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On Moduli Spaces of Parabolic Vector Bundles of Rank 2 over $\mathbb{CP}^1$

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1. Introduction

Let $S \subset \mathbb{CP}^1$ be a finite subset such that $\#S \geq 5$. Fix an integer d. Let $\mathcal{M}_S(d) = \mathcal{M}_S$ be the moduli space of parabolic semistable vector bundles $E_* \to \mathbb{CP}^1$ of rank 2 and degree d with parabolic structure over S such that for each point $s \in S$ the parabolic weights of E_* at s are 0 and $1/2$. In [4], geometric realizations of the variety $\mathcal{M}_S$ were obtained by the third author (under the assumption that $\#S$ is even).

Our aim here is to address the following Torelli type question:

> Take two subsets S_1 and S_2 such that the variety $\mathcal{M}_{S_1}$ is isomorphic to $\mathcal{M}_{S_2}$. Does this imply that the multi-pointed curve $(\mathbb{CP}^1, S_1)$ is isomorphic to $(\mathbb{CP}^1, S_2)$?

The following theorem proved here (see Theorem 4.2) shows that this indeed is the case.

Theorem 1.1. *Take two finite subsets S_1 and S_2 of $\mathbb{CP}^1$ of cardinality ≥ 5. The variety $\mathcal{M}_{S_1}$ is isomorphic to $\mathcal{M}_{S_2}$ if and only if there is an automorphism*

$$\varphi \colon \mathbb{CP}^1 \to \mathbb{CP}^1$$

such that $\varphi(S_1) = S_2$.

If $\#S = 4$, then the moduli space $\mathcal{M}_S$ is isomorphic to $\mathbb{CP}^1$. Therefore, the assumption in Theorem 1.1 that there are at least five parabolic points is necessary.

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2. Hitchin Map and Unstable Locus

Let

$$S \subset \mathbb{CP}^1$$

be a finite subset of the complex projective line such that

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$$n := \#S \geq 5.$$

Fix an integer d. We consider parabolic vector bundles

$$E_* \to \mathbb{CP}^1$$

satisfying the following conditions:

- $\operatorname{rank}(E_*) = 2$;
- $\operatorname{degree}(E) = d$, where E is the vector bundle underlying E_*;
- the parabolic divisor of E_* is S; and
- for each point $s \in S$, the parabolic weights of E_s are $\{0, 1/2\}$.

Therefore,

$$\text{par-deg}(E_*) = d + \frac{n}{2},$$

where par-deg(E_*) is the parabolic degree of E_*.

Let $\mathcal{M}_S = \mathcal{M}_S(d)$ denote the moduli space of parabolic semistable vector bundles of the type just described; see [5]. This moduli space $\mathcal{M}_S$ is a normal projective variety, defined over $\mathbb{C}$, of dimension $n - 3$.

Let

$$\mathcal{M}_S^s \subset \mathcal{M}_S \tag{2.1}$$

be the Zariski open dense subset that parameterizes the stable parabolic vector bundles of the given type. The complement $\mathcal{M}_S \setminus \mathcal{M}_S^s$ is a finite set because there are only finitely many polystable parabolic vector bundles of the given type. This open subset $\mathcal{M}_S^s$ coincides with the smooth locus of $\mathcal{M}_S$.

We note that if $\#S = n$ is odd then, for any $E_* \in \mathcal{M}_S$,

$$\frac{\text{par-deg}(E_*)}{\operatorname{rank}(E)} = \frac{a}{2} + \frac{1}{4},$$

where a is an integer. Since the parabolic degree of a line subbundle of E is an integral multiple of $1/2$, it follows that E_* is actually parabolic stable. Consequently, $\mathcal{M}_S$ is a smooth projective variety whenever n is odd.

Let E_* be any parabolic vector bundle of the numerical type considered here (it need not be parabolic semistable). A *Higgs field* on E_* is a section

$$\theta \in H^0(\mathbb{CP}^1, \operatorname{End}(E) \otimes K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)),$$

where E, as before, is the vector bundle underlying E_* such that, for each point $s \in S$, the endomorphism

$$\theta(s) \in \operatorname{End}(E_s)$$

is nilpotent with respect to the quasiparabolic filtration of E_s (see [1, Sec. 6] for more details); if $\ell \subset E_s$ is the quasiparabolic filtration, then the nilpotency condition means that $\theta(s)(E_s) \subset \ell$ and $\theta(s)(\ell) = 0$. Note that from the Poincaré adjunction formula it follows that the fiber of the line bundle $K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)$ over any point $s \in S$ is identified with $\mathbb{C}$. A *parabolic Higgs bundle* is a pair of the form (E_*, θ), where E_* is a parabolic vector bundle and θ is a Higgs field on E_*.

REMARK 2.1. If θ is a Higgs field on E_*, then trace(θ) is a section of $K_{\mathbb{CP}^1}$ because $\theta(s)$ is nilpotent for each $s \in S$. Since $H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}) = 0$, we conclude that trace$(\theta) = 0$.

A parabolic Higgs bundle (E_*, θ) is called *stable* (resp., *semistable*) if, for all line subbundles $L \subset E$ with $\theta(L) \subset L \otimes K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)$, the inequality

$$\text{par-deg}(L_*) < \text{par-deg}(E_*)/2 \quad (\text{resp., par-deg}(L_*) \le \text{par-deg}(E_*)/2)$$

holds, where L_* is the parabolic line bundle defined by L equipped with the induced parabolic structure.

