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On the Beurling–Ahlfors Transform's Weak-type Constant

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1. Introduction

The Beurling–Ahlfors transform, denoted by S, is defined on $L^p(\mathbb{C})$, $1 \le p < \infty$, by

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(w)}{(z-w)^2}\,dw,$$

where dw is the 2-dimensional Lebesgue measure and the integral is understood as a Cauchy principal value. By the theory of Calderón and Zygmund this is a bounded operator on $L^p(\mathbb{C})$ for $1 < p < \infty$. In fact one can show the Fourier multiplier associated with the operator is $\bar{\xi}/\xi$, and so it is an isometry on $L^2(\mathbb{C})$ by Plancharel's theorem. Its norm on the other L^p spaces is unknown and is an area of interest, especially in the field of quasiconformal mappings as

$$S(\bar{\partial} f) = \partial f$$

gives a connection between the z and $\bar{z}$ derivatives. The well-known Iwaniec conjecture [4] asserts that

$$\|S\|_p = p^* - 1 := \max\{p, p/(p-1)\} - 1.$$

The current best estimate of

$$\|S\|_p \le 1.575(p^* - 1)$$

is due to Bañeulos and Janakiraman [2].

For a function f on a measure space (X,μ) we define the weak "norm" of f as

$$\|f\|_w := \sup_{\lambda>0} \mu(|f| \ge \lambda)\lambda.$$

For an operator T defined on $L^1(X,\mu)$, but not necessarily bounded, define

$$\|T\|_w := \sup_{f\in L^1,\, f\neq 0} \frac{\|Tf\|_w}{\|f\|_1}.$$

An $L^1(X,\mu)$ bounded operator T has $\|T\|_w < \infty$ by Chebyshev's inequality, but an operator with $\|T\|_w < \infty$ is not necessarily bounded in $L^1(X,\mu)$. If $\|T\|_w$ is finite, we say that T is *weak-type bounded* with constant $\|T\|_w$.

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It is an interesting problem to find $\|S\|_w$ (see [1, Chap. 4] for motivation). Perhaps the first result along these lines is due to Bañuelos and Janakiraman [3]. They introduce an operator Λ, called Λ_0 in [3], which is defined as

$$\Lambda f(x) := \frac{1}{x}\int_0^x f(y)\,dy - f(x), \quad 0 < x < \infty. \tag{1}$$

They show that Λ is an isometry on $L^2(0,\infty)$. If $f \in L^1(0,\infty)$ and $F(z) = f(|z|^2)$ for $z \in \mathbb{C}$, then $F \in L^1(\mathbb{C})$,

$$SF(z) = \frac{\bar{z}}{z}\Lambda f(|z|^2), \quad \|SF\|_w = \pi\|\Lambda f\|_w, \quad \text{and} \quad \|F\|_1 = \pi\|f\|_1$$

where the quantities are taken in their respective spaces.

THEOREM A [3]. $\|\Lambda\|_w = \frac{1}{\log 2}$ *and so* $\|S\|_w \geq \frac{1}{\log 2}$.

A function that gives the extremal value in Theorem A is $f(x) = -\chi_{(0,2]}(x)\cdot\log x$ where χ represents the indicator function. Then $\Lambda f \equiv 1$ on $(0,2]$ and then jumps down and decreases for $(2,\infty)$.

The main purpose of this paper is to prove a companion result for Theorem A. Let

$$\Lambda^* f(x) := \int_x^\infty \frac{f(y)}{y}\,dy - f(x), \quad 0 < x < \infty,$$

be an operator defined on $L^p(0,\infty)$ for all $1 \leq p < \infty$. First we show that Λ^* is the adjoint of Λ. Let $f \in L^1(0,\infty)$ and $g \in L^\infty(0,\infty)$. Then

$$\begin{aligned}\langle f, \Lambda g\rangle &= \int_0^\infty f(x)\left[\frac{1}{x}\int_0^x g(y)\,dy - g(x)\right]dx \\ &= \int_0^\infty g(y)\int_y^\infty \frac{f(x)}{x}\,dx\,dy - \int_0^\infty f(y)g(y)\,dy = \langle \Lambda^* f, g\rangle.\end{aligned}$$

Our first result is to note that Λ^* provides a way to calculate S for a class of radial functions.

