gamma_t)$ such that the central fiber $f^{-1}(0)$ is a member of λ_0^P and the other fibers $f^{-1}(t)$ $(t \neq 0)$ are members of λ^P.

We give the list of lattice types of Z-splitting pairs that generate all other lattice types by specialization. It turns out that the lineages of lattice types via specialization are classified by the class order as defined next.

DEFINITION 3.9. The *class order* of a Z-splitting pair (B, Γ) (or of a lattice type $\lambda^P(B, \Gamma)$ of Z-splitting pairs) is the order of $[\tilde{\Gamma}^+]$ in $G_B = \Lambda_B / \Sigma_B$.

Table 3.2 Lattice Types $\lambda_{\alpha,\tau}$ in γ_α for $\alpha \in \{\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}\}$

α	R_B	degs	τ	z_1	z_2	G_B	F_B
$\mathfrak{A}$	$3A_5$	$[3,3]$	l	1	0	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
			n	0	0	$\mathbb{Z}/2\mathbb{Z}$	0
$\mathfrak{B}$	A_3+2A_7	$[2,4]$	l	1	0	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
			c	0	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
			n	0	0	$\mathbb{Z}/2\mathbb{Z}$	0
$\mathfrak{C}$	$2A_4+A_9$	$[1,5]$	l	1	0	$\mathbb{Z}/10\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$
			n	0	0	$\mathbb{Z}/2\mathbb{Z}$	0
$\mathfrak{D}$	$A_3+A_5+A_{11}$	$[2,4]$	l	1	1	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$

In what follows, the index τ in lattice types $\lambda_{\alpha,\tau}$ takes symbolic values n, l, or c, which stand (respectively) for "none", "line", and "conic".

The classification of Z-splitting lines is as follows.

Definition 3.10. For $\alpha \in \{\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}\}$, let γ_α be the configuration type given in the following table.

α	R_B	degs	
$\mathfrak{A}$	$3A_5$	$[3,3]$	(the cubics are smooth)
$\mathfrak{B}$	A_3+2A_7	$[2,4]$	(the quartic has A_3)
$\mathfrak{C}$	$2A_4+A_9$	$[1,5]$	(the quintic has $2A_4$)
$\mathfrak{D}$	$A_3+A_5+A_{11}$	$[2,4]$	(the quartic has A_5)

Proposition 3.11. *Let α be one of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$.*

(1) *The lattice types $\lambda_{\alpha,\tau}$ in the configuration type γ_α are given in Table 3.2. The invariants $z_1(\lambda_{\alpha,\tau})$, $z_2(\lambda_{\alpha,\tau})$, G_B, and F_B of these lattice types are also given in this table.*

(2) *Let B be a lattice-generic member of $\lambda_{\alpha,l}$ such that there exists a unique Z-splitting line Γ for B. Then Γ passes through the three singular points of B and the cyclic group G_B is generated by $[\tilde{\Gamma}^+]$.*

Definition 3.12. Let B and Γ be as in Proposition 3.11(2). We put

$$\lambda^P_{\mathrm{lin},d} := \lambda^P(B,\Gamma),$$

where "lin" denotes "line" and d is the order of G_B; that is, $d = 6, 8, 10, 12$ according as $\alpha = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$.

These lattice types $\lambda^P_{\mathrm{lin},d}$ are the originators of the lineages of lattice types of Z-splitting lines.

Theorem 3.13. *Let (B,Γ) be a Z-splitting pair with* $\deg\Gamma = 1$. *Then the class order d of $\lambda^P(B,\Gamma)$ is 6, 8, 10 or 12, and $\lambda^P(B,\Gamma)$ is a specialization of the lattice type $\lambda^P_{\mathrm{lin},d}$.*

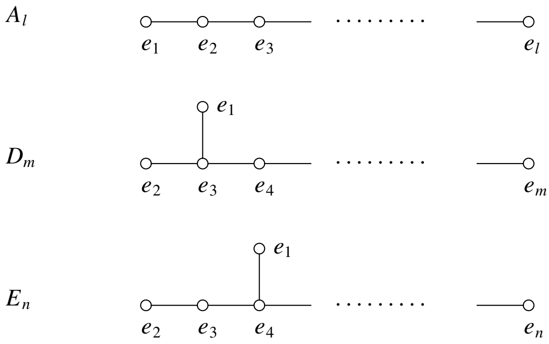

Figure 3.1 Dynkin diagram

The classification of Z-splitting conics is as follows.

DEFINITION 3.14. For $\alpha \in \{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}\}$, let γ_α be the configuration type given in the following table.

α	R_B	degs	
$\mathfrak{a}$	$6A_2$	[6]	
$\mathfrak{b}$	$2A_1 + 4A_3$	[2, 4]	(the quartic has $2A_1$)
$\mathfrak{c}$	$4A_4$	[6]	
$\mathfrak{d}$	$2A_1 + 2A_2 + 2A_5$	[2, 4]	(the quartic has $2A_2$)
$\mathfrak{e}$	$3A_6$	[6]	
$\mathfrak{f}$	$A_1 + A_3 + 2A_7$	[2, 4]	(the quartic has $A_1 + A_3$)

DEFINITION 3.15. Let P be a singular point of B, and let $e_1, \dots, e_r$ be the exceptional (-2)-curves on X_B over P indexed in such a way that the dual graph is given in Figure 3.1. Let $\tilde{\Gamma}^+$ be a lift of a smooth splitting curve Γ. Suppose that $P \in \Gamma$. Since Γ is smooth and splitting, there exists a unique e_j among $e_1, \dots, e_r$ that intersects $\tilde{\Gamma}^+$ (see Lemma 5.4). We put $\tau_P(\tilde{\Gamma}^+) := j$. If $P \notin \Gamma$, we put $\tau_P(\tilde{\Gamma}^+) := 0$ and $\tau_P(\tilde{\Gamma}^-) := 0$.

PROPOSITION 3.16. *Let α be one of* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}, \mathfrak{f}$.

(1) *The lattice types $\lambda_{\alpha,\tau}$ in the configuration type γ_α are given in Table 3.3, together with the invariants $z_1(\lambda_{\alpha,\tau})$, $z_2(\lambda_{\alpha,\tau})$, G_B, and F_B.*

(2) *Let B be a lattice-generic member of $\lambda_{\alpha,c}$. Then the Z-splitting conics Γ for B are given in Table 3.4, where* ord *is the class order of (B, Γ), and $\tau_P(\tilde{\Gamma}^+)$ is described under an appropriate choice of numbering of the exceptional (-2)-curves and the lift of Γ.*

DEFINITION 3.17. Let B be as in Proposition 3.16(2), and let Γ be a Z-splitting conic for B such that $[\tilde{\Gamma}^+]$ generates G_B. We put

$$\lambda^P_{\mathrm{con},d} := \lambda^P(B, \Gamma),$$

where "con" denotes "conic" and d is the order of G_B.

Table 3.3 Lattice Types $\lambda_{\alpha,\tau}$ in γ_α for $\alpha \in \{\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{f}\}$

α	R_B	degs	τ	z_1	z_2	G_B	F_B
$\mathfrak{a}$	$6A_2$	[6]	c	0	1	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
			n	0	0	0	0
$\mathfrak{b}$	$2A_1+4A_3$	$[2,4]$	c	0	1	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
			n	0	0	$\mathbb{Z}/2\mathbb{Z}$	0
$\mathfrak{c}$	$4A_4$	[6]	c	0	2	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$
			n	0	0	0	0
$\mathfrak{d}$	$2A_1+2A_2+2A_5$	$[2,4]$	c	0	2	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
			n	0	0	$\mathbb{Z}/2\mathbb{Z}$	0
$\mathfrak{e}$	$3A_6$	[6]	c	0	3	$\mathbb{Z}/7\mathbb{Z}$	$\mathbb{Z}/7\mathbb{Z}$
			n	0	0	0	0
$\mathfrak{f}$	$A_1+A_3+2A_7$	$[2,4]$	c	0	3	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
			l	1	0	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$
			n	0	0	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Table 3.4 Z-Splitting Conics of $\lambda_{\alpha,c}$ for $\alpha \in \{\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{f}\}$

α	Γ	ord	$\tau_P(\tilde{\Gamma}^+)$					
$\mathfrak{a}$			A_2	A_2	A_2	A_2	A_2	A_2
	Γ	3	1	1	1	1	1	1
$\mathfrak{b}$			A_1	A_1	A_3	A_3	A_3	A_3
	Γ	4	1	1	1	1	1	1
$\mathfrak{c}$			A_4	A_4	A_4	A_4		
	Γ_1	5	1	1	2	2		
	Γ_2	5	2	2	4	4		
$\mathfrak{d}$			A_1	A_1	A_2	A_2	A_5	A_5
	Γ_1	6	1	1	2	2	1	1
	Γ_2	3	0	0	1	1	2	2
$\mathfrak{e}$			A_6	A_6	A_6			
	Γ_1	7	1	2	3			
	Γ_2	7	2	4	6			
	Γ_3	7	3	6	2			
$\mathfrak{f}$			A_1	A_3	A_7	A_7		
	Γ_1	8	1	1	1	5		
	Γ_2	4	0	2	2	2		
	Γ_3	8	1	3	3	7		

REMARK 3.18. For $d = 5, 7, 8$, the lattice type $\lambda^P_{\mathrm{con},d} = \lambda^P(B, \Gamma)$ does not depend on the choice of Γ as long as $[\tilde{\Gamma}^+]$ generates G_B.

These lattice types $\lambda^P_{\mathrm{con},d}$ are the originators of the lineages of lattice types of Z-splitting conics.

THEOREM 3.19. *Let (B, Γ) be a Z-splitting pair with* $\deg \Gamma = 2$. *Then the class order d of $\lambda^P(B, \Gamma)$ is 3, 4, 5, 6, 7, or 8, and $\lambda^P(B, \Gamma)$ is a specialization of the lattice type $\lambda^P_{\mathrm{con},d}$.*

REMARK 3.20. The simple sextics in $\lambda^P_{\mathrm{con},3}$ are the classical torus sextics, which have been studied in detail by many authors (see e.g. [19]). The simple sextics in $\lambda^P_{\mathrm{con},5}$ are studied by Degtyarev in [10] and [12]. The simple sextics in $\lambda^P_{\mathrm{con},7}$ are studied by Degtyarev in [10] and by Degtyarev and Oka in [13].

3.3. Generators of F_B and Z-Splitting Cubic Curves

THEOREM 3.21. *Let B be a lattice-generic member of a lattice type $\lambda = \lambda(B)$.*

(1) *The finite abelian group $F_B = \Lambda_B/\Theta_B$ is generated by the classes of lifts of smooth Z-splitting curves of degree ≤ 3; that is, we have*

$$\Lambda_B = \Theta_B + \langle \mathcal{Z}_1(B) \rangle + \langle \mathcal{Z}_2(B) \rangle + \langle \mathcal{Z}_3(B) \rangle. \tag{3.1}$$

(2) *If $z_1(\lambda) > 0$ or $z_2(\lambda) > 0$, then F_B is nontrivial and is generated by the classes of lifts of Z-splitting curves of degree ≤ 2.*

The generators $\langle \mathcal{Z}_3(B) \rangle$ are indispensable in (3.1), as the following example $\lambda_{QC,n}$ shows.

PROPOSITION 3.22. *Let γ_{QC} be the configuration type of simple sextics $B = Q + C$ with* $\operatorname{degs} B = [2, 4]$, $R_B = 3A_1 + 4A_3$, *and the quartic curve Q having* $3A_1$.

(1) *The configuration type γ_{QC} contains exactly two lattice types, $\lambda_{QC,n}$ and $\lambda_{QC,c}$, which are distinguished as follows:*

$$z_1(\lambda_{QC,c}) = 0, \quad z_2(\lambda_{QC,c}) = 1, \qquad z_1(\lambda_{QC,n}) = 0, \quad z_2(\lambda_{QC,n}) = 0.$$

These lattice types have isomorphic G_B and F_B; for a member B of γ_{QC}, G_B is cyclic of order 4 and F_B is of order 2.

(2) *Let $B = Q + C$ be a lattice-generic member of $\lambda_{QC,c}$, and let Γ be the unique Z-splitting conic for B. Then G_B is generated by $[\tilde{\Gamma}^+]$.*

(3) *Let $B' = Q' + C'$ be a lattice-generic member of $\lambda_{QC,n}$ such that $\mathcal{Z}_1(B') = \mathcal{Z}_2(B') = \emptyset$. Then $\mathcal{Z}_3(B')$ consists of two elements $[\tilde{E}^+]$ and $[\tilde{E}^-]$, and $G_{B'}$ is generated by $[\tilde{E}^+]$. Let E be the image of a general member of the elliptic pencil $|\tilde{E}^+|$, which is a smooth Z-splitting cubic curve. Then E passes through every point of* $\operatorname{Sing} B'$ *and is tangent to each of Q' and C'.*

We need Z-splitting curves to generate $F_{B'} \neq 0$. We put

$$\lambda^P_{QC,n} := \lambda^P(B', E),$$

where (B', E) is the Z-splitting pair in Proposition 3.22(3). The lattice type $\lambda^P_{QC,n}$ is the ancestor of all lattice types for which we need Z-splitting cubic curves to generate F_B.

THEOREM 3.23. *Let λ_0 be a lattice type of simple sextics with a lattice-generic member B_0. Suppose that $z_1(\lambda_0) = 0$ and $z_2(\lambda_0) = 0$ but $F_{B_0} \neq 0$.*

(1) *The set $\mathcal{Z}_3(B_0)$ consists of two elements $[\tilde{E}_0^+]$ and $[\tilde{E}_0^-]$, and G_{B_0} is cyclic of order 4 and generated by $[\tilde{E}_0^+]$.*

(2) *Let E_0 be the image of a general member of the elliptic pencil $|\tilde{E}_0^+|$. Then the lattice type $\lambda^P(B_0, E_0)$ is a specialization of the lattice type $\lambda^P_{QC,n}$ defined previously.*

4. Classification of Lattice Types of Simple Sextics

4.1. Fundamental System of Roots

Let L be an even negative-definite lattice, and let D_L be the set of roots in L. We denote by ${}^0\mathrm{Hom}(L,\mathbb{R})$ the space of all linear forms $t\colon L \to \mathbb{R}$ such that $t(d) \neq 0$ holds for any $d \in D_L$. For $t \in {}^0\mathrm{Hom}(L,\mathbb{R})$, we put

$$(D_L)_t^+ := \{d \in D_L \mid t(d) > 0\}.$$

An element $d \in (D_L)_t^+$ is said to be *decomposable* if there exist $d_1, d_2 \in (D_L)_t^+$ such that $d = d_1 + d_2$; otherwise, we say that d is *indecomposable*. The proof of the following well-known fact is found, for example, in Ebeling [14, Prop. 1.4].

PROPOSITION 4.1. *The set F_t of indecomposable elements in $(D_L)_t^+$ is a fundamental system of roots in L. Conversely, if F is a fundamental system of roots in L, then there exists a linear form $t' \in {}^0\mathrm{Hom}(L,\mathbb{R})$ such that F is equal to the set $F_{t'}$ of indecomposable elements in $(D_L)_{t'}^+$.*

We call F_t the *fundamental system of roots associated with* $t\colon L \to \mathbb{R}$.

COROLLARY 4.2. *There exists a one-to-one correspondence between the set of fundamental systems of roots in L and the set of connected components of ${}^0\mathrm{Hom}(L,\mathbb{R})$.*

REMARK 4.3. A fundamental system of roots F in L is associated with $t \in {}^0\mathrm{Hom}(L,\mathbb{R})$ if and only if $(t,d) > 0$ holds for any $d \in F$.

4.2. The Kähler Cone and Polarizations of a $K3$ Surface

Let X be a $K3$ surface, and let ω_X be a basis of $H^{2,0}(X)$. We put

$$\begin{aligned}
H_X &:= \{x \in H^2(X,\mathbb{R}) \mid (x,\omega_X) = 0\},\\
D_X &:= \{d \in \mathrm{NS}(X) \mid d^2 = -2\},\\
\Gamma_X &:= \{x \in H_X \mid x^2 > 0\},\\
{}^0\Gamma_X &:= \{x \in \Gamma_X \mid (x,d) \neq 0 \text{ for all } d \in D_X\}.
\end{aligned}$$

We have $H_X = H^2(X,\mathbb{R}) \cap H^{1,1}(X)$ and $\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H_X$. We also have

$$\Gamma_X = \Gamma_X^+ \sqcup (-\Gamma_X^+) \quad \text{(disjoint)},$$

where Γ_X^+ is the connected component of Γ_X that contains a Kähler class of X.

