Michigan Math. J. 59 (2010), 505–515

Nonlinearity of Morphisms in Non-Archimedean and Complex Dynamics

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Dedicated to the memory of Professor Juha Heinonen

1. Introduction

One of the aims of this paper is extending the fundamental Cremer theorem from the iteration theory of one complex variable to the setting of higher-dimensional dynamics over more general valued-fields, not necessarily $\mathbb{C}$. We note that analytic function theory over such fields was already well prepared in the fundamental work [A] around 1960.

Let K be a commutative *algebraically closed* field that is complete and non-trivial with respect to an absolute value (or valuation) $|\cdot|$. Then $|\cdot|$ is said to be *non-Archimedean* if, for all $z, w \in K$, $|z - w| \le \max\{|z|, |w|\}$. Otherwise, $|\cdot|$ is said to be *Archimedean,* in which case K is topologically isomorphic to $\mathbb{C}$ (with Hermitian norm). We extend $|\cdot|$ to K^ℓ ($\ell \in \mathbb{N}$) as the maximum norm $|Z| = |Z|_\ell = \max_{j=1,\dots,\ell}|z_j|$ for $Z = (z_1, \dots, z_\ell)$. We consider the polydisk

$$\bar{P}(Z_0, r) = \bar{P}^\ell(Z_0, r) := \{Z \in K^\ell;\ |Z - Z_0| \le r\}$$

for $Z_0 \in K^\ell$ and $r > 0$. The extended $|\cdot|_\ell$ is non-Archimedean if and only if the original $|\cdot|_1$ is also, and in this case

$$\operatorname{int} \bar{P}(Z_0, r) = \bar{P}(Z_0, r).$$

We denote the origin in K^ℓ by $O = O_\ell$. In the Archimedean case $K = \mathbb{C}$, $\mathbb{C}^\ell$ also has the Hermitian norm $\|\cdot\| = \|\cdot\|_\ell$ ($\asymp |\cdot|_\ell$ uniformly).

Let $\pi\colon K^{n+1} \setminus \{O\} \to \mathbb{P}^n(K)$ be the canonical projection. Set the integer $\ell(n) = \binom{n+1}{2}$ so that $\bigwedge^2 K^{n+1} \cong K^{\ell(n)}$ (cf. [Ko, Sec. 8.1]). We equip $\mathbb{P}^n(K)$ with the *chordal distance* $[z, w]$ between $z, w \in \mathbb{P}^n(K)$, defined as

$$[z, w] := \begin{cases} \dfrac{|Z \wedge W|_{\ell(n)}}{|Z|_{n+1}|W|_{n+1}} \le 1 & (|\cdot| \text{ is non-Archimedean}), \\[2ex] \dfrac{\|Z \wedge W\|_{\ell(n)}}{\|Z\|_{n+1}\|W\|_{n+1}} \le 1 & (|\cdot| \text{ is Archimedean}), \end{cases} \tag{1.1}$$

where $Z \in \pi^{-1}(z)$ and $W \in \pi^{-1}(w)$. For $z_0 \in \mathbb{P}^n(K)$ and $r > 0$, we consider the ball

$$\bar{B}(z_0, r) := \{z \in \mathbb{P}^n(K);\ [z, z_0] \le r\}.$$

Received October 27, 2008. Revision received May 27, 2010.

Let $f\colon \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a (finite) *morphism*—that is, there is a homogeneous polynomial map $F\colon K^{n+1} \to K^{n+1}$, which is called a *lift* of f, such that $F^{-1}(O) = \{O\}$ and

$$\pi \circ F = f \circ \pi. \tag{1.2}$$

The (algebraic) degree $d = \deg f$ is that of F as a homogeneous polynomial map. The *Fatou set* $F(f)$ is the largest open set at each point of which the family $\{f^k; k \in \mathbb{N}\}$ is equicontinuous, and the *Julia set* is $J(f) := \mathbb{P}^n(K) \setminus F(f)$. In the Archimedean case, $J(f) \neq \emptyset$ if $d \geq 2$. In the non-Archimedean case, $J(f)$ may be empty even if $d \geq 2$ (cf. [KS, Thm. 10, Rem. 12]). One of the results is the following.