THEOREM 1. *If* $f \in L^p(\mathbb{R}^+,\mathbb{R})$, $1 \leq p < \infty$, *and* $F(z) = (z/\bar{z})f(|z|^2)$, *then*

$$SF(z) = -\Lambda^* f(|z|^2).$$

This relationship gives us another way to calculate a lower bound for the weak-type norm $\|S\|_w$.

THEOREM 2. $\|\Lambda^*\|_w = \frac{1}{\log 2}$, *and so* $\|S\|_w \geq \frac{1}{\log 2}$.

Note that there is no obvious reason for the quantities $\|\Lambda\|_w$ and $\|\Lambda^*\|_w$ to be the same. A function that is extremal for $\|\Lambda^*\|_w$ is

$$f(x) = \frac{\chi_{(1/2,1]}(x)}{x}, \quad \text{which gives} \quad \Lambda^* f(x) = \chi_{[0,1/2]}(x) - \chi_{(1/2,1]}(x). \tag{2}$$

As $\|f\|_1 = \log 2$ and $\|\Lambda^* f\|_w = 1$ the value $1/\log 2$ is achieved.

In order to prove Theorem 2, we recast the question as a nontrivial question about Λ. We first note, by exploiting our adjoint relationship, that the image of any f in $L^1[0,\infty)$ under Λ^* has zero mean:

$$\int_0^\infty \Lambda^* f\,dx = \langle \Lambda^* f, 1\rangle = \langle f, \Lambda 1\rangle = \langle f, 0\rangle = 0. \tag{3}$$

Now note that when looking for $\|\Lambda^*\|_w$, it is enough to look at the f that have a limit at 0 that is noninfinite. As Λ^* is unaffected for all $x > \varepsilon$ if we change f on $(0,\varepsilon)$ but keep the same absolute sum from 0 to ε, the only difference in our ratio defining $\|\Lambda^*\|_w$ would be a difference of at most ε in the numerator, and so we may restrict f to having a noninfinite limit at zero. This is our first reduction. Now we note that for such f, $\Lambda(\Lambda^* f) = f$:

$$\begin{aligned}\Lambda(\Lambda^* f)(u) &= \frac{1}{u}\int_0^u\left(\int_x^\infty \frac{f(t)}{t}\,dt - f(x)\right)dx - \left(\int_u^\infty \frac{f(t)}{t}\,dt - f(u)\right)\\ &= \frac{1}{u}\int_0^u\int_x^\infty \frac{f(t)}{t}\,dt\,dx - \frac{1}{u}\int_0^u f(x)\,dx - \int_u^\infty \frac{f(t)}{t} + f(u).\end{aligned}$$

We now integrate the first integral in the last line above by parts to obtain

$$\int_0^u\int_x^\infty \frac{f(t)}{t}\,dt\,dx = u\int_u^\infty \frac{f(t)}{t}\,dt - 0 + \int_0^u f(x)\,dx$$

where the integration by parts is justified by our reduction on f. Plugging this calculation into last expression above for $\Lambda(\Lambda^* f)(u)$ yields $f(u)$. So on the function class to which we have reduced for finding $\|\Lambda^*\|_w$ we also have that $\Lambda \circ \Lambda^* = \mathrm{Id}$. So, using this fact and (3), Theorem 2 is equivalent to the following statement.

THEOREM 3. *If $f \in L^1(\mathbb{R}^+,\mathbb{R})$ with $\int_0^\infty f(x)\,dx = 0$, then*

$$\frac{|\{x \in \mathbb{R}^+ : |f(x)| \geq \lambda\}|\lambda}{\|\Lambda f\|_1} \leq \frac{1}{\log 2}.$$

There exist f and λ for which equality can be attained.

In Section 2 we prove Theorem 1, and in Section 3 we prove Theorem 3 and hence Theorem 2. We note the following conjecture from [3].

CONJECTURE. $\|S\|_w = \frac{1}{\log 2}$.

