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# The Rationality of the Moduli Space of Genus-4 Curves Endowed with an Order-3 Subgroup of Their Jacobian

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## 0. Introduction

Let $C$ be a smooth, irreducible complex projective curve of genus $g$ and let $\eta \in \mathrm{Pic}^0(C)$ be a nontrivial $n$th root of the trivial bundle $\mathcal{O}_C$. For several different reasons, special attention has been paid, now and in the recent past, to the moduli spaces $\mathcal{R}_{g,n}$ of pairs $(C, \eta)$ as above and to its possible compactifications (see e.g. [CapCasC]).

For instance, they are generalizations of the case $n = 2$, the so-called *Prym moduli spaces,* usually denoted by $\mathcal{R}_g$. Since they are related to the theory of Prym varieties, the interest in this case occupies a prominent position. In particular, many results on the Kodaira dimension of $\mathcal{R}_g$ are now available, while classical geometric descriptions of $\mathcal{R}_g$ exist for $g \leq 7$. More precisely, let us mention that Farkas and Ludwig [FLu] proved that $\mathcal{R}_g$ is of general type for $g \geq 14$ and $g \neq 15$. On the other hand, unirational parameterizations of $\mathcal{R}_g$ are known for $g \leq 7$ [Cat, D, Do, ILoS, Ve1, Ve2].

One can also consider the moduli spaces $\mathcal{R}_{g,\langle n\rangle}$ of pairs $(C, \mathbb{Z}/n\mathbb{Z})$, where $C$ is a smooth, irreducible complex projective curve of genus $g$ and $\mathbb{Z}/n\mathbb{Z}$ is a cyclic subgroup of order $n$ of $\mathrm{Pic}^0(C)$. As $\mathcal{R}_{g,\langle 2\rangle} = \mathcal{R}_{g,2}$, these mouli spaces are generalizing the Prym moduli spaces in a (slightly) different way. In contrast to the case $n = 2$, not very much is known about $\mathcal{R}_{g,n}$ and $\mathcal{R}_{g,\langle n\rangle}$ for $n > 2$. In particular, the (probably short) list of all pairs $(g, n)$ such that $\mathcal{R}_{g,n}$ and $\mathcal{R}_{g,\langle n\rangle}$ have negative Kodaira dimension is not known.

The rationality of $\mathcal{R}_{g,n}$ and $\mathcal{R}_{g,\langle n\rangle}$ has been proved in some cases of very low genus: the case of $\mathcal{R}_4$ is a result of Catanese [Cat]. The rationality of $\mathcal{R}_3$ was proved by Katsylo in [Ka]. Independent proofs are also due to Catanese and to Dolgachev; see [D] (also for $\mathcal{R}_2$). Recently, the rationality of $\mathcal{R}_{3,3}$ and of $\mathcal{R}_{3,\langle 3\rangle}$ has been proven by Catanese and the first author [BCat].

To complete the picture, we recall that $\mathcal{R}_{1,n}$ is an irreducible curve for every prime $n$ and that its geometric genus is well known.

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Usually, for reaching one of the previous rationality results, beautiful geometric properties of low-genus canonical curves come to the rescue. This is the case also for this paper: we will use the classical geometry of cubic surfaces in $\mathbb{P}^3$ to prove the following statement.

THEOREM 0.1. *The moduli space $\mathcal{R}_{4,\langle 3\rangle}$ is rational.*

Let us very briefly describe our proof.

We consider the moduli space $\mathcal{P}$ of the sets $\underline{x}$ of six general (unordered) points in $\mathbb{P}^2$. Equivalently, $\mathcal{P}$ is the moduli space of pairs $(S, \sigma)$ such that $S \subset \mathbb{P}^3$ is a smooth cubic surface and $\sigma\colon S \to \mathbb{P}^2$ is the blow-up of $\underline{x}$. Let $C \in |\omega_S^{-2}|$ be smooth, and let $L \in |\sigma^* \mathcal{O}_{\mathbb{P}^2}(1)|$. It turns out that $C$ is a canonical curve of genus 4 endowed with the line bundle

$$\eta_C := \omega_C(-L) \in \mathrm{Pic}^0(C).$$

We say that $(C, \eta_C)$ defines a point of $\mathcal{R}_{4,3}$ if $C$ is smooth, $\eta_C$ is nontrivial, and $\eta_C^3 \cong \mathcal{O}_C$. Of course, we are interested in the locally closed set

$$Q_{\underline{x}} := \{C \in |\omega_S^{-2}| \mid (C, \eta_C) \text{ defines a point of } \mathcal{R}_{4,3}\}.$$

To study this set, we make use (as in [BaVe]) of the cup product map

$$\mu\colon H^1(\omega_S^{-1}(-3L)) \otimes H^0(\omega_S^{-2}) \to H^1(\omega_S^{-3}(-3L)).$$

Let $v \otimes s \in H^1(\omega_S^{-1}(-3L)) \otimes H^0(\omega_S^{-2})$ be a nonzero decomposable vector and let $C = \mathrm{div}(s)$: we show that $v \otimes s \in \mathrm{Ker}\, \mu$ if and only if $(C, \eta_C)$ defines a point of $\mathcal{R}_{4,3}$. Then we consider the Segre product

$$\Sigma := \mathbb{P}(H^1(\omega_S^{-1}(-3L))) \times |\omega_S^{-2}| \subset \mathbb{P} := \mathbb{P}(H^1(\omega_S^{-1}(-3L)) \otimes H^0(\omega_S^{-2}))$$

and the intersection scheme

$$\mathbb{M}^o_{\underline{x}} := \mathbb{P}(\mathrm{Ker}\, \mu) \cap \Sigma^o \subset \mathbb{P},$$

where $\Sigma^o := \{(z, C) \in \Sigma \mid C \text{ is smooth}\}$. It turns out that the projection $(z, C) \to C$ induces a biregular map

$$q_{\underline{x}}\colon \mathbb{M}^o_{\underline{x}} \to Q_{\underline{x}}.$$

Moreover, by the previous remarks, there exists a natural rational map

$$\phi_{\underline{x}}\colon \mathbb{M}^o_{\underline{x}} \to \mathcal{R}_{4,3}$$

sending $(z, C)$ to the moduli point of the pair $(C, \eta_C)$.

As a first step, we show that $\mathbb{M}^o_{\underline{x}}$ is integral and of the expected dimension 5. Let $\mathbb{M}_{\underline{x}}$ be the Zariski closure of $\mathbb{M}^o_{\underline{x}}$. Then we show that the projection

$$p_{\underline{x}}\colon \mathbb{M}_{\underline{x}} \to \mathbb{P}(H^1(\omega_S^{-1}(-3L)))$$

is a locally trivial $\mathbb{P}^1$-bundle over a suitable nonempty open set.

As a second step, we globalize the construction to the moduli space $\mathcal{P}$. On a dense open set $\mathcal{P}^o \subset \mathcal{P}$, we define vector bundles $\mathcal{E}$ and $\mathcal{H}$ such that:

- the fibre of $\mathcal{E}$ at the moduli point of $\underline{x}$ is $H^1(\omega_S^{-1}(-3L))$;
- the fibre of $\mathcal{H}$ at the moduli point of $\underline{x}$ is $H^0(\omega_S^{-2})$.

Let $\mathbb{E} := \mathbb{P}(\mathcal{E})$ and $\mathbb{H} := \mathbb{P}(\mathcal{H})$ be the respective projectivizations. In the fibre product $\mathbb{E} \times_{\mathcal{P}^o} \mathbb{H}$ we construct then an integral variety

$$\mathbb{M} \subset \mathbb{E} \times_{\mathcal{P}^o} \mathbb{H}$$

such that, at the moduli point of $\underline{x}$, the fibre of the projection $\pi : \mathbb{M} \to \mathcal{P}^o$ is the Zariski closure $\mathbb{M}_{\underline{x}}$ of $\mathbb{M}^o_{\underline{x}}$. This allows us to define a rational map

$$\phi \colon \mathbb{M} \to \mathcal{R}_{4,3}$$

whose restriction to the fibre $\mathbb{M}_{\underline{x}}$ is the rational map $\phi_{\underline{x}}$ just considered.

As a third step we show that:

(1) $\phi \colon \mathbb{M} \to \mathcal{R}_{4,3}$ is birational;
(2) $\mathbb{M}$ is birational to $\mathcal{P} \times \mathbb{P}^4 \times \mathbb{P}^1$.

Indeed, (2) follows because $\mathbb{E}$ is a $\mathbb{P}^4$-bundle over $\mathcal{P}^o$ and because $\mathbb{M}$ is birational to a $\mathbb{P}^1$-bundle over $\mathbb{E}$.

Finally, we observe that $\mathcal{R}_{4,\langle 3\rangle} = \mathcal{R}_{4,3}/\langle i\rangle$, where $i \colon \mathcal{R}_{4,3} \to \mathcal{R}_{4,3}$ is the involution mapping $(C, \eta)$ to $(C, \eta^{-1})$. Moreover, this involution is given on $\mathcal{P} \times \mathbb{P}^5$ by the Schlaefli involution $j \colon \mathcal{P} \to \mathcal{P}$ on the first factor and the identity on the second factor. The quotient $\mathcal{P}/\langle j\rangle$ turns out to be the moduli space of a double six, a well-known configuration of lines in a cubic surface (see Section 2). Now, the rationality of $\mathcal{R}_{4,\langle 3\rangle}$ is a consequence of the following theorem.

THEOREM 0.2. *The moduli space $\mathcal{P}/\langle j\rangle$ of double sixes in $\mathbb{P}^2$ is rational.*

This result is due to Coble [Cob, p. 176], and a modern proof of it has been communicated to us by Igor Dolgachev. We reproduce in the last section Dolgachev's proof; we would like to thank him warmly for allowing us to include his proof and for further useful remarks on this paper.