THEOREM 1 (nonlinearity of morphisms). *Let $f\colon \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a morphism of degree $d > 0$. If there exist a ball $\bar{B}(z_0, r) \subset \mathbb{P}^n(K)$ and a morphism $g\colon \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ such that*

$$\liminf_{k\to\infty} \frac{1}{d^k} \log \sup_{\bar{B}(z_0,r)} [f^k, g] = -\infty, \tag{1.3}$$

then either f is linear (i.e., $d = 1$) or $J(f) = \emptyset$.

REMARK 1.4. From the proof, we may replace the second assertion $J(f) = \emptyset$ by the quantitative one: for *any* ball $\bar{B}(z_0, r) \subset \mathbb{P}^n(K)$, (1.3) holds. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ Liouville enough, the linear map $f_\alpha(z) = e^{2i\pi\alpha} z$ ($J(f) = \emptyset$) satisfies (1.3) for $g = \mathrm{Id}_{\mathbb{P}^1(\mathbb{C})}$ (and any $\bar{B}(z_0, r) \subset \mathbb{P}^n(\mathbb{C})$).

We next give applications of Theorem 1.

Analytic Linearization over a Field K

Consider the K-algebra

$$\mathcal{O}_n \cong K\{X_1, \dots, X_n\} = \Big\{ f = \sum c_I X^I;\ \limsup_{|I|\to\infty} |c_I|^{1/|I|} =: r_f^{-1} < \infty \Big\}$$

of all germs of analytic functions at O, where $I = (i_1, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index, $X_1^{i_1} \cdots X_n^{i_n}$ is denoted by X^I, and we put $|I| := i_1 + \cdots + i_n$. For the germ $\phi = (f_1, \dots, f_n) \in (\mathcal{O}_n)^n$ of an analytic *map,* we put $r_\phi := \min_{i=1,\dots,n} r_{f_i}$ and identify the linear part of ϕ at O with

$$A_\phi := \left(\frac{\partial f_i}{\partial X_j}(O) \right)_{i,j=1,\dots,n} \in M(n, K) \cong \mathrm{End}(K^n).$$

We also denote the operator norm on $M(n, K)$ by $|\cdot|$.

A germ $\phi = (f_1, \dots, f_n) \in (\mathcal{O}_n)^n$ fixing O is (analytically) *linearizable* if there is an $H \in (\mathcal{O}_n)^n$ fixing O such that $A_H = I_n$ (unit matrix) and H satisfies the *Schröder* (or Poincáre) equation

$$\phi \circ H = H \circ A_\phi. \tag{1.5}$$

From Siegel and Sternberg [Sie; St] and its non-Archimedean version [HeY], ϕ is linearizable if A_ϕ is diagonalizable and the eigenvalues $\lambda_1, \dots, \lambda_n$ of A_ϕ satisfy the *Diophantine* condition: there exist $C > 0$ and $\beta \geq 0$ such that, for every $I = (i_1, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$ with $|I| \geq 1$,

$$|(\lambda_1^{i_1} \cdots \lambda_n^{i_n}) - 1| \geq \frac{C}{|I|^\beta}.$$

The case $|A_\phi| < 1$ over $\mathbb{C}$ is studied in a quite general setting in [BeDM].

Consider the inverse of a coordinate chart,

$$\sigma : K^n \ni (z_1, \dots, z_n) \mapsto (1 : z_1 : \cdots : z_n) \in \mathbb{P}^n(K),$$

that is *locally uniformly bi-Lipschitz* at O. Let $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a morphism fixing $z_0 \in \mathbb{P}^n(K)$. Assuming that $z_0 = \sigma(O)$ without loss of generality, we say f is *linearizable* at z_0 if the germ $\phi_f \in (\mathcal{O}_n)^n$ of the analytic map $\sigma^{-1} \circ f \circ \sigma : \bar{P}^n(O, r) \to K^n$ is linearizable.

THEOREM 2 (nonresonance). *Let $f : \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a morphism of degree $d \geq 2$ that fixes $z_0 \in \mathbb{P}^n(K)$, and suppose that $J(f) \neq \emptyset$. If f is linearizable at z_0 and $|A_{\phi_f}| \leq 1$, then*

$$\liminf_{k \to \infty} \frac{1}{d^k} \log |(A_{\phi_f})^k - I_n| > -\infty. \tag{1.6}$$

If in addition A_{ϕ_f} is diagonalizable, then its eigenvalues $\lambda_1, \dots, \lambda_n$ satisfy

$$\liminf_{k \to \infty} \frac{1}{d^k} \log \max_{j=1,\dots,n} |\lambda_j^k - 1| > -\infty. \tag{1.7}$$

REMARK 1.8. The boundedness (1.6) is regarded as a higher-dimensional version of the Cremer condition [C, p. 157].