So Theorem 2 gives further evidence that this conjecture may be true. We also note that Theorem A and Theorem 3 are different results as the role of f and Λf are reversed.

2. Proof of Theorem 1

We first consider that with $\mathbb{D} = \{z : |z| < 1\}$, we have the pair

$$\rho_1 = \frac{z}{\bar{z}} \cdot \chi_{\mathbb{D}}(z) \quad \text{and} \quad S\rho_1(z) = (1 + \log(|z|^2))\chi_{\mathbb{D}}(z). \tag{4}$$

This was first noticed by Iwaniec in [5]. The pair ρ_1 and $S\rho_1$ determine the action of S on the class of functions in Theorem 1 from above. By a change of variables, one can show that

$$\rho_R(z) = \frac{z}{\bar{z}} \cdot \chi_{B_R}(z),$$

where B_R is the disc of radius $R > 0$, is transformed to

$$S\rho_R(z) = S\rho_1\left(\frac{z}{R}\right) = \left(1 + \log\left(\left|\frac{z}{R}\right|^2\right)\right)\chi_{B_R}(z).$$

As a consequence, we know how the Beurling–Ahlfors transform acts on linear combinations of these functions, and by continuity of our operator S, on L^p limits of the functions. This is detailed for a similar class in [3] and we will not reproduce the argument here. We now show the relationship between f and F in Theorem 1. Let $f(x) = \chi_{[0,R)}(x)$. Then for $u \leq R$,

$$-\Lambda^* f(u) = \left(1 + \log\left(\frac{u}{R}\right)\right)\chi_{[0,R)}(u).$$

Hence as

$$\rho_R(z) = \frac{z}{\bar{z}} \cdot \chi_{[0,R)}(|z|) = \frac{z}{\bar{z}} \cdot \chi_{[0,R^2)}(|z|^2)$$

we get

$$S\rho_R(z) = -\Lambda^*(\chi_{[0,R^2)})(|z|^2),$$

which, by the density of linear combinations, proves Theorem 1.

3. Proof of Theorem 3

We have already noted by (2) that the value $1/\log 2$ can be achieved. We now must show that this is best possible. To do this we make some assumptions that allow us to work in an orderly fashion. By linearity we may assume $\lambda = 1$. Note that without loss of generality, we can assume that f has compact support. Also our ratio

$$\frac{|\{|f| \geq 1\}|}{\|\Lambda f\|_1}$$

is invariant under dilation so we may assume that our support is contained in $[0, 1]$. Also functions of this type may be approximated by functions $g_m = \sum_{i=1}^m x_i \chi_{[(i-1)/m, i/m)}(x)$ as $m \to \infty$. In order to make our calculations simpler, we dilate these approximants by m, so that our functions have support on $[0, m]$. So we start with, for $m \geq 2$, a function of the form

$$f(x) = \sum_{i=1}^{m} x_i \chi_{[i-1,i)}(x); \quad \text{and} \quad \sum_{i=1}^{m} x_i = 0. \tag{5}$$

Then one can calculate

$$\Lambda f(x) = \frac{x_1 - x_2}{x}\chi_{[1,2)}(x) + \cdots + \frac{x_1 + \cdots + x_{m-1}(m-1)x_m}{x}\chi_{[m-1,m)} \tag{6}$$

and so

$$\|\Lambda f\|_1 = |x_1 - x_2| \log 2 + \cdots + |x_1 + \cdots + x_{m-1} - (m-1)x_m| \log\left(\frac{m}{m-1}\right). \tag{7}$$

So if we show that the quantity

$$R(f) := \frac{|\{|f| \geq 1\}|}{\|\Lambda f\|_1} \tag{8}$$

is less than or equal to $1/\log 2$, then Theorem 3 is proved. We first note that if one has two different functions f and g supported on $[0, m]$ such that $\int_0^n f = \int_0^n g$ for some $n < m$ and $f = g$ on $[n, m]$, then Λf and Λg are identical on $[n, m]$, as one can see from the preceding formulas. This fact will allow us to make modifications on progressively larger intervals without changing our function "too much".