DEFINITION 4.4. The *Kähler cone* $\mathcal{K}_X$ of X is the set of vectors $\kappa \in H_X$ satisfying $(D,\kappa) > 0$ for any effective divisor D on X.

Every Kähler class of X is contained in $\mathcal{K}_X$. Conversely, as a corollary of Theorem 6.2 we see that every vector in $\mathcal{K}_X$ is a Kähler class on X.

The following proposition is an immediate consequence of Definition 4.4.

PROPOSITION 4.5. *A vector $v \in \mathrm{NS}(X)$ is numerically effective (nef) if and only if v is contained in the closure of the Kähler cone $\mathcal{K}_X$ in H_X.*

We set

$$\Delta_X := \{d \in D_X \mid d \text{ is effective}\}.$$

By the Riemann–Roch theorem, we see that D_X is a disjoint union of Δ_X and $-\Delta_X$. For $d \in D_X$, we put

$$d^\perp := \{x \in H_X \mid (x,d) = 0\}$$

and call $d^\perp$ the *wall* associated with $d \in D_X$. The family of walls $\{d^\perp \mid d \in D_X\}$ is locally finite in the cone Γ_X and partitions Γ_X into the connected components of ${}^0\Gamma_X$. The following proposition is well known (see e.g. [5, Chap. VIII, Cor. 3.9]).

PROPOSITION 4.6. *The Kähler cone $\mathcal{K}_X \subset H_X$ is the unique connected component of $\Gamma_X^+ \cap {}^0\Gamma_X$ such that $(x,d) > 0$ holds for every $d \in \Delta_X$ and every $x \in \mathcal{K}_X$.*

A line bundle $\mathcal{L}$ on X is called a *polarization* if $\mathcal{L}$ is nef, $\mathcal{L}^2 > 0$, and the complete linear system $|\mathcal{L}|$ has no fixed components. If $\mathcal{L}$ is a polarization, then $|\mathcal{L}|$ has no base points by [21, Cor. 3.2], and hence defines a morphism

$$\Phi_{|\mathcal{L}|} : X \to \mathbb{P}^N,$$

where $N = \dim|\mathcal{L}|$.

PROPOSITION 4.7. *A vector $v \in \mathrm{NS}(X)$ is the class of a polarization if and only if $v^2 > 0$, v is nef, and the set $\{x \in \mathrm{NS}(X) \mid (v,x) = 1, x^2 = 0\}$ is empty.*

Proof. See Nikulin [17, Prop. 0.1] and the argument in the proof of (4) $\Rightarrow$ (1) in Urabe [30, Prop. 1.7]. □

Let $\mathcal{L}$ be a polarization on X. The orthogonal complement $[\mathcal{L}]^\perp$ of $\langle[\mathcal{L}]\rangle$ in $\mathrm{NS}(X)$ is negative definite by the Hodge index theorem. Hence we can easily prove the following (see [26, Prop. 2.4]).

PROPOSITION 4.8. *The set of classes of (-2)-curves that are contracted by $\Phi_{|\mathcal{L}|}$ is equal to the fundamental system of roots in $[\mathcal{L}]^\perp$ associated with the linear form $t_\kappa : [\mathcal{L}]^\perp \to \mathbb{R}$ given by $t_\kappa(v) := (v,\kappa)$, where κ is a vector in the Kähler cone $\mathcal{K}_X$.*

COROLLARY 4.9. *Let $U \subset H_X$ be a sufficiently small open ball with center $[\mathcal{L}]$. Then $U \cap \mathcal{K}_X$ is an open cone with the vertex $[\mathcal{L}]$ and with the faces being the walls $d^{\perp}$, where d are the (-2)-curves contracted by $\Phi_{|\mathcal{L}|}$.*

4.3. Lattice Types of Simple Sextics

We denote by $\mathbb{L}$ the $K3$ lattice—that is, an even unimodular lattice of signature $(3, 19)$—which is unique up to isomorphisms. We put

$$\Omega_{\mathbb{L}} := \{[\omega] \in \mathbb{P}_*(\mathbb{L} \otimes \mathbb{C}) \mid (\omega, \omega) = 0,\ (\omega, \bar{\omega}) > 0\},$$

which is a complex manifold of dimension 20 with two connected components. A *marked $K3$ surface* is a pair (X, ϕ) of a $K3$ surface X and an isomorphism $\phi\colon H^2(X, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L}$ of lattices. There exists a universal family $(\pi_1\colon \mathcal{X}_1 \to \mathcal{M}_1, \Phi_1)$ of marked $K3$ surfaces over a *non-Hausdorff* smooth complex manifold $\mathcal{M}_1$ of dimension 20, where Φ_1 is an isomorphism $R^2\pi_{1*}\mathbb{Z} \cong \mathcal{M}_1 \times \mathbb{L}$ of locally constant systems of lattices over $\mathcal{M}_1$. (See [5, Chap. VIII, Sec. 12] or [6].) For $t \in \mathcal{M}_1$, we have a point

$$\tau_1(t) := [\phi_t(\omega_{X_t})] \in \Omega_{\mathbb{L}},$$

where (X_t, ϕ_t) is the marked $K3$ surface corresponding to t and ω_{X_t} is a basis of $H^{2,0}(X_t)$. We call $\tau_1(t)$ the *period point of* (X_t, ϕ_t). It is well known that the *period map*

$$\tau_1\colon \mathcal{M}_1 \to \Omega_{\mathbb{L}}$$

is holomorphic and surjective [5, Chap. VIII, Sec. 12; 6].

Yang [32] presented an algorithm to classify all lattice data that can be realized as lattice data of simple sextics. His method is based on the following proposition, which was proved by the surjectivity of τ_1 and Propositions 4.7 and 4.8.

PROPOSITION 4.10 (Urabe [30; 31]). *Lattice data $[\mathcal{E}, h, \Lambda]$ is isomorphic to lattice data of simple sextics if and only if $[\mathcal{E}, h, \Lambda]$ satisfies the following*:

(i) *the lattice Λ can be embedded primitively in $\mathbb{L}$*;
(ii) $\{x \in \Lambda \mid (x, h) = 0, x^2 = -2\} = \{x \in \langle \mathcal{E} \rangle \mid x^2 = -2\}$; *and*
(iii) $\{x \in \Lambda \mid (x, h) = 1, x^2 = 0\} = \emptyset$.

COMPUTATION 4.11. Let R be an ADE-type of rank ≤ 19. We determine all lattice data of simple sextics B with $R_B = R$. We put $\Sigma := \langle h \rangle \oplus \langle \mathcal{E} \rangle$, where $h^2 = 2$ and $\mathcal{E}$ is the fundamental system of roots of type R. We then calculate the *discriminant form* of Σ. (See [16, Sec. 1] for the definition of the discriminant form of an even lattice.) We then make the complete list of isotropic subgroups H of the discriminant form of Σ.

For each isotropic subgroup H, we calculate the even overlattice $\Lambda(H)$ of Σ corresponding to H by [16, Prop. 1.4.1]. We then determine whether or not $\Lambda = \Lambda(H)$ satisfies conditions (ii) and (iii) in Proposition 4.10 by the method described in [25, Sec. 4] and next determine whether or not $\Lambda(H)$ can be embedded primitively into $\mathbb{L}$ by means of [16, Thm. 1.12.1] or by the method of p-excess due to

Conway and Sloane [9, Chap. 15], as described in [26, Sec. 3] (see also [8, Chaps. 8 and 9]).

We conclude that $[\mathcal{E}, h, \Lambda(H)]$ is realized as lattice data of simple sextics B with $R_B = R$ if and only if $\Lambda(H)$ satisfies the conditions in Proposition 4.10.

More precisely, the family of simple sextics B with $\ell(B) \cong [\mathcal{E}, h, \Lambda]$ is described as follows. Suppose that lattice data $[\mathcal{E}, h, \Lambda]$ satisfies conditions (i), (ii), and (iii) of Proposition 4.10. We choose a primitive embedding

$$\psi\colon \Lambda \hookrightarrow \mathbb{L}$$

and consider Λ as a primitive sublattice of $\mathbb{L}$. In particular, we have $\mathcal{E} \subset \mathbb{L}$ and $h \in \mathbb{L}$.

REMARK 4.12. The primitive embedding of Λ in $\mathbb{L}$ is not unique in general. In fact, by choosing different primitive embeddings of Λ in $\mathbb{L}$, we often obtain distinct connected components of the equisingular family (see Degtyarev [11]). More strongly, we have obtained examples of pairs of simple sextics B_1 and B_2 such that $B_1 \sim_{\mathrm{lat}} B_2$ but $B_1 \not\sim_{\mathrm{emb}} B_2$ by considering different primitive embeddings of Λ (see [1; 27; 28]). See also Section 8.2.

For $[\omega] \in \Omega_{\mathbb{L}}$, we put

$$\mathrm{NS}^{[\omega]} := \{x \in \mathbb{L} \mid (x, \omega) = 0\},$$

which is a primitive sublattice of $\mathbb{L}$. We then put

$$\Omega_{\psi^\perp} := \{[\omega] \in \Omega_{\mathbb{L}} \mid (\omega, x) = 0 \text{ for all } x \in \Lambda\} \subset \Omega_{\mathbb{L}}$$

and denote by $\Omega^{\diamond}_{\psi^\perp}$ the set of all $[\omega] \in \Omega_{\psi^\perp}$ such that $\mathrm{NS}^{[\omega]}$ satisfies the following conditions, which correspond to properties (ii) and (iii) for Λ in Proposition 4.10:

$$\{x \in \mathrm{NS}^{[\omega]} \mid (x, h) = 0,\ x^2 = -2\} = \{x \in \langle \mathcal{E} \rangle \mid x^2 = -2\}; \tag{4.1}$$

$$\{x \in \mathrm{NS}^{[\omega]} \mid (x, h) = 1,\ x^2 = 0\} = \emptyset. \tag{4.2}$$

Note that the complement of $\Omega^{\diamond}_{\psi^\perp}$ in $\Omega_{\psi^\perp}$ is a locally finite family of complex analytic subspaces. From the surjectivity of τ_1 and Propositions 4.7 and 4.8, we easily obtain the following result.

PROPOSITION 4.13. *For any point $p \in \Omega^{\diamond}_{\psi^\perp}$, there exists a simple sextic B with a marking $\phi\colon H^2(X_B, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L}$ such that $\phi(h_B) = h$, $\phi(\mathcal{E}_B) = \mathcal{E}$, $\phi(\Lambda_B) = \Lambda$, and the period point of (X_B, ϕ) is p.*

Conversely, if B is a simple sextic with a marking $\phi\colon H^2(X_B, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L}$ and if $\psi'\colon \Lambda_B \xrightarrow{\sim} \Lambda$ is an isomorphism of lattice data from $\ell(B)$ to the lattice data $[\mathcal{E}, h, \Lambda]$, then the period point of (X_B, ϕ) is contained in $\Omega^{\diamond}_{\psi^\perp}$, where $\psi\colon \Lambda \hookrightarrow \mathbb{L}$ is the primitive embedding obtained from $\phi|_{\Lambda_B}\colon \Lambda_B \hookrightarrow \mathbb{L}$ via ψ'.

We then put

$$\Omega^{\diamond\diamond}_{\psi^\perp} := \{[\omega] \in \Omega^{\diamond}_{\psi^\perp} \mid \mathrm{NS}^{[\omega]} = \Lambda\}.$$

If $p \in \Omega^{\diamond\diamond}_{\psi^\perp}$, then the corresponding simple sextic B is lattice-generic. It is obvious that $\Omega^{\diamond\diamond}_{\psi^\perp}$ is dense in $\Omega^{\diamond}_{\psi^\perp}$. Hence we obtain the following corollary.

COROLLARY 4.14. *Given a simple sextic B, we can obtain a lattice-generic simple sextic B' by an arbitrarily small equisingular deformation of B.*

5. Algorithms for a Lattice Type

Let B be a simple sextic. *Throughout this section, we assume that B is lattice-generic* (except in Corollary 5.26). In particular, every splitting curve is pre-Z-splitting. We present an algorithm to determine the configuration type and the sets $\mathcal{Z}_1(B)$, $\mathcal{Z}_2(B)$, and $\mathcal{Z}_3(B)$ from the lattice data $\ell(B) = [\mathcal{E}_B, h_B, \Lambda_B]$ of B.

Recall that, for a splitting curve Γ, we denote by $\tilde{\Gamma}^+, \tilde{\Gamma}^- \subset X_B$ the lifts of Γ. For an irreducible component B_i of B, we denote by $\tilde{B}_i \subset X_B$ the reduced part of the strict transform of B_i; that is, we put $\tilde{B}_i := \tilde{B}_i^+ = \tilde{B}_i^-$.

We denote by $j_B \colon W_B \to \mathbb{P}^2$ the *Jung–Horikawa embedded resolution* (canonical embedded resolution) of $B \subset \mathbb{P}^2$, which is the minimal succession of blow-ups such that the strict transform of B is smooth and such that any distinct irreducible components of the total transform of B with *odd* multiplicities do not intersect (see [5, Chap. III, Sec. 7]). Then we have the *finite* double covering $\tilde{\pi}_B \colon X_B \to W_B$ that makes the following diagram commutative:

$$\begin{array}{ccc} X_B & \xrightarrow{\rho_B} & Y_B \\ {\scriptstyle\tilde{\pi}_B}\big\downarrow & & \big\downarrow{\scriptstyle\pi_B} \\ W_B & \xrightarrow{j_B} & \mathbb{P}^2. \end{array}$$

For $P \in \operatorname{Sing} B$, let $\mathcal{E}_P = \{e_1, \dots, e_r\}$ be the set of exceptional (-2)-curves on X_B over P, which are indexed as in Figure 3.1. For simplicity, we use the same letter for an exceptional (-2)-curve and its class, and consider $\mathcal{E}_P$ as a subset of Σ_B. Then $e_1, \dots, e_r$ form the fundamental system of roots in the sublattice $\langle \mathcal{E}_P \rangle$ of Σ_B associated with a Kähler class of X_B. We denote by $e_1^\vee, \dots, e_r^\vee$ the dual basis of the dual lattice $\langle \mathcal{E}_P \rangle^\vee \subset \langle \mathcal{E}_P \rangle \otimes \mathbb{Q}$. We have an orthogonal direct-sum decomposition

$$\Sigma_B = \langle h_B \rangle \oplus \bigoplus_{P \in \operatorname{Sing} B} \langle \mathcal{E}_P \rangle.$$

Recall that Λ_B is the primitive closure of Σ_B in $H^2(X_B, \mathbb{Z})$. We consider the decomposition

$$\Lambda_B \otimes \mathbb{Q} = \Sigma_B \otimes \mathbb{Q} = \langle h_B \rangle \otimes \mathbb{Q} \oplus \bigoplus \langle \mathcal{E}_P \rangle \otimes \mathbb{Q}. \tag{5.1}$$

For $x \in \Lambda_B$, we denote by $x_h \in \langle h_B \rangle \otimes \mathbb{Q}$ and $x_P \in \langle \mathcal{E}_P \rangle \otimes \mathbb{Q}$ the components of x under the direct-sum decomposition (5.1). The following is obvious.

LEMMA 5.1. *Let D be an effective divisor on X_B such that $(D, h_B) = 0$. Then we have $[D] \in \langle \mathcal{E}_B \rangle^+$. In particular, we have $[D] \in \Sigma_B$ and $[D]_P \in \langle \mathcal{E}_P \rangle^+$ for any $P \in \operatorname{Sing} B$.*

DEFINITION 5.2. We say that a vector $x \in \Lambda_B$ is *v-smooth at* $P \in \operatorname{Sing} B$ if $x_P = 0$ or $x_P = e_i^\vee$ for some i. We say that x is *v-smooth* if x is v-smooth at every $P \in \operatorname{Sing} B$. (The "$v$" in *v-smooth* stands for "vector".)

DEFINITION 5.3. Let $m_P(e_i^\vee)$ denote the multiplicity of the curve $\tilde{\pi}_B(e_i) \subset W_B$ in the total transform of B in W_B. We also put $m_P(0) := 0$. Thus we have $m_P(x_P)$ for a vector $x \in \Lambda_B$ that is v-smooth at P.

LEMMA 5.4. *Let $\tilde{\Gamma}$ be a lift of a splitting curve Γ, and let P be a point of* $\operatorname{Sing} B$. *Suppose that $P \notin \Gamma$ or that Γ is smooth at P. Then the vector $[\tilde{\Gamma}] \in \Lambda_B$ is v-smooth at P and $m_P([\tilde{\Gamma}]_P)$ is even.*

This lemma is proved together with the following one.