This work also profited of some interesting discussions on the subject with Fabrizio Catanese during a visit of the second author in Bayreuth. Finally, we wish to thank the referee for his useful remarks and corrections.

## 1. Plane Sextics of Genus 4

Let $C$ be a smooth, irreducible, complete curve of genus $g \geq 3$ and let $\eta \in \mathrm{Pic}^0(C)$ be a nontrivial line bundle. We are interested in the linear system $|\omega_C \otimes \eta^{-1}|$ and its associated rational map

$$\phi_\eta \colon C \to \mathbb{P}^{g-2}.$$

We first derive some basic properties of $|\omega_C \otimes \eta^{-1}|$.

PROPOSITION 1.1. *The following conditions are equivalent*:

(1) $|\omega_C \otimes \eta^{-1}|$ *has a base point* $p$;
(2) $h^0(\eta(p)) = 1$;
(3) $\eta \cong \mathcal{O}_C(q - p)$ *for some* $q \in C \setminus \{p\}$.

*Proof.* (1) $\Leftrightarrow$ (2): $p$ is a base point of $|\omega_C \otimes \eta^{-1}|$ if and only if (iff)

$$\dim|(\omega_C \otimes \eta^{-1})(-p)| = \dim|\omega_C \otimes \eta^{-1}| = g-2$$

iff $h^1(\eta(p)) = g-1$ iff $h^0(\eta(p)) = 1$.

(2) $\Leftrightarrow$ (3): This is obvious. □

The next (well-known) statement complements the previous one.

PROPOSITION 1.2. *Let $b$ be the base divisor of $|\omega_C(p-q)|$, where $p, q \in C$ and $p \neq q$. Then either $b = p$ or $C$ is hyperelliptic, $b$ is of degree 2, and $b = p + i(q)$, where $i$ is the hyperelliptic involution.*

From now on we will restrict ourselves to the following case:

$$g = 4, \qquad \eta^{\otimes 3} \cong \mathcal{O}_C.$$

PROPOSITION 1.3. *Let $C$ be a general curve of genus 4 and let $\eta$ be a nonzero 3-torsion element of* $\operatorname{Pic}^0(C)$. *Then $|\omega_C \otimes \eta^{-1}|$ has no base points.*

*Proof.* By Proposition 1.1, $|\omega_C \otimes \eta^{-1}|$ has a base point iff $\eta \cong \mathcal{O}_C(q-p)$ with $p, q \in C$. In particular, $3p \sim 3q$. Let $f: C \to \mathbb{P}^1$ be the map defined by $|3p| = |3q|$; then $f$ has double ramification in $p$ and $q$. By Hurwitz's formula, the degree of the ramification divisor $R$ of $f$ is 12. Then, since $2p + 2q \leq R$, $f$ has at most ten branch points. Hence, by Riemann's existence theorem, the map $f$ depends on at most seven moduli. This implies the statement. □

Therefore we will assume from now on that $|\omega_C \otimes \eta^{-1}|$ *has no base points.* Consider

$$\Gamma_\eta := \phi_\eta(C) \subset \mathbb{P}^2.$$

Since $\phi_\eta$ is a morphism, we have the following possibilities:

- $\phi_\eta: C \to \Gamma_\eta$ has degree 1 and $\Gamma_\eta$ is an integral sextic;
- $\phi_\eta: C \to \Gamma_\eta$ has degree 2 and $\Gamma_\eta$ is an integral cubic;
- $\phi_\eta: C \to \Gamma_\eta$ has degree 3 and $\Gamma_\eta$ is an integral conic.

All these cases actually occur, but we will show that for a general curve $C$, $\Gamma_\eta$ is a sextic and $\operatorname{Sing}\Gamma_\eta$ consists of six ordinary double points in general position.

To this purpose we consider the image $S \subset \operatorname{Pic}^2(C)$ of the Abel map

$$a: C^{(2)} \to \operatorname{Pic}^2(C).$$

Then the cohomology class of $S$ is $\theta^2/2$, where $\theta$ is the class of a theta divisor in $\operatorname{Pic}^2(C)$. In particular, we have $S^2 = 6$. For any $\eta \in \operatorname{Pic}^0(C)$, let $S_\eta$ be the translate of $S$ by $\eta$. Moreover, let

$$Z_\eta := a^* S_\eta$$

be the pull-back of $S_\eta$ under the Abel map. By the transversality of a general translate, $S_\eta$ is transversal to $a$ for general $\eta$. Therefore, in this case, the scheme $Z_\eta$ consists of six smooth and distinct points.

LEMMA 1.4. *For $d \in C^{(2)}$, the following conditions are equivalent*:

(1) *$\phi_\eta$ contracts $d$ to a point*;
(2) *$h^0(\eta(d)) \geq 1$*;
(3) *$d \in Z_\eta$.*

*Proof.* By assumption, $|\omega_C \otimes \eta^{-1}|$ has no base points. Then $\phi_\eta$ contracts $d$ to a point iff $h^0((\omega_C \otimes \eta^{-1})(-d)) \geq 2$. On the other hand, $h^0((\omega_C \otimes \eta^{-1})(-d)) \geq 2$ iff $h^0(\eta(d)) \geq 1$ iff $d \in Z_\eta$. □

In the following lemma we prove that, with respect to $Z_\eta$ and $\phi_\eta$, a general hyperelliptic curve $C$ has a sufficiently general behavior.

LEMMA 1.5. *Let $C$ be a general hyperelliptic curve of genus 4 and let $\eta$ be any nontrivial line bundle on $C$ such that $\eta^3 \cong \mathcal{O}_C$. Then*:

(1) *each element $d \in Z_\eta$ is a smooth divisor of degree 2*;
(2) *$\phi_\eta : C \to \Gamma_\eta$ is a birational morphism*;
(3) *$Z_\eta$ is a smooth, 0-dimensional scheme supported in six points*;
(4) *$\Gamma_\eta$ is a sextic with an ordinary singular point of multiplicity 4.*

*Moreover, conditions* (1)–(3) *hold also for a general $C$ of genus 4.*

*Proof.* (1) Assume by contradiction that $d \in Z_\eta$ is not smooth. Then it follows that $d = 2p$ for some $p \in C$. On the other hand, Lemma 1.4 implies that $\eta(d) \cong \mathcal{O}_C(d')$, where $d'$ is an effective divisor. Therefore we have $d' = q + r$ for some $q, r \in C$. Since $\eta^3$ is trivial, the divisors $6p$ and $3q + 3r$ generate a pencil $P$. To prove (1), we show that a pencil like $P$ cannot exist on a general hyperelliptic curve $C$.

First, let us show that $P$ has no base points: if $P$ has a base point $x$, then $x = p$ and $p \in \{q, r\}$. Hence $3p$ is a fixed component of $P$. Moreover, $|3p|$ is a pencil of degree 3 and, since $C$ is hyperelliptic, it has a base point. But then $4p$ is a fixed component of $P$ and $p = q = r$: a contradiction.

Now let $|h|$ be the hyperelliptic pencil of $C$. We consider the morphism $\psi : C \to \mathbb{P}^1 \times \mathbb{P}^1$, defined by $P \times |h|$, and the curve $\Gamma := \psi(C)$. Two cases are possible:

(A) $\Gamma$ has bidegree $(6, 2)$ and $\psi : C \to \Gamma$ is birational;
(B) $\Gamma$ has bidegree $(3, 1)$ and $\psi : C \to \Gamma$ has degree 2.

In both cases we show that $C$ cannot be a general hyperelliptic curve.

(A) Note that $\psi(q)$ and $\psi(r)$ are in the same line $l$ of bidegree $(0, 1)$ and $\psi(p) \notin l$. Indeed, the pull-back by $\psi$ of the ruling of lines of bidegree $(0, 1)$ is the pencil $P$ generated by the divisors $6p$ and $3q + 3r$. Now let us fix coordinates $(X_0 : X_1) \times (Y_0 : Y_1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ such that

$$l = \{Y_1 = 0\}, \qquad \psi(p) = \{X_0 - X_1 = Y_0 = 0\},$$
$$\{\psi(q), \psi(r)\} = \{X_0 X_e = Y_1 = 0\},$$

where $e = 1$ if $\psi(q) \neq \psi(r)$ and $e = 0$ if $\psi(q) = \psi(r)$. After this choice, $\Gamma$ is an element of the 8-dimensional linear system $\Sigma$ defined by the equation

$$aY_0^2X_0^3X_e^3 + bY_1^2(X_0 - X_1)^6 + \sum_{i=0,\dots,6} c_iY_0Y_1X_0^iX_1^{6-i} = 0.$$

A general element of $\Sigma$ is smooth; moreover, $\dim \Sigma = 8$. On the other hand, $p_a(\Gamma) = 5$, whence $\Gamma$ is singular and belongs to the 7-dimensional discriminant hypersurface $\Delta \subset \Sigma$. Note that the stabilizer $G \subset \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ of $\Sigma$ has dimension $\geq 1$, whence $\dim \Delta/G \leq 6$. But then $C$ has at most 6 moduli and it is not general hyperelliptic.

(B) $\Gamma$ is a smooth rational curve of type $(3,1)$. Note that $6p = \psi^*l_1$ and $3q + 3r = \psi^*l_2$, where $l_1, l_2$ are lines of type $(0,1)$. The latter equality implies $l_2 \cdot \Gamma = 3v$ for some $v \in l_2$, hence $q + r = \psi^*v$. The first one implies $l_1 \cdot \Gamma = 3u$ for some $u \in l_1$, hence $2p = \psi^*u$. Since $\Gamma$ is rational, it follows that $\psi^*(u - v) = 2p - (q + r) \sim 0$. Then $\eta \cong \mathcal{O}_C$ and this case is excluded.

(2) A non-birational $\phi_\eta$ ramifies at some $p \in C$. Hence $\phi_\eta$ contracts $2p$ and, by Lemma 1.4, $2p \in Z_\eta$. By (1) this is impossible for a general $C$.