Let $\mathcal{N}_S(H)$ denote the moduli space of semistable parabolic Higgs bundles of rank 2 and degree d over $\mathbb{CP}^1$ with parabolic structure over S and having parabolic weights 0 and $1/2$ at each point of S. This $\mathcal{N}_S(H)$ is a normal quasiprojective variety defined over $\mathbb{C}$ of dimension $2n - 6$. Consider the total space $T^*\mathcal{M}_S^s$ of the cotangent bundle of the moduli space $\mathcal{M}_S^s$ defined in (2.1). We have a natural embedding

$$\iota\colon T^*\mathcal{M}_S^s \to \mathcal{N}_S(H) \tag{2.2}$$

because, for any $E_* \in \mathcal{M}_S^s$, the cotangent space $T^*_{E_*}\mathcal{M}_S^s$ is the space of all Higgs fields on E_*. The image $\iota(T^*\mathcal{M}_S^s)$ is a Zariski open dense subset of $\mathcal{N}_S(H)$.

Let

$$H\colon \mathcal{N}_S(H) \to H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)) \tag{2.3}$$

be the Hitchin map that sends any (E_*, θ) to trace(θ^2) [3]; the condition that $\theta(s)$ is nilpotent ensures that trace(θ^2) lies inside the subspace

$$H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)) \subset H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(2S)).$$

Let

$$f\colon K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S) \to \mathbb{CP}^1 \tag{2.4}$$

be the natural projection. For any $v \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S))$, let

$$\rho_v\colon K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S) \to K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(2S)$$

be the morphism of varieties defined by $\omega \mapsto \omega^{\otimes 2} - v(f(\omega))$, where f is defined in (2.4). The scheme-theoretic inverse image $(\rho_v)^{-1}(0_X)$, where 0_X is the image of the zero section, is called the *spectral curve* for v.

For a general point

$$v \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)),$$

the corresponding spectral curve C_v in the total space of $K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)$ is a connected smooth projective curve of genus $n - 3$, and the fiber $H^{-1}(v)$ is identified with $\text{Pic}^{d+n-2}(C_v)$.

Consider the morphism

$$f_v\colon C_v \to \mathbb{CP}^1 \tag{2.5}$$

obtained by restricting the projection f in (2.4). The parabolic vector bundle corresponding to any $\xi \in \text{Pic}^{d+n-2}(C_v)$ has the direct image $f_{v*}\xi$ as the underlying vector bundle. The subset of $\mathbb{CP}^1$ over which f_v is ramified contains S. Therefore, for any

$$\xi \in \mathrm{Pic}^{d+n-2}(C_v)$$

and any $s \in S$, the fiber $(f_{v*}\xi)_s$ has a line given by the locally defined sections of ξ that vanish at the reduced point $f^{-1}(s)$. The quasiparabolic filtration on $f_{v*}\xi$ over s is defined by this line.

PROPOSITION 2.2. *Take any $v \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S))$ such that the corresponding spectral curve C_v is smooth and connected. The codimension of the complement*

$$\mathrm{Pic}^{d+n-2}(C_v) \setminus (\iota(T^*\mathcal{M}_S^s) \cap \mathrm{Pic}^{d+n-2}(C_v)) \subset \mathrm{Pic}^{d+n-2}(C_v)$$

is at least 2, where ι is the embedding in (2.2).

Proof. Take any $\xi \in \mathrm{Pic}^{d+n-2}(C_v)$ such that the corresponding parabolic vector bundle is not stable. The parabolic vector bundle corresponding to ξ will be denoted by V_*. We recall that

$$V := f_{v*}\xi \to \mathbb{CP}^1$$

is the holomorphic vector bundle underlying V_*, where f_v is the projection in (2.5).

For convenience, write $d+n-2 = \delta$. Since V_* is not parabolic stable, we have a short exact sequence of vector bundles

$$0 \to L \xrightarrow{\sigma} V \to V/L \to 0 \tag{2.6}$$

such that, for the parabolic line bundle L_* defined by the subbundle L equipped with the induced parabolic structure, the inequality

$$\text{par-deg}(L_*) \geq \text{par-deg}(V_*)/2 = (2d+n)/4 = (2\delta - n + 4)/4 \tag{2.7}$$

holds. Set

$$\hat{L} := f_v^* L.$$

Let

$$\phi\colon \hat{L} \to \xi \tag{2.8}$$

be the composition of homomorphisms

$$\hat{L} = f_v^* L \xrightarrow{f_v^*\sigma} f_v^* f_{v*}\xi \to \xi,$$

where σ is the homomorphism in (2.6) and $f_v^* f_{v*}\xi \to \xi$ is the natural homomorphism. Since ϕ does not vanish identically, we have

$$\mathrm{degree}(\hat{L}) \leq \mathrm{degree}(\xi) = \delta. \tag{2.9}$$

Take a point $s \in S$. If L_* has parabolic weight $1/2$ at s, then the homomorphism ϕ in (2.8) vanishes at the point $f_v^{-1}(s) \in C_v$. Let

$$\beta \in \frac{1}{2}\mathbb{Z}$$

be the parabolic weight of L_*. From the preceding observation we have

$$\#(\mathrm{Div}(\phi) \cap f_v^{-1}(S)) \geq 2\beta. \tag{2.10}$$

Using (2.7),

$$\mathrm{degree}(\hat{L}) = 2 \cdot \mathrm{degree}(L) \geq 2 \cdot \left(\frac{2\delta - n + 4}{4} - \beta\right) = \delta - \frac{n-4}{2} - 2\beta. \tag{2.11}$$