LEMMA 5.5. *Let $\Gamma \subset \mathbb{P}^2$ be a smooth splitting curve not contained in B. Let $\Gamma^W \subset W_B$ and $B^W \subset W_B$ be the strict transforms of Γ and B, respectively, by $j_B \colon W_B \to \mathbb{P}^2$, and let $\tilde{B} \subset X_B$ be the strict transform of B by $\tilde{\rho}_B \colon X_B \to \mathbb{P}^2$. Then we have*

$$(\tilde{\Gamma}^+, \tilde{\Gamma}^-)_X = (\tilde{\Gamma}^+, \tilde{B})_X = (\tilde{\Gamma}^-, \tilde{B})_X = (\Gamma^W, B^W)_W/2,$$

where $(\cdot,\cdot)_X$ and $(\cdot,\cdot)_W$ denote the intersection numbers on X_B and on W_B, respectively.

Proof of Lemmas 5.4 and 5.5. The statement of Lemma 5.4 is obviously true in the case where $P \notin \Gamma$. The proof of Lemma 5.4 for the case where Γ is an irreducible component of B is given in Remark 5.7.

Suppose that Γ is splitting, is not contained in B, and passes through P. Let $F_1, \dots, F_m \subset W_B$ be the exceptional curves over P of j_B, and let m_k be the multiplicity of F_k in the total transform of B by j_B. We denote by $T \subset W_B$ a sufficiently small tubular neighborhood of $j_B^{-1}(P)$ and put $\tilde{T} := \tilde{\pi}_B^{-1}(T) \subset X_B$. If $\left(\sum F_j, \Gamma^W\right)_W > 1$, then the image Γ of Γ^W by j_B would be singular at P. Hence there exists a unique irreducible component F_i such that $(F_i, \Gamma^W) = 1$ and $(F_j, \Gamma^W) = 0$ for $j \neq i$. Let Q be the intersection point of F_i and Γ^W. Note that Γ^W is smooth at Q and intersects F_i transversely at Q. Suppose that $Q \notin B^W$, so that Γ^W is disjoint from B^W in T. Then, since Γ is splitting, the multiplicity m_i is even and $\tilde{\pi}_B^{-1}(Q)$ consists of two distinct points. Hence $\tilde{\Gamma}^+$, $\tilde{\Gamma}^-$, and $\tilde{B}$ are mutually disjoint in $\tilde{T}$, and Lemma 5.4 holds by $m_P([\tilde{\Gamma}]_P) = m_i$. Suppose that $Q \in B^W$, and let n_Q be the intersection multiplicity of B^W and Γ^W at Q. Since Γ is splitting, $m_i + n_Q$ must be even. Since $B^W \cap F_i \neq \emptyset$, it follows that m_i is even. Therefore $n_Q > 1$ and hence B^W intersects F_i transversely at Q; in other words, P is not of type A_l with l even. Thus the pull-back of F_i by $\tilde{\pi}_B$ is irreducible, and Lemma 5.4 holds by $m_P([\tilde{\Gamma}]_P) = m_i$. In this case, the intersection multiplicity of $\tilde{\Gamma}^+$ and $\tilde{\Gamma}^-$ (or of $\tilde{\Gamma}^+$ and $\tilde{B}$, or of $\tilde{\Gamma}^-$ and $\tilde{B}$) at the point of X_B over Q is equal to $n_Q/2$. □

REMARK 5.6. If P is of type A_l, then the multiplicity $m_P(e_i^\vee)$ is even for any i. If P is of other type, then $m_P(e_i^\vee)$ is even if and only if e_i is subject to the following restrictions.

- If P is of type D_{2k}, then i is even or 1 or 2.
- If P is of type D_{2k+1}, then i is odd or 1 or 2.
- If P is of type E_6, then $i \neq 1$.
- If P is of type E_7, then $i \neq 2, 4, 6$.
- If P is of type E_8, then $i \neq 2, 4, 6, 8$.

REMARK 5.7. Let B_i be an irreducible component of B that contains $P \in \operatorname{Sing} B$ and is smooth at P. Then the component $[\tilde{B}_i]_P \in \langle \mathcal{E}_P \rangle \otimes \mathbb{Q}$ is given as follows.

- If P is of type A_{2k-1}, then $[\tilde{B}_i]_P = e_k^\vee$.
- If P is of type D_{2k}, then $[\tilde{B}_i]_P = e_1^\vee$ or $[\tilde{B}_i]_P = e_2^\vee$ or $[\tilde{B}_i]_P = e_{2k}^\vee$.
- If P is of type D_{2k+1}, then $[\tilde{B}_i]_P = e_{2k+1}^\vee$.
- If P is of type E_7, then $[\tilde{B}_i]_P = e_7^\vee$.

If P is of another type, then every local irreducible components of B at P is singular.

By Remark 5.7, we obtain the following lemma.

LEMMA 5.8. *Let B_i be an irreducible component of B that contains $P \in \operatorname{Sing} B$ and is smooth at P. Then $[\tilde{B}_i]_P \in \langle \mathcal{E}_P \rangle \otimes \mathbb{Q}$ is not contained in $\langle \mathcal{E}_P \rangle$.*

The next lemma is elementary, but it plays a crucial role in the sequel.

LEMMA 5.9. (1) *For every $e_i^\vee$, we have $(e_i^\vee)^2 < 0$ and $e_i^\vee \notin \langle \mathcal{E}_P \rangle^+$.*

(2) *Suppose that $e_i^\vee - e_j^\vee \in \langle \mathcal{E}_P \rangle^+$. Then $(e_i^\vee)^2 > (e_j^\vee)^2$ or $e_i^\vee = e_j^\vee$.*

(3) *If $e_i^\vee$ is contained in $\langle \mathcal{E}_P \rangle$ and if $m_P(e_i^\vee)$ is even, then $(\iota_B(e_i^\vee), e_i^\vee) < -9/2$ holds.*

Proof. We have to prove this lemma only for the negative-definite root lattices of type A_l ($l = 1, \ldots, 19$), D_m ($m = 4, \ldots, 19$), and E_n ($n = 6, 7, 8$). Hence the assertions can be proved by case-by-case calculations. For the proof, we use Remark 5.6. The involution ι_B is calculated by Remark 5.10. (The author does not know any conceptual proof of this lemma.) □

REMARK 5.10. The involution ι_B on Λ_B is determined by the ADE-type of R_B. We have $\iota_B(h_B) = h_B$. The action of ι_B on $\mathcal{E}_P$ is described as follows.

- If P is of type A_l, then $\iota_B(e_i) = e_{l+1-i}$.
- If P is of type D_{2k}, then ι_B acts on $\mathcal{E}_P$ identically.
- If P is of type D_{2k+1}, then ι_B interchanges e_1 and e_2 and fixes $e_3, \ldots, e_{2k+1}$.
- If P is of type E_6, then $\iota_B(e_1) = e_1$ and $\iota_B(e_i) = e_{8-i}$ for $i = 2, \ldots, 6$.
- If P is of type E_7 or E_8, then ι_B acts on $\mathcal{E}_P$ identically.

COROLLARY 5.11. *Let $x \in \Lambda_B$ and $y \in \Lambda_B$ be v-smooth vectors. If $(x, h_B) = (y, h_B)$ and $x^2 = y^2$ hold and if $x - y$ is effective, then $x = y$.*

Proof. Since $x - y$ is effective and $(x - y, h_B) = 0$, we have $x_P - y_P \in \langle \mathcal{E}_P \rangle^+$ for every $P \in \operatorname{Sing} B$ by Lemma 5.1. Suppose that $x \neq y$, and let $P \in \operatorname{Sing} B$ be a point such that $x_P \neq y_P$. Since x and y are v-smooth, each of x_P and y_P is

0 or $e_i^\vee$ for some i. If $y_P = 0$, then $x_P \neq 0$ and $x_P \in \langle \mathcal{E}_P \rangle^+$, which contradicts Lemma 5.9(1). If $y_P \neq 0$ then we have $x_P^2 > y_P^2$ by Lemma 5.9(1) and (2), which contradicts $x^2 = y^2$. □

PROPOSITION 5.12. *Let $x \in \Lambda_B$ be a v-smooth vector with $(x, h_B) = 1$ and $x^2 = -2$. Then x is the class of a (-2)-curve that is mapped isomorphically to a line on $\mathbb{P}^2$.*

Proof. By the Riemann–Roch theorem for X_B, we have an effective divisor D on X_B such that $x = [D]$. Since $(x, h_B) = 1$, there exists a unique irreducible component C of D such that $(C, h_B) = 1$. Note that C is mapped isomorphically to a line on $\mathbb{P}^2$ and hence the image of C is a splitting line. Therefore, $[C]^2 = -2$ and $[C]$ is v-smooth by Lemma 5.4. By Corollary 5.11, we have $x = [C]$. □

We put

$$\begin{aligned} \mathcal{L}_B &:= \{x \in \Lambda_B \mid x \text{ is } v\text{-smooth}, \ (x, h_B) = 1, \ x^2 = -2\}, \\ \mathcal{L}_B^b &:= \{x \in \mathcal{L}_B \mid \iota_B(x) = x\}, \quad \text{and} \\ \mathcal{L}_B^l &:= \{x \in \mathcal{L}_B \mid \iota_B(x) \neq x\}. \end{aligned}$$

COROLLARY 5.13. *The map $B_i \mapsto [\tilde{B}_i]$ induces a bijection from the set of irreducible components B_i of B of degree 1 to the set $\mathcal{L}_B^b$.*

COROLLARY 5.14. *The set $\mathcal{L}_B^l$ is equal to the set $\mathcal{Z}_1(B)$ of the classes of lifts of Z-splitting lines.*

Next we proceed to the study of Z-splitting conics.

PROPOSITION 5.15. *Let $\tilde{C} \subset X_B$ be a curve that is mapped isomorphically to a smooth conic C on $\mathbb{P}^2$. Then $[\tilde{C}] \notin \Sigma_B$.*

Proof. We put $x := [\tilde{C}]$. Suppose that C is an irreducible component of B. Then $x_P \neq 0$ for some $P \in \operatorname{Sing} B$ and hence $x \notin \Sigma_B$ by Lemma 5.8. Suppose that C is not contained in B. Then $(\iota_B(x), x) \geq 0$. Since C is smooth, x_P is v-smooth with $m_P(x_P)$ being even for every $P \in \operatorname{Sing} B$. Since $(x, h_B) = 2$, we have $(\iota_B(x_h), x_h) = x_h^2 = 2$ and hence

$$(\iota_B(x), x) = 2 + \sum_P (\iota_B(x_P), x_P) \geq 0. \tag{5.2}$$

Suppose that $x \in \Sigma_B$ and hence $x_P \in \langle \mathcal{E}_P \rangle$ for any $P \in \operatorname{Sing} B$. For any $P \in C \cap \operatorname{Sing} B$, we have $x_P \neq 0$ and hence $(\iota_B(x_P), x_P) < -9/2$ by Lemma 5.9(3). By (5.2), we therefore have $C \cap \operatorname{Sing} B = \emptyset$ and hence $(\iota_B(x), x) = 2$. However, we have $(\iota_B(x), x) = 6$ because $(B, C) = 12$ on $\mathbb{P}^2$. Thus we get a contradiction. □

PROPOSITION 5.16. *Let $x \in \Lambda_B$ be a v-smooth vector such that $(x, h_B) = 2, x^2 = -2$, and $x \notin \Sigma_B$. Then one and only one of the following statements holds:*

(i) *there exist* $l_1, l_2 \in \mathcal{L}_B$ *such that* $x - (l_1 + l_2) \in \langle \mathcal{E}_B \rangle^+$; *or*
(ii) x *is the class of a* (-2)*-curve* $\tilde{C}$ *that is a lift of a splitting conic* C *on* $\mathbb{P}^2$.

Proof. Note that x is the class of an effective divisor of X_B. We denote by $|D|$ the complete linear system of effective divisors D such that $x = [D]$. The irreducible decomposition of each $D \in |D|$ is either

$$D = \tilde{C}_1 + \tilde{C}_2 + \sum e_i \quad \text{with} \;\; (\tilde{C}_1, h_B) = (\tilde{C}_2, h_B) = 1 \;\; \text{and} \;\; e_i \in \mathcal{E}_B \tag{5.3}$$

or

$$D = \tilde{C} + \sum e_i \quad \text{with} \;\; (\tilde{C}, h_B) = 2 \;\; \text{and} \;\; e_i \in \mathcal{E}_B. \tag{5.4}$$

Suppose that there exists a $D \in |D|$ for which (5.3) holds. Since B is assumed to be lattice-generic, we have $[\tilde{C}_1], [\tilde{C}_2] \in \Lambda_B$. Since $\tilde{C}_1$ and $\tilde{C}_2$ are mapped isomorphically to lines on $\mathbb{P}^2$, the vectors $[\tilde{C}_1]$ and $[\tilde{C}_2]$ are v-smooth with the square norm -2. Therefore $[\tilde{C}_1]$ and $[\tilde{C}_2]$ are in $\mathcal{L}_B$ and thus case (i) occurs.

Suppose that there exists a $D \in |D|$ for which (5.4) holds. The image of $\tilde{C}$ in $\mathbb{P}^2$ is either a line or a smooth conic. If the image were a line, then $\tilde{C}$ would be a strict transform of the line and hence $[\tilde{C}]$ would be contained in Σ_B, which contradicts the assumption. Therefore $\tilde{C}$ is a lift of a splitting conic C. In particular, $[\tilde{C}] \in \Lambda_B$ is a v-smooth vector with $[\tilde{C}]^2 = -2$. By Corollary 5.11 we have $x = [\tilde{C}]$, and therefore case (ii) occurs.

Suppose that both cases (i) and (ii) occur. Then there exists a $D_1 \in |D|$ for which (5.3) holds and there exists a $D_2 \in |D|$ for which (5.4) holds. By the preceding argument, the existence of D_2 implies that x is the class of a lift $\tilde{C}$ of a splitting conic C and, in particular, that $|D|$ consists of a single member $\tilde{C}$, which contradicts the existence of D_1. Hence only one of (i) or (ii) occurs. □

We put:

$$\mathcal{C}'_B := \{x \in \Lambda_B \mid x \text{ is } v\text{-smooth}, \, (x, h_B) = 2, \, x^2 = -2, \, x \notin \Sigma_B\},$$

$$\mathcal{C}_B := \{x \in \mathcal{C}'_B \mid \text{for any } l_1, l_2 \in \mathcal{L}_B, \text{ we have } x - (l_1 + l_2) \notin \langle \mathcal{E}_B \rangle^+\};$$

$$\mathcal{C}^b_B := \{x \in \mathcal{C}_B \mid \iota_B(x) = x\},$$

$$\mathcal{C}^l_B := \{x \in \mathcal{C}_B \mid \iota_B(x) \neq x\}.$$

COROLLARY 5.17. *The map* $B_i \mapsto [\tilde{B}_i]$ *induces a bijection from the set of irreducible components* B_i *of* B *of degree 2 to the set* $\mathcal{C}^b_B$.

COROLLARY 5.18. *The set* $\mathcal{C}^l_B$ *is equal to the set* $\mathcal{Z}_2(B)$ *of the classes of lifts of* Z*-splitting conics.*

Next we study Z-splitting cubic curves. We put

$$\mathcal{G}_B := \{g \in \Lambda_B \mid g^2 = 0, \, (g, h_B) = 3, \text{ and } (g, v) \geq 0 \text{ for any } v \in \mathcal{E}_B \cup \mathcal{L}_B\},$$

$$\mathcal{G}^b_B := \{g \in \mathcal{G}_B \mid \iota_B(g) = g\},$$

$$\mathcal{G}^l_B := \{g \in \mathcal{G}_B \mid \iota_B(g) \neq g\}.$$

LEMMA 5.19. *Every $g \in \mathcal{G}_B$ is the class $[\tilde{E}]$ of a member of an elliptic pencil $|\tilde{E}|$ on X_B.*

Proof. We have an effective divisor D such that $g = [D]$ and $\dim|D| > 0$. We decompose $|D|$ into the movable part $|M|$ and the fixed part Ξ. Since $\dim|M| > 0$, we have $(M, h_B) \geq 2$ and hence $(\Xi, h_B) \leq 1$. Therefore every irreducible component C of Ξ either is an element of $\mathcal{E}_B$ or is mapped isomorphically to a line of $\mathbb{P}^2$. In the latter case, we have $[C] \in \mathcal{L}_B$. Hence $(C, g) \geq 0$ holds for any irreducible component C of Ξ by the definition of $\mathcal{G}_B$. Therefore g is nef. Then, by Nikulin [17, Prop. 0.1], we have $\Xi = \emptyset$ and there exists an elliptic pencil $|\tilde{E}|$ on X_B such that $|D| = m|\tilde{E}|$ for some integer $m > 0$. From $(g, h_B) = 3$, we obviously have $m = 1$. □

By Proposition 3.6, we see that every $g \in \mathcal{Z}_3(B)$ is nef and hence satisfies $(g, v) \geq 0$ for any $v \in \mathcal{E}_B \cup \mathcal{L}_B$. Combining Proposition 3.6 and Lemma 5.19, we obtain the following statement.