Let us define some notation:

$$k_n := \int_0^n f = \sum_{i=1}^n x_i,$$

$$m_f(j) := |\{|f| \geq 1\} \cap [0, j)| \quad \text{and} \quad M_f(j) := |\{|f| \geq 1\} \cap [j, \infty)|,$$

$$\nu_f(j) := \int_0^j |\Lambda_f|\, dx \quad \text{and} \quad N_f(j) := \int_j^\infty |\Lambda_f|\, dx.$$

Therefore

$$R(f) = \frac{m_f(j) + M_f(j)}{\nu_f(j) + N_f(j)} \quad \text{for all} \;\; j \in [0, m).$$

We also note the following.

Observation A. Let $a, b, c, d \geq 0$. If $\frac{a}{b} \leq \frac{1}{\log 2}$ and $\frac{c}{d} \geq \frac{1}{\log 2}$ then $\frac{a}{b} \leq \frac{a+c}{b+d}$. If $\frac{a}{b} > \frac{1}{\log 2}$ and $\frac{c}{d} \geq \frac{1}{\log 2}$ then $\frac{a+c}{b+d} > \frac{1}{\log 2}$. Of course, $\frac{1}{\log 2}$ can be replaced by any other positive number.

3.1. *Modification on* $[0, 2)$

Let

$$f_2 := f \cdot \chi_{[2,m)} + \begin{cases} \frac{k_2}{2} \cdot \chi_{[0,2)} & \text{if } |k_2| \geq 2, \\ \chi_{\left[0, \frac{k_2+2}{2}\right)} - \chi_{\left[\frac{k_2+2}{2}, 2\right)} & \text{if } 0 \leq k_2 < 2, \\ -\chi_{\left[0, \frac{|k_2|+2}{2}\right)} + \chi_{\left[\frac{|k_2|+2}{2}, 2\right)} & \text{if } -2 < k_2 < 0, \end{cases}$$

and note that $f = f_2$ on $[2, m)$, $\int_0^2 f = \int_0^2 f_2$, $M_f(2) = M_{f_2}(2)$, and $N_f(2) = N_{f_2}(2)$. Without loss of generality, by multiplying by -1 we can assume k_2 is nonnegative. We now examine a series of cases.

Case 1: $k_2 \geq 2$. First, we note that if $k_2 \geq 2$, then $2 = m_{f_2}(2) \geq m_f(2)$ and as $\Lambda f_2 = 0$ on $[0, 2)$, we also have $0 = \nu_{f_2}(2) \leq \nu_f(2)$. So $R(f) \leq R(f_2)$.

Case 2: $0 \leq k_2 < 2$ *and* $m_f(2) = 2$. Then $|f| \geq 1$ on $[0, 2)$ and so both $|x_1|$ and $|x_2|$ are greater than or equal to 1. As $k_2 < 2$, x_1 and x_2 must have opposite signs; thus by (7) we know that $\nu_f(2) \geq 2\log 2$. But $\Lambda f_2(x) = \frac{2a}{x}\chi_{[a,2)}(x)$ on $[0,2)$, where $a = \frac{k_2+2}{2}$, so that $\nu_{f_2}(2) = (k_2+2)\log\left(\frac{4}{k_2+2}\right)$. One may check that $\nu_{f_2}(2) \leq \nu_f(2)$. As $m_f(2) = m_{f_2}(2) = 2$, we conclude that $R(f) \leq R(f_2)$.

Case 3: $0 \leq k_2 < 2$ *and* $m_f(2) = 0$. As $m_{f_2}(2) = 2$ we know $m_{f_2}(2) - m_f(2) = 2$. Also

$$\nu_{f_2}(2) - \nu_f(2) = (k_2+2)\log\left(\frac{4}{k_2+2}\right) - |x_1 - x_2|\log 2.$$