So,

$$\begin{aligned} \mathrm{degree}(\mathrm{Div}(\phi)) &= \mathrm{degree}(\xi) - \mathrm{degree}(\hat{L}) \\ &\leq \delta - \delta + \frac{n-4}{2} + 2\beta = \frac{n-4}{2} + 2\beta. \end{aligned} \tag{2.12}$$

Note that from (2.11) and (2.9),

$$\frac{\delta}{2} \geq \mathrm{degree}(L) \geq \frac{\delta}{2} - \frac{n-4}{4} - \beta.$$

Hence $\mathrm{degree}(L)$ can take only finitely many values. Since $\hat{L} = f_v^*L$, the isomorphism class of $\hat{L}$ is uniquely determined by the integer $\mathrm{degree}(L)$. Hence from (2.10) and (2.12) we conclude that all $\xi \in \mathrm{Pic}^{\delta}(C_v)$ such that corresponding parabolic vector bundle in $\mathcal{M}_S$ is not stable are parameterized by a scheme of dimension $\leq [n/2] - 2$, where $[n/2] \in \mathbb{N}$ is the integral part of $n/2$.

Hence the codimension of

$$\mathrm{Pic}^{d+n-2}(C_v) \setminus (\iota(T^*\mathcal{M}_S^s) \cap \mathrm{Pic}^{d+n-2}(C_v)) \subset \mathrm{Pic}^{d+n-2}(C_v)$$

is at least $n - 3 - ([n/2] - 2)$. Finally,

$$n - 3 - ([n/2] - 2) = n - [n/2] - 1 \geq 2$$

(recall that $n \geq 5$). This completes the proof of the proposition. □

Take any algebraic function ψ on $T^*\mathcal{M}_S^s$. From Proposition 2.2 it follows that ψ is constant on $\iota(T^*\mathcal{M}_S^s) \cap \mathrm{Pic}^{d+n-2}(C_v)$. Hence ψ factors through the Hitchin map $H|_{T^*\mathcal{M}_S^s}$ in (2.3).

3. Theta Divisor and the Pullback of the Anticanonical Bundle

As in Proposition 2.2, take $v \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S))$ such that the corresponding spectral curve C_v is connected and smooth. Let

$$p\colon Z := \iota(T^*\mathcal{M}_S^s) \cap \mathrm{Pic}^{d+n-2}(C_v) \to \mathcal{M}_S^s \tag{3.1}$$

be the restriction of the natural projection $T^*\mathcal{M}_S^s \to \mathcal{M}_S^s$. From Proposition 2.2 we know that the inclusion map $Z \hookrightarrow \mathrm{Pic}^{d+n-2}(C_v)$ induces an isomorphism of Picard groups.

Let

$$\Theta \in H^2(\mathrm{Pic}^{d+n-2}(C_v), \mathbb{Z})$$

be the canonical polarization given by the cup product on $H^1(C_v, \mathbb{Z})$.

LEMMA 3.1. *For the projection p in* (3.1),

$$p^* c_1(T\mathcal{M}_S^s) = 4^{n-3} \cdot \Theta.$$

Proof. Fix a Weierstrass point $x_0 \in C_v$; so the map f_v in (2.5) is ramified at x_0. There is a unique Poincaré line bundle

$$\mathcal{L} \to C_v \times \mathrm{Pic}^{d+n-2}(C_v)$$

such that the restriction of $\mathcal{L}$ to $\{x_0\} \times \mathrm{Pic}^{d+n-2}(C_v)$ is a trivial line bundle. Consider the direct image

$$\mathcal{W} := (f_v \times \mathrm{Id}_{\mathrm{Pic}^{d+n-2}(C_v)})_* \mathcal{L} \to \mathbb{CP}^1 \times \mathrm{Pic}^{d+n-2}(C_v). \tag{3.2}$$

Since f_v is ramified over S, for each point $(s, \xi) \in S \times \mathrm{Pic}^{d+n-2}(C_v)$, the fiber $\mathcal{W}_{(s,\xi)}$ has a filtration

$$\ell \subset \mathcal{W}_{(s,\xi)}$$

given by the locally defined sections of $\mathcal{L}|_{C_v \times \{\xi\}}$ that vanish at the point $\hat{s} := f_v^{-1}(s)_{\mathrm{red}} \in C_v$. Therefore, the line ℓ is naturally identified with the fiber $(K_{C_v})_{\hat{s}} \otimes \mathcal{L}_{(\hat{s},\xi)}$, and the quotient line $\mathcal{W}_{(s,\xi)}/\ell$ is identified with $\mathcal{L}_{(\hat{s},\xi)}$.