COROLLARY 5.20. *We have $\mathcal{G}_B^l = \mathcal{Z}_3(B)$.*

PROPOSITION 5.21. *Suppose that B does not have any irreducible components of degree ≤ 2. Then B is irreducible if and only if $\mathcal{G}_B^b = \emptyset$.*

Proof. Suppose that B is reducible. Then B is a union of two irreducible cubic curves E_0 and E_∞. Note that, for each $P \in E_0 \cap E_\infty$, either E_0 or E_∞ is smooth at P. Let $\mathcal{P} \subset |\mathcal{O}_{\mathbb{P}^2}(3)|$ be the pencil spanned by E_0 and E_∞. Examining the Jung–Horikawa resolution $j_B \colon W_B \to \mathbb{P}^2$ explicitly, we see that j_B resolves the base points of $\mathcal{P}$; hence we obtain an elliptic fibration

$$\phi_{\mathcal{P}} \colon W_B \to \mathbb{P}^1$$

on W_B such that, by $j_B \colon W_B \to \mathbb{P}^2$, the general fiber of $\phi_{\mathcal{P}}$ is mapped to a member of $\mathcal{P}$ and $\phi_{\mathcal{P}}^{-1}(0)$ and $\phi_{\mathcal{P}}^{-1}(\infty)$ are mapped to E_0 and E_∞, respectively. Moreover, the branching locus of $\tilde{\pi}_B \colon X_B \to W_B$ is contained in $\phi_{\mathcal{P}}^{-1}(0) \cup \phi_{\mathcal{P}}^{-1}(\infty)$. Indeed, suppose that E_0 is smooth at $P \in E_0 \cap E_\infty$, and let $F_1, \ldots, F_m$ be the exceptional curves of j_B over P. There exists a unique F_i among them that intersects the strict transform of E_0. This component F_i becomes a section of $\phi_{\mathcal{P}}$, and the other components are mapped to ∞ by $\phi_{\mathcal{P}}$. The multiplicity of F_i in the total transform of B is even, so $\tilde{\pi}_B$ does not ramify along the section F_i.

Thus we have an elliptic fibration $\psi_{\mathcal{P}} \colon X_B \to \mathbb{P}^1$ that fits in a commutative diagram

$$\begin{array}{ccc} X_B & \xrightarrow{\tilde{\pi}_B} & W_B \\ {\scriptstyle \psi_{\mathcal{P}}}\downarrow & & \downarrow{\scriptstyle \phi_{\mathcal{P}}} \\ \mathbb{P}^1 & \xrightarrow{\bar{\pi}_B} & \mathbb{P}^1, \end{array}$$

where $\bar{\pi}_B \colon \mathbb{P}^1 \to \mathbb{P}^1$ is the double covering branching at $0 \in \mathbb{P}^1$ and $\infty \in \mathbb{P}^1$. Let $\tilde{E} \subset X_B$ be the general fiber of the elliptic fibration $\psi_{\mathcal{P}} \colon X_B \to \mathbb{P}^1$. Since $\tilde{E}$ is nef, we see that $g := [\tilde{E}] \in \Lambda_B$ is an element of $\mathcal{G}_B^b$.

Conversely, suppose that $g \in \mathcal{G}^b_B$. By Lemma 5.19 we have an elliptic fibration $\psi\colon X_B \to \mathbb{P}^1$ such that the class of its general fiber $\tilde{E}$ is g. Since $\iota_B(g) = g$, the involution ι_B preserves this elliptic fibration. Therefore $\psi\colon X_B \to \mathbb{P}^1$ is obtained from an elliptic fibration $\phi\colon W_B \to \mathbb{P}^1$ on $W_B = X_B/\langle\iota_B\rangle$ by the base change $\bar{\pi}\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree 2. Since the branch points of $\bar{\pi}$ consist of two points, the branch curve of $\tilde{\pi}_B\colon W_B \to X_B$ is contained in the union of two fibers of $\phi\colon W_B \to \mathbb{P}^1$, each of which is mapped to a cubic irreducible component of B. □

REMARK 5.22. Suppose that $g \in \mathcal{G}^b_B$. Note that a point $P \in \operatorname{Sing} B$ of type A_1 is an intersection point of the irreducible components E_0 and E_∞ of B if and only if $g_P \neq 0$. Therefore we can recover the configuration type of B from g.

REMARK 5.23. There are additional necessary conditions for $\operatorname{degs} B$ to be $[3,3]$, which are helpful in calculation. If $\operatorname{degs} B = [3,3]$, then R_B consists of the following ADE-types: A_2, A_{2k-1}, D_5, D_{2k}, and E_7. Moreover, for a point $P \in \operatorname{Sing} B$ of type t_P, the component g_P of the vector $g \in \mathcal{G}^b_B$ should be the one indicated in the following table.

t_P	A_2	A_1	A_{2k-1} $(k>1)$	D_5	D_4	D_{2k} $(k>2)$	E_7
g_P	0	0 or $e_1^\vee$	$e_k^\vee$	$e_5^\vee$	$e_1^\vee, e_2^\vee$, or $e_4^\vee$	$e_1^\vee$ or $e_2^\vee$	$e_7^\vee$

We now interpret these geometric results as lattice-theoretic results.

DEFINITION 5.24. A fundamental system of roots is called *irreducible* if the corresponding Dynkin diagram is connected.

Let $\ell = [\mathcal{E}, h, \Lambda]$ be lattice data. We put

$$\Sigma := \langle h\rangle \oplus \langle\mathcal{E}\rangle.$$

We denote by $\operatorname{sing}\ell$ the set of irreducible components of $\mathcal{E}$, and we let

$$\mathcal{E} = \bigsqcup_{P \in \operatorname{sing}\ell} \mathcal{E}_P$$

be the irreducible decomposition of $\mathcal{E}$. We then have an orthogonal direct-sum decomposition

$$\Lambda \otimes \mathbb{Q} = \langle h\rangle \otimes \mathbb{Q} \oplus \bigoplus \langle\mathcal{E}_P\rangle \otimes \mathbb{Q}.$$

We say that $x \in \Lambda$ is $\mathbf{v}$-*smooth at* $P \in \operatorname{sing}\ell$ if the component $x_P \in \langle\mathcal{E}_P\rangle \otimes \mathbb{Q}$ of x is either 0 or equal to some $e_i^\vee \in \langle\mathcal{E}_P\rangle^\vee$, where $e_1^\vee, \dots, e_r^\vee$ is the basis of $\langle\mathcal{E}_P\rangle^\vee$ dual to the basis $\mathcal{E}_P = \{e_1, \dots, e_r\}$ of $\langle\mathcal{E}_P\rangle$. We say that x is $\mathbf{v}$-*smooth* if $x \in \Lambda$ is $\mathbf{v}$-smooth at every $P \in \operatorname{sing}\ell$.

REMARK 5.25. The notion "$\mathbf{v}$-smooth" is the lattice-theoretic version of the geometric notion "v-smooth" defined in Definition 5.2.

We also define an involution ι of $\Lambda \otimes \mathbb{Q}$ by Remark 5.10 with Λ_B replaced by Λ and ι_B replaced by ι. Then we can define the subsets

$$\mathcal{L}^l(\ell),\ \mathcal{L}^b(\ell),\ \mathcal{C}^l(\ell),\ \mathcal{C}^b(\ell),\ \mathcal{G}^l(\ell),\ \mathcal{G}^b(\ell)$$

of Λ in the same way as we defined the sets $\mathcal{L}^l(B)$, $\mathcal{L}^b(B)$, $\mathcal{C}^l(B)$, $\mathcal{C}^b(B)$, $\mathcal{G}^l(B)$, and $\mathcal{G}^b(B)$ but with Λ_B replaced by Λ, h_B replaced by h, Σ_B replaced by Σ, v-smooth replaced by $\mathbf{v}$-smooth, and ι_B replaced by ι. If $\phi\colon \Lambda_B \xrightarrow{\sim} \Lambda$ is an isomorphism of lattice data from $\ell(B)$ to ℓ, then ϕ maps $\mathcal{L}^l(B), \mathcal{L}^b(B), \mathcal{C}^l(B), \mathcal{C}^b(B)$, $\mathcal{G}^l(B)$, and $\mathcal{G}^b(B)$ to (respectively) $\mathcal{L}^l(\ell)$, $\mathcal{L}^b(\ell)$, $\mathcal{C}^l(\ell)$, $\mathcal{C}^b(\ell)$, $\mathcal{G}^l(\ell)$, and $\mathcal{G}^b(\ell)$ bijectively. In other words, these subsets of Λ_B are determined only by the lattice data of B.

Thus we have shown that the configuration type of a lattice-generic simple sextic B is determined by the lattice type of B. Hence we obtain the following result, which was proved by Yang in [32].

COROLLARY 5.26. *Let B_1 and B_2 be simple sextics (not necessarily lattice-generic) of the same lattice type. Then $B_1 \sim_{\mathrm{cfg}} B_2$ holds.*

Proof. There exist lattice-generic simple sextics B_1' and B_2' such that $B_1' \sim_{\mathrm{eqs}} B_1$ and $B_2' \sim_{\mathrm{eqs}} B_2$. Since $B_1' \sim_{\mathrm{lat}} B_2'$, we have $B_1' \sim_{\mathrm{cfg}} B_2'$ by the previous arguments. Thus $B_1 \sim_{\mathrm{cfg}} B_2$ follows. □

We have also shown that the subsets $\mathcal{Z}_1(B)$, $\mathcal{Z}_2(B)$, and $\mathcal{Z}_3(B)$ of Λ_B for a lattice-generic simple sextic B are determined only by the lattice type of B, and hence Theorem 3.2 is proved.

COMPUTATION 5.27. We have already obtained the complete list of lattice data of simple sextics by Computation 4.11. For each piece $\ell = [\mathcal{E}, h, \Lambda]$ of the lattice data in this list, we make the following calculation.

We compute the subsets $\mathcal{L}^l(\ell), \mathcal{L}^b(\ell), \mathcal{C}^l(\ell), \mathcal{C}^b(\ell)$ of Λ. If $\mathcal{L}^b(\ell) = \mathcal{C}^b(\ell) = \emptyset$, then we calculate $\mathcal{G}^b(\ell)$. Thus we determine the configuration type containing the lattice type of the lattice data ℓ. We then calculate

$$\Theta := \begin{cases} \Sigma + \langle \mathcal{L}^b(\ell)\rangle + \langle \mathcal{C}^b(\ell)\rangle & \text{if } \mathcal{L}^b(\ell) \neq \emptyset \text{ or } \mathcal{C}^b(\ell) \neq \emptyset, \\ \Sigma + \langle \mathcal{G}^b(\ell)\rangle & \text{if } \mathcal{L}^b(\ell) = \mathcal{C}^b(\ell) = \emptyset, \end{cases}$$

and $F_\ell := \Lambda/\Theta$.

Suppose that $\mathcal{L}^l(\ell) \neq \emptyset$ or $\mathcal{C}^l(\ell) \neq \emptyset$. We confirm that $F_\ell \neq 0$ and that the equality $\Lambda = \Theta + \langle \mathcal{L}^l(\ell)\rangle + \langle \mathcal{C}^l(\ell)\rangle$ holds. Suppose that $\mathcal{L}^l(\ell) = \mathcal{C}^l(\ell) = \emptyset$ but $F_\ell \neq 0$. We then calculate $\mathcal{G}^l(\ell)$ and confirm that $\mathcal{G}^l(\ell)$ consists of two elements, that $\Lambda = \Theta + \langle \mathcal{G}^l(\ell)\rangle$ holds, and that Λ/Σ is cyclic of order 4.

REMARK 5.28. In order to determine whether or not two lattice types are contained in the same configuration type, we have to use the *combinatorial* definition of the configuration type; this is given in [4, Rem. 3], for example.

By this calculation, we prove Theorem 3.5, Theorem 3.21, and the first part of Theorem 3.23. We also obtain the complete list of lattice data of Z-splitting pairs (B, Γ) with $\deg\Gamma \leq 2$ or with $z_1(\lambda(B)) = z_2(\lambda(B)) = 0$, $F_B \neq 0$, and Γ smooth cubic. Our next task is to determine the relation of specializations among the lattice data of Z-splitting pairs.

6. Specialization of Lattice Types

For the study of specialization of lattice types, we need to refine the period map $\tau_1\colon \mathcal{M}_1 \to \Omega_{\mathbb{L}}$ (see [5, Chap. VIII] or [6]). Consider the real vector bundle $R^2\pi_{1*}\mathbb{R}$ of rank 22 over the non-Hausdorff moduli space $\mathcal{M}_1$, where $\pi_1\colon \mathcal{X}_1 \to \mathcal{M}_1$ is the universal family of (marked) $K3$ surfaces. A point of this vector bundle is given by (t, x), where $t \in \mathcal{M}_1$ and $x \in H^2(X_t, \mathbb{R})$. We then put

$$\mathcal{M}_2 := \{(t, \kappa) \in R^2\pi_{1*}\mathbb{R} \mid \kappa \text{ is a Kähler class of } X_t\};$$

that is, $\mathcal{M}_2$ is the base space of the universal family of the triples (X, ϕ, κ), where (X, ϕ) is a marked $K3$ surface and κ is a Kähler class of X.

For a point $[\omega]$ of $\Omega_{\mathbb{L}}$, we put

$$\begin{aligned}
H^{[\omega]} &:= \{x \in \mathbb{L} \otimes \mathbb{R} \mid (x, \omega) = 0\},\\
\mathrm{NS}^{[\omega]} &:= H^{[\omega]} \cap \mathbb{L} \quad \text{(as defined in Section 5)},\\
D^{[\omega]} &:= \{d \in \mathrm{NS}^{[\omega]} \mid d^2 = -2\},\\
\Gamma^{[\omega]} &:= \{x \in H^{[\omega]} \mid x^2 > 0\},\\
{}^0\Gamma^{[\omega]} &:= \{x \in \Gamma^{[\omega]} \mid (x, d) \neq 0 \text{ for all } d \in D^{[\omega]}\}.
\end{aligned}$$

We then put

$$\begin{aligned}
H\Omega_{\mathbb{L}} &:= \{([\omega], x) \in \Omega_{\mathbb{L}} \times (\mathbb{L} \otimes \mathbb{R}) \mid x \in H^{[\omega]}\},\\
K\Omega_{\mathbb{L}} &:= \{([\omega], x) \in \Omega_{\mathbb{L}} \times (\mathbb{L} \otimes \mathbb{R}) \mid x \in \Gamma^{[\omega]}\},\\
{}^0K\Omega_{\mathbb{L}} &:= \{([\omega], x) \in \Omega_{\mathbb{L}} \times (\mathbb{L} \otimes \mathbb{R}) \mid x \in {}^0\Gamma^{[\omega]}\}.
\end{aligned}$$

We have a commutative diagram

$$\begin{tikzcd}
{}^0K\Omega_{\mathbb{L}} \arrow[r, hook] \arrow[dr, "\Pi_\Omega"'] & K\Omega_{\mathbb{L}} \arrow[r, hook] \arrow[d] & H\Omega_{\mathbb{L}} \arrow[dl] \\
& \Omega_{\mathbb{L}}, &
\end{tikzcd} \tag{6.1}$$

where the maps to $\Omega_{\mathbb{L}}$ are the projection onto the first factor. Note that $K\Omega_{\mathbb{L}}$ and $H\Omega_{\mathbb{L}}$ are locally trivial fiber spaces over $\Omega_{\mathbb{L}}$. We have the following lemma.