(4) Let $|h|$ be the $g_2^1$ of $C$; then $\omega_C \cong \mathcal{O}_C(3h)$. This implies that $|\omega_C \otimes \eta^{-1}| = |h + b|$, where $b \in |\eta^{-1}(2h)|$. Here $b$ has degree 4 and $\phi_\eta(b)$ is a point $o$. Then, since $\phi_\eta$ is birational, $o$ is a 4-tuple point of $\Gamma_\eta$. By the genus formula, $\mathrm{Sing}\,\Gamma_\eta = \{o\}$ and, by (1), $b$ is smooth. Hence $o$ is an ordinary 4-tuple point.

(3) If $b = p_1 + p_2 + p_3 + p_4$ is the divisor considered above, then the conclusion is definitely clear: by (4) $Z_\eta$ is a scheme supported in the six points $p_i + p_j$, $1 \leq i < j \leq 4$. On the other hand, we know that $Z_\eta$ has length 6. So $Z_\eta$ is smooth.

Finally we remark that statements (1)–(3) define an open subset $U$ in the moduli space of curves of genus 4, where the statements are true. By the previous part of the proof, $U$ is not empty. This completes the proof. □

LEMMA 1.6. *If $C$ is general, then $\Gamma_\eta$ has no point of multiplicity $m \geq 3$.*

*Proof.* We can assume that $\Gamma_\eta$ is a non-hyperelliptic sextic. Then each point $o \in \Gamma_\eta$ has multiplicity $m \leq 3$. To exclude the case $m = 3$, we consider the theta divisor $\Theta := \{N \in \mathrm{Pic}^3(C) \mid h^0(N) \geq 1\}$ and observe the following.

*Claim.* $\Gamma_\eta$ has a triple point iff $L \otimes \eta \in \Theta$ iff $M \otimes \eta^{-1} \in \Theta$, where $|L|$ and $|M|$ are the only trigonal pencils on $C$.

*Proof of Claim.* Note that the following conditions are equivalent:

(1) $\Gamma_\eta$ has a triple point $o$;
(2) there exists an effective $d \in \mathrm{Div}^3(C)$ such that $\phi_\eta(d) = o$;
(3) there exists an effective $d \in \mathrm{Div}^3(C)$ such that $h^0(\omega_C \otimes \eta^{-1}(-d)) = 2$;
(4) there exists an effective $d \in \mathrm{Div}^3(C)$ such that $h^0(\eta(d)) = 2$.

On the other hand, a non-hyperelliptic $C$ has at most two trigonal line bundles, say $L$ and $M$. Therefore (2) holds iff $\{L, M\} = \{\omega_C \otimes \eta^{-1}(-d), \eta(d)\}$ iff $L \otimes \eta \in \Theta$ iff $M \otimes \eta^{-1} \in \Theta$. This proves the claim.

The previous conditions cannot hold for all $\eta \in \mathrm{Pic}_3^0(C) - \{\mathcal{O}_C\}$. Indeed, this would imply that the set $L + \mathrm{Pic}_3^0(C) := \{\eta \otimes L,\ \eta \in \mathrm{Pic}_3^0(C)\}$ is in the theta divisor $\Theta$ of $\mathrm{Pic}^3(C)$. Equivalently, $\mathrm{Pic}_3^0(C)$ would be contained in

$$\Theta_0 := \{N \otimes L^{-1},\ N \in \Theta\} = \{N^{-1} \otimes M,\ N \in \Theta\},$$

where the latter equality follows from

$$N \in \Theta \iff \omega_C \otimes N^{-1} \in \Theta \text{ and } \omega_C \cong L \otimes M.$$

But it is well known that, in the embedding defined by $3\Theta_0$, the set $\mathrm{Pic}^0_3(C) - \{\mathcal{O}_C\}$ is not in a hyperplane. Hence there exists a nontrivial $\eta$ such that $h^0(\eta(d)) \le 1$ for each effective $d \in \mathrm{Div}^3(C)$. Since $\mathcal{R}_{4,3}$ is irreducible, this property holds for all $\eta \in \mathrm{Pic}^0_3(C) - \{\mathcal{O}_C\}$ if $C$ is general. $\square$

The next definition is well known; it will be important in the sequel.

Definition 1.1. Six distinct points of $\mathbb{P}^2$ are in general position if no conic contains all of them and no line contains three of them.

We are now ready to prove the main result of this section.

Theorem 1.7. *Let $C$ be a general curve of genus 4 and let $\eta$ be any nontrivial line bundle on $C$ such that $\eta^3 \cong \mathcal{O}_C$. Then*:

(1) $\phi_\eta : C \to \Gamma_\eta$ *is a birational morphism*;
(2) $\mathrm{Sing}\,\Gamma_\eta$ *consists of six ordinary nodes in general position.*

*Proof.* From the previous lemmas we know that $\phi_\eta$ is a birational morphism and that $\Gamma_\eta$ is a sextic with at most double points. Let $Z_\eta$ be the scheme considered in Lemma 1.4. We know from Lemma 1.5 that $Z_\eta$ consists of six distinct and smooth effective divisors of degree 2 on $C$ and that each of them is contracted by $\phi_\eta$ to a double point of $\Gamma_\eta$. It is easy to deduce that $\mathrm{Sing}\,\Gamma_\eta$ consists of six ordinary nodes: $x_1, \dots, x_6$. It remains to show that they are in general position.

Assume that a conic $B$ contains $\mathrm{Sing}\,\Gamma_\eta$. It is easy to deduce that, for any line $L \subset \mathbb{P}^2$, $\phi^* L \in |\omega_C|$, whence $|\omega_C| = |\omega_C \otimes \eta^{-1}|$: a contradiction.

The condition that no line contains three points of $\mathrm{Sing}\,\Gamma_\eta$ is equivalent to the following condition on $Z_\eta$: for any three distinct elements $u, v, t \in Z_\eta$, their sum $u + v + t$ is not in $|\omega_C \otimes \eta^{-1}|$. So it suffices to prove the latter condition for at least one pair $(C, \eta)$. Let $C$ be general hyperelliptic; from the proof of Lemma 1.5(4) we know that

$$Z_\eta = \{x_i + x_j;\ 1 \le i < j \le 4\} \quad \text{for some } \ b = x_1 + x_2 + x_3 + x_4$$

and that $|\omega_C \otimes \eta^{-1}| = |b + h|$, where $|h|$ is the hyperelliptic pencil of $C$. Let $u + v + t \in |\omega_C \otimes \eta^{-1}|$ for some $u, v, t \in Z_\eta$; then $u + v + t \sim b + h$. Moreover, $u + v + t - b$ is effective because $\mathrm{Supp}\, b \subset \mathrm{Supp}\, u \cup \mathrm{Supp}\, v \cup \mathrm{Supp}\, t$. Hence $u + v + t - b = p' + p''$, where $p', p'' \in \mathrm{Supp}\, b$ and $b \equiv h + p' + p''$. But then $|\omega_C \otimes \eta^{-1}| = |2h + p' + p''|$ and $p', p''$ are base points: a contradiction. $\square$

We end this section by giving some definitions and fixing some further notation that will be used in the sequel.

Definition 1.2. Let $\mathrm{Hilb}_6(\mathbb{P}^2)$ be the Hilbert scheme of six points in $\mathbb{P}^2$. Then $\mathcal{X}$ is the open subset of $\mathrm{Hilb}_6(\mathbb{P}^2)$ parameterizing those schemes $\underline{x}$ that are supported in six points in general position.

DEFINITION 1.3. $\mathcal{R}_{4,3}$ is the moduli space of pairs $(C, \eta)$ such that:

(1) $C$ is a smooth, irreducible projective curve of genus 4;
(2) $\eta$ is a nonzero 3-torsion element of $\mathrm{Pic}^0(C)$.

Here $\mathcal{R}_{4,3}$ is an irreducible quasi-projective variety of dimension 9. Throughout the paper we will keep the previous notation.

REMARK 1.1. Note that we have a natural involution $i$ on $\mathcal{R}_{4,3}$ sending $(C, \eta)$ to $(C, \eta^{-1})$. We will denote the quotient $\mathcal{R}_{4,3}/i$ by $\mathcal{R}_{4,\langle 3\rangle}$. Obviously, $\mathcal{R}_{4,\langle 3\rangle}$ is the moduli space of pairs of smooth, irreducible projective curves of genus 4 together with a cyclic subgroup of $\mathrm{Pic}^0(C)$ of order 3.

## 2. A Cup Product Map on the Cubic Surface

It is important to point out that Theorem 1.7 relates the family of pairs $(C, \eta)$ to *smooth* cubic surfaces in $\mathbb{P}^3$.

More precisely, assume that $(C, \eta)$ defines a general point of $\mathcal{R}_{4,3}$. Then $\mathrm{Sing}\,\Gamma_\eta$ is an element of $\mathcal{X}$—that is, its points are in *general position*. Let $\sigma\colon S \to \mathbb{P}^2$ be the blow-up of $\mathrm{Sing}\,\Gamma_\eta$. Then $S$ is a del Pezzo surface of degree 3; in other words, its anticanonical divisor $-K_S$ is *semiample* and $K_S^2 = 3$. It is well known that, for the blow-up of $\mathbb{P}^2$ in six distinct points, the following conditions are equivalent:

- the six points are in general position;
- the anticanonical divisor is very ample.

Thus we will assume that our *$S$ is anticanonically embedded in $\mathbb{P}^3$ as a smooth cubic surface*. We also remark that

(1) $C \in |-2K_S|$,
(2) $\eta \cong \mathcal{O}_C(-K_S - L)$, and
(3) $E \cdot C = \sum_{d \in Z_\eta} d$,

where $L \cong \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$ and $E$ is the exceptional divisor of $\sigma$. In particular, $C$ is a quadratic section of $S$ and a canonical curve of genus 4.