Let

$$\mathcal{E} \subset \mathrm{End}(\mathcal{W}) = \mathcal{W} \otimes \mathcal{W}^* \to \mathbb{CP}^1 \times \mathrm{Pic}^{d+n-2}(C_v) \tag{3.3}$$

be the locally free subsheaf of $\mathrm{End}(\mathcal{W})$ defined by the sheaf of trace-0 endomorphisms that preserve the aforementioned filtration over $S \times \mathrm{Pic}^{d+n-2}(C_v)$. Note that

$$\mathrm{End}(\mathcal{W}) = \mathrm{ad}(\mathcal{W}) \oplus \mathcal{O}_{\mathbb{CP}^1 \times \mathrm{Pic}^{d+n-2}(C_v)},$$

where $\mathrm{ad}(\mathcal{W})$ is the subbundle of $\mathrm{End}(\mathcal{W})$ defined by the sheaf of trace-0 endomorphisms. Let

$$\iota_{\hat{S}} \colon \hat{S} := f_v^{-1}(S)_{\mathrm{red}} \hookrightarrow C_v$$

be the inclusion map. So

$$\mathcal{A}_0 := (\iota_{\hat{S}} \times \mathrm{Id}_{\mathrm{Pic}^{d+n-2}(C_v)})_* (\iota_{\hat{S}} \times \mathrm{Id}_{\mathrm{Pic}^{d+n-2}(C_v)})^* K_{C_v} \tag{3.4}$$

is a torsion sheaf on $C_v \times \mathrm{Pic}^{d+n-2}(C_v)$ with support $\hat{S} \times \mathrm{Pic}^{d+n-2}(C_v)$. Note that $\mathcal{A}_0$ is the restriction to $\hat{S} \times \mathrm{Pic}^{d+n-2}(C_v)$ of the pullback of K_{C_v} to $C_v \times \mathrm{Pic}^{d+n-2}(C_v)$. Using our description of the lines ℓ and $\mathcal{W}_{(s,\xi)}/\ell$, from (3.3) we get a short exact sequence of sheaves

$$0 \to \mathcal{E} \to \mathrm{End}(\mathcal{W}) \to \mathcal{A}_0 \oplus \mathcal{O}_{\mathbb{CP}^1 \times \mathrm{Pic}^{d+n-2}(C_v)} \to 0, \tag{3.5}$$

where $\mathcal{A}_0$ is defined in (3.4).

Let

$$q \colon \mathbb{CP}^1 \times \mathrm{Pic}^{d+n-2}(C_v) \to \mathrm{Pic}^{d+n-2}(C_v) \tag{3.6}$$

be the natural projection. Consider the map p in (3.1). The pulled-back tangent bundle $p^* T\mathcal{M}_S^s$ is identified with $R^1 q_* \mathcal{E}$, where $\mathcal{E}$ is defined in (3.3). We note that

$$q_* \mathcal{E} = 0$$

because a stable parabolic vector bundle is simple, meaning that all automorphisms of a stable parabolic vector bundle preserving the quasiparabolic filtrations are scalar multiplications.

Note that since the restriction of $\mathcal{L}$ to $\{x_0\} \times \mathrm{Pic}^{d+n-2}(C_v)$ is trivial, the restriction of $\mathcal{L}$ to $\{x\} \times \mathrm{Pic}^{d+n-2}(C_v)$ is topologically trivial for all $x \in C_v$.

Since $R^1 q_* \mathcal{E} = p^* T\mathcal{M}_S^s$ and $\det q_* \mathcal{E}$ is trivial, we conclude that

$$p^* \det T\mathcal{M}_S^s = p^* \bigwedge^{n-3} T\mathcal{M}_S^s = (\det R^1 q_* \mathcal{E}) \otimes (\det q_* \mathcal{E})^*. \tag{3.7}$$

From (3.5),

$$c_i(R^j q_* \mathcal{E}) = c_i(R^j q_* \operatorname{End}(\mathcal{W})) \in H^{2i}(\mathrm{Pic}^{d+n-2}(C_v), \mathbb{Q})$$

for all $i, j \geq 0$. Hence, from (3.7),

$$p^* c_1(T\mathcal{M}_S^s) = c_1(R^1 q_* \operatorname{End}(\mathcal{W})) - c_1(q_* \operatorname{End}(\mathcal{W})). \tag{3.8}$$

Define

$$F := f_v \times \mathrm{Id}_{\mathrm{Pic}^{d+n-2}(C_v)}.$$

From the definition of $\mathcal{W}$ (see (3.2)) and the projection formula, we conclude that

$$\operatorname{End}(\mathcal{W}) = F_*(\mathcal{L} \otimes F^* \mathcal{W}^*). \tag{3.9}$$

Let

$$\hat{q} \colon C_v \times \mathrm{Pic}^{d+n-2}(C_v) \to \mathrm{Pic}^{d+n-2}(C_v) \tag{3.10}$$

be the natural projection. Since f_v is a finite map, from (3.9) we have

$$\begin{aligned}(\det R^1 q_* \operatorname{End}(\mathcal{W})) &\otimes (\det q_* \operatorname{End}(\mathcal{W}))^* \\ &= \det R^1 \hat{q}_*(\mathcal{L} \otimes F^* \mathcal{W}^*) \otimes (\det \hat{q}_*(\mathcal{L} \otimes F^* \mathcal{W}^*))^*,\end{aligned}$$

where q is the projection in (3.6).

Hence, from (3.8),

$$p^* c_1(T\mathcal{M}_S^s) = c_1(\det R^1 \hat{q}_*(\mathcal{L} \otimes F^* \mathcal{W}^*)) - c_1(\det \hat{q}_*(\mathcal{L} \otimes F^* \mathcal{W}^*)). \tag{3.11}$$

Let

$$\eta \colon C_v \to C_v \tag{3.12}$$

be the nontrivial Galois involution of the covering f_v; so η is the hyperelliptic involution. Define

$$\hat{\eta} := \eta \times \mathrm{Id}_{\mathrm{Pic}^{d+n-2}(C_v)}. \tag{3.13}$$