If $\nu_{f_2}(2) - \nu_f(2)$ is nonpositive, then with

$$m_f(2) + M_f(2) = a, \qquad \nu_f(2) + N_f(2) = b,$$
$$m_{f_2}(2) - m_f(2) = 2 = c, \qquad \nu_{f_2}(2) - \nu_f(2) = d$$

we have $R(f) = \frac{a}{b} < \frac{a+c}{b+d} = R(f_2)$. If instead $d > 0$, then

$$\frac{c}{d} = \frac{m_{f_2}(2) - m_f(2)}{\nu_{f_2}(2) - \nu_f(2)} \geq \frac{2}{(k_2+2)\log\left(\frac{4}{k_2+2}\right)} \geq \frac{1}{\log 2}$$

for all $k_2 \in [0,2)$. We conclude from Observation A that if $R(f) = \frac{a}{b} \leq \frac{1}{\log 2}$ then $R(f) \leq \frac{a+c}{b+d} = R(f_2)$, while if $R(f) > \frac{1}{\log 2}$ then $R(f_2) > \frac{1}{\log 2}$ as well.

Case 4: $0 \leq k_2 < 2$ *and* $m_f(2) = 1$. We know $m_{f_2}(2) - m_f(2) = 1$ and

$$d = \nu_{f_2}(2) - \nu_f(2) = (k_2+2)\log\left(\frac{4}{k_2+2}\right) - |x_1 - x_2|\log 2.$$

As in the previous case, we wish to examine

$$\frac{c}{d} = \frac{m_{f_2}(2) - m_f(2)}{\nu_{f_2}(2) - \nu_f(2)} = \frac{1}{(k_2+2)\log\left(\frac{4}{k_2+2}\right) - |x_1 - x_2|\log 2}.$$

If $d \leq 0$, then the argument in Case 3 shows again that $R(f) < R(f_2)$. One may show that if $d > 0$ then the expression $c/d \geq 1/\log 2$. This may be checked by noting that exactly one of $|x_1|$ or $|x_2|$ is greater than or equal to 1 and then maximizing d with this assumption.

So in all possible cases of f and f_2, either $R(f) \leq R(f_2)$ or $R(f)$ and $R(f_2)$ are both strictly greater than $1/\log 2$.

3.2. *Modification on* $[0, n+1)$

After comparing f to f_2 in Section 3.1, we now wish to compare f_2 to f_3, then f_3 to f_4, and so on. We will compare the general f_n to f_{n+1} for $2 \leq n \leq m-1$. First, the definitions are

$$f_n := f \cdot \chi_{[n,m)} + \begin{cases} \frac{k_n}{n}\chi_{[0,n)} & \text{if } |k_n| \geq n, \\ \chi_{\left[0,\frac{k_n+n}{2}\right)} - \chi_{\left[\frac{k_n+n}{2},n\right)} & \text{if } 0 \leq k_n < n, \\ -\chi_{\left[0,\frac{|k_n|+n}{2}\right)} + \chi_{\left[\frac{|k_n|+n}{2},n\right)} & \text{if } -n < k_n < 0, \end{cases}$$

and

$$
\begin{aligned}
f_{n+1} &:= f \cdot \chi_{[n+1,m)} \\
&\quad + \begin{cases} \frac{k_{n+1}}{n+1}\chi_{[0,n+1)} & \text{if } |k_{n+1}| \geq n+1, \\ \chi_{\left[0,\frac{k_{n+1}+n+1}{2}\right)} - \chi_{\left[\frac{k_{n+1}+n+1}{2},n+1\right)} & \text{if } 0 \leq k_{n+1} < n+1, \\ -\chi_{\left[0,\frac{|k_{n+1}|+n+1}{2}\right)} + \chi_{\left[\frac{|k_{n+1}|+n+1}{2},n+1\right)} & \text{if } -n-1 < k_{n+1} < 0. \end{cases}
\end{aligned}
$$

Our cases will correspond to various possibilities for k_n and x_{n+1}.

Case 1: $|k_{n+1}| \geq n+1$. In this case $m_{f_{n+1}} = n+1$ and $\nu_{f_{n+1}} = 0$. Hence $R(f_n) \leq R(f_{n+1})$.

For the rest of the cases, by multiplying f by -1 if necessary, we may assume $0 \leq k_{n+1} < n+1$.