Singular Domain over the Field $\mathbb{C}$

Suppose now that $|\cdot|$ is Archimedean, and identify K with $\mathbb{C}$.

Let $f : \mathbb{P}^n = \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n$ be a morphism, which is now holomorphic, of degree $d \geq 2$. Each component D of $F(f)$—a so-called *Fatou component* of f—is Stein and Kobayashi hyperbolic [U1]. In particular, D is holomorphically separable and the biholomorphic automorphism $\mathrm{Aut}(D)$ is a Lie group.

If there is an $(f^{k_j}) \subset (f^k)$ that converges to Id_D locally uniformly on D, then $f^p(D) = D$ for some $p \in \mathbb{N}$ (D is *cyclic*) and, moreover, $f^p|_D \in \mathrm{Aut}(D)$. Following Fatou [Fa, Sec. 28], we call such D a *singular domain* (un domaine *singulier*) of f, which is also known as a *Siegel* domain [FSi2] and a *rotation* domain [U2]. We find several nice (higher-dimensional) examples in [Mi].

When $n = 1$, a singular domain D is either a Siegel disk or an Herman ring. When $n \geq 2$, the following partial analogue is known. Let G be the closed subgroup generated by $f^p|_D$ in $\mathrm{Aut}(D)$, let G_0 be the component of G containing Id_D, and put

$$q := \min\{j \in p\mathbb{N};\ f^j|_D \in G_0\}.$$

Then there is a Lie group isomorphism $G_0 \to \mathbb{T}^s$ for some $s \in \{1, \dots, n\}$ that maps $f^q|_D$ to $(e^{2i\pi\alpha_1}, \dots, e^{2i\pi\alpha_s})$ for some $\alpha_1, \dots, \alpha_s \in \mathbb{R} \setminus \mathbb{Q}$ (see [FSi2; Mi; U2]). In the maximal case $s = n$, we say that the singular domain D is *of maximal type*.

A singular domain D of maximal type is exactly a generalization of one-dimensional Siegel disks and Herman rings: put $\lambda_j := e^{2i\pi\alpha_j}$ $(j = 1, \dots, n)$ and $\Lambda := \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in M(n, \mathbb{C}) \cong \mathrm{End}(\mathbb{C}^n)$. By [BaBDa, Thm. 1], there is a biholomorphism (*linearization map*) Φ from a Reinhardt domain $U \subset \mathbb{C}^n$ to D such that the Schröder equation

$$f^q \circ \Phi = \Phi \circ \Lambda$$

holds on U. We have the following result.

THEOREM 3 (a priori bound). *Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$. If a singular domain D of f is of maximal type, then* (*with notation as before*) *D satisfies*

$$\lim_{k\to\infty} \frac{1}{d^{qk}} \log \max_{j=1,\dots,n} |\lambda_j^k - 1| = 0. \tag{1.9}$$

In the case of $n = 1$, every singular domain of f is of maximal type. In this case, (1.9) is essentially proved in [FSi1, p. 169] by pluripotential theory and in [O, Main Thm. 3] by a Nevanlinna theoretical argument. Both proofs contain some one-dimensional arguments that are not easily extended to higher dimensions.

The proofs of Theorems 1, 2, and 3 are motivated by the theory of arithmetic height and do not rely on pluripotential theory.

Finally, we give a *vanishing* result on the Valiron deficiency

$$\delta_V(\mathrm{Id}_{\mathbb{P}^n}, (f^k)) := \limsup_{k\to\infty} \frac{1}{d^k} \int_{\mathbb{P}^n} \log \frac{1}{[f^k, \mathrm{Id}]} \, \mathrm{d}\omega_{\mathrm{FS}}^n$$

of (f^k) for $\mathrm{Id}_{\mathbb{P}^n}$ (cf. [DrO]). Here we denote the Fubini–Study Kähler form on $\mathbb{P}^n$ by ω_{FS}.