Let

$$\hat{S} \subset f_v^{-1}(S)_{\mathrm{red}} \times \mathrm{Pic}^{d+n-2}(C_v) \subset C_v \times \mathrm{Pic}^{d+n-2}(C_v) =: \mathcal{Z}$$

be the reduced divisor. Consider the natural surjective homomorphism

$$F^* \mathcal{W} \to \mathcal{L} \to 0$$

on $\mathcal{Z}$. Its kernel is identified with $\hat{\eta}^* \mathcal{L} \otimes \mathcal{O}_{\mathcal{Z}}(-\hat{S})$, where $\hat{\eta}$ is defined in (3.13). Therefore, we have a short exact sequence of vector bundles over $\mathcal{Z}$:

$$0 \to \mathcal{L}^* \to F^* \mathcal{W}^* \to (\hat{\eta}^* \mathcal{L}^*) \otimes \mathcal{O}_{\mathcal{Z}}(\hat{S}) \to 0.$$

Tensoring this with $\mathcal{L}$, we get the short exact sequence of vector bundles

$$0 \to \mathcal{O}_{\mathcal{Z}} \to \mathcal{L} \otimes F^*\mathcal{W}^* \to \mathcal{L} \otimes (\hat{\eta}^*\mathcal{L}^*) \otimes \mathcal{O}_{\mathcal{Z}}(\hat{S}) \to 0. \tag{3.14}$$

For a vector bundle $E' \to C_v \times \mathrm{Pic}^{d+n-2}(C_v) =: \mathcal{Z}$, define

$$\mathrm{Det}(E') := (\det R^1\hat{q}_*E') \otimes (\det \hat{q}_*E')^*,$$

where $\hat{q}$ is the projection in (3.10).

Now, from (3.14) and (3.11),

$$p^*c_1(T\mathcal{M}_S^s) = c_1(\mathrm{Det}(\mathcal{L} \otimes (\hat{\eta}^*\mathcal{L}^*) \otimes \mathcal{O}_{\mathcal{Z}}(\hat{S}))). \tag{3.15}$$

From the short exact sequence of coherent sheaves

$$0 \to \mathcal{L} \otimes \hat{\eta}^*\mathcal{L}^* \to \mathcal{L} \otimes (\hat{\eta}^*\mathcal{L}^*) \otimes \mathcal{O}_{\mathcal{Z}}(\hat{S}) \to \mathcal{O}_{\hat{S}} \to 0$$

on $C_v \times \mathrm{Pic}^{d+n-2}(C_v)$, we conclude that

$$\mathrm{Det}(\mathcal{L} \otimes (\hat{\eta}^*\mathcal{L}^*) \otimes \mathcal{O}_{\mathcal{Z}}(\hat{S})) = \mathrm{Det}(\mathcal{L} \otimes \hat{\eta}^*\mathcal{L}^*).$$

So, from (3.15),

$$p^*c_1(T\mathcal{M}_S^s) = c_1(\mathrm{Det}(\mathcal{L} \otimes \hat{\eta}^*\mathcal{L}^*)). \tag{3.16}$$

Now note that the involution $\hat{\eta}$ lifts to the line bundle $\mathcal{L} \otimes \hat{\eta}^*\mathcal{L}$. The isotropy subgroups, for the action of $\mathbb{Z}/2\mathbb{Z}$, act trivially on the fibers of $\mathcal{L} \otimes \hat{\eta}^*\mathcal{L}$. Hence $\mathcal{L} \otimes \hat{\eta}^*\mathcal{L}$ descends to a line bundle on $\mathcal{Z}/\hat{\eta} = \mathbb{CP}^1 \times \mathrm{Pic}^{d+n-2}(C_v)$. Since the restriction of $\mathcal{L}$ to $\{x_0\} \times \mathrm{Pic}^{d+n-2}(C_v)$ is a trivial line bundle and x_0 is fixed by f_v, the restriction of $\mathcal{L} \otimes \hat{\eta}^*\mathcal{L}$ to $\{x_0\} \times \mathrm{Pic}^{d+n-2}(C_v)$ is also trivial. We further note that any line bundle on $\mathbb{CP}^1 \times \mathrm{Pic}^{d+n-2}(C_v)$ is of the form $L_1 \boxtimes L_2$. Hence $\mathcal{L} \otimes \hat{\eta}^*\mathcal{L}$ is the pullback of a line bundle on $\mathbb{CP}^1$. In other words,

$$\hat{\eta}^*\mathcal{L}^* = \mathcal{L} \otimes \gamma^*\mathcal{O}_{\mathbb{CP}^1}(a), \tag{3.17}$$

where $a \in \mathbb{Z}$ and γ is the composition of the projection $C_v \times \mathrm{Pic}^{d+n-2}(C_v) \to C_v$ with the map f_v.

From (3.16) and (3.17),

$$p^*c_1(T\mathcal{M}_S^s) = c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2} \otimes \gamma^*\mathcal{O}_{\mathbb{CP}^1}(a))). \tag{3.18}$$

We will now compare $c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2}))$ with $c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2} \otimes \gamma^*\mathcal{O}_{\mathbb{CP}^1}(a)))$.