REMARK 2.1. Though $\mathrm{Sing}\,\Gamma_\eta$ is a set of points in general position, still we did not prove that it is a *general point* of $\mathcal{X}$, so that $S$ is a *general* smooth cubic surface. The proof of this property is a relevant step of this section (see Theorem 2.9).

In the sequel we will deal with *any element* $\underline{x} \in \mathcal{X}$ supported on the set $\{x_1, \dots, x_6\}$. Let $\sigma\colon S \to \mathbb{P}^2$ be the blow-up of $\underline{x}$ and let $E_i = \sigma^{-1}(x_i)$, $i = 1, \dots, 6$. As usual, we will have the line bundle

$$L := \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$$

and the *exceptional divisor* $E := E_1 + \cdots + E_6$ of $\sigma$.

DEFINITION 2.1. For any $C \in |-2K_S|$ we define:

- $s_C :=$ any nonzero vector of $H^0(\mathcal{O}_S(-2K_S))$ vanishing on $C$;
- $\eta_C := \mathcal{O}_C(-K_S - L)$;
- $n_C := C \cdot E$.

It is easily checked that $\eta^{\otimes 3}(n_C) \cong \mathcal{O}_C(-2K_S)$.

PROPOSITION 2.1. *For any $C \in |-2K_S|$, the sheaf $\eta_C$ is nontrivial.*

*Proof.* Consider the long exact sequence associated to the exact sequence

$$0 \to \mathcal{O}_S(K_S - L) \to \mathcal{O}_S(-K_S - L) \to \eta_C \to 0.$$

We have $h^1(\mathcal{O}_S(K_S - L)) = 0$. Since no conic contains $\underline{x}$, we have also $h^0(\mathcal{O}_S(-K_S - L)) = 0$. Then $h^0(\eta_C) = 0$ and $\eta_C$ is nontrivial. □

Note that $\sigma|_C$ is the map defined by $|\omega_C \otimes \eta_C^{-1}|$. Therefore, if $(C, \eta)$ is a pair as in Section 1 and $\underline{x} = \operatorname{Sing} \Gamma_\eta$, one has $\eta \cong \eta_C$.

We want to understand the family of smooth curves $C \in |-2K_S|$ such that $\eta_C^3 \cong \mathcal{O}_C$. To this purpose, we analyze the cup product

$$\cup\colon H^0(\mathcal{O}_S(-2K_S)) \otimes H^1(\mathcal{O}_S(-E)) \to H^1(\mathcal{O}_S(-2K_S - E)).$$

First we observe that there is a standard exact sequence

$$0 \to \mathcal{O}_S(-E) \to \mathcal{O}_S(-2K_S - E) \to \eta_C^3 \to 0$$

just because $-C - 3K_S - 3L \sim -E$. Second, we consider the associated long exact sequence and recall that the induced map

$$\mu_C\colon H^1(\mathcal{O}_S(-E)) \to H^1(\mathcal{O}_S(-2K_S - E))$$

is the cup product with $s_C$. Let

$$s_C^\perp := \{v \in H^1(\mathcal{O}_S(-E)) \mid v \cup s_C = 0\}$$

be the $\cup$-orthogonal space of $s_C$. The next property is immediate.

PROPOSITION 2.2. *Let $C \in |-2K_S|$. Then $H^0(\eta_C^3) = s_C^\perp$.*

*Proof.* Since $h^0(\mathcal{O}_S(-2K_S - E)) = 0$, the preceding long exact sequence implies that $H^0(\eta_C^3) = \operatorname{Ker} \mu_C = s_C^\perp$. □

Moreover, we have the following result.

PROPOSITION 2.3. *Let $C \in |-2K_S|$ be a smooth curve. Then the following conditions are equivalent*:

(1) *$\eta_C$ is a nonzero 3-torsion element of* $\operatorname{Pic}^0(C)$;
(2) *the $\cup$-orthogonal space to $s_C$ has dimension 1*;
(3) *$n_C$ is the base locus of a pencil $Q \subset |-2K_S|$.*

*Proof.* (1) $\Leftrightarrow$ (2): This is clear from the previous remarks.

(2) $\Rightarrow$ (3): By $\eta^{\otimes 3}(n_C) \cong \mathcal{O}_C(-2K_S)$ and $\eta^{\otimes 3} \cong \mathcal{O}_C$, it follows that $\mathcal{O}_C(n_C) \cong \mathcal{O}_C(-2K_S)$. By $h^1(\mathcal{O}_S) = 0$, the restriction map

$$H^0(\mathcal{O}_S(-2K_S)) \to H^0(\mathcal{O}_C(-2K_S))$$

is surjective. Hence $n_C$ is cut out on $C$ by a member of $|-2K_S|$.

(3) $\Rightarrow$ (2): This follows by reversing the argument. □

SCHLAEFLI'S DOUBLE SIX. We need to consider the six exceptional lines defined as follows:

$F_i :=$ strict transform of the conic through $\underline{x} - \{x_i\}$ under $\sigma$, $i = 1, \dots, 6$.

The twelve lines $E_1, \dots, E_6, F_1, \dots, F_6$ form a configuration of lines on $S$ that is well known as a *Schlaefli's double six*. In particular, the divisor

$$F := F_1 + \cdots + F_6$$

is the exceptional divisor of a second blow-up $\hat{\sigma}\colon S \to \mathbb{P}^2$ defined by

$$|\hat{L}| := |5L - 2E|.$$

It is immediately checked that $\mathcal{O}_C(-K_S - \hat{L}) \cong \mathcal{O}_C(K_S + L) \cong \eta_C^{-1}$. Let us also point out that $\hat{\sigma}|_C\colon C \to \mathbb{P}^2$ is the morphism associated to $|\omega_C \otimes \eta_C|$.

REMARK 2.2. For later use we observe that, for $(C, \eta) \in \mathcal{R}_{4,3}$, $\operatorname{Sing}\Gamma_\eta \cup \operatorname{Sing}\Gamma_{\eta^{-1}}$ gives rise to a Schlaefli's double six on $S$.

Since $h^1(\mathcal{O}_S(-E)) = 5$, we fix from now on the following notation:

- $\mathbb{P}^4 := \mathbb{P}(H^1(S, \mathcal{O}_S(-E)))$;
- $\bar{v} \in \mathbb{P}^4$ is the point defined by $v \in H^1(S, \mathcal{O}_S(-E)) \setminus \{0\}$.

Now we study the $\cup$-orthogonal space $v^\perp$ of a vector $v \in H^1(\mathcal{O}_S(-E))$. Note that $v$ corresponds to a vector of $\operatorname{Ext}^1(\mathcal{O}_S(-2K_S), \mathcal{O}_S(-2K_S - E))$. Therefore, a general $v$ defines an extension

$$0 \to \mathcal{O}_S(-2K_S - E) \to \mathcal{V} \to \mathcal{O}_S(-2K_S) \to 0,$$

where $\mathcal{V}$ is a rank-2 vector bundle on $S$. It is easy to compute that

$$\det \mathcal{V} \cong \mathcal{O}_S(F).$$

Passing to the long exact sequence, the coboundary map

$$\partial_v\colon H^0(\mathcal{O}_S(-2K_S)) \to H^1(\mathcal{O}_S(-E - 2K_S))$$

is the cup product with $v$. Therefore it follows that

$$v^\perp = \operatorname{Ker}(\partial_v) \cong H^0(\mathcal{V}).$$

From the same long exact sequence we see that $h^2(\mathcal{V}) = 0$. Therefore, applying Riemann–Roch to $\mathcal{V}$, we conclude that

$$\dim v^\perp = h^0(\mathcal{V}) \geq 2.$$

DEFINITION 2.2. Let $\bar{v} \in \mathbb{P}^4$ be the point defined by the vector $v$. Then

$$P_{\bar{v}} := \mathbb{P}(v^\perp) \; (= \mathbb{P}(H^0(\mathcal{V}))).$$

Here $P_{\bar{v}}$ is a linear system contained in $|-2K_S|$, and by the previous remarks we have

$$\dim P_{\bar{v}} \geq 1.$$

In order to describe $P_{\bar{v}}$, we need to understand its position with respect to the linear subspaces $\Lambda_i \subset |-2K_S|$, which are defined as

$$\Lambda_i := \{C \in |-2K_S| \mid F_i \subset C\}, \quad i = 1, \dots, 6.$$

We have the following result.

THEOREM 2.4. *Assume that $P_{\bar{v}}$ has no fixed component. Then*

(1) *$P_{\bar{v}}$ is a pencil,*
(2) *$P_{\bar{v}} \cap \Lambda_i$ is a point for each $i = 1, \dots, 6$, and*
(3) *the base locus of $P_{\bar{v}}$ is a quadratic section of $F$.*

*Proof.* We know that $\det \mathcal{V} \cong \mathcal{O}_S(F)$ and $h^0(\mathcal{V}) \geq 2$. Let us consider $s \in H^0(\mathcal{V}) \setminus \{0\}$ and its scheme of zeros $Z_s$. Since $c_2(\mathcal{V}) = 0$, either $Z_s = \emptyset$ or $\dim Z_s = 1$.

*Claim.* If $P_{\bar{v}}$ does not have a fixed component, then there exists a $s \in H^0(\mathcal{V}) \setminus \{0\}$ such that $Z_s = \emptyset$.

We will prove the claim in a moment. We can now assume that $Z_s = \emptyset$. Then $s$ defines an exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{V} \to \mathcal{O}_S(F) \to 0.$$

Since $S$ is regular, the associated long exact sequence yields $h^0(\mathcal{V}) = 2$. Hence $\mathcal{P}_{\bar{v}}$ is a pencil and (1) follows.