THEOREM 4 (a vanishing theorem). *If every singular domain of f is of maximal type, then*

$$\delta_V(\mathrm{Id}_{\mathbb{P}^n}, (f^k)) = 0. \tag{1.10}$$

We expect that the conclusion (1.10) still remains true with no maximality assumption on singular domains.

2. Proof of Theorem 1

Let $f\colon \mathbb{P}^n(K) \to \mathbb{P}^n(K)$ be a *morphism* of degree $d \geq 2$ and F a lift of f. We gather some basic facts about the *Green function* associated to F and its application. For the details, see [HuP] in the Archimedean case and [BakRu, Sec. 3; KS, Sec. 2] in the non-Archimedean case.

The homogeneity of F and elimination theory (cf. [V]) yield that

$$\sup_{K^{n+1}\setminus\{O\}} \left| \frac{1}{d} \log|F| - \log|\cdot| \right| < \infty,$$

so that $((\log|F^k|)/d^k)$ is a Cauchy sequence, and the limit

$$G^F := \lim_{k\to\infty} \frac{1}{d^k} \log|F^k| \colon K^{n+1} \setminus \{O\} \to \mathbb{R} \tag{2.1}$$

is called the *Green function* of F. More precisely, the convergence is uniform, so that G^F is continuous on $K^{n+1} \setminus \{O\}$ and

$$\sup_{K^{n+1}\setminus\{O\}} |G^F - \log|\cdot|| < \infty. \tag{2.2}$$

In the Archimedean case, by [U1, Thm. 2.2], $Z_0 \in \pi^{-1}(F(f))$ if and only if G^F is locally pluriharmonic at Z_0. As a consequence, we have the following result.

THEOREM 2.3 [U1, Thm. 2.2]. *Suppose that $|\cdot|$ is Archimedean. If there is an infinite subfamily of $\{f^k;\ k \in \mathbb{N}\}$ equicontinuous at every point of a neighborhood of $z_0 \in \mathbb{P}^n(K)$, then $z_0 \in F(f)$.*

In non-Archimedean case, by [KS, Thm. 23], $z_0 \in F(f)$ if and only if

$$\pi_*(G^F - \log|\cdot|) \colon \mathbb{P}^n(K) \to \mathbb{R}$$

is locally constant at z_0. We point out that their proof yields more.

THEOREM 2.4. *Suppose that $|\cdot|$ is non-Archimedean. If there is an infinite subfamily of $\{f^k\}$ equicontinuous at $z_0 \in \mathbb{P}^n(K)$, then $z_0 \in F(f)$.*

Now we prove Theorem 1. Take f, g, and $\bar{B}(z_0, r)$ (satisfying (1.3)) as in Theorem 1, and let $F, G \colon K^{n+1} \to K^{n+1}$ be lifts of f and g, respectively. Since $\pi : K^{n+1} \to \mathbb{P}^n(K)$ is surjective, open, and continuous, there exist $Z_0 \in \pi^{-1}(z_0)$ and $s > 0$ such that $\pi(\bar{P}(Z_0, s)) \subset \bar{B}(z_0, r)$. Supposing that $d = \deg f \geq 2$ and $J(f) \neq \emptyset$, we will derive a contradiction.

First, we consider the case that $\operatorname{int} \bar{P}(Z_0, s) \cap \pi^{-1}(J(f)) \neq \emptyset$. Let $(k_i) \subset \mathbb{N}$ be an infinite sequence such that

$$\lim_{i\to\infty} \frac{1}{d^{k_i}} \log \sup_{\pi(\bar{P}(Z_0,s))} [f^{k_i}, g] = \liminf_{k\to\infty} \frac{1}{d^k} \log \sup_{\pi(\bar{P}(Z_0,s))} [f^k, g].$$

By Theorems 2.3 and 2.4, there exists a $w_0 \in \pi(\operatorname{int} \bar{P}(Z_0, s))$ where $\{f^{k_i}\}$ is not equicontinuous. Hence there exist $(k_j) \subset (k_i)$ and $(w_j) \subset \mathbb{P}^n(K)$ such that $\lim_{j\to\infty} w_j = w_0$ and $\liminf_{j\to\infty}[f^{k_j}(w_j), f^{k_j}(w_0)] > 0$; then the continuity of g at w_0 implies

$$\liminf_{j\to\infty} \sup_{\pi(\bar{P}(Z_0,s))} [f^{k_j}, g] > 0.$$

Therefore, in this case, we have proved that

$$\liminf_{k\to\infty} \frac{1}{d^k} \log \sup_{\pi(\bar{P}(Z_0,s))} [f^k, g] \geq 0. \tag{2.5}$$