First assume that $a > 0$. Fix a reduced effective divisor $D_0 \subset C_v$ such that $\mathcal{O}_{C_v}(D_0) = f_v^*\mathcal{O}_{\mathbb{CP}^1}(a)$. Consider the short exact sequence of sheaves

$$0 \to \mathcal{L}^{\otimes 2} \to \mathcal{L}^{\otimes 2} \otimes \gamma^*\mathcal{O}_{\mathbb{CP}^1}(a) \to (\mathcal{L}^{\otimes 2} \otimes \gamma^*\mathcal{O}_{\mathbb{CP}^1}(a))|_{D_0 \times \mathrm{Pic}^{d+n-2}(C_v)} \to 0$$

on $C_v \times \mathrm{Pic}^{d+n-2}(C_v)$. We have seen that the restriction of $\mathcal{L}$ to $\{x\} \times \mathrm{Pic}^{d+n-2}(C_v)$ is topologically trivial for all $x \in C_v$. Therefore, from the preceding short exact sequence of sheaves it follows that

$$c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2})) = c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2} \otimes \gamma^*\mathcal{O}_{\mathbb{CP}^1}(a))) \in H^2(\mathrm{Pic}^{d+n-2}(C_v), \mathbb{Q}).$$

Next assume that $a < 0$, and consider the short exact sequence of sheaves

$$0 \to \mathcal{L}^{\otimes 2} \otimes \gamma^*\mathcal{O}_{\mathbb{CP}^1}(a) \to \mathcal{L}^{\otimes 2} \to (\mathcal{L}^{\otimes 2})|_{D_0 \times \mathrm{Pic}^{d+n-2}(C_v)} \to 0,$$

where $D_0 \subset C_v$ is a reduced effective divisor such that $\mathcal{O}_{C_v}(D_0) = f_v^* \mathcal{O}_{\mathbb{CP}^1}(-a)$. Using this short exact sequence yields, as before, that

$$c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2})) = c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2} \otimes \gamma^* \mathcal{O}_{\mathbb{CP}^1}(a))) \in H^2(\mathrm{Pic}^{d+n-2}(C_v), \mathbb{Q}).$$

Therefore, from (3.18),

$$p^* c_1(T\mathcal{M}_S^s) = c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2})). \tag{3.19}$$

Take any Poincaré line bundle $\mathcal{L}_b$ on $C_v \times \mathrm{Pic}^b(C_v)$ such that the restriction of $\mathcal{L}_b$ to $\{x\} \times \mathrm{Pic}^b(C_v)$ is topologically trivial for some (hence all) $x \in C_v$. Let

$$q_b \colon C_v \times \mathrm{Pic}^b(C_v) \to \mathrm{Pic}^b(C_v)$$

be the natural projection. Then it is known that

$$c_1((\det R^1 q_{b*} \mathcal{L}_b) \otimes (\det q_{b*} \mathcal{L}_b)^*) \in H^2(\mathrm{Pic}^b(C_v), \mathbb{Q})$$

coincides with the canonical polarization on $\mathrm{Pic}^b(C_v)$.

Consider the map

$$\varphi_0 \colon \mathrm{Pic}^{d+n-2}(C_v) \to \mathrm{Pic}^{2(d+n-2)}(C_v)$$

defined by $\xi \mapsto \xi^{\otimes 2}$. The cited property of the canonical polarization implies that

$$c_1(\mathrm{Det}(\mathcal{L}^{\otimes 2})) = \varphi_0^* \Theta, \tag{3.20}$$

where

$$\Theta \in H^2(\mathrm{Pic}^{2(d+n-2)}(C_v), \mathbb{Q})$$

is the canonical polarization. Since $\dim \mathrm{Pic}^{d+n-2}(C_v) = n-3$, from (3.19) and (3.20) we conclude that

$$p^* c_1(T\mathcal{M}_S^s) = 4^{n-3} \cdot \Theta.$$

This completes the proof of the lemma. □

A theorem due to Lefschetz asserts that r times a principal polarization on an abelian variety is very ample if $r \geq 3$ (see [2, p. 317]). Therefore, from Lemma 3.1 and Proposition 2.2 we conclude that the line bundle

$$p^* \det T\mathcal{M}_S^s \in \mathrm{Pic}(\mathrm{Pic}^{d+n-2}(C_v)) = \mathrm{Pic}(\mathcal{Z})$$

is very ample (see (3.1) for $\mathcal{Z}$). Hence we can reconstruct $\mathrm{Pic}^{d+n-2}(C_v)$ from $\mathcal{Z}$ by taking its closure in the complete linear system $|p^* \det T\mathcal{M}_S^s|$. Therefore, starting from $\mathcal{M}_S$ we can reconstruct the Hitchin fibration (see (2.3)) over a Zariski open dense subset of $H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S))$.

If we know r times a principal polarization on an abelian variety, where r is a given nonzero integer, then we can uniquely recover the principal polarization. Therefore, the standard Torelli theorem gives the following.

> Starting from $\mathcal{M}_S$ we can reconstruct the family of spectral curves over a Zariski open dense subset of $H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S))$.

4. Infinitesimal Deformations of the Spectral Curve

The total space of the line bundle $K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)$ will be denoted by $\mathcal{Y}$. Consider the short exact sequence of vector bundles on $\mathcal{Y}$,

$$0 \to f^*(K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)) \to T\mathcal{Y} \xrightarrow{df} f^*T\mathbb{CP}^1 \to 0, \tag{4.1}$$

where df is the differential of the projection f in (2.4). The sequence (4.1) implies that

$$\bigwedge^2 T\mathcal{Y} = f^*\mathcal{O}_{\mathbb{CP}^1}(S). \tag{4.2}$$

As in Section 3, take $v \in H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S))$ such that the corresponding spectral curve C_v is connected and smooth. Let

$$\tau : C_v \hookrightarrow \mathcal{Y} \tag{4.3}$$

be the inclusion map of the spectral curve.