To prove (2), tensor the preceding exact sequence by $\mathcal{O}_S(-F_i)$ and consider the associated long exact sequence. Since $h^1(\mathcal{O}_S(-F_i)) = 0$ and $F_i$ is a component of $F$, it follows that $h^0(\mathcal{V}(-F_i)) = h^0(\mathcal{O}_S(F - F_i)) = 1$. On the other hand, the extension defined by $v$ induces the following commutative diagram with exact lines:

$$\begin{array}{ccccc} H^0(\mathcal{O}_S(-2K_S - E - F_i)) & \longrightarrow & H^0(\mathcal{V}(-F_i)) & \xrightarrow{u} & H^0(\mathcal{O}_S(-2K_S - F_i)) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(\mathcal{O}_S(-2K_S - E)) & \longrightarrow & H^0(\mathcal{V}) & \longrightarrow & H^0(\mathcal{O}_S(-2K_S)). \end{array}$$

Here $u$ is injective because $h^0(\mathcal{O}_S(-2K_S - E - F_i)) = 0$. The vertical arrows are injective, too. Then the image of $H^0(\mathcal{V}(-F_i))$ in $H^0(\mathcal{O}_S(-2K_S))$ is the unique element $C \in P_{\bar{v}}$ that contains $F_i$. This implies (2).

*Proof of Claim.* Assume that $\dim Z_s = 1$ for all $s \in H^0(\mathcal{V}) - \{0\}$. Let $s \in H^0(\mathcal{V}) - \{0\}$ and write $Z_s = Z_{0,s} + D_s$, where $Z_{0,s}$ is 0-dimensional and $D_s$ is a curve. Consider the exact sequence

$$0 \to \mathcal{O}_S(D_s) \to \mathcal{V} \to \mathcal{I}_{Z_{0,s}}(F - D_s) \to 0,$$

where $\mathcal{I}_{Z_{0,s}}$ is the ideal sheaf of $Z_{0,s}$ in $S$. Applying the same argument as before but replacing $F_i$ by $D_s$, it follows that each $C \in P_{\bar{v}}$ contains a curve $D_s$. On the other hand, $D_s \neq C$: if not, $\mathcal{O}_S(-2K_S)$ would be a subbundle of $\mathcal{V}$, which is impossible for $s \neq 0$. Thus we have shown that each $C \in P_{\bar{v}}$ is reducible. For a general $C \in P_{\bar{v}}$, let $d := -K_S \cdot B$ be the minimal degree of an integral curve $B \subset C$. Then one has $d \leq 3$. If $d = 1$, then $B$ is a line and a fixed component of $P_{\bar{v}}$; if $d = 2$, then $B$ is a conic and $\dim|B| = 1$. It is easy to see that the family of reducible curves $C \in |-2K_S|$ containing some conic of $|B|$ is exactly the Segre embedding of $\mathbb{P}^4 \times \mathbb{P}^1 := |B| \times |-2K_S - B|$. Note that $P_{\bar{v}}$ is a linear space in $\mathbb{P}^4 \times \mathbb{P}^1$ and that the projection $P_{\bar{v}} \to |B|$ is a surjective morphism. This implies that $P_{\bar{v}} = \{E\} \times |B|$ for some $E \in |-2K_S - B|$. Hence $P_{\bar{v}}$ is a pencil and $E$ is its fixed component. If $d = 3$ and $B$ is a skew cubic, then $\dim|B| = 2$. The argument is completely analogous to the previous one: we leave it to the reader. The last case is when $d = 3$ and $B$ is a plane section of $S$. Then any $C \in P_{\bar{v}}$ is the union of two elements $B, B'$ of $|-K_S|$. It is well known that then either one of them is a fixed component or the pencil $P_{\bar{v}}$ consists of a pencil of $|-K_S|$ containing $B$ and $B'$. Since $B$ is integral, the latter pencil has no fixed components. Moreover, by monodromy, one cannot distinguish between $B$ and $B'$ when $C$ varies in $P_{\bar{v}}$. This implies that both $B$ and $B'$ are in $D_s$ and hence that $D_s = B + B' = C$. Therefore $\mathcal{O}_S(-2K_S)$ is a subbundle of $\mathcal{V}$, a case already excluded. Thus we have proved the Claim.

To prove (3), note that the base locus $m$ of $P_{\bar{v}}$ has degree 12. By (2), $P_{\bar{v}}$ contains some $C_i = D_i + F_i$, $i = 1, \ldots, 6$. Let $C \in P_{\bar{v}}$ be an element not containing $F_1, \ldots, F_6$. Then $m_i := C \cdot F_i$ is a subscheme of $m$. The same holds for $C \cdot F = \sum m_i$, since the $F_i$ are disjoint. Hence $m = C \cdot F$ for degree reasons. □

DEFINITION 2.3. For any $C \in |-2K_S|$ we define $m_C := C \cdot F$.

The next result simply summarizes some useful equivalent conditions.

PROPOSITION 2.5. *Let $C \in |-2K_S|$ be smooth. Then the following conditions are equivalent*:

(1) *$\eta_C$ is a nontrivial 3-torsion element of* $\operatorname{Pic}^0(C)$;
(2) *$C \in P_{\bar{v}}$ for some $\bar{v} \in \mathbb{P}^4$*;
(3) *$n_C$ is the base locus of a pencil $Q \subset |-2K_S|$*;
(4) *$m_C$ is the base locus of a pencil $P$, and $P = P_{\bar{v}}$ for some $\bar{v} \in \mathbb{P}^4$.*

*Proof.* Conditions (1), (2), and (3) are equivalent by Proposition 2.3. To prove their equivalence with (4), we recall that $\hat{L} = 5L - 2E$ defines a morphism $\hat{\sigma} : S \to \mathbb{P}^2$, which is the contraction of the previously defined divisor $F$. It is clear that $\mathcal{O}_C(-K_S - \hat{L}) \cong \eta_C^{-1}$. Then, if we apply Proposition 2.3 to $\hat{\sigma}$ and $\eta_C^{-1}$, it follows that $\eta_C^{-1}$ is a nontrivial 3-torsion element of $\operatorname{Pic}^0(C)$ iff (4) holds. Proposition 2.3 also implies the last statement. □

PROPOSITION 2.6. *There exists a $\bar{v} \in \mathbb{P}^4$ such that $P_{\bar{v}}$ is a pencil with no fixed component and the general element $C \in P_{\bar{v}}$ is smooth.*

*Proof.* We use Schlaefli's double six configuration $E_1, \ldots, E_6, F_1, \ldots, F_6$. Observe that $|-2K_S|$ contains the elements

$$C_0 := E_1 + E_2 + E_3 + F_4 + F_5 + F_6,$$
$$C_\infty := F_1 + F_2 + F_3 + E_4 + E_5 + E_6.$$

Obviously, the pencil $P$ generated by $C_0$ and $C_\infty$ has no fixed components. With a small additional effort one shows that its base locus $m$ is a smooth quadratic section of $F$. Hence a general $C \in P$ is smooth and $m = m_C$. Then Proposition 2.5 implies that $P = P_{\bar{v}}$ for some $\bar{v} \in \mathbb{P}^4$. □

COROLLARY 2.7. *The cup product*

$$\cup\colon H^1(\mathcal{O}_S(-E)) \otimes H^0(\mathcal{O}_S(-2K_S)) \to H^1(\mathcal{O}_S(-E-2K_S))$$

*is surjective for any* $\underline{x} \in \mathcal{X}$.

*Proof.* Let $\cup_v\colon \langle v\rangle \otimes H^0(\mathcal{O}_S(-2K_S)) \to H^1(\mathcal{O}_S(-E-2K_S))$ be the cup product with a general $v \in H^1(\mathcal{O}_S(-E))$. By Theorem 2.4 and Proposition 2.6, $\dim \operatorname{Ker} \cup_v = v^\perp = 2$. Since

$$\dim(\langle v\rangle \otimes H^0(\mathcal{O}_S(-2K_S))) = 10 \quad \text{and} \quad h^1(\mathcal{O}_S(-E-2K_S)) = 8,$$

it follows that $\cup_v$ is surjective. Hence $\cup$ is surjective. □

We mention without proof, since it is not needed in the sequel, a method to describe those $P_{\bar{v}}$ having fixed components.

PROPOSITION 2.8. *Let $D$ be the fixed curve of a linear system $P_{\bar{v}}$. Then $D$ is contained in a quadratic section of $S$. Moreover, $\mathcal{V}$ fits in an exact sequence*

$$0 \to \mathcal{O}_S(D) \to \mathcal{V} \to \mathcal{I}_Z(F-D) \to 0,$$

*where* $\dim Z = 0$, *$\mathcal{I}_Z$ is the ideal sheaf of $Z$, and* $\deg Z + D(F-D) = 0$.

REMARK 2.3. For instance, let $\bar{v}_i \in \mathbb{P}^4$ define the natural extension

$$0 \to \mathcal{O}_S(-2K_S - E) \to \mathcal{O}_S(-2K_S - E + E_i) \oplus \mathcal{O}_S(-2K_S - E_i) \\ \to \mathcal{O}_S(-2K_S) \to 0.$$

In this case, we have $D = E_i$ for the fixed curve of $P_{\bar{v}_i}$ and $Z = 0$. Moreover,

$$P_{\bar{v}_i} = |-2K_S - E_i| = \mathbb{P}^6.$$

All this suggests that we should study the projectivized set of decomposable tensors $v \otimes s$ such that $v \cup s = 0$ and $\operatorname{div}(s)$ is smooth. To describe this set, we consider:

- $\mathbb{P}^{49} := \mathbb{P}(H^1(-E) \otimes H^0(-2K_S))$;
- the Segre embedding $\mathbb{P}^4 \times |-2K_S| \subset \mathbb{P}^{49}$;
- the linear subspace $\mathbb{P}(\operatorname{Ker} \cup) \subset \mathbb{P}^{49}$.

Then we give the following definition.