We prepare a comparison estimate (2.6) in what follows. For every $k \in \mathbb{N}$, $F^k \wedge G \colon K^{n+1} \to K^{\ell(n)}$ is a polynomial map of degree $d^k + \deg g$. For every $Z_0 \in K^{n+1}$ and every $s > 0$, from homogeneous expansion of $F^k \wedge G$ at Z_0, we have by Cauchy estimate,

$$|(F^k \wedge G)(Z)| \le \sum_{I \in \mathbb{Z}_{\ge 0}^{n+1},\, |I| \le d^k + \deg g} \frac{\sup_{\bar{P}(Z_0,s)} |F^k \wedge G|}{s^{|I|}} |Z - Z_0|^{|I|}$$

in the Archimedean case; by the maximum modulus principle (cf. (3.1)), we obtain

$$|(F^k \wedge G)(Z)| \le \max_{I \in \mathbb{Z}_{\ge 0}^{n+1},\, |I| \le d^k + \deg g} \frac{\sup_{\bar{P}(Z_0,s)} |F^k \wedge G|}{s^{|I|}} |Z - Z_0|^{|I|}$$

in the non-Archimedean case. In each case, if $\bar{P}(Z_0, s) \subset K^{n+1} \setminus \{O\}$ then, for every $Z' \notin \bar{P}(Z_0, s) \cup \{O\}$ and every $r' > 0$ small enough, we have

$$\begin{aligned} &\frac{1}{d^k} \log \sup_{\bar{P}(Z',r')} |F^k \wedge G| \\ &\qquad \le \frac{1}{d^k} \log \sup_{\bar{P}(Z_0,s)} |F^k \wedge G| + \left(1 + \frac{\deg g}{d^k}\right) \log \frac{|Z' - Z_0| + r'}{s} + o(1), \end{aligned}$$

where the $o(1)$ term appears only when K is Archimedean and equals

$$d^{-k} \log\Bigl(\sum_{I \in \mathbb{Z}_{\ge 0}^{n+1},\, |I| \le d^k + \deg g} 1 \Bigr) = O(kd^{-k}) \quad \text{as } k \to \infty.$$

Since $\|\cdot\| \asymp |\cdot|$ uniformly, we may replace $|F^k \wedge G|$ by $\|F^k \wedge G\|$ in the Archimedean case. This, together with (1.1), (1.2), and (2.1), implies the comparison estimate

$$\begin{aligned} &\liminf_{k \to \infty} \frac{1}{d^k} \log \sup_{\pi(\bar{P}(Z',r'))} [f^k, g] + \inf_{\bar{P}(Z',r')} G^F \\ &\quad \le \liminf_{k \to \infty} \frac{1}{d^k} \log \sup_{\pi(\bar{P}(Z_0,s))} [f^k, g] + \sup_{\bar{P}(Z_0,s)} G^F + \log \frac{|Z' - Z_0| + r'}{s}. \end{aligned} \tag{2.6}$$

Now we suppose that $\operatorname{int} \bar{P}(Z_0, s) \cap \pi^{-1}(J(f)) = \emptyset$. Decreasing $s > 0$ if necessary, we also assume that $\bar{P}(Z_0, s) \cap \pi^{-1}(J(f)) = \emptyset$; then there is a $Z' \in \pi^{-1}(J(f)) \setminus \bar{P}(Z_0, s)$. If $r' > 0$ is small enough, then by (2.6) (and (2.5) for $Z_0 = Z', s = r'$),

$$\begin{aligned} &0 + \inf_{\bar{P}(Z',r')} G^F \\ &\qquad \le \liminf_{k \to \infty} \frac{1}{d^k} \log \sup_{\pi(\bar{P}(Z_0,s))} [f^k, g] + \sup_{\bar{P}(Z_0,s)} G^F + \log \frac{|Z' - Z_0| + r'}{s}, \end{aligned}$$

which together with (2.2) implies that

$$-\infty < \liminf_{k \to \infty} \frac{1}{d^k} \log \sup_{\pi(\bar{P}(Z_0,s))} [f^k, g].$$

The proof of Theorem 1 is now complete.