LEMMA 6.1 [5, Chap. VIII, Cor. 9.2]. *The space ${}^0K\Omega_{\mathbb{L}}$ is open in $K\Omega_{\mathbb{L}}$, and hence the projection Π_Ω is an open immersion.*

Let t be a point of $\mathcal{M}_1$, and let

$$[\omega_t] := \tau_1(t) \in \Omega_{\mathbb{L}}$$

be the period point of (X_t, ϕ_t). Then the marking $\phi_t\colon H^2(X_t, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L}$ maps H_{X_t} to $H^{[\omega_t]}$, Γ_{X_t} to $\Gamma^{[\omega_t]}$, $\mathrm{NS}(X_t)$ to $\mathrm{NS}^{[\omega_t]}$, and D_{X_t} to $D^{[\omega_t]}$; hence ϕ_t maps ${}^0\Gamma_{X_t}$

to ${}^{0}\Gamma^{[\omega_t]}$. Since every Kähler class of X_t is contained in the Kähler cone $\mathcal{K}_{X_t} \subset {}^{0}\Gamma_{X_t}$, we can define a map

$$\tau_2 \colon \mathcal{M}_2 \to {}^{0}K\Omega_{\mathbb{L}},$$

which is called the *refined period map,* by

$$\tau_2(t,\kappa) := (\tau_1(t), \phi_t(\kappa)).$$

Then we obtain a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_2 & \xrightarrow{\tau_2} & {}^{0}K\Omega_{\mathbb{L}} \\ {\scriptstyle \Pi_{\mathcal{M}}}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_{\Omega}} \\ \mathcal{M}_1 & \xrightarrow{\tau_1} & \Omega_{\mathbb{L}}, \end{array} \tag{6.2}$$

where the vertical arrows $\Pi_{\mathcal{M}}$ and Π_{Ω} are the forgetful maps.

The following statement plays a crucial role in the study of specialization of lattice types.

THEOREM 6.2 [5, Chap. VIII, Thms. 12.3 and 14.1]. *The refined period map τ_2 is an isomorphism.*

The specialization of lattice types of simple sextics and Z-splitting pairs can be described by geometric embeddings of lattice data.

DEFINITION 6.3. Let $\ell = [\mathcal{E}, h, \Lambda]$ and $\ell_0 = [\mathcal{E}_0, h_0, \Lambda_0]$ be lattice data. By a *geometric embedding* of ℓ into ℓ_0 we mean a primitive embedding $\sigma \colon \Lambda \hookrightarrow \Lambda_0$ of the lattice Λ into the lattice Λ_0 that satisfies $\sigma(h) = h_0$ and $\sigma(\mathcal{E}) \subset \langle \mathcal{E}_0 \rangle^{+}$.

DEFINITION 6.4. Let $\ell^P = [\mathcal{E}, h, \Lambda, S]$ and $\ell_0^P = [\mathcal{E}_0, h_0, \Lambda_0, S_0]$ be extended lattice data. A geometric embedding of ℓ^P into ℓ_0^P is a geometric embedding $\sigma \colon \Lambda \hookrightarrow \Lambda_0$ of $[\mathcal{E}, h, \Lambda]$ into $[\mathcal{E}_0, h_0, \Lambda_0]$ such that

$$\sigma(S) \subset S_0 + \langle \mathcal{E}_0 \rangle^{+} := (v_0^{+} + \langle \mathcal{E}_0 \rangle^{+}) \cup (v_0^{-} + \langle \mathcal{E}_0 \rangle^{+}), \quad \text{where } S_0 = \{v_0^{\pm}\}.$$

Let $f \colon \mathcal{B} \to \Delta$ be an analytic family of simple sextics, where f is the projection from $\mathcal{B} \subset \mathbb{P}^2 \times \Delta$ to Δ and $B_t := f^{-1}(t)$ is a simple sextic on $\mathbb{P}^2 \times \{t\}$ for any $t \in \Delta$. Suppose that f is equisingular over $\Delta^{\times}$. We define a geometric embedding

$$\sigma_{\mathcal{B},t} \colon \Lambda_{B_t} \hookrightarrow \Lambda_{B_0}$$

of the lattice data $\ell(B_t)$ with $t \neq 0$ into the lattice data $\ell(B_0)$ as follows. We consider the double cover

$$\mathcal{Y}_{\mathcal{B}} \to \mathbb{P}^2 \times \Delta$$

branching exactly along $\mathcal{B}$. Note that every fiber of $\mathcal{Y}_{\mathcal{B}} \to \Delta$ is birational to a $K3$ surface. Therefore, by Kulikov [15], there exists a birational transformation $\mathcal{X}_{\mathcal{B}} \to \mathcal{Y}_{\mathcal{B}}$ such that the composite holomorphic map

$$\pi_{\mathcal{B}} \colon \mathcal{X}_{\mathcal{B}} \to \Delta$$

is a smooth family of $K3$ surfaces. Note that the fiber of $\pi_{\mathcal{B}}$ over $t \in \Delta$ is isomorphic to X_{B_t}. Note also that $\mathcal{X}_{\mathcal{B}}$ has a line bundle $\mathcal{L}_{\mathcal{B}}$ such that the class of the restriction of $\mathcal{L}_{\mathcal{B}}$ to $X_{B_t} = \pi_{\mathcal{B}}^{-1}(t)$ is equal to $h_{B_t} \in H^2(X_{B_t}, \mathbb{Z})$ for any $t \in \Delta$. Then we have a trivialization

$$R^2\pi_{\mathcal{B}*}\mathbb{Z} \cong \Delta \times \mathbb{L},$$

which induces markings $H^2(X_{B_t}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L}$ for any $t \in \Delta$. Using this trivialization, we obtain a primitive embedding $\sigma_{\mathcal{B},t} \colon \Lambda_{B_t} \hookrightarrow \Lambda_{B_0}$ of lattices by the specialization homomorphism

$$H^2(X_{B_t}, \mathbb{Z}) \xrightarrow{\sim} H^2(X_{B_0}, \mathbb{Z}).$$

This $\sigma_{\mathcal{B},t}$ induces a geometric embedding of the lattice data $\ell(B_t)$ for $t \neq 0$ into the lattice data $\ell(B_0)$. Indeed, $\sigma_{\mathcal{B},t}$ maps h_{B_t} to h_{B_0} because the polarizations on X_{B_t} form a family $\mathcal{L}_{\mathcal{B}}$. Moreover, any exceptional (-2)-curve on X_{B_t} $(t \neq 0)$ degenerates into an effective divisor on X_{B_0}, whose reduced irreducible components must be exceptional (-2)-curves on X_{B_0} because its degree with respect to the polarization h_{B_0} is 0.

Proposition 6.5. *Let $\{(B_t, \Gamma_t)\}_{t\in\Delta}$ be an analytic family of Z-splitting pairs that is equisingular over $\Delta^\times$. Then the geometric embedding $\sigma_{\mathcal{B},t}$ of $\ell(B_t)$ with $t \neq 0$ into $\ell(B_0)$ yields a geometric embedding of the extended lattice data $\ell^P(B_t, \Gamma_t)$ with $t \neq 0$ into the extended lattice data $\ell^P(B_0, \Gamma_0)$.*

Proof. Since Γ_t degenerates into Γ_0, the curve $\tilde{\Gamma}_t^+ \subset X_{B_t}$ for $t \neq 0$ degenerates into an effective divisor on X_{B_0} that is the sum of $\tilde{\Gamma}_0^+$ (or $\tilde{\Gamma}_0^-$) and some exceptional (-2)-curves on X_{B_0}. Hence the geometric embedding $\sigma_{\mathcal{B},t} \colon \Lambda_{B_t} \hookrightarrow \Lambda_{B_0}$ of $\ell(B_t)$ into $\ell(B_0)$ constructed previously satisfies $\sigma_{\mathcal{B},t}([\tilde{\Gamma}_t^+]) \in [\tilde{\Gamma}_0^+] + \langle \mathcal{E}_{B_0}\rangle^+$ or $\sigma_{\mathcal{B},t}([\tilde{\Gamma}_t^+]) \in [\tilde{\Gamma}_0^-] + \langle \mathcal{E}_{B_0}\rangle^+$. □

Corollary 6.6. *Let λ_0^P and λ^P be lattice types of Z-splitting pairs, and let ℓ_0^P and ℓ^P be the corresponding extended lattice data. If λ_0^P is a specialization of λ^P, then there exists a geometric embedding of ℓ^P into ℓ_0^P.*

Since a geometric embedding $\sigma \colon \Lambda_B \hookrightarrow \Lambda_{B_0}$ of $\ell^P(B, \Gamma)$ into $\ell^P(B_0, \Gamma_0)$ induces a homomorphism of finite abelian groups $G_B \to G_{B_0}$ that maps $([\tilde{\Gamma}^+] \bmod \Sigma_B) \in G_B$ to $([\tilde{\Gamma}_0^+] \bmod \Sigma_{B_0}) \in G_{B_0}$ or $([\tilde{\Gamma}_0^-] \bmod \Sigma_{B_0}) \in G_{B_0}$, we obtain our next corollary.

Corollary 6.7. *If $\lambda_0^P = \lambda^P(B_0, \Gamma_0)$ is a specialization of $\lambda^P = \lambda^P(B, \Gamma)$, then the class order of λ_0^P is a divisor of the class order of λ^P.*

In order to show that the existence of a geometric embedding of lattice data with certain properties is *sufficient* for the existence of the specialization, we prepare two easy lemmas.

Let $\pi \colon \mathcal{X} \to \Delta$ be a smooth family of $K3$ surfaces. We put $X_t := \pi^{-1}(t)$.

LEMMA 6.8. *Let s be a section of $R^2\pi_*\mathbb{Z}$. If $s_t := s|_{X_t} \in H^2(X_t, \mathbb{Z})$ is contained in $H^{1,1}(X_t)$ for any $t \in \Delta$, then there exists a line bundle $\mathcal{L}_{\mathcal{X}}$ on $\mathcal{X}$ such that the class of the restriction $\mathcal{L}_t := \mathcal{L}_{\mathcal{X}}|_{X_t}$ is equal to s_t.*

Proof. This follows immediately from the commutative diagram

$$\begin{tikzcd} H^1(\mathcal{X}, \mathcal{O}^\times) \arrow[r] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r] \arrow[d, "\wr"] & H^2(\mathcal{X}, \mathcal{O}) \arrow[d, "\wr"] \\ & H^0(\Delta, R^2\pi_*\mathbb{Z}) \arrow[r] & H^0(\Delta, R^2\pi_*\mathcal{O}), \end{tikzcd}$$

where the horizontal sequences are induced from the exponential exact sequence $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$. □

LEMMA 6.9. *Let $\mathcal{L}_{\mathcal{X}}$ be a line bundle on $\mathcal{X}$, and let $\mathcal{L}_t := \mathcal{L}_{\mathcal{X}}|_{X_t}$ for $t \in \Delta$. If $h^1(X_0, \mathcal{L}_0) = 0$ and $h^0(X_0, \mathcal{L}_0) > 0$, then there exists a linear subspace $V \subset H^0(\mathcal{X}, \mathcal{L}_{\mathcal{X}})$ of dimension equal to $h^0(X_0, \mathcal{L}_0)$ such that, after replacing Δ with a smaller disc if necessary, the restriction homomorphism $H^0(\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \to H^0(X_t, \mathcal{L}_t)$ maps V isomorphically onto $H^0(X_t, \mathcal{L}_t)$ for any $t \in \Delta$.*

Proof. From $h^0(X_0, \mathcal{L}_0) > 0$, we have $h^2(X_0, \mathcal{L}_0) = 0$. By the semi-continuity theorem, the assumption $h^1(X_0, \mathcal{L}_0) = 0$ implies that both $h^1(X_t, \mathcal{L}_t) = 0$ and $h^0(X_t, \mathcal{L}_t) = h^0(X_0, \mathcal{L}_0)$ for t in a sufficiently small neighborhood of 0 because $\mathcal{L}_t^2 \in \mathbb{Z}$ is constant. Hence, by replacing Δ with a smaller disc if necessary, we can assume that $H^1(\mathcal{X}, \mathcal{L}_{\mathcal{X}}) = 0$ and hence $H^1(\mathcal{X}, \mathcal{L}_{\mathcal{X}}(-X_t)) = 0$ holds for any $t \in \Delta$ because $\mathcal{L}_{\mathcal{X}} \cong \mathcal{L}_{\mathcal{X}}(-X_t)$ on $\mathcal{X}$. Therefore, the restriction homomorphism $H^0(\mathcal{X}, \mathcal{L}_{\mathcal{X}}) \to H^0(X_t, \mathcal{L}_t)$ is surjective for any $t \in \Delta$. □

The following proposition seems to be well known. However, we present a complete proof because it illustrates how the refined period map is used for the study of specializations of simple sextics and it also sets up various tools necessary for the proof of Proposition 6.16.

PROPOSITION 6.10. *Let $\ell_0 = [\mathcal{E}_0, h_0, \Lambda_0]$ and $\ell = [\mathcal{E}, h, \Lambda]$ be lattice data of simple sextics. Suppose that a simple sextic B_0 with an isomorphism $\alpha_0 \colon \Lambda_0 \xrightarrow{\sim} \Lambda_{B_0}$ of lattice data from ℓ_0 to $\ell(B_0)$ is given. If a geometric embedding $\sigma \colon \Lambda \hookrightarrow \Lambda_0$ of ℓ into ℓ_0 is given, then we can construct an analytic family $f \colon \mathcal{B} \to \Delta$ consisting of simple sextics $B_t = f^{-1}(t)$ and isomorphisms*

$$\alpha_t \colon \Lambda \xrightarrow{\sim} \Lambda_{B_t}$$

of lattice data from ℓ to $\ell(B_t)$ for $t \neq 0$ that satisfy the following conditions:

(i) *the central fiber $f^{-1}(0)$ of f is the given simple sextic B_0;*
(ii) *f is equisingular over $\Delta^\times$;*
(iii) *for $t \neq 0$, the composite $\alpha_0^{-1} \circ \sigma_{\mathcal{B},t} \circ \alpha_t \colon \Lambda \hookrightarrow \Lambda_0$ is equal to the given geometric embedding σ of ℓ into ℓ_0; and*
(iv) *the locus of all $t \in \Delta$ such that B_t is lattice-generic is dense in Δ.*

Proof. For simplicity, we put

$$X_0 := X_{B_0}.$$

We fix a marking

$$\phi_0\colon H^2(X_0,\mathbb{Z}) \xrightarrow{\sim} \mathbb{L}.$$

By α_0 and ϕ_0, we obtain a primitive embedding

$$\psi\colon \Lambda_0 \hookrightarrow \mathbb{L};$$

by the composition of σ and ψ, we obtain a primitive embedding

$$\psi\circ\sigma\colon \Lambda \hookrightarrow \mathbb{L}.$$

From now on, we consider Λ and Λ_0 as primitive sublattices of $\mathbb{L}$ by $\psi\circ\sigma$ and ψ, respectively:

$$\Lambda \subset \Lambda_0 \subset \mathbb{L}.$$

In particular, we have $h = h_0 = \phi_0(h_{B_0}) \in \mathbb{L}$ as well as $\mathcal{E}_0 = \phi_0(\mathcal{E}_{B_0}) \subset \mathbb{L}$ and $\mathcal{E} \subset \langle\mathcal{E}_0\rangle^+ \subset \mathbb{L}$. Moreover, we have inclusions of complex submanifolds

$$\Omega_{\psi^\perp} \subset \Omega_{(\psi\circ\sigma)^\perp} \subset \Omega_{\mathbb{L}}.$$

Let $[\eta_0] \in \Omega_{\mathbb{L}}$ be the period point of the marked $K3$ surface (X_0,ϕ_0). Then $[\eta_0]$ is a point of $\Omega_{\psi^\perp}$. We choose an analytic embedding

$$\delta\colon \Delta \hookrightarrow \Omega_{(\psi\circ\sigma)^\perp}$$

of the open unit disk $\Delta \subset \mathbb{C}$ into a sufficiently small neighborhood of $[\eta_0]$ such that (a) $\delta(0) = [\eta_0]$, (b) $\delta^{-1}(\Omega^{\diamond}_{(\psi\circ\sigma)^\perp}) = \Delta\setminus\{0\}$ holds, and (c) $\delta^{-1}(\Omega^{\diamond\diamond}_{(\psi\circ\sigma)^\perp})$ is dense in Δ. (These properties can be achieved because $\Omega^{\diamond}_{(\psi\circ\sigma)^\perp}$ is open in $\Omega_{(\psi\circ\sigma)^\perp}$ and $\Omega^{\diamond\diamond}_{(\psi\circ\sigma)^\perp}$ is dense in $\Omega_{(\psi\circ\sigma)^\perp}$.) We write

$$\delta(t) = [\eta_t] \in \Omega_{\mathbb{L}}.$$

Consider the pull-back

$$\begin{tikzcd} {}^0K\Omega_\delta \arrow[r, hook] \arrow[dr] & K\Omega_\delta \arrow[r, hook] \arrow[d] & H\Omega_\delta \arrow[dl] \\ & \Delta & \end{tikzcd} \tag{6.3}$$

of the diagram (6.1) by $\delta\colon \Delta \hookrightarrow \Omega_{(\psi\circ\sigma)^\perp} \hookrightarrow \Omega_{\mathbb{L}}$. For simplicity, we put

$$H := H^{[\eta_0]} \quad\text{and}\quad \Gamma := \Gamma^{[\eta_0]}.$$

Then we have trivializations

$$K\Omega_\delta \cong \Delta\times\Gamma \quad\text{and}\quad H\Omega_\delta \cong \Delta\times H \tag{6.4}$$

over Δ that extend the identity maps over $0 \in \Delta$ and such that the inclusion $K\Omega_\delta \hookrightarrow H\Omega_\delta$ is the Cartesian product of the identity map of Δ and the inclusion $\Gamma \hookrightarrow H$. Since $([\omega],h) \in K\Omega_{\mathbb{L}}$ for any $[\omega] \in \Omega_{(\psi\circ\sigma)^\perp}$, we have a section $t \mapsto (\delta(t),h)$ of $K\Omega_\delta \to \Delta$.