DEFINITION 2.4. $\mathbb{T}_{\underline{x}} := \mathbb{P}(\operatorname{Ker} \cup) \cap \mathbb{P}^4 \times |-2K_S|$.

It is clear from the definition that

$$\mathbb{T}_{\underline{x}} = \{(\bar{v}, C) \in \mathbb{P}^4 \times |-2K_S| \mid h^0(\eta_C^3) \geq 1\}$$

and also that, for any $\bar{v} \in \mathbb{P}^4$, one has

$$\mathbb{T}_{\underline{x}} \cap \{\bar{v}\} \times |-2K_S| = \{\bar{v}\} \times P_{\bar{v}}.$$

It turns out that $\mathbb{T}_{\underline{x}}$ is reducible: this is clear when one considers the example of Remark 2.3 and Theorem 2.9. However, we are now ready to show that there exists a unique irreducible component containing all pairs $(\bar{v}, C) \in \mathbb{T}_{\underline{x}}$ such that $C$ is smooth. To prove this, one more definition will be convenient.

DEFINITION 2.5.

- $\mathbb{M}_{\underline{x}} :=$ the Zariski closure of $\{(\bar{v}, C) \in \mathbb{T}_{\underline{x}} \mid \dim P_{\bar{v}} = 1\}$;
- $\mathbb{M}^o_{\underline{x}} := \{(\bar{v}, C) \in \mathbb{T}_{\underline{x}} \mid C \text{ is smooth}\}$;
- $p_{\underline{x}} : \mathbb{M}_{\underline{x}} \to \mathbb{P}^4$ and $q_{\underline{x}} : \mathbb{M}_{\underline{x}} \to |-2K_S|$ are the projection maps.

By Proposition 2.6, $\mathbb{M}_{\underline{x}}$ and $\mathbb{M}^o_{\underline{x}}$ are not empty. Notice also that $\mathbb{M}^o_{\underline{x}} \subset \mathbb{M}_{\underline{x}}$. Indeed, let $(\bar{v}, C) \in \mathbb{M}^o_{\underline{x}}$; then $C \in P_{\bar{v}}$. Since $C$ is integral, $P_{\bar{v}}$ has no fixed component. Hence, by Theorem 2.4, $\dim P_{\bar{v}} = 1$. Clearly, $\mathbb{M}^o_{\underline{x}}$ is open in $\mathbb{M}_{\underline{x}}$.

We are now ready to conclude this section.

THEOREM 2.9. *For any $\underline{x} \in \mathcal{X}$, the projection $p_{\underline{x}} : \mathbb{M}_{\underline{x}} \to \mathbb{P}^4$ is surjective. Moreover, $p_{\underline{x}}$ is a locally trivial $\mathbb{P}^1$-bundle over a nonempty open set of $\mathbb{P}^4$. In particular, $\mathbb{M}_{\underline{x}}$ is irreducible and rational.*

*Proof.* The fibre of $p_{\underline{x}}$ at $\bar{v}$ is the linear space $P_{\bar{v}}$. By definition, $\mathbb{M}_{\underline{x}}$ is the Zariski closure of the union of the fibres of minimal dimension 1. Hence $\mathbb{M}_{\underline{x}}$ is a union of irreducible components of $\mathbb{T}_{\underline{x}}$. Each of them has dimension $\geq 5$. Indeed, by Corollary 2.7, the cup product map $\cup$ is surjective. Then the codimension of $\mathbb{P}(\mathrm{Ker} \cup)$ in $\mathbb{P}^{49}$ is $h^1(\mathcal{O}_S(-2K_S - E)) = 8$. Hence, counting dimensions, each irreducible component of $\mathbb{T}_{\underline{x}}$ has dimension $\geq 5$. But over a dense open set $U$ of $p_{\underline{x}}(\mathbb{M}_{\underline{x}})$, the fibre of $p_{\underline{x}}$ is $\mathbb{P}^1$. Hence $p_{\underline{x}}(\mathbb{M}_{\underline{x}}) = \mathbb{P}^4$ and $\mathbb{M}_{\underline{x}}$ is irreducible. Moreover, $\mathbb{M}_{\underline{x}}$ is a $\mathbb{P}^1$-bundle over $U$. □

## 3. Moduli of Plane Models of Cubic Surfaces

Starting from a pair $(C, \eta)$, we came up with a set $\underline{x} \in \mathcal{X}$ of six points of $\mathbb{P}^2$ in general position. It is now time to globalize our constructions over the moduli space of $\underline{x}$. This space can be viewed as the GIT-quotient $\mathcal{X}/\mathrm{PGL}(3)$, and it will be discussed in the next section. On the other hand, it is well known that this space is birational to another moduli space, the space of pairs defined as follows.

DEFINITION 3.1. A pair $(S, L)$ is a *plane model of a cubic surface* if:

- $S$ is a del Pezzo surface of degree 3;
- $-K_S$ is very ample;
- $L \in \mathrm{Pic}(S)$, $L^2 = 1$, and $K_S \cdot L = -3$.

DEFINITION 3.2. The moduli space of pairs $(S, L)$ will be denoted by $\mathcal{P}$.

As is well known, every line bundle $L \in \mathrm{Pic}(S)$ satisfying the listed conditions defines a map $\sigma : S \to \mathbb{P}^2$ that is the blow-up of a set $\underline{x} \in \mathcal{X}$. It is also well known that the assignment $(S, L) \to \underline{x}$ induces a map

$$\mathcal{P} \to \mathcal{X}/\mathrm{PGL}(3),$$

which is a birational morphism.

For the blow-up $\sigma : S \to \mathbb{P}^2$, we will keep our usual conventions:

- $\underline{x} = \{x_1 \dots x_6\}$ = fundamental locus of $\sigma^{-1}$;
- $E$ = the exceptional divisor of $\sigma$;
- $E_i = \sigma^{-1}(x_i)$;
- $S$ is embedded in $\mathbb{P}^3$ by $|-K_S|$.

There are 72 plane models of the same cubic $S$, and they come in pairs: each pair defines a Schlaefli's double six.

REMARK 3.1. As already mentioned, a smooth curve $C$ of genus 4 together with a subgroup of order 3 in $\mathrm{Pic}^0(C)$ defines a Schlaefli's double six.

Let $(S_1, L_1)$ and $(S_2, L_2)$ be plane models of cubic surfaces.

DEFINITION 3.3. A *morphism of plane models of cubic surfaces*

$$\psi : (S_1, L_1) \to (S_2, L_2)$$

is a morphism $\psi : S_1 \to S_2$ such that $L_2 = \psi^* L_1$. We will say that

(1) $\psi$ is an isomorphism if $\psi$ is biregular and that
(2) $\psi$ is an automorphism if, furthermore, $(S_1, L_1) = (S_2, L_2)$.

PROPOSITION 3.1. *For a general plane model $(S, L)$ of a cubic surface, the only automorphism is the identity.*

*Proof.* Let $\psi : (S, L) \to (S, L)$ be an automorphism. The assumption $\psi^* L \cong L$ implies $\psi^* E = E$. Then $\psi$ induces a map $\bar{\psi} \in \mathrm{Aut}(\mathbb{P}^2)$ such that $\bar{\psi}(\underline{x}) = \underline{x}$. This implies $\bar{\psi} = \mathrm{id}_{\mathbb{P}^2}$ and hence $\psi = \mathrm{id}_S$. □

The moduli space $\mathcal{P}$ contains a nonempty open set

$$\mathcal{P}^o \subset \mathcal{P},$$

which is the moduli space of pairs $(S, L)$ with trivial automorphism group. From the general theory of moduli of del Pezzo surfaces and their explicit construction, as in [CoVL], it follows that on $\mathcal{P}^o$ there exists a universal family representing the moduli functor. This is a pair $(\mathcal{S}, \mathcal{L})$, where $\mathcal{S}$ is a variety endowed with a morphism $\pi : \mathcal{S} \to \mathcal{P}^o$ and a line bundle $\mathcal{L}$. We call such a pair the *universal plane model of a cubic surface*. In particular, it has the following properties:

- the fibre of $\pi$ at a moduli point of $(S, L)$ is $S$;
- the restriction of $\mathcal{L}$ to this fibre $S$ is $L$.

Let $\omega$ be the relative dualizing sheaf of $\pi$. Then we consider the sheaves

(i) $\mathcal{H} := \pi_*(\omega^{-2})$,
(ii) $\mathcal{E} := R^1\pi_*(\mathcal{L}^{-3} \otimes \omega^{-1})$, and
(iii) $\mathcal{F} := R^1\pi_*(\omega^{-3} \otimes \mathcal{L}^{-3})$.

Let $u$ be the moduli point of $(S, L)$. For the fibre over $u$, we have:

(i) $\mathcal{H}_u \cong H^0(S, \mathcal{O}(-2K_S))$;
(ii) $\mathcal{E}_u \cong H^1(S, \mathcal{O}(-E))$;
(iii) $\mathcal{F}_u \cong H^1(S, \mathcal{O}(-2K_S - E))$.

Recall that $h^0(\mathcal{O}_S(-2K_S)) = 10$, $h^1(\mathcal{O}_S(-E)) = 5$, and $h^1(\mathcal{O}_S(-E - 2K_S)) = 8$. Therefore, by Grauert's theorem, it follows that $\mathcal{H}$, $\mathcal{E}$, and $\mathcal{F}$ are vector bundles of rank 10, 5, and 8, respectively. Finally we consider the morphism

$$\mu \colon \mathcal{E} \otimes \mathcal{H} \to \mathcal{F},$$

which fibrewise over $u$ induces the cup product

$$\cup \colon H^0(S, \mathcal{O}_S(-2K_S)) \otimes H^1(S, \mathcal{O}_S(-E)) \to H^1(S, \mathcal{O}_S(-2K_S - E)).$$

By Corollary 2.7, $\cup$ is surjective. Hence $\mu$ is surjective and the kernel of $\mu$ is a vector bundle: we will denote it as $\mathcal{K}$.