3. Proof of Theorem 2

Consider a germ

$$h = \sum c_I X^I \in \mathcal{O}_n.$$

For every $r \in (0, r_h) \cap |K^*|$ $(\limsup_{|I|\to\infty} |c_I|^{1/|I|} =: r_h^{-1})$, h induces a (rigid analytic) function

$$\bar{P}^n(O,r) \ni Z \mapsto h(Z) = \sum c_I Z^I \in K,$$

which is (uniformly) Lipschitz continuous on $\bar{P}^n(O,r)$.

In the non-Archimedean case, the *maximum modulus principle*

$$(|h|_r :=) \sup_I |c_I| r^{|I|} = \sup_{Z \in \bar{P}^n(O,r)} |h(Z)| \tag{3.1}$$

holds and so the Lipschitz constant of h can be chosen as $|h|_r/r$ (see [BoGR, Sec. 5.1.4; Hs, Prop. 1.1; KS, Lemma 21]).

Now we prove Theorem 2. We continue to use the same notation as in Section 1. Suppose there is an $H \in (\mathcal{O}_n)^n$ fixing O such that

$$A_H = I_n \quad \text{and} \quad ((\sigma^{-1} \circ f \circ \sigma) \circ H =)\ \phi_f \circ H = H \circ A_{\phi_f}.$$

Fix $r \in (0, r_{\phi_f})$ so small that $\sigma\colon \bar{P}^n(O,r) \to \sigma(\bar{P}^n(O,r))$ (we normalized as $\sigma(O) = z_0$) is bi-Lipschitz, and choose $s \in (0, r_H)$ such that $H(\bar{P}^n(O,s)) \subset \bar{P}^n(O,r)$. From the assumption $|A_{\phi_f}| \le 1$, it follows that for every $k \in \mathbb{N}$, $(A_{\phi_f})^k(\bar{P}^n(O,s)) \subset \bar{P}^n(O,s)$ and so

$$f^k \circ (\sigma \circ H) = (\sigma \circ H) \circ (A_{\phi_f})^k$$

on $\bar{P}^n(O,s)$. Since $\sigma \circ H\colon \bar{P}^n(O,s) \to \mathbb{P}^n(K)$ is Lipschitz, we have

$$\begin{aligned}
&\liminf_{k\to\infty} \frac{1}{d^k} \log \sup_{(\sigma\circ H)(\bar{P}^n(O,s))} [f^k, \mathrm{Id}_{\mathbb{P}^n(K)}] \\
&\quad \le \liminf_{k\to\infty} \frac{1}{d^k} \log \sup_{\bar{P}^n(O,s)} [(\sigma \circ H) \circ (A_{\phi_f})^k, \sigma \circ H] \\
&\quad \le \liminf_{k\to\infty} \frac{1}{d^k} \sup_{\bar{P}^n(O,s)} \log |(A_{\phi_f})^k - I_n| \le \liminf_{k\to\infty} \frac{1}{d^k} \log |(A_{\phi_f})^k - I_n|,
\end{aligned}$$

which together with Theorem 1 completes the proof so long as $H(\bar{P}^n(O,s))$ is open (then so is $(\sigma \circ H)(\bar{P}^n(O,s)))$ for $s > 0$ small enough. Indeed, we have the following statement.

THEOREM 3.2 (inverse function theorem; see [A, Sec. 10, Thm. 10]). *Let $H = (f_1, \dots, f_n) \in (\mathcal{O}_n)^n$ fix O. If* $\det A_H \ne 0$, *then for $s > 0$ small enough, $H\colon \bar{P}^n(O,s) \to H(\bar{P}^n(O,s))$ is a bianalytic homeomorphism.*

The proof of Theorem 2 is now complete.

4. Proof of Theorem 3

Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$, and let D be a singular domain of f with $f^p(D) = D$. Suppose that D is of maximal type; we continue to use the same notation ($q \in p\mathbb{N}$, $\lambda_1, \dots, \lambda_n \in S^1$, $\Lambda \in M(n, \mathbb{C})$, $\Phi\colon U \to D$) as in Section 1.

