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2D Inviscid Heat Conductive Boussinesq Equations on a Bounded Domain

Kun Zhao

1. Introduction

One major challenge in fluid dynamics is the question of global existence and large-time asymptotic behavior of solutions to certain initial value (Cauchy) problems or initial-boundary value problems (IBVP) for modeling equations. For decades, the question of global existence/finite time blow-up of smooth solutions for the three-dimensional incompressible Euler or Navier–Stokes equations has been one of the most outstanding open problems in applied analysis. The answer to this question will play an important role in understanding core problems in fluid dynamics such as the onset of turbulence. Enormous efforts have been made on this subject, but the resolution of some basic issues is still missing. The main difficulty is to understand the vortex stretching effect in 3D flows. As part of the effort to understand the vortex stretching effect in 3D flows, various simplified model equations have been proposed. Among these models, the 2D Boussinesq system is known to be one of the most commonly used because it is analogous to the 3D incompressible Euler or Navier–Stokes equations for axisymmetric swirling flow, and it shares a similar vortex stretching effect as that in the 3D incompressible flow. Better understanding of the 2D Boussinesq system will undoubtedly shed light on the understanding of 3D flows (cf. [21]).

In this paper, we consider the 2D inviscid heat conductive Boussinesq equations

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \theta \mathbf{e}_2, \\ \theta_t + U \cdot \nabla \theta = \kappa \Delta \theta, \\ \nabla \cdot U = 0, \end{cases} \tag{1.1}$$

where $U = (u, v)$ is the velocity vector field, P is the scalar pressure, θ is the scalar temperature, the constant $\kappa > 0$ models thermal diffusion, and $\mathbf{e}_2 = (0,1)^{\mathrm{T}}$. In this paper, we consider (1.1) in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. The system is supplemented by the following initial and boundary conditions:

$$\begin{cases} (U,\theta)(\mathbf{x},0) = (U_0,\theta_0)(\mathbf{x