We choose the trivialization (6.4) in such a way that $K\Omega_\delta \cong \Delta \times \Gamma$ maps this section to the constant section $t \mapsto (t, h)$ of $\Delta \times \Gamma \to \Delta$. For a vector $d \in \mathbb{L}$ with $d^2 = -2$ and a point $[\omega] \in \Omega_{\mathbb{L}}$ with $(\omega, d) = 0$, we put

$$W(d) := \{x \in \mathbb{L} \otimes \mathbb{R} \mid (x, d) = 0\} \quad \text{and} \quad d^{\perp}_{[\omega]} := W(d) \cap H^{[\omega]}.$$

Then $d^{\perp}_{[\omega]}$ is a hyperplane of the real vector space $H^{[\omega]}$. Since $\langle \mathcal{E} \rangle \subset \langle \mathcal{E}_0 \rangle^{+}$, we see that $\langle \mathcal{E} \rangle$ is a sublattice of $\langle \mathcal{E}_0 \rangle$, and hence the set $D_{\langle \mathcal{E} \rangle}$ of roots in $\langle \mathcal{E} \rangle$ is a subset of the set $D_{\langle \mathcal{E}_0 \rangle}$ of roots in $\langle \mathcal{E}_0 \rangle$:

$$D_{\langle \mathcal{E} \rangle} \subset D_{\langle \mathcal{E}_0 \rangle}.$$

We have

$$D_{\langle \mathcal{E} \rangle} \subset D^{[\eta_t]} \text{ for any } t \in \Delta \quad \text{and} \quad D_{\langle \mathcal{E}_0 \rangle} \subset D^{[\eta_0]}.$$

More precisely, we have

$$D_{\langle \mathcal{E} \rangle} = \{d \in D^{[\eta_t]} \mid h \in d^{\perp}_{[\eta_t]}\} \quad \text{for } t \neq 0, \tag{6.5}$$

because $\delta(t) \in \Omega^{\diamond}_{(\psi \circ \sigma)^{\perp}}$ for $t \neq 0$, and

$$D_{\langle \mathcal{E}_0 \rangle} = \{d \in D^{[\eta_0]} \mid h \in d^{\perp}_{[\eta_0]}\}. \tag{6.6}$$

We choose the trivialization (6.4) in such a way that, for each $d \in D_{\langle \mathcal{E} \rangle}$, the isomorphism $H\Omega_\delta \cong \Delta \times H$ maps the family of walls $\{([\eta_t], x) \in H\Omega_\delta \mid x \in d^{\perp}_{[\eta_t]}\}$ over Δ to the constant family $\Delta \times d^{\perp}_{[\eta_0]}$. We denote by ${}^0(\Delta \times \Gamma)$ the open subset of $\Delta \times \Gamma$ that corresponds to the open subset ${}^0K\Omega_\delta \subset K\Omega_\delta$ by the trivialization, and we put

$$\mathcal{W} := (\Delta \times \Gamma) \setminus {}^0(\Delta \times \Gamma).$$

Recall that the complement of ${}^0K\Omega_\delta$ in $K\Omega_\delta$ is the union of walls

$$\{([\eta_t], x) \in K\Omega_\delta \mid x \in d^{\perp}_{[\eta_t]} \text{ for some } d \in D^{[\eta_t]}\}.$$

Therefore, by the description (6.5) and (6.6) of walls passing through h, if $\mathbb{B} \subset \Gamma$ is a sufficiently small ball with the center h then

$$(\Delta \times \mathbb{B}) \cap \mathcal{W} = \bigcup_{d \in D_{\langle \mathcal{E} \rangle}} (\Delta \times d'^{\perp}_{[\eta_0]}) \cup \bigcup_{d \in D_{\langle \mathcal{E}_0 \rangle} \setminus D_{\langle \mathcal{E} \rangle}} (\{0\} \times d'^{\perp}_{[\eta_0]}),$$

where $d'^{\perp}_{[\eta_0]} := d^{\perp}_{[\eta_0]} \cap \Gamma$. In other words, the projection

$${}^0(\Delta \times \Gamma) \cap (\Delta \times \mathbb{B}) \to \Delta$$

is a constant family of cones in the ball $\mathbb{B}$ partitioned by the walls associated with $d \in D_{\langle \mathcal{E} \rangle}$ over $\Delta^{\times}$, where the central fiber is partitioned further by the walls associated with $d \in D_{\langle \mathcal{E}_0 \rangle} \setminus D_{\langle \mathcal{E} \rangle}$.

We have a unique connected component of the central fiber

$${}^0(\Delta \times \Gamma) \cap (\{0\} \times \mathbb{B}) \subset \{0\} \times \Gamma = \Gamma^{[\eta_0]}$$

that is mapped to the Kähler cone $\mathcal{K}_{X_{B_0}} \subset {}^0\Gamma_{X_{B_0}}$ of X_{B_0} via the marking ϕ_0. We choose a point $(0, v_0)$ from this connected component. Then $v_0 \in \Gamma^{[\eta_0]}$ corresponds to a Kähler class of X_{B_0} via the marking ϕ_0. In particular, we have

$$(v_0, e) > 0 \quad \text{for any} \ \ e \in \mathcal{E}_0.$$

Since $\mathcal{E} \subset \langle \mathcal{E}_0 \rangle^+$, we have

$$(v_0, e) > 0 \quad \text{for any} \ \ e \in \mathcal{E}. \tag{6.7}$$

By the foregoing description of ${}^0(\Delta \times \Gamma) \cap (\Delta \times \mathbb{B})$, we see that $(t, v_0) \in \Delta \times \Gamma$ is a point of ${}^0(\Delta \times \Gamma)$ for any $t \in \Delta$. We denote by

$$\tilde{\delta} \colon \Delta \to {}^0K\Omega_\delta$$

the section of $K\Omega_\delta \to \Delta$ corresponding to the constant section $t \mapsto (t, v_0)$ of ${}^0(\Delta \times \Gamma) \to \Delta$, and we let

$$\tilde{\delta}_{\mathcal{M}} \colon \Delta \to \mathcal{M}_2$$

be the map corresponding to $\tilde{\delta}$ via τ_2. We denote by (X_t, ϕ_t, κ_t) the marked $K3$ surface (X_t, ϕ_t) with a Kähler class κ_t corresponding to $\tilde{\delta}_{\mathcal{M}}(t) \in \mathcal{M}_2$. Let $h_{X_t} \in H^2(X_t, \mathbb{Z})$ be the vector such that $\phi_t(h_{X_t}) = h$. Since $\eta_t \perp h$, we have $h_{X_t} \in \mathrm{NS}(X_t)$. Suppose that $t \neq 0$. Since h is contained in the closure of the connected component of ${}^0\Gamma^{[\eta_t]}$ containing $\phi_t(\kappa_t)$, the class $h_{X_t} \in \mathrm{NS}(X_t)$ is nef by Proposition 4.5. By Proposition 4.7 and $\delta(t) \in \Omega^{\diamond}_{(\psi\circ\sigma)^\perp}$, condition (4.2) in the definition of $\Omega^{\diamond}_{(\psi\circ\sigma)^\perp}$ implies that h_{X_t} is the class of a polarization $\mathcal{L}_t$ of degree 2 on X_t. Note that we have $(\kappa_t, e) > 0$ for any $e \in \phi_t(\mathcal{E})$ by (6.7). By $\delta(t) \in \Omega^{\diamond}_{(\psi\circ\sigma)^\perp}$ again, condition (4.1) in the definition of $\Omega^{\diamond}_{(\psi\circ\sigma)^\perp}$ implies that $\phi_t^{-1}(\mathcal{E})$ is a fundamental system of roots in $\langle h_{X_t} \rangle^\perp \subset \mathrm{NS}(X_t)$ associated with the Kähler class κ_t. Consequently, Proposition 4.8 implies that $\phi_t^{-1}(\mathcal{E})$ is equal to the set of classes of (-2)-curves contracted by $\Phi_{|\mathcal{L}_t|} \colon X_t \to \mathbb{P}^2$. Let B_t be the branch curve of $\Phi_{|\mathcal{L}_t|}$. Then the markings $\phi_t \colon H^2(X_t, \mathbb{Z}) \cong \mathbb{L}$ yield isomorphisms of lattices from $\Lambda_{B_t} \subset H^2(X_t, \mathbb{Z})$ to $\Lambda \subset \mathbb{L}$ that induce isomorphisms of lattice data $\ell(B_t) \cong \ell$ for $t \neq 0$. We define $\alpha_t \colon \Lambda \xrightarrow{\sim} \Lambda_{B_t}$ to be the inverse of this isomorphism.

We will show that, making Δ smaller if necessary, these simple sextics B_t form an analytic family. Let $\pi_{\tilde{\delta}} \colon \mathcal{X}_{\tilde{\delta}} \to \Delta$ be the family of X_t, which is the pull-back of the universal family $\pi_1 \colon \mathcal{X}_1 \to \mathcal{M}_1$ by $\Pi_{\mathcal{M}} \circ \tilde{\delta}_{\mathcal{M}}$. Then $t \mapsto h_{X_t}$ gives a section of $R^2\pi_{\tilde{\delta}*}\mathbb{Z}$. By Lemma 6.8, there exists a line bundle $\mathcal{L}_{\mathcal{X}}$ on $\mathcal{X}_{\tilde{\delta}}$ such that the restriction $\mathcal{L}_{\mathcal{X}}|_{X_t}$ is equal to the polarization $\mathcal{L}_t$ given previously for any $t \in \Delta$. Note that $h^0(X_0, \mathcal{L}_0) = 3$ and $h^1(X_0, \mathcal{L}_0) = 0$ by Nikulin [17, Prop. 0.1]. Therefore, shrinking Δ if necessary, we have a 3-dimensional subspace V of $H^0(\mathcal{X}_{\tilde{\delta}}, \mathcal{L}_{\mathcal{X}})$ such that the restriction homomorphism maps V onto $H^0(X_t, \mathcal{L}_t)$ isomorphically for any $t \in \Delta$. In particular, the linear system V has no base points. Considering the morphism

$$\Phi_V \colon \mathcal{X}_{\tilde{\delta}} \to \mathbb{P}^2$$

induced by V, we obtain an analytic family of morphisms $X_t \to \mathbb{P}^2$ with the branch curve $B_t \subset \mathbb{P}^2$ and hence an analytic family of simple sextics over Δ. It is obvious that this analytic family and the isomorphisms $\alpha_t \colon \Lambda \xrightarrow{\sim} \Lambda_{B_t}$ of lattice data from ℓ to $\ell(B_t)$ for $t \neq 0$ have the required properties. $\square$

By Proposition 6.10 together with the construction of the geometric embedding $\sigma_{\mathcal{B},t}$, we obtain the following result.

COROLLARY 6.11. *Let λ_0 and λ be lattice types of simple sextics, and let ℓ_0 and ℓ be the corresponding lattice data. Then λ_0 is a specialization of λ if and only if there exists a geometric embedding of ℓ into ℓ_0.*

REMARK 6.12. By the theory of adjacency of singularities [2; 29], we see that if $\lambda(B_0)$ is a specialization of $\lambda(B)$ then the Dynkin diagram of R_B is a subgraph of the Dynkin diagram of R_{B_0}.

Let B be a simple sextic and let $D := C_1 + \cdots + C_m$ be an effective divisor on X_B, where $C_1, \ldots, C_m$ are reduced and irreducible. A *subcurve* of D is, by definition, a divisor

$$C := C_{i_1} + \cdots + C_{i_n},$$

where $\{C_{i_1}, \ldots, C_{i_n}\}$ is a (possibly empty) subset of $\{C_1, \ldots, C_m\}$.

LEMMA 6.13. *Let $D := C_1 + \cdots + C_m$ be an effective divisor on X_B. We put $h^1(D) := \dim H^1(X_B, \mathcal{O}(D))$.*

(1) *Suppose that $D^2 = -2$. For $h^1(D) = 0$ to hold, it is sufficient that $C^2 \leq -2$ hold for any nonempty subcurve C of D.*

(2) *Suppose that $D^2 = 0$ and $(D, h_B) = 3$. For $h^1(D) = 0$ to hold, it is sufficient that $C^2 \leq 0$ hold for any nonempty subcurve C of D.*

Proof. Let $|M|$ be the movable part of $|D|$, where M is a subcurve of D. Suppose that $D^2 = -2$. If $h^1(D) > 0$, then we have $|M| \neq \emptyset$ and hence $M^2 \geq 0$. Suppose that $D^2 = 0$ and $(D, h_B) = 3$. If $h^1(D) > 0$, then either $M^2 > 0$ or $|M| = m|E|$ with $m > 1$ for some elliptic pencil $|E|$. Since $(D, h_B) = 3$, we would have $(E, h_B) = 1$ in the latter case, which is absurd. □

We translate this geometric fact into a lattice-theoretic sufficient condition that can be easily checked by a computer.

DEFINITION 6.14. Let $\ell^P = [\mathcal{E}, h, \Lambda, \{v^\pm\}]$ be extended lattice data, and let

$$w := v^+ + \sum m_e e \quad (e \in \mathcal{E},\ m_e \geq 0)$$

be an element of $v^+ + \langle \mathcal{E} \rangle^+$. We say that $u \in \Lambda$ is a *subcurve vector of w* if u is $n_v v^+ + \sum n_e e$ with $m_e \geq n_e \geq 0$ for any $e \in \mathcal{E}$ and any $n_v = 0$ (or $n_v = 1$).

Suppose that $w^2 = -2$ or that $w^2 = 0$ and $(w, h) = 3$. We say that w *satisfies the vanishing-h^1 condition* if $u^2 \leq w^2$ holds for any nonzero subcurve vector u of w. We define the vanishing-h^1 condition for elements w of $v^- + \langle \mathcal{E} \rangle^+$ in the same way.

DEFINITION 6.15. We say that a geometric embedding σ of $\ell^P = [\mathcal{E}, h, \Lambda, \{v^\pm\}]$ into $\ell_0^P = [\mathcal{E}_0, h_0, \Lambda_0, \{v_0^\pm\}]$ satisfies the vanishing-h^1 condition if $\sigma(v^+) \in \{v_0^\pm\} + \langle \mathcal{E}_0 \rangle^+$ satisfies the vanishing-h^1 condition.