For every $\varepsilon > 0$, there exist $Z_0 \in \mathbb{C}^{n+1} \setminus \{O\}$ and $s, R > 0$ such that

$$\bar{P}(Z_0, s) \subset \pi^{-1}(D) \ (\subset \pi^{-1}(F(f))), \qquad P(Z_0, R) \cap \pi^{-1}(J(f)) \neq \emptyset,$$

$$0 < \log \frac{R}{s} < \frac{\varepsilon}{3}, \qquad \sup_{Z, W \in \bar{P}(Z_0, R)} |G^F(Z) - G^F(W)| < \frac{\varepsilon}{3}.$$

After choosing $Z' \in P(Z_0, R) \cap \pi^{-1}(J(f))$ and $r' > 0$ small enough, from (2.6) (and (2.5) for $Z_0 = Z'$, $s = r'$), we have

$$0 - \frac{\varepsilon}{3} < \liminf_{k \to \infty} \frac{1}{d^k} \log \sup_{\pi(\bar{P}(Z_0, s))} [f^k, \mathrm{Id}_{\mathbb{P}^n}] + 2 \cdot \frac{\varepsilon}{3}. \tag{4.1}$$

Put $V := \pi(\bar{P}(Z_0, s))$ $(\subset D)$. Since the restriction of Φ^{-1} to $\overline{\bigcup_{k \in \mathbb{N}} f^k(V)}$ is bi-Lipschitz, from (4.1) we obtain

$$\begin{aligned} -\varepsilon &< \liminf_{k \to \infty} \frac{1}{d^{qk}} \log \sup_V [f^{qk}, \mathrm{Id}_{\mathbb{P}^n}] \\ &= \liminf_{k \to \infty} \frac{1}{d^{qk}} \log \sup_{\Phi^{-1}(V)} |\Lambda^k - \mathrm{Id}_{\mathbb{C}^n}| \\ &= \liminf_{k \to \infty} \frac{1}{d^{qk}} \log \max_{j=1,\dots,n} |\lambda_j^k - 1|. \end{aligned}$$

The proof of Theorem 3 is now complete.

5. Proof of Theorem 4

Let $f\colon \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$, and let F be a lift of f. In this section, we denote the Lebesgue measure on $\mathbb{C}^{n+1}$ by m.

On every compact set in $\mathbb{C}^{n+1}$, by (2.1) and (2.2) we have that $((\log\|F^k\|)/d^k)$ is uniformly bounded; hence, by (1.1), $((\log\|F^k \wedge \mathrm{Id}\|)/d^k)$ is uniformly bounded from above.

Let $(k_i) \subset \mathbb{N}$ be an infinite sequence. If $((\log\|F^{k_i} \wedge \mathrm{Id}\|)/d^{k_i})$ converges to $-\infty$ uniformly on the compact set $\{\|Z\| = 1\}$, then it follows from (1.1), (2.1), and (2.2) that

$$\lim_{i \to \infty} \frac{1}{d^{k_i}} \log[f^{k_i}, \mathrm{Id}_{\mathbb{P}^n}] = -\infty$$

uniformly on $\pi(\{\|Z\| = 1\}) = \mathbb{P}^n$. Together with Theorem 2.3, this implies that $\mathbb{P}^n = F(f)$, which is a contradiction.

The bounds on $((\log\|F^{k_i} \wedge \mathrm{Id}\|)/d^{k_i})$ yield, by [H, Thm. 4.1.9(a)] (see also [Az, Thm. 1.1.1]), a subsequence $(k_j) \subset (k_i)$ such that the plurisubharmonic limit

$$\phi := \lim_{j\to\infty} \frac{1}{d^{k_j}} \log\|F^{k_j} \wedge \mathrm{Id}\| \tag{5.1}$$

exists in $L^1_{\mathrm{loc}}(\mathbb{C}^{n+1}, m)$. By (1.1) and (2.1), we then have $\phi \le G^F$.