Case 2: $0 \leq k_{n+1} < n+1$ *and* $x_{n+1} \geq 1$. This is an unbounded region with the following lines corresponding to its boundary: $k_n + x_{n+1} = n+1$, $k_n + x_{n+1} = 0$, and $x_{n+1} = 1$. The first line corresponds to our assumption that $k_{n+1} < n+1$, the second to our assumption that $0 \leq k_{n+1}$, and the third to $x_{n+1} \geq 1$. We will show that on this entire region, $R(f_n) \leq R(f_{n+1})$.

As $m_{f_n}(n+1)$ and $m_{f_{n+1}}(n+1)$ are both equal to $n+1$ as $|x_{n+1}| \geq 1$ on this region, and we have concocted f_n and f_{n+1} to have equal $M(n+1)$ and $N(n+1)$ values, we must only compare the νs directly. We record our formulas, which can be found using (1):

$$
\nu_{f_{n+1}}(n+1) = (k_n + x_{n+1} + n + 1)\log\left(\frac{2(n+1)}{k_n + x_{n+1} + n + 1}\right)
$$

and

$$
\begin{aligned}
&\nu_{f_n}(n+1) \\
&\quad = \chi_{[0,n)}(|k_n|)\cdot(|k_n|+n)\log\left(\frac{2n}{|k_n|+n}\right) + |k_n - nx_{n+1}|\log\left(\frac{n+1}{n}\right).
\end{aligned}
$$

We must show that on region A,

$$
\nu_{f_{n+1}}(n+1) \leq \nu_{f_n}(n+1). \tag{9}
$$

Note, using $k_n + x_{n+1} \geq 0$ that if k_n is fixed and x_{n+1} increases then $\nu_{f_{n+1}}(n+1)$ decreases. Also, since $k_n \leq nx_{n+1}$ for all points in Case 2, $\nu_{f_n}(n+1)$ increases as x_{n+1} increases. So if we show that (9) holds for the points in our region of interest where x_{n+1} is smallest for a given k_n, we will have our desired inequality on all of the region for Case 1. There are two lines for which we must check (9): $0 = k_n + x_{n+1}$ where $x_{n+1} > 1$ and $-1 \leq k_n < n$ where $x_{n+1} = 1$. The first is easy to check. The second we do as a subcase.

Subcase 2.1: $-1 \leq k_n < n$ *and* $x_{n+1} = 1$. We first record that for this subcase

$$
\nu_{f_{n+1}}(n+1) = (k_n + n + 2)\log\left(\frac{2(n+1)}{k_n + n + 2}\right)
$$

and

$$\nu_{f_n}(n+1) = (|k_n| + n)\log\left(\frac{2n}{|k_n| + n}\right) + |k_n - n|\log\left(\frac{n+1}{n}\right).$$

Let us start with $-1 \le k_n \le 0$. Then $\nu_{f_n}(n+1) = (n-k_n)\log\left(\frac{2(n+1)}{n-k_n}\right)$, and (9) may be easily checked.

The other segment, $0 \le k_n < n$ is a bit more difficult. It is easy to check that (9) holds for a given n with a calculator, but this is not sufficient. Note that on this segment, $f_n(x)$ and $f_{n+1}(x)$ take only two values for $x \in [0, n+1)$: 1 and -1. The function $f_n(x)$ is 1 from 0 to $(k_n+n)/2$, -1 from $(k_n+n)/2$ to n, and 1 again from n to $n+1$. The function $f_{n+1}(x)$ is 1 from 0 to $(k_n+n+2)/2$ and -1 from $(k_n+n+2)/2$ to $n+1$. It is simply a matter of comparing functions of these type. We first note that $(k_n+n)/2$ is larger than $n/2$, and some integer multiple of $(k+n)/2$ is arbitrarily close to an integer, so by the fact that our quantities are invariant under dilation, it suffices to prove the following lemma.

LEMMA 1. *Let f be* 1 *on* $[0, j)$, -1 *on* $[j, N)$, *and again* 1 *on* $[N, N+l)$ *where* $j \ge N/2$. *Let g be equal to* 1 *on* $[0, j+l)$ *and* -1 *on* $[j+l, N+l)$. *Then* $\|\Lambda f\|_1 \ge \|\Lambda g\|_1$.