We fix the following notation for the induced projective bundles:

- $\mathbb{K} := \mathbb{P}(\mathcal{K})$;
- $\mathbb{P} := \mathbb{P}(\mathcal{E} \otimes \mathcal{H})$;
- $\mathbb{E} := \mathbb{P}(\mathcal{E})$;
- $\mathbb{H} := \mathbb{P}(\mathcal{H})$.

Since $\mathcal{F}$ has rank 8, $\mathbb{K}$ has codimension 8 in $\mathbb{P}$. We denote by

$$\mathbb{K}_u,\ \mathbb{P}_u,\ \mathbb{E}_u,\ \mathbb{H}_u$$

(respectively) the fibres at $u$ of the projective bundles $\mathbb{K}$, $\mathbb{P}$, $\mathbb{E}$, $\mathbb{H}$. We also need to consider the set of decomposable tensors

$$\Sigma := \mathbb{E} \times_{\mathcal{P}^o} \mathbb{H} \subset \mathbb{P}.$$

This is the subvariety of $\mathbb{P}$ whose fibre at $u$ is the Segre product

$$\mathbb{E}_u \times \mathbb{H}_u \subset \mathbb{P}_u.$$

In particular, $\Sigma$ is endowed with the two natural projections

$$p \colon \Sigma \to \mathbb{E} \quad \text{and} \quad q \colon \Sigma \to \mathbb{H}.$$

If $u$ is the moduli point of $(S, L)$, then $\mathbb{H}_u = |-2K_S|$. Hence a point of $\mathbb{H}$ is a pair $(u, C)$, where $u$ is as before and $C \in |-2K_S|$. In particular, $\mathbb{H}$ contains the following open set:

$$\mathcal{U} := \{(u, C) \in \mathbb{H} \mid C \text{ is smooth}\}.$$

DEFINITION 3.4. $\mathbb{M}$ is the Zariski closure of

$$\mathbb{M}^o := \mathbb{K} \cap \mathbb{E} \times_{\mathcal{P}^o} \mathcal{U}.$$

Here $\mathbb{M}^o$ is a scheme over $\mathcal{P}^o$. Let $\mathbb{M}^o_u$ be its fibre at $u \in \mathcal{P}^o$; then

$$\mathbb{M}^o_u = \mathbb{K}_u \cap \mathbb{E}_u \times U,$$

where $U = \{C \in |-2K_S| \mid C \text{ is smooth}\}$.

We already met $\mathbb{M}^o_u$. Indeed, $u$ is the moduli point of $(S, L)$, and $|L|$ defines the blow-up $\sigma\colon S \to \mathbb{P}^2$ of a given set $\underline{x} \in \mathcal{X}$. In Definition 2.5 we considered

$$\mathbb{M}^o_{\underline{x}} = \mathbb{P}(\operatorname{Ker} \cup) \cap \mathbb{P}^4 \times U,$$

where $U$ is the same as before. But it is clear that $\mathbb{E}_u = \mathbb{P}^4 = \mathbb{P}(H^1(\mathcal{O}_S(-E)))$ and that $\mathbb{K}_u = \mathbb{P}(\operatorname{Ker} \cup)$. Hence $\mathbb{M}^o_u = \mathbb{M}^o_{\underline{x}}$.

In Section 2 we described $\mathbb{M}^o_{\underline{x}}$ and $\mathbb{M}_{\underline{x}}$. So we are now in position to describe $\mathbb{M}$ and its projection map

$$p|_{\mathbb{M}}\colon \mathbb{M} \to \mathbb{E}.$$

THEOREM 3.2. *$\mathbb{M}$ is irreducible and dominates $\mathbb{E}$ via $p|_{\mathbb{M}}$. This map is a locally trivial $\mathbb{P}^1$-bundle over a nonempty open set of $\mathbb{E}$; that is, $\mathbb{M}$ is birational to $\mathcal{P} \times \mathbb{P}^4 \times \mathbb{P}^1$.*

*Proof.* Note that, at a general point $(u, C) \in \mathbb{E}$, the fibre of $p$ is $\mathbb{M}_{\underline{x}}$. The statement then follows from the description of $\mathbb{M}_{\underline{x}}$ given in Theorem 2.9. We omit for brevity several standard details. □

## 4. A Birational Model for $\mathcal{R}_{4,3}$ and the Rationality of $\mathcal{R}_{4,\langle 3\rangle}$

The aim of this section is to prove that $\mathbb{M}$ is birational to the moduli space $\mathcal{R}_{4,3}$ of étale triple covers of genus-4 curves. Therefore, by Theorem 3.2 we have that $\mathcal{R}_{4,3}$ is birational to $\mathcal{P} \times \mathbb{P}^5$.

REMARK 4.1. It is an obvious consequence of our construction that the involution $i\colon \mathcal{R}_{4,3} \to \mathcal{R}_{4,3}$, $i(C, \eta) = (C, \eta^{-1})$, corresponds to the involution $(j, \mathrm{id})\colon \mathcal{P} \times \mathbb{P}^5 \to \mathcal{P} \times \mathbb{P}^5$, where $j$ is the Schlaefli involution on $\mathcal{P}$.

To begin, we observe that a point in the open set $\mathbb{M}^o$ (defined in the previous section) is a triple $(u, \bar{v}, C)$, where $u$ is the moduli point of a plane model of a cubic $(S, L)$ or, equivalently, of six unordered points $\underline{x} \in \mathcal{X}$ in general position in $\mathbb{P}^2$ and $(\bar{v}, C) \in \mathbb{M}_u = \mathbb{M}^o_{\underline{x}}$. This is equivalent to saying that $C$ is a smooth element of $|-2K_S|$ and that

$$v \cup s_C = 0,$$

where $v \in H^1(\mathcal{O}_S(-E))$ defines the point $\bar{v} \in \mathbb{E}_u$ and $s_C \in H^0(\mathcal{O}_S(-2K_S))$ is an equation of $C$. We proved in Section 2 that $\eta := \mathcal{O}_C(-L - K_S)$ is a nonzero 3-torsion element of $\operatorname{Pic}^0(C)$. Hence $\mathbb{M}$ comes equipped with a natural rational map

$$\alpha\colon \mathbb{M} \to \mathcal{R}_{4,3}$$

sending the triple $(u, \bar{v}, C)$ to the moduli point of $(C, \eta)$. Conversely, we want now to show that the triple $(u, \bar{v}, C)$ is uniquely reconstructed from the pair $(C, \eta)$. This implies that there exists a second rational map

$$\beta\colon \mathcal{R}_{4,3} \to \mathbb{M}$$

that is inverse to $\alpha$. We proceed in several steps.

*Step 1*

We observe that the image of $|\omega_C \otimes \eta^{-1}|$ is a plane sextic $\Gamma_\eta$ such that $\operatorname{Sing}\Gamma_\eta \equiv \underline{x} \bmod \mathrm{PGL}(3)$. This follows because $\eta \cong \mathcal{O}_C(-K_S - L)$. Since $\omega_C \cong \mathcal{O}_C(-K_S)$, we have $|\omega_C \otimes \eta_C^{-1}| = |\mathcal{O}_C(L)|$. Hence, up to projective automorphisms, $\Gamma_\eta = \sigma(C)$ and therefore $\operatorname{Sing}\Gamma_\eta = \underline{x} \bmod \mathrm{PGL}(3)$.

*Step 2*

Starting from $(C, \eta)$, we first associate to it the moduli point $u$ of the pair $(S, L)$ or, equivalently, the moduli point of $\operatorname{Sing}\Gamma_\eta$. Next, we reconstruct uniquely the curve $C$ in the linear system $|-2K_S|$. More precisely, we have to show that there exists a unique $D \in |-2K_S|$ such that $D \cong C$ and $\mathcal{O}_D(-K_S - L) \cong \eta$. This follows from the next lemma.

LEMMA 4.1. *Assume that $C_1, C_2 \in |-2K_S|$ are smooth biregular curves such that $\varepsilon_1 \cong \varepsilon_2$, where $\varepsilon_i = \mathcal{O}_{C_i}(-K_S - L)$. Then $C_1 = C_2$.*

*Proof.* Applying the same proof as the one of Proposition 2.1, it follows that $\varepsilon_i$ is nontrivial. Let $\Gamma_i = \sigma(C_i)$. Then, under our assumption, $\Gamma_1$ and $\Gamma_2$ are projectively equivalent. Since $\operatorname{Sing}\Gamma_1 = \operatorname{Sing}\Gamma_2 = \underline{x}$, there exists an $a \in \operatorname{Aut}(\mathbb{P}^2)$ such that $a(\underline{x}) = \underline{x}$. Since the pair $(S, L)$ has no automorphism, $a$ is the identity and hence $C_1 = C_2$. $\square$

REMARK 4.2. Let $\mathrm{Pic}_{0,4}$ be the universal Picard variety of genus 4 and degree 0 and let $s\colon |-2K_S| \to \mathrm{Pic}_{0,4}$ be the rational map sending $C$ to the moduli point of the pair $(C, \mathcal{O}_C(-K_S - L))$. By Lemma 4.1, $s$ is injective on the open set $U$ of smooth curves.

*Step 3*

So far we have reconstructed uniquely from $(C, \eta)$ the moduli point $u$ and a copy of $C$ in $|-2K_S|$ such that $\eta \cong \mathcal{O}_C(-K_S - L)$. Finally, the point $\bar{v}$ is also uniquely reconstructed: consider the standard exact sequence

$$0 \to \mathcal{O}_S(-E) \to \mathcal{O}_S(-E - 2K_S) \to \eta^3 \to 0.$$

Passing to the long exact sequence, the image of $H^0(\eta^3)$ via the coboundary map is exactly the vector space $\bar{v}$ generated by $v$.