We assume that $\{\phi - G^F < 0\} \neq \emptyset$. Since $\phi - G^F$ is upper semicontinuous, we can choose $\bar{P}(Z_0, r) \subset \{\phi - G^F < 0\}$; then, by a version of the Hartogs lemma [H, Thm. 4.1.9(b)], it follows that

$$\limsup_{j\to\infty} \sup_{\bar{P}(Z_0,r)} \left(\frac{1}{d^{k_j}} \log\|F^{k_j} \wedge \mathrm{Id}\| - G^F \right) \le \sup_{\bar{P}(Z_0,r)} (\phi - G^F) < 0.$$

Hence, by (1.1) and (2.1) we have

$$\limsup_{j\to\infty} \sup_{\pi(\bar{P}(Z_0,r))} \frac{1}{d^{k_j}} \log[f^{k_j}, \mathrm{Id}_{\mathbb{P}^n}] < 0. \tag{5.2}$$

Therefore, $\pi(Z_0)$ is contained in a Fatou component D of f, which must be a singular domain of f with, say, $f^p(D) = D$.

Suppose that D is of maximal type; we continue to use the same notation ($G_0 \subset G \subset \mathrm{Aut}(D)$, $q \in p\mathbb{N}$, $\lambda_1, \dots, \lambda_n \in S^1$, $\Lambda \in M(n, \mathbb{C})$, $\Phi \colon U \to D$) as in Section 1. From (5.2) (and the identity theorem), $f^{k_j}|_D$ tends to Id_D locally uniformly on D; thus, for every $j \in \mathbb{N}$ large enough, we have ($k_j \in p\mathbb{N}$ and) $f^{k_j}|_D \in G_0$.

Lemma 5.3. $G_0 \cap \{f^k|_D; k \in p\mathbb{N}\} = \{f^k|_D; k \in q\mathbb{N}\}$.

Proof. If $f^k|_D \in G_0$, then writing $k = Qq + r$ ($Q \in \mathbb{N} \cup \{0\}$, $r \in \{0, 1, \dots, q-1\} \cap p\mathbb{N}$) yields $f^r|_D = f^k|_D \circ f^{-Qq}|_D \in G_0$, so that $r = 0$ from the minimality of q. The reverse inclusion is clear. □

Put $\tilde{k}_j := k_j/q \in \mathbb{N}$, and choose $z_0 \in D$ such that $\Phi^{-1}(z_0) = (w_1, \dots, w_n)$ satisfies $\min_{j=1,\dots,n} |w_j| > 0$. Because the restriction of Φ^{-1} to $\overline{\{f^k(z_0); k \in \mathbb{N}\}}$ is bi-Lipschitz, we have

$$\begin{aligned} 0 &> \limsup_{j\to\infty} \frac{1}{d^{k_j}} \log[f^{k_j}(z_0), z_0] \\ &= \limsup_{j\to\infty} \frac{1}{d^{q\tilde{k}_j}} \log|\Lambda^{\tilde{k}_j}(\Phi^{-1}(z_0)) - \Phi^{-1}(z_0)| \\ &\ge \liminf_{k\to\infty} \frac{1}{d^{qk}} \log \max_{j=1,\dots,n} |\lambda_j^k - 1|, \end{aligned}$$

which contradicts (1.9).

We have proved that if every singular domain of f is of maximal type, then in $L^1_{\mathrm{loc}}(\mathbb{C}^{n+1}, m)$,

$$\lim_{k\to\infty} \frac{1}{d^k} \log\|F^k \wedge \mathrm{Id}\| = G^F.$$

This, together with (1.1) and (2.1), implies (1.10) by a change of variables under the projection π.

The proof of Theorem 4 is now complete.

REMARK 5.4. The argument deriving (5.1) and (5.2) from [H, Thm. 4.1.9] is similar to that in [FSi1, p. 169]. If we choose the $(k_i) \subset \mathbb{N}$ so that

$$\lim_{i\to\infty} \frac{1}{d^{k_i}} \int_{\mathbb{P}^n} \log \frac{1}{[f^{k_i}, \mathrm{Id}]} \, \mathrm{d}\omega_{\mathrm{FS}}^n = \delta_V((f^k), \mathrm{Id}_{\mathbb{P}^n}),$$

then (5.1) implies—by an argument similar to the one in the proof of Theorem 4—that, even if f has a singular domain that is not of maximal type,

$$\delta_V((f^k), \mathrm{Id}_{\mathbb{P}^n}) < \infty. \tag{5.5}$$

This finiteness proves (1.7) for the Archimedean case $K = \mathbb{C}$.

ACKNOWLEDGMENTS. The author thanks Professor Tetsuo Ueda for valuable comments and the referee for careful scrutiny. This research was partially supported by JSPS Grant-in-Aid for Young Scientists (B), 19740068.

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