PROPOSITION 6.16. *Let $\ell^P = [\mathcal{E}, h, \Lambda, \{v^\pm\}]$ and $\ell_0^P = [\mathcal{E}_0, h_0, \Lambda_0, \{v_0^\pm\}]$ be the lattice data of Z-splitting pairs (B, Γ) and (B_0, Γ_0), respectively. Suppose that Γ and Γ_0 are smooth of degree ≤ 3. Then the lattice type $\lambda^P(B_0, \Gamma_0)$ is a specialization of the lattice type $\lambda^P(B, \Gamma)$ if there exists a geometric embedding $\sigma : \Lambda \hookrightarrow \Lambda_0$ of ℓ^P into ℓ_0^P that satisfies the vanishing-h^1 condition.*

Proof. By Remark 2.27, we can assume that the representatives (B, Γ) and (B_0, Γ_0) of $\lambda^P(B, \Gamma)$ and $\lambda^P(B_0, \Gamma_0)$ are lattice-generic. We fix a marking

$$\phi_0 \colon H^2(X_{B_0}, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L}.$$

We then consider Λ_0 as a primitive sublattice of $\mathbb{L}$ in such a way that the marking ϕ_0 induces an isomorphism

$$\phi_0 \colon \Lambda_{B_0} \xrightarrow{\sim} \Lambda_0$$

of lattice data from $\ell^P(B_0)$ to ℓ_0^P. By Proposition 6.10, we have an analytic family $\{B_t\}_{t\in\Delta}$ of simple sextics constructed from the geometric embedding $\sigma : \Lambda \hookrightarrow \Lambda_0$ of $\ell = [\mathcal{E}, h, \Lambda]$ into $\ell_0 = [\mathcal{E}_0, h_0, \Lambda_0]$ and the isomorphism ϕ_0. Let

$$\pi_{\tilde{\delta}} \colon \mathcal{X}_{\tilde{\delta}} \to \Delta$$

be the smooth family of $K3$ surfaces constructed in the proof of Proposition 6.10. Then $X_t := \pi_{\tilde{\delta}}^{-1}(t)$ is equal to X_{B_t} and is equipped with markings

$$\phi_t \colon H^2(X_t, \mathbb{Z}) \xrightarrow{\sim} \mathbb{L}$$

that are continuously varying with t. We have lifts $\tilde{\Gamma}_0^\pm$ of Γ_0 on $X_0 = X_{B_0}$. Our aim is to deform $\tilde{\Gamma}_0^\pm$ to curves on X_t that are the lifts of Z-splitting curves for B_t.

By construction, the markings ϕ_t induce isomorphisms of lattices

$$\phi_t \colon \Lambda_{B_t} \xrightarrow{\sim} \Lambda$$

for $t \neq 0$ that induce an isomorphism of lattice data $\ell(B_t) \cong \ell$. Moreover, the specialization homomorphism

$$H^2(X_t, \mathbb{Z}) \xrightarrow{\sim} H^2(X_0, \mathbb{Z})$$

induces the geometric embedding $\sigma : \Lambda \hookrightarrow \Lambda_0$ of ℓ to ℓ_0 under the isomorphisms ϕ_t $(t \neq 0)$ and ϕ_0. Then $v^+ \in \Lambda$ with $\sigma(v^+) \in \Lambda_0$ gives rise to a section $\tilde{v}$ of the locally constant system $R^2\pi_{\tilde{\delta}*}\mathbb{Z}$ on Δ; namely, $\tilde{v}_t := \tilde{v}|_{X_t} \in H^2(X_t, \mathbb{Z})$ is mapped by ϕ_t to v^+ for $t \neq 0$ and to $\sigma(v^+)$ for $t = 0$. In particular, we have $\tilde{v}_t \in H^{1,1}(X_t)$ for any $t \in \Delta$; hence, by Lemma 6.8, there exists a line bundle $\mathcal{D}$ on $\mathcal{X}_{\tilde{\delta}}$ such that the class of $\mathcal{D}_t := \mathcal{D}|_{X_t}$ is equal to $\tilde{v}_t$. Since $[\mathcal{E}, h, \Lambda, \{v^\pm\}]$ is the lattice data of (B, Γ), the assumption that Γ is smooth of degree ≤ 3 implies that v^+ satisfies either $(v^+)^2 = -2$ or both $(v^+)^2 = 0$ and $(v^+, h) = 3$; hence $\sigma(v^+) \in \Lambda_0$ also satisfies

$$(\sigma(v^+))^2 = -2 \quad \text{or} \quad (\sigma(v^+))^2 = 0 \ \text{ and } \ (\sigma(v^+), h_0) = 3.$$

Therefore Lemma 6.13 can be applied, and the assumption that $\sigma(v^+)$ satisfies the vanishing-h^1 condition implies

$$H^1(X_0, \mathcal{D}_0) = 0.$$

After we interchange v_0^+ and v_0^- (and hence $\tilde{\Gamma}_0^+$ and $\tilde{\Gamma}_0^-$) if necessary, there exist a finite number of exceptional (-2)-curves e_i on X_0 such that

$$\tilde{v}_0 = \left[\tilde{\Gamma}_0^+ + \sum e_i\right].$$

Let s_0 be the section of the invertible sheaf $\mathcal{O}\big(\tilde{\Gamma}_0^+ + \sum e_i\big)$ on X_0 such that $s_0 = 0$ defines the divisor $\tilde{\Gamma}_0^+ + \sum e_i$. By Lemma 6.9, there exists a section $s \in H^0(\mathcal{X}_{\tilde{\delta}}, \mathcal{D})$ such that its restriction to X_0 is s_0. We put $s_t := s|_{X_t}$ for $t \neq 0$ and let $\tilde{\Gamma}_t$ be the curve on X_t cut out by $s_t = 0$. Since $\phi_t([\tilde{\Gamma}_t]) = v^+ \in \Lambda$, we have $[\tilde{\Gamma}_t] \in \Lambda_{B_t}$. Since $[\mathcal{E}, h, \Lambda, \{v^\pm\}]$ is the lattice data of (B, Γ) and since $[\tilde{\Gamma}^\pm] \in \mathcal{Z}_n(B)$ with $n = \deg \Gamma = (v^+, h) \leq 3$, we see that if B_τ is lattice-generic with $\tau \neq 0$ then

$$[\tilde{\Gamma}_\tau] \in \mathcal{Z}_n(B_\tau)$$

holds by Theorem 3.2. In particular, if $n < 3$ then $\tilde{\Gamma}_t$ is a (-2)-curve. When $n = 3$ we replace s by $s + s'$, where

$$s' \in H^0(\mathcal{X}_{\tilde{\delta}}, \mathcal{D}(-X_0)) = H^0(\mathcal{X}_{\tilde{\delta}}, \mathcal{D}) \otimes \mathcal{O}_\Delta(-0)$$

is chosen generally, and we assume that $\tilde{\Gamma}_t$ is irreducible. We denote by Γ_t the image of $\tilde{\Gamma}_t$ by the double covering $X_{B_t} \to \mathbb{P}^2$. Then Γ_t is a smooth Z-splitting curve that degenerates to Γ_0. Since the lattice data of (B_t, Γ_t) for $t \neq 0$ is isomorphic to ℓ^P, the analytic family $(B_t, \Gamma_t)_{t\in\Delta}$ of Z-splitting pairs gives rise to the specialization of ℓ^P to ℓ_0^P. □

COMPUTATION 6.17. By Computation 5.27, we have obtained the complete list $\mathtt{LD}_n$ of lattice data of Z-splitting pairs (B, Γ) with $n := \deg \Gamma \leq 2$ as well as the complete list $\mathtt{LD}_3$ of lattice data of Z-splitting pairs (B, Γ) with $z_1(\lambda(B)) = z_2(\lambda(B)) = 0$, $F_B \neq 0$, and Γ a smooth cubic.

For each $\ell^P = [\mathcal{E}, h, \Lambda, S]$ in $\mathtt{LD}_1$ (resp. $\mathtt{LD}_2$), we calculate the class order d of ℓ^P (i.e., the order of $v \in S$ in the finite abelian group $\Lambda/(\langle h\rangle \oplus \langle \mathcal{E}\rangle))$ and confirm that d is either 6, 8, 10, or 12 (resp., 3, 4, 5, 6, 7, or 8).

For each $n = 1, 2$ and the class order d, we denote by $\mathtt{LD}_{n,d}$ the set of lattice data $\ell^P \in \mathtt{LD}_n$ with the class order d and denote by $l_{n,d}^P$ the member of $\mathtt{LD}_{n,d}$ whose total Milnor number $\mu_B = \operatorname{rank}\langle\mathcal{E}\rangle$ is *minimal*. It turns out that the condition that μ_B be minimal determines $l_{n,d}^P$ uniquely and that the corresponding lattice types are equal to $\lambda_{\mathrm{lin},d}^P$ or $\lambda_{\mathrm{con},d}^P$, given in Definition 3.12 or 3.17 according as $n = 1$ or 2. Then, for each ℓ^P in $\mathtt{LD}_{n,d}$ that is not $l_{n,d}^P$, we search for a geometric embedding of $l_{n,d}^P$ into ℓ^P that satisfies the vanishing-h^1 condition and confirm that there exists at least one such embedding. Thus Theorems 3.13 and 3.19 are proved.

We also confirm that there exists unique lattice data l_{QC}^P in $\mathtt{LD}_3$ with μ_B being minimal; that the lattice type corresponding to l_{QC}^P is $\lambda_{QC,n}$; and that, for each piece of lattice data ℓ^P in $\mathtt{LD}_3$ that is not l_{QC}^P, there exists at least one geometric embedding of l_{QC}^P into ℓ^P that satisfies the vanishing-h^1 condition. Thus the second half of Theorem 3.23 is also proved.

Table 7.1 The Isotropic Subgroups H_i

	Generators	
H_0	0	0
H_1	$[[0, 4, 4, 0]]$	Cyclic of order 2
H_2	$[[1, 1, 1, 1]]$	Cyclic of order 8
H_3	$[[2, 2, 2, 0]]$	Cyclic of order 4

7. Demonstration

We demonstrate the calculations for the ADE-type $A_3 + 2A_7$. Let $\langle \mathcal{E} \rangle$ be the negative-definite root lattice of type $A_3 + 2A_7$ with a distinguished fundamental system of roots

$$\mathcal{E} = \{t_1, t_2, t_3\} \perp \{e_1, \dots, e_7\} \perp \{e'_1, \dots, e'_7\},$$

where $\{t_1, t_2, t_3\}$ is of type A_3 with $(t_i, t_{i+1}) = 1$ for $i = 1, 2$ and where $\{e_1, \dots, e_7\}$ and $\{e'_1, \dots, e'_7\}$ are of type A_7 with $(e_i, e_{i+1}) = (e'_i, e'_{i+1}) = 1$ for $i = 1, \dots, 6$. The automorphism group $\mathrm{Aut}(\mathcal{E})$ of $\mathcal{E}$ is isomorphic to $\{\pm 1\} \times (\{\pm 1\} \wr \mathfrak{S}_2)$, where the first factor is the involution $t_1 \leftrightarrow t_3$ of A_3 and where $\{\pm 1\} \wr \mathfrak{S}_2$ is the wreath product of the involution $e_i \leftrightarrow e_{8-i}$ of A_7 and the permutation of the components of $2A_7$. We put

$$\Sigma = \langle \mathcal{E} \rangle \oplus \langle h \rangle,$$

where $h^2 = 2$. Then the discriminant group $\Sigma^\vee / \Sigma$ of Σ is

$$\langle \bar{t}_3^\vee \rangle \oplus \langle \bar{e}_7^\vee \rangle \oplus \langle \bar{e}_7^{\prime\vee} \rangle \oplus \langle \bar{h}^\vee \rangle \cong (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}),$$

where $\bar{x} = x \bmod \Sigma$. Here the discriminant form $q \colon \Sigma^\vee / \Sigma \to \mathbb{Q}/2\mathbb{Z}$ of Σ is given by

$$q(w, x, y, z) = -\frac{3}{4} w^2 - \frac{7}{8} x^2 - \frac{7}{8} y^2 + \frac{1}{2} z^2 \bmod 2\mathbb{Z},$$

where $(w, x, y, z) = w\bar{t}_3^\vee + x\bar{e}_7^\vee + y\bar{e}_7^{\prime\vee} + z\bar{h}^\vee$. We determine all isotropic subgroups H such that the corresponding overlattice $\Lambda = \Lambda(H)$ satisfies the three conditions in Proposition 4.10. Up to the action of $\mathrm{Aut}(\mathcal{E})$, these subgroups are given in Table 7.1. Hence there exist four lattice types $\lambda(H_i)$ of simple sextics B with $R_B = A_3 + 2A_7$. We denote by $B(H_i)$ a lattice-generic member of $\lambda(H_i)$.

Next we calculate the subsets $\mathcal{L}(H_i) := \mathcal{L}_{B(H_i)}$ and $\mathcal{C}(H_i) := \mathcal{C}_{B(H_i)}$ of $\Lambda(H_i)$ for each H_i and then deduce information about the geometry of $B(H_i)$. From now on, vectors in $\Lambda(H_i) \subset \Sigma^\vee$ are written with respect to the basis

$$t_1^\vee, \dots, t_3^\vee, e_1^\vee, \dots, e_7^\vee, e_1^{\prime\vee}, \dots, e_7^{\prime\vee}, h^\vee$$

of $\Sigma^\vee$ that is *dual* to $\mathcal{E} \cup \{h\}$.

(H_0) We have $\mathcal{L}(H_0) = \emptyset$ and $\mathcal{C}(H_0) = \emptyset$. Hence $B(H_0)$ is irreducible. (If $\operatorname{degs} B(H_0) = [3, 3]$, then the two cubic irreducible components would intersect with multiplicity 10.) Moreover, we have $z_1(\lambda(H_0)) = z_2(\lambda(H_0)) = 0$.

(H_1) We have $\mathcal{L}(H_1) = \emptyset$ and $\mathcal{C}(H_1) = \{u\}$, where

$$u := [0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,2].$$

Since u is invariant under the involution on $\Lambda(H_1)$, we have degs $B(H_1) = [2,4]$ with the irreducible component of degree 2 passing through two A_7 points and disjoint from the tacnode A_3. Moreover, we have $z_1(\lambda(H_1)) = z_2(\lambda(H_1)) = 0$. This lattice type is denoted by $\lambda_{\mathfrak{B},n}$ in Proposition 3.11.

(H_2) We have $\mathcal{L}(H_2) = \{v, \iota_B(v)\}$ and $\mathcal{C}(H_2) = \{u\}$, where

$$v := [1,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,1] \neq \iota_B(v)$$

and u is the same vector as in (H_1). Hence we have $B(H_2) \sim_{\mathrm{cfg}} B(H_1)$ as well as $z_1(\lambda(H_2)) = 1$ and $z_2(\lambda(H_2)) = 0$. This lattice type is denoted by $\lambda_{\mathfrak{B},l}$. The class v of the lift of a Z-splitting line is of order 8 in the discriminant group $\Sigma^\vee/\Sigma$. Because there are no Z-splitting lines of class order 8 for simple sextics of total Milnor number < 17, it follows that the Z-splitting line for $B(H_2)$ is the originator of the lineage of Z-splitting lines of class order 8 whose lattice type is denoted by $\lambda^P_{\mathrm{lin},8}$.

(H_3) We have $\mathcal{L}(H_3) = \emptyset$ and $\mathcal{C}(H_3) = \{u, w, \iota_B(w)\}$, where

$$w := [0,1,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,2] \neq \iota_B(w)$$

and u is the same vector as in (H_1). Hence we have $B(H_3) \sim_{\mathrm{cfg}} B(H_1)$ as well as $z_1(\lambda(H_3)) = 0$ and $z_2(\lambda(H_3)) = 1$. This lattice type is denoted by $\lambda_{\mathfrak{B},c}$. The class $w = [\tilde{\Gamma}]$ of the lift of a Z-splitting conic Γ is of order 4 in the discriminant group $\Sigma^\vee/\Sigma$. The conic Γ is tangent to the quartic irreducible component of $B(H_3)$ at the three singular points of $B(H_3)$.

Next we describe the originator of the lineage of Z-splitting conics of class order 4 and how the Z-splitting conic for $B(H_3)$ is obtained from this originator by specialization.