This is a matter of computing and counting. One can find that

$$\|\Lambda f\|_1 = 2j\left(\log\left(\frac{j+1}{j}\right) + \log\left(\frac{j+2}{j+1}\right) + \cdots + \log\left(\frac{N}{N-1}\right)\right) + 2(N-j)\left(\log\left(\frac{N+1}{N}\right) + \cdots + \log\left(\frac{N+l}{N+l-1}\right)\right)$$

and

$$\|\Lambda g\|_1 = 2(j+l)\left(\log\left(\frac{j+l+1}{j+l}\right) + \cdots + \log\left(\frac{N+l}{N+l-1}\right)\right).$$

Consider the sequence of logarithms as a sequence of decreasing weights. The total of the quantities multiplied by the weights is the same for each of the above as $2j(N-j) + 2(N-j)l = 2(j+l)((N+l)-(j+l))$. As $2(N-j) \le 2j \le 2(j+l)$ we've simply moved our quantities to lower weights for g than for f. Hence $\|\Lambda f\|_1 \ge \|\Lambda g\|_1$ and our lemma is proved.

From the lemma we see that (9) holds in our Subcase 2.1 when $0 \le k_n < n$ and now Case 2 is completed as (9) implies $R(f_n) \le R(f_{n+1})$.

Case 3: $0 \le k_{n+1} < n+1$ *and* $x_{n+1} \le -1$. This case consists of an unbounded region with the following lines constituting its boundary: $k_n + x_{n+1} = n+1$, $k_n + x_{n+1} = 0$, and $x_{n+1} = -1$. Note that again $m_{f_n}(n+1) = n+1$, so it suffices again to prove (9). We will break this region into two different subcases.

Subcase 3.1: $n \le k_n$ *and* $x_{n+1} \le -1$. On this unbounded subregion we have the following explicit formulas:

$$\nu_{f_{n+1}}(n+1) = (k_n + x_{n+1} + n + 1)\log\left(\frac{2(n+1)}{k_n + x_{n+1} + n + 1}\right)$$

and

$$\nu_{f_n}(n+1) = (k_n - nx_{n+1})\log\left(\frac{n+1}{n}\right).$$

Consider the lines of constant $k_n + x_{n+1} = c$ in this region. By inspection $\nu_{f_{n+1}}(n+1)$ is constant on these lines. One can show that $\nu_{f_n}(n+1)$ increases as we move down lines $k_n + x_{n+1} = c$ with increasing k_n. So it suffices to check (9) only on the lines which make the boundary of our subregion: the line $x_{n+1} = -1$ and the line $k_n = n$. This is another elementary calculation.

Subcase 3.2: $1 \le k_n < n$ *and* $x_{n+1} \le -1$. With $\nu_{f_{n+1}}(n+1)$ the same as in Subcase 3.1 and using $1 \le k_n < n$, we compute

$$\nu_{f_n}(n+1) = (k_n + n)\log\left(\frac{2n}{k_n + n}\right) + |k_n - nx_{n+1}|\log\left(\frac{n+1}{n}\right).$$

On the line $x_{n+1} = -1$ in our subregion, the functions f_n and f_{n+1} are the same. So (9) holds on that line.

As noted before, on lines of constant $c = x_{n+1} + k_n$, $\nu_{f_{n+1}}(n+1)$ is constant. One may show that $\nu_{f_n}(n+1)$ increases along these lines in our subregion as k_n increases. Hence $\nu_{f_{n+1}}(n+1) \le \nu_{f_n}(n+1)$ holds for this subcase.

So in all possibilities for Case 3, we have that $\nu_{f_{n+1}}(n+1) \le \nu_{f_n}(n+1)$. As the rest of the quantities determining $R(f_{n+1})$ and $R(f_n)$ are the same in the region for Case 3, we conclude that $R(f_n) \le R(f_{n+1})$.