As in Section 3, let

$$\hat{S} = f_v^{-1}(S)_{\text{red}} \subset C_v \tag{4.4}$$

be the reduced divisor, where f_v as in (2.5) is the restriction of f to C_v. Let

$$N_{C_v} := (\tau^*T\mathcal{Y})/TC_v \tag{4.5}$$

be the normal bundle, where τ is defined in (4.3).

Take any point $s \in S$. Note that all the spectral curves pass through the point $(s, 0) \in \mathcal{Y}$. Also, the restriction of the projection f (see (2.4)) to any spectral curve is ramified over s. Therefore, the tangent space, at v, of the family of spectral curves is parameterized by

$$H^0(C_v, N_{C_v} \otimes_{\mathcal{O}_{C_v}} \mathcal{O}_{C_v}(-2\hat{S})),$$

where $\hat{S}$ is the divisor in (4.4), and N_{C_v} is the normal bundle in (4.5). Hence the infinitesimal deformation map for the family of spectral curves is an injective homomorphism

$$T_v H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)) \to H^0(C_v, N_{C_v} \otimes \mathcal{O}_{C_v}(-2\hat{S})). \tag{4.6}$$

We note that $T_v H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)) = H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S))$ and

$$\dim H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)) = n - 3.$$

We will prove that the homomorphism in (4.6) is an isomorphism by showing that

$$\dim H^0(C_v, N_{C_v} \otimes \mathcal{O}_{C_v}(-2\hat{S})) = n - 3. \tag{4.7}$$

Let

$$\mathcal{T} \subset \tau^*T\mathcal{Y} \tag{4.8}$$

be the inverse image of $N_{C_v} \otimes_{\mathcal{O}_{C_v}} \mathcal{O}_{C_v}(-2\hat{S}) \subset N_{C_v}$ by the quotient map $\tau^*T\mathcal{Y} \to N_{C_v}$ in (4.5). In other words, $\mathcal{T}$ fits in the short exact sequence

$$0 \to \mathcal{T} \to \tau^*T\mathcal{Y} \to N_{C_v}/N_{C_v} \otimes_{\mathcal{O}_{C_v}} \mathcal{O}_{C_v}(-2\hat{S}) \to 0. \tag{4.9}$$

Since $(f \circ \tau)^* \mathcal{O}_{\mathbb{CP}^1}(S) = f_v^* \mathcal{O}_{\mathbb{CP}^1}(S) = \mathcal{O}_{C_v}(2\hat{S})$, from (4.2) and (4.9) it follows that

$$\bigwedge^2 \mathcal{T} = \mathcal{O}_{C_v}. \tag{4.10}$$

Consider the natural inclusion of TC_v in $\tau^* T\mathcal{Y}$. From the construction of $\mathcal{T}$ in (4.8) we conclude that this inclusion map yields a short exact sequence of vector bundles

$$0 \to TC_v \to \mathcal{T} \to N_{C_v} \otimes_{\mathcal{O}_{C_v}} \mathcal{O}_{C_v}(-2\hat{S}) \to 0 \tag{4.11}$$

over C_v. From (4.10) and (4.11) we know that

$$N_{C_v} \otimes_{\mathcal{O}_{C_v}} \mathcal{O}_{C_v}(-2\hat{S}) = K_{C_v}. \tag{4.12}$$

Since genus$(C_v) = n - 3$, from the isomorphism in (4.12) we conclude that (4.7) holds. Hence the injective homomorphism in (4.6) is an isomorphism. In other words,

$$\begin{aligned} H^0(\mathbb{CP}^1, K_{\mathbb{CP}^1}^{\otimes 2} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)) &= H^0(C_v, N_{C_v} \otimes \mathcal{O}_{C_v}(-2\hat{S})) \\ &= H^0(C_v, K_{C_v}). \end{aligned} \tag{4.13}$$

Let

$$\begin{aligned} 0 \to H^0(C_v, \mathcal{T}) \to H^0(C_v, N_{C_v} \otimes \mathcal{O}_{C_v}(-2\hat{S})) \\ = H^0(C_v, K_{C_v}) \xrightarrow{\alpha} H^1(C_v, TC_v) \end{aligned} \tag{4.14}$$

be the long exact sequence of cohomologies associated to the short exact sequence of sheaves in (4.11) (see also (4.13)). The homomorphism α in (4.14) is the infinitesimal deformation map for the family of spectral curves.

LEMMA 4.1. *For the homomorphism α in* (4.14),

$$\dim \alpha(H^0(C_v, K_{C_v})) = n - 4.$$

Proof. First note that $\dim H^0(C_v, K_{C_v}) = n - 3$. Also, kernel$(\alpha) \neq 0$, because the automorphisms of the line bundle $K_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(S)$ given by the nonzero scalar multiplications produce deformations of the embedded spectral curve that do not change the isomorphism class of the curve. Hence

$$\dim \alpha(H^0(C_v, K_{C_v})) \leq n - 4.$$

Consider the short exact sequence of vector bundles on $\mathcal{Y}$ in (4.1). Let

$$0 \to (f_v^* K_{\mathbb{CP}^1}) \otimes \mathcal{O}_{C_v}(2\hat{S}) \to \tau^* T\mathcal{Y} \to f_v^* T\mathbb{CP}^1 \to 0$$

be the restriction of it to C_v; the divisor $\hat{S}$ is defined in (4.4), and τ is defined in (4.3). This exact sequence gives a short exact sequence of vector bundles

$$0 \to (f_v^* K_{\mathbb{CP}^1}) \otimes \mathcal{O}_{C_v}(\hat{S}) \to \mathcal{T} \to (f_v^* T\mathbb{CP}^1) \otimes \mathcal{O}_{C_v}(-\hat{S}) \to 0, \tag{4.15}$$

where $\mathcal{T}$ is defined in (4.8).