By Steps 1–3, the triple $(u, \bar{v}, C)$ is uniquely defined by the isomorphism class of $(C, \eta)$. Hence there exists a rational map

$$\beta \colon \mathcal{R}_{4,3} \to \mathbb{M}$$

sending the moduli point of $(C, \eta)$ to $(u, \bar{v}, C)$. We conclude as follows.

THEOREM 4.2. *$\alpha \colon \mathbb{M} \to \mathcal{R}_{4,3}$ is birational. In particular, $\mathcal{R}_{4,3} \simeq \mathcal{P} \times \mathbb{P}^5$.*

*Proof.* Since $\mathbb{M}$ and $\mathcal{R}_{4,3}$ are integral and of the same dimension, $\alpha$ and $\beta$ are each the inverse of the other. The second part then follows from Theorem 3.2. □

We conclude this section with the following statement.

THEOREM 4.3. *$\mathcal{R}_{4,\langle 3\rangle}$ is rational.*

*Proof.* We have $\mathcal{R}_{4,\langle 3\rangle} \simeq \mathcal{R}_{4,3}/i \simeq \mathcal{P}/j \times \mathbb{P}^5$, where $j$ is the Schlaefli involution. Moreover, $\mathcal{P}/j$ is rational, as shown in the next section. □

## 5. The Rationality of the Moduli Space of Double Sixes in $\mathbb{P}^2$

Recall that $\mathcal{P}$ is birational to the GIT-quotient $\mathcal{X}/\mathrm{PGL}(3)$. In this section, we prove the rationality of the quotient of the moduli space of six unordered points in $\mathbb{P}^2$ by the Schlaefli involution. In other words, we show that the moduli space of double sixes (modulo PGL(3)) in $\mathbb{P}^2$ is rational. A proof of this result is due to Coble [Cob, p. 176]. A modern proof was kindly communicated to us by Igor Dolgachev. In the remaining part of this section, we present Dolgachev's proof. This completes the proof of the rationality of $\mathcal{R}_{4,\langle 3\rangle}$.

THEOREM 5.1. *The moduli space of double sixes in $\mathbb{P}^2$ is rational.*

*Proof.* Consider the GIT-quotient $\mathcal{Y} := \mathcal{X}/\mathrm{PGL}(3)$. Let $\Delta$ be the big diagonal in $(\mathbb{P}^2)^6$. Let $P_2^6 = ((\mathbb{P}^2)^6 \setminus \Delta)//\mathrm{SL}(3)$ be the GIT-quotient with respect to the symmetric linearization of the action of SL(3). Obviously, $\mathcal{Y}$ is birationally isomorphic to $P_2^6/\mathfrak{S}_6$. Recall from [DO] that

$$P_2^6 \cong \operatorname{Proj} R_2^6,$$

where $R_2^6$ is the graded algebra with $d$-homogeneous part $(R_2^6)_d = (V(d)^{\otimes 6})^{\mathrm{SL}(3)}$ and $V(d) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$.

Let $V(1)_i$ be the $i$th copy of the space $V(1)$ and let $t_1^{(i)}, t_2^{(i)}, t_3^{(i)}$ be a basis of $V(1)_i$. For any subset $I = \{i_1, i_2, i_3\}$ of $\{1, \dots, 6\}$, denote by $D_I$ the determinant of the matrix

$$\begin{pmatrix} t_1^{(i_1)} & t_2^{(i_1)} & t_3^{(i_1)} \\ t_1^{(i_2)} & t_2^{(i_2)} & t_3^{(i_2)} \\ t_1^{(i_3)} & t_2^{(i_3)} & t_3^{(i_3)} \end{pmatrix}.$$

We consider $D_I$ as an element of $V(1)_{i_1} \otimes V(1)_{i_2} \otimes V(1)_{i_3}$. By the fundamental theorem of invariant theory, the vector space $(R_2^6)_d$ is spanned by the (tensor) products of $D_I$ such that each $k \in \{1, \dots, 6\}$ appears exactly $d$ times. An explicit computation (due to Coble; see [DO]) shows that the graded algebra $R_2^6$ is generated by five elements of degree 1 given by

$$x_0 = D_{123}D_{456}, \quad x_1 = D_{124}D_{356}, \quad x_2 = D_{125}D_{346},$$
$$x_3 = D_{134}D_{256}, \quad x_4 = D_{135}D_{246}$$

and one element of degree 2:

$$x_5 = D_{123}D_{145}D_{246}D_{356} - D_{124}D_{135}D_{236}D_{456}.$$

There is one relation

$$\begin{aligned} x_5^2 = {} & (-x_2x_3 + x_1x_4 + x_0x_1 + x_0x_4 - x_0x_2 - x_0x_3 - x_0^2)^2 \\ & - 4x_0x_1x_4(-x_0 + x_1 - x_2 - x_3 + x_4). \end{aligned}$$

After change of a basis,

$$(y_0, y_1, y_2, y_3, y_4, y_5) = (x_0, x_1, x_4, -x_0 - x_2, -x_0 - x_3, x_5),$$

the relation becomes

$$y_5^2 = (y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4y_0y_1y_2(y_0 + y_1 + y_2 + y_3 + y_4). \quad (*)$$

Note that the polynomial on the right-hand side defines a quartic hypersurface in $\mathbb{P}^4$ isomorphic to the dual of the 10-nodal Segre cubic 3-fold. This hypersurface is also isomorphic to a compactification of the moduli space of principally polarized abelian surfaces with level-2 structure (see [Ig]).

Now let us see how the permutation group $\mathfrak{S}_6$ acts on the generators. Using the straightening algorithm, we find that the following representatives of the conjugacy classes of $\mathfrak{S}_6$ act on the the space $(R_2^6)_1$—that is, on $(x_0, \dots, x_4)$—as follows.

$$\begin{aligned}
(12)\colon &\ (-x_0, -x_1, -x_2, x_0 - x_1 + x_3, -x_0 - x_2 + x_4).\\
(12)(34)\colon &\ (-x_1, -x_0, x_2, -x_0 + x_1 - x_3, -x_0 - x_3 + x_4).\\
(12)(34)(56)\colon &\ (x_1, x_0, x_0 - x_1 + x_2, x_0 - x_1 + x_3, x_4).\\
(123)\colon &\ (x_0, x_0 - x_1 + x_3, -x_0 - x_2 + x_4, -x_1, -x_2).\\
(1234)\colon &\ (x_0 - x_1 + x_3, x_0, x_0 + x_2 - x_4, x_1, x_0 + x_3 - x_4).\\
(1234)(56)\colon &\ (-x_0 + x_1 - x_3, -x_0, x_0 - x_1 + x_2 + x_3 - x_4, -x_1, -x_4).\\
(12345)\colon &\ (-x_0 + x_1 - x_3, x_0 + x_2 - x_4, x_0, x_0 + x_3 - x_4, x_1).\\
(123)(45)\colon &\ (-x_0, -x_0 - x_2 + x_4, x_0 - x_1 + x_3, -x_2, -x_1).\\
(123456)\colon &\ (x_0 - x_1 + x_3, -x_0 - x_2 + x_4, -x_0 + x_1 - x_2 - x_3 + x_4,\\
&\ \ -x_0 - x_3 + x_4, x_4).\\
(123)(456)\colon &\ (x_0, x_0 + x_2 - x_4, x_0 - x_1 + x_2 + x_3 - x_4, x_2, x_0 - x_1 + x_2).
\end{aligned}$$

This allows us to compute the characters of the representation of $\mathfrak{S}_6$ on $(R_2^6)_1$, and we find that this representation is isomorphic to the 5-dimensional irreducible $\mathbb{U}_5$ that is obtained from the 5-dimensional *standard* representation of $\mathfrak{S}_6$ by composition with an outer automorphism of $\mathfrak{S}_6$.

One checks that the polynomial

$$(y_0y_1 + y_0y_2 + y_1y_2 - y_3y_4)^2 - 4y_0y_1y_2(y_0 + y_1 + y_2 + y_3 + y_4)$$

is invariant. Therefore, under the isomorphism of representations $(R_2^6)_1 \cong \mathbb{U}_5$, this polynomial corresponds to a linear combination of the elementary symmetric polynomials $\sigma_2$ and $\sigma_4$ in variables $z_0, \ldots, z_4$. Since $\mathfrak{S}_6$ acts on $R_2^6$ by automorphisms of the graded ring, the subspace $(R_6^2)_2$ is invariant and the relation $(*)$ is $\mathfrak{S}_6$-invariant. This shows that $\mathfrak{S}_6$ acts on $x_5$ by changing it to $\pm x_5$. It is immediately checked that $x_5$ is not invariant; hence it is transformed via the sign representation of $\mathfrak{S}_6$. There is another skew invariant of $\mathbb{U}_5$, the square root $D$ of the discriminant $\delta$ of a general polynomial of degree 6, whence we get a further invariant of the $\mathfrak{S}_6$ action—namely, $Dx_5$. We have the equation $(Dx_5)^2 = \delta(\lambda_1\sigma_2^2 + \lambda_2\sigma_4)$, which is invariant. Therefore, we see that $X_6^2$ is isomorphic to a hypersurface $F_{34} \subset \mathbb{P}(2,3,4,5,6,17)$.

Now we take the quotient of $X_6^2$ by the Schlaefli involution $j$. By [DO], the Schlaefli involution is just the association morphism $a_{2,6}\colon R_6^2 \to R_6^2$, which is given by $x_i \mapsto x_i$ for $0 \le i \le 4$, and $D_{123}D_{145}D_{246}D_{356} \mapsto D_{124}D_{135}D_{236}D_{456}$. This implies that $x_5 \mapsto -x_5$ and the square root $D$ of the discriminant $\delta$ is invariant. This implies that $X_6^2/j$ is isomorphic to $\mathbb{P}(2,3,4,5,6)$, whence rational. □

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