Case 4: $0 \le k_{n+1} < n + 1$ *and* $|x_{n+1}| < 1$. We start by noting that $n + 1 = m_{f_{n+1}}(n+1) > m_{f_n}(n+1) = n$. Recall Case 4 of Section 3.1: it required us to use Observation A because $m_{f_2}(2) > m_f(2)$. We are in a similar situation. As $m_{f_{n+1}}(n+1) - m_{f_n}(n+1) = 1$ we need to show that

$$d := \nu_{f_{n+1}}(n+1) - \nu_{f_n}(n+1) \le \log 2 \tag{10}$$

in our region in question. If $d \le 0$, then $R(f_n) < R(f_{n+1})$ as we are adding to the numerator and subtracting from the denominator of R. If $d > 0$, we show (10) and then invoke Observation A. For this case the most important line for our analysis is $x_{n+1} = k_n/n$ for $k_n \in [0, n)$ because that line corresponds to the "critical line" of $\nu_{f_n}(n+1)$ because the absolute-value term changes sign there. We calculate d on this critical line by plugging in k_n/n for x_{n+1}:

$$\left(k_n + \frac{k_n}{n} + n + 1\right)\log\left(\frac{2(n+1)}{k_n + \frac{k_n}{n} + n + 1}\right) - (k_n + n)\log\left(\frac{2n}{k_n + n}\right). \tag{11}$$

One may check that (10) holds on our critical line $x_{n+1} = k_n/n$. Now recall that $\nu_{f_{n+1}}(n+1)$ is constant on lines of constant $k_n + x_{n+1} = c$; if $\nu_{f_n}(n+1)$ increases along these lines as we move away from the critical line, then (10) will hold on the entirety of our region in question. This is easily checked. We now invoke Observation A to conclude that either $R(f_n) \le R(f_{n+1})$ or both $R(f_n)$ and $R(f_{n+1})$ are greater than $1/\log 2$ and so Case 4 is concluded.

In all cases for $n \geq 2$ we concluded either $R(f_n) \leq R(f_{n+1})$ or both $R(f_n)$ and $R(f_{n+1})$ are greater than $1/\log 2$. So continuing until $n = m$, if $R(f) > 1/\log 2$, it follows that $R(f_m) > 1/\log 2$. But $R(f_m) = 1/\log 2$ as it is a dilate of (2). Hence the alternative is true: $R(f) < 1/\log 2$.

4. A Conjecture

The foregoing theorems about Λ^*, combined with [3] tell the whole "weak"-story for Λ and its adjoint. There are other operators that coincide with the Beurling–Ahlfors transform on certain radial functions. Let a natural number $m \geq 0$ be given. In [3] it is shown that given an $f \in L^2(0, \infty)$ one may set $g(z) = f(|z|^2)$ and $F(z) = \frac{\bar{z}^m}{|z|^m} g(z)$. Then with

$$\Lambda_m f(u) = \frac{m+1}{u^{1+m/2}} \int_0^u f(x) x^{m/2}\, dx - f(u),$$

one has the relation

$$SF(z) = \frac{\bar{z}^{m+2}}{|z|^{m+2}} \Lambda_m f(|z|^2).$$

Note that $\Lambda_0 = \Lambda$. So it is conceivable that one could analyze Λ_m and come up with a larger lower bound for the weak-type constant $\|S\|_w$. However, the author believes the following is true.

CONJECTURE 1. *For $m \in \{1, 2, \ldots\}$,*

$$\|\Lambda_m\|_w = \frac{m 2^{2/(2+m)}}{(2+m)(2 - 2^{2/(2+m)})}.$$

This value is achieved by the function

$$f_m(x) = \left[\left(1 + \frac{2}{m}\right) - x^{m/2}\right] \chi_{[0,x_0)}(x) \quad \textit{where } x_0 = 2^{2/(2+m)} \left(\frac{2+m}{m}\right)^{2/m}.$$

The author arrived at this conjecture by taking an optimal sum of $(1 + 2/m)$ and $x^{m/2}$ for f, as the image of the former under Λ_m is the constant 1 and the latter vanishes under Λ_m. The author believes this is best possible because the quantity for $\|\Lambda_m\|_w$ above, if considered a function on the positive reals, has a limit of $1/\log 2$ at 0. This suggests that no more information about $\|S\|_w$ can be obtained with these Λ_m operators.

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