Any simple sextic of total Milnor number < 14 does not have Z-splitting conics of class order 4, and there exists a unique lattice type $\lambda_{\mathfrak{b},c}$ of total Milnor number 14 whose lattice-generic member B' has a Z-splitting conic Γ of class order 4. The ADE-type of the lattice type is $2A_1+4A_3$. Consider the negative-definite root lattice $\langle \mathcal{E}' \rangle$ of type $2A_1 + 4A_3$ with a distinguished fundamental system of roots

$$\mathcal{E}' := \{a^{(1)}\} \perp \{a^{(2)}\} \perp \{b^{(1)}, c^{(1)}, d^{(1)}\} \perp \cdots \perp \{b^{(4)}, c^{(4)}, d^{(4)}\},$$

where $\{a^{(\nu)}\}$ is of type A_1 and $\{b^{(\nu)}, c^{(\nu)}, d^{(\nu)}\}$ is of type A_3 with $(b^{(\nu)}, c^{(\nu)}) = (c^{(\nu)}, d^{(\nu)}) = 1$. We put

$$\Sigma' = \langle \mathcal{E}' \rangle \oplus \langle h \rangle.$$

The discriminant group of Σ' is isomorphic to

$$(\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^4 \oplus (\mathbb{Z}/2\mathbb{Z}),$$

with

$$q'(x_1, x_2, y_1, y_2, y_3, y_4, z) = -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{3}{4}y_1^2 - \frac{3}{4}y_2^2 - \frac{3}{4}y_3^2 - \frac{3}{4}y_4^2 + \frac{1}{2}z^2 \bmod 2\mathbb{Z}.$$

The overlattice $\Lambda_{B'}$ of the lattice type $\lambda_{\mathfrak{b},c}$ corresponds to the isotropic subgroup

$$H' := \langle [1,1,1,1,1,1,0] \rangle,$$

which is cyclic of order 4. We denote vectors of $\Lambda_{B'} \subset (\Sigma')^\vee$ with respect to the basis of $(\Sigma')^\vee$ dual to the basis $\mathcal{E}' \cup \{h\}$ of Σ'. Then the classes of the lifts of the Z-splitting conic Γ' for the lattice-generic member B' of $\lambda_{\mathfrak{b},c}$ are equal to

$$w' := [1,1,0,0,1,0,0,1,0,0,1,0,0,1,2]$$

and $\iota_B(w')$. Let $\sigma : (\Sigma')^\vee \to \Sigma^\vee$ be the homomorphism given by the matrix

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a & -a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -b & -a & -c & b & a & c & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & -a & -b & -b & -a & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c & a & b & b & -a & -b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -c & -a & -b & c & a & b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & a & -b & 0 & 0 & 0 & 0 & 0 & 0 & -b & -a & -c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -b & a & b & 0 & 0 & 0 & 0 & 0 & 0 & b & a & -b & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -c & -a & -b & 0 & 0 & 0 & 0 & 0 & 0 & -b & a & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

where $a = 1/2$, $b = 1/4$, and $c = 3/4$. It can be easily checked that $\sigma(h) = h$, that $\sigma(\mathcal{E}') \subset \langle \mathcal{E} \rangle^+$, and that σ embeds the lattice $\Lambda_{B'} \subset (\Sigma')^\vee$ into the lattice $\Lambda_{B(H_3)} \subset \Sigma^\vee$ primitively. Moreover, we have

$$\sigma(w') = w + t_2 + e'_2.$$

We can easily see that $\sigma(w') = w + t_2 + e'_2$ satisfies the vanishing-h^1 condition. Therefore, $\lambda^P(B(H_3), \Gamma)$ is a specialization of $\lambda^P(B', \Gamma')$.

REMARK 7.1. There are six configuration types and seven lattice types with ADE-type $2A_1 + 4A_3$.

REMARK 7.2. This triple $\{\lambda_{\mathfrak{B},c}, \lambda_{\mathfrak{B},l}, \lambda_{\mathfrak{B},n}\}$ is the example of lattice Zariski triple with the smallest total Milnor number.

REMARK 7.3. Let B_τ be a sextic in the lattice type $\lambda_{\mathfrak{B},\tau}$, where $\tau = c, l, n$, and let $B_\tau = C_\tau \cup Q_\tau$ be the irreducible decomposition of B_τ with $\deg Q_\tau = 4$. Consider the normalization

$$\nu\colon \tilde{Q}_\tau \to Q_\tau$$

of the quartic curve Q_τ with one tacnode. Then $\tilde{Q}_\tau$ is a curve of genus 1. Let $p, q \in \tilde{Q}_\tau$ be the inverse images of the tacnode, and let $s, t \in \tilde{Q}_\tau$ be the inverse images of the two A_7-singular points $C_\tau \cap Q_\tau$. Then, in the elliptic curve $\mathrm{Pic}^0(\tilde{Q}_\tau)$, the order of the class of the divisor $p + q - s - t$ on $\tilde{Q}_\tau$ is 4, 2, or 1 according as $\tau = c, l$, or n.

8. Miscellaneous Facts and Final Remarks

8.1. Numerical Criterion of the Pre-Z-Splittingness

DEFINITION 8.1. Let Γ be a smooth splitting curve for B that is not contained in B. Let P be a singular point of B. We define $\sigma_P(\Gamma) \in \mathbb{Q}$ as follows. If $P \notin \Gamma$, we put $\sigma_P(\Gamma) := 0$. Suppose that $P \in \Gamma$. If P is of type A_l, then

$$\sigma_P(\Gamma) := -m^2/(l+1), \quad \text{where } m = \min(\tau_P(\tilde{\Gamma}^+), l + 1 - \tau_P(\tilde{\Gamma}^+)).$$

(Recall that $\tau_P(\tilde{\Gamma}^+)$ is defined in Definition 3.15.) If P is of type D_m, then

$$\sigma_P(\Gamma) := \begin{cases} -m/4 & \text{if } m \text{ is even and } \tau_P(\tilde{\Gamma}^+) = 1 \text{ or } 2, \\ 1/2 - m/4 & \text{if } m \text{ is odd and } \tau_P(\tilde{\Gamma}^+) = 1 \text{ or } 2, \\ \tau_P(\tilde{\Gamma}^+) - m - 1 & \text{if } \tau_P(\tilde{\Gamma}^+) \geq 3. \end{cases}$$

If P is of type E_n, then $\sigma_P(\Gamma)$ is defined by the following table.

$\tau_P(\tilde{\Gamma}^+)$	1	2	3	4	5	6	7	8
E_6	-2	$-2/3$	$-8/3$	-6	$-8/3$	$-2/3$		
E_7	$-7/2$	-2	-6	-12	$-15/2$	-4	$-3/2$	
E_8	-8	-4	-14	-30	-20	-12	-6	-2

Using Remark 5.10, we can easily check that $\sigma_P(\Gamma)$ does not depend on the choice of the lift $\tilde{\Gamma}^+$.

PROPOSITION 8.2. *Let $\tilde{B} \subset X_B$ be the reduced part of the strict transform of B. Suppose that Γ is a smooth splitting curve for B that is not contained in B. We put*

$$t_\Gamma := (\tilde{B}, \tilde{\Gamma}^+) = (\tilde{B}, \tilde{\Gamma}^-).$$

Then the following inequality holds:

$$(\deg \Gamma)^2/2 + \textstyle\sum_P \sigma_P(\Gamma) \leq t_\Gamma. \tag{8.1}$$

The splitting curve Γ is pre-Z-splitting if and only if the equality holds in (8.1).

Proof. Let $N_{\mathbb{Q}}$ denote the orthogonal complement of the subspace $\Sigma_B \otimes \mathbb{Q} = \Lambda_B \otimes \mathbb{Q}$ in $\mathrm{NS}(X_B) \otimes \mathbb{Q}$. Then the intersection pairing is negative definite on $N_{\mathbb{Q}}$, and the involution ι_B on $\mathrm{NS}(X_B) \otimes \mathbb{Q}$ acts on $N_{\mathbb{Q}}$ by the multiplication by -1. We have a decomposition

$$[\tilde{\Gamma}^{+}] = \frac{\deg \Gamma}{2} h + \sum \gamma_P + n,$$

where $\gamma_P \in \langle \mathcal{E}_P \rangle \otimes \mathbb{Q}$ and $n \in N_{\mathbb{Q}}$. Then we have

$$t_\Gamma = ([\tilde{\Gamma}^{+}], [\tilde{\Gamma}^{-}]) = \frac{(\deg \Gamma)^2}{2} + \sum (\gamma_P, \iota_B(\gamma_P)) - n^2$$

by Lemma 5.5. The value $\sigma_P(\Gamma)$ is defined in such a way that $\sigma_P(\Gamma) = (\gamma_P, \iota_B(\gamma_P))$ holds. Since $n^2 \leq 0$ and $n^2 = 0$ holds if and only if $n = 0$, we obtain the proof. □

EXAMPLE 8.3. Let f and g be general homogeneous polynomials of degree 2 and 3, respectively. The splitting conic $\Gamma = \{f = 0\}$ for a torus sextic $B_{\mathrm{trs}} = \{f^3 + g^2 = 0\}$ is Z-splitting because we have $\deg \Gamma = 2$, $t_\Gamma = 0$, and $\sigma_P(\Gamma) = -1/3$ for each ordinary cusp P of B_{trs}.

REMARK 8.4. As a corollary of the classifications of Z-splitting pairs, we obtain the following. Let (B, Γ) be a lattice-generic Z-splitting pair with $\deg \Gamma \leq 2$. Then $B \cap \Gamma$ is contained in $\operatorname{Sing} B$, and $\tilde{\Gamma}_{+} \cap \tilde{\Gamma}_{-} = \emptyset$.

8.2. Relation between $\sim_{\mathrm{emb}}$ and $\sim_{\mathrm{lat}}$

In many lattice Zariski k-ples, the distinct lattice types have different embedding topology.

THEOREM 8.5. *Suppose that B and B' satisfy $B \sim_{\mathrm{cfg}} B'$. If G_B and $G_{B'}$ have different orders, then $B \not\sim_{\mathrm{emb}} B'$.*

Proof. We can assume that B and B' are lattice-generic. We consider the *transcendental lattices* of X_B and $X_{B'}$ defined by

$$T_B := (\mathrm{NS}(X_B) \hookrightarrow H^2(X_B, \mathbb{Z}))^{\perp}, \qquad T_{B'} := (\mathrm{NS}(X_{B'}) \hookrightarrow H^2(X_{B'}, \mathbb{Z}))^{\perp}.$$

From $B \sim_{\mathrm{cfg}} B'$, we have $R_B = R_{B'}$; hence $\operatorname{disc} \Sigma_B = \operatorname{disc} \Sigma_{B'}$ holds, where disc denotes the discriminant of the lattice. Combining this with $|G_B| \neq |G_{B'}|$, we obtain $\operatorname{disc} \Lambda_B \neq \operatorname{disc} \Lambda_{B'}$. Since $H^2(X_B, \mathbb{Z})$ and $H^2(X_{B'}, \mathbb{Z})$ are unimodular, we obtain

$$\operatorname{disc} T_B \neq \operatorname{disc} T_{B'}.$$

Then $B \not\sim_{\mathrm{emb}} B'$ follows because the transcendental lattice of X_B is a topological invariant of $(\mathbb{P}^2, B)$ for a lattice-generic B, a fact that was proved in [27] and [28]. □

REMARK 8.6. We have not yet obtained any examples of pairs $[B_1, B_2]$ of simple sextics with $B_1 \not\sim_{\mathrm{lat}} B_2$ but $B_1 \sim_{\mathrm{emb}} B_2$. For the example of the lattice Zariski couple $\lambda_{QC,c}$ and $\lambda_{QC,n}$ in Proposition 3.22 we have $|G_B| = |G_{B'}| = 4$, where $B \in \lambda_{QC,c}$ and $B' \in \lambda_{QC,n}$, and hence Theorem 8.5 does not apply. It would be an interesting problem to study the topology of simple sextics in $\lambda_{QC,c}$ and $\lambda_{QC,n}$.

8.3. Examples of Many Z-Splitting Conics

For any lattice type $\lambda(B)$ of simple sextics, we have $z_1(\lambda(B)) \leq 1$. On the other hand, we have lattice types $\lambda(B)$ of simple sextics such that $z_2(\lambda(B)) = 12$

or $z_2(\lambda(B)) = 6$. (These two are the largest and the second-largest values for $z_2(\lambda(B))$.)

Suppose that $z_2(\lambda(B)) = 12$. Then B is a nine-cuspidal sextic. The configuration type of nine-cuspidal sextics B consists of a single lattice type, and the group G_B is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Moreover, the class orders of the twelve Z-splitting conics for B are all 3. A nine-cuspidal sextic B is the dual curve of a smooth cubic curve C, and the nine cusps are in one-to-one correspondence with the inflection points of C. In particular, the set $\operatorname{Sing} B$ has a natural structure of the 2-dimensional affine space over $\mathbb{F}_3$. Each Z-splitting conic Γ passes through six points of $\operatorname{Sing} B$, and the complement $\operatorname{Sing} B \setminus (\operatorname{Sing} B \cap \Gamma)$ is an affine line of $\operatorname{Sing} B$. Thus there is a one-to-one correspondence between the set of Z-splitting conics for B and the set of affine lines of $\operatorname{Sing} B$.

Suppose that $z_2(\lambda(B)) = 6$. Then B is a union of three smooth conics with $R_B = 6A_3$. The configuration type of simple sextics B with $\operatorname{degs} B = [2, 2, 2]$ and $R_B = 6A_3$ consists of a single lattice type, and the group G_B is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Moreover, the class orders of the six Z-splitting conics for B are all 4. Let $B = C_1 + C_2 + C_3$ be a simple sextic in this lattice type. There exists a one-to-one correspondence between the six Z-splitting conics for B and the six tacnodes of B that is described as follows. Let $P \in \operatorname{Sing} B$ be a tacnode that is a tangent point of two distinct conics C_i and C_j. Then there exists a unique Z-splitting conic that does not pass through P but (a) is tangent to both C_i and C_j at the other tacnode $P' \in \operatorname{Sing} B$ on $C_i \cap C_j$ and (b) passes through the other four tacnodes on C_k $(k \neq i, j)$.

8.4. Degeneration of Z-Splitting Conics

Consider the following two lattice types of simple sextics:

$$\lambda_{\mathfrak{A},l} = \lambda_{\text{lin},6} \quad (R_B = 3A_5,\ \operatorname{degs} B = [3,3],\ z_1(\lambda_{\mathfrak{A},l}) = 1);$$
$$\lambda_{\mathfrak{a},c} = \lambda_{\text{con},3} \quad (R_B = 6A_2,\ \operatorname{degs} B = [6],\ z_2(\lambda_{\mathfrak{a},c}) = 1).$$

It is well known that any member of $\lambda_{\mathfrak{a},c} = \lambda_{\text{con},3}$ is defined by an equation of $(2, 3)$-torus type,

$$B : f^3 + g^2 = 0 \quad (\deg f = 2,\ \deg g = 3),$$

whereas it is easy to see that any member of $\lambda_{\mathfrak{A},l} = \lambda_{\text{lin},6}$ is defined by an equation of $(2, 6)$-torus type,

$$B' : l^6 + g^2 = 0 \quad (\deg l = 1,\ \deg g = 3).$$

If the quadratic polynomial f degenerates into l^2, then B degenerates into B' and the Z-splitting conic $\Gamma = \{f = 0\}$ for B degenerates into the double of the Z-splitting line $\Gamma' = \{l = 0\}$ for B'. We can therefore regard the Z-splitting line Γ' as the reduced part of a *nonreduced* Z-splitting conic.

It seems that any Z-splitting line can be obtained as the reduced part of a nonreduced Z-splitting conic as just described. For example, it is quite plausible that there exist the following specializations—from the lattice type λ with $z_2(\lambda) = 1$

to the lattice type λ' with $z_1(\lambda') = 1$—that transform the Z-splitting conic for λ to the double of the Z-splitting line for λ'.

λ	λ'
$\lambda_{\mathfrak{b},c} = \lambda_{\mathrm{con},4}$ ($R_B = 2A_1 + 4A_3$, degs $B = [2,4]$)	$\lambda_{\mathfrak{B},l} = \lambda_{\mathrm{lin},8}$ ($R_B = A_3 + 2A_7$, degs $B = [2,4]$)
$\lambda_{\mathfrak{c},c} = \lambda_{\mathrm{con},5}$ ($R_B = 4A_4$, degs $B = [6]$)	$\lambda_{\mathfrak{C},l} = \lambda_{\mathrm{lin},10}$ ($R_B = 2A_4 + A_9$, degs $B = [1,5]$)
$\lambda_{\mathfrak{d},c} = \lambda_{\mathrm{con},6}$ ($R_B = 2A_1 + 2A_2 + 2A_5$, degs $B = [2,4]$)	$\lambda_{\mathfrak{D},l} = \lambda_{\mathrm{lin},12}$ ($R_B = A_3 + A_5 + A_{11}$, degs $B = [2,4]$)

The adjacency of ADE-types in these conjectural specializations are all of the type $2A_l \to A_{2l+1}$. However, the existence of these specializations has not yet been confirmed.

8.5. Z-Splitting Curves in Positive Characteristics

The study of Z-splitting curves stems from the research of supersingular $K3$ surfaces in characteristic 2. In [24] we developed the theory of Z-splitting curves for purely inseparable double covers of $\mathbb{P}^2$ by supersingular $K3$ surfaces in characteristic 2. The configuration of Z-splitting curves for such a covering is described by a binary linear code of length 21. Using this theory, we have described the stratification of the moduli of polarized supersingular $K3$ surfaces of degree 2 in characteristic 2 by the Artin invariant.

Using the structure theorem of the Néron–Severi lattices of supersingular $K3$ surfaces that is given by Rudakov and Sharfarevich [20], we can construct the theory of Z-splitting curves for supersingular double sextics in odd characteristics. Note that every supersingular $K3$ surface can be obtained as double sextics [23; 25].

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