Since degree$((f_v^* T\mathbb{CP}^1) \otimes \mathcal{O}_{C_v}(-\hat{S})) = 4 - n < 0$, from (4.15) we have

$$H^0(C_v, \mathcal{T}) = H^0(C_v, (f_v^* K_{\mathbb{CP}^1}) \otimes \mathcal{O}_{C_v}(\hat{S})). \tag{4.16}$$

Let

$$D_W \subset C_v$$

be the set of Weierstrass points. So we have $\hat{S} \subset D_W$. The complement $D_W \setminus \hat{S}$ will be denoted by D'. From the differential df_v of the map f_v we have

$$f_v^* K_{\mathbb{CP}^1} = K_{C_v} \otimes \mathcal{O}_{C_v}(-D_W).$$

Hence

$$(f_v^* K_{\mathbb{CP}^1}) \otimes \mathcal{O}_{C_v}(\hat{S}) = K_{C_v} \otimes \mathcal{O}_{C_v}(-D'). \tag{4.17}$$

By Serre duality and the Riemann–Roch theorem,

$$\dim H^0(C_v, K_{C_v} \otimes \mathcal{O}_{C_v}(-D')) = \dim H^0(C_v, \mathcal{O}_{C_v}(D')). \tag{4.18}$$

Take a meromorphic function ζ on C_v that is holomorphic on $C_v \setminus D'$ and has poles of order ≤ 1 on the points of D'. So $\zeta - \zeta \circ \eta$ vanishes on $\hat{S}$, where η, as in (3.12), is the hyperelliptic involution. Since $\#\hat{S} > \#D'$, we conclude that $\zeta - \zeta \circ \eta = 0$. Therefore, ζ must be a constant function. In other words,

$$\dim H^0(C_v, \mathcal{O}_{C_v}(D')) = 1.$$

Hence, from (4.16), (4.17), and (4.18) we conclude that

$$H^0(C_v, \mathcal{T}) = 1.$$

Therefore, $\dim \operatorname{kernel}(\alpha) = 1$, where α is the homomorphism in (4.14). This completes the proof of the lemma. $\square$

The hyperelliptic involution of a hyperelliptic curve is unique. The quotient by the hyperelliptic involution of a hyperelliptic curve of genus $n - 3$ is a curve of genus 0 equipped with $2n - 4$ unordered marked points. The isomorphism class of a hyperelliptic curve is uniquely recovered from the isomorphism class of the corresponding multi-pointed curve of genus 0.

So, when the spectral curve C_v moves in the family, the corresponding $(2n-4)$-pointed curve of genus 0 moves. Since the n parabolic points S are contained in the $2n - 4$ marked points, the dimension of the image of the infinitesimal deformation map is at most $2n - 4 - n = n - 4$. From Lemma 4.1 we know that the dimension of the image of the corresponding infinitesimal deformation map is, in fact, $n - 4$. If a set T of n points other than the set of parabolic points can be made to remain fixed in the family of isomorphism classes of genus-0 curves with unordered $2n - 4$ marked points given by the spectral curves, then first note that the intersection of this set T with the set of parabolic points S has cardinality ≥ 4. Hence, there are no nontrivial automorphisms of $\mathbb{CP}^1$ that fix $(\#S \cap T)$ points. Therefore, the dimension of the image of the infinitesimal deformation map is at most the cardinality of the complement (in the set of $2n - 4$ ramification points) of the union $S \cup T$. If T is different from S, this contradicts the fact that the dimension of the image of the infinitesimal deformation map is $n - 4$.

From this it follows that we can recover the isomorphism class of the n-pointed curve $(\mathbb{CP}^1, S)$ starting from the family of spectral curves. More precisely, let $M_{0,n}$ denote the moduli space of smooth curves of genus 0 with n unordered marked points. From the parameter space of the smooth connected spectral curves, we have a multi-valued forgetful map to $M_{0,n}$ that sends a spectral C to $(C/\langle\iota\rangle, S_C)$, where

$$\iota\colon C \to C$$

is the hyperelliptic involution and $S_C \subset C/\langle\iota\rangle$ is a set of n points contained in the image of the Weierstrass points of C. So this multi-valued map is actually $\binom{2n-4}{n}$-valued. Among these $\binom{2n-4}{n}$ (locally defined) functions, there is exactly one function that is constant, and the image of the constant function coincides with the point of $M_{0,n}$ given by $(\mathbb{CP}^1, S)$.

We remarked at the end of Section 3 that the family of spectral curves over a Zariski open subset can be recovered from $\mathcal{M}_S$. Hence we have proved the following theorem.

THEOREM 4.2. *Take two finite subsets S_1 and S_2 of $\mathbb{CP}^1$ of cardinality ≥ 5. Let $\mathcal{M}_{S_1}(d)$ (resp., $\mathcal{M}_{S_2}(d)$) be the corresponding moduli spaces of semistable parabolic vector bundles of rank 2 and degree d. Then the variety $\mathcal{M}_{S_1}(d)$ is isomorphic to $\mathcal{M}_{S_2}(d)$ if and only if there is an automorphism of $\mathbb{CP}^1$ that takes the subset S_1 surjectively to S_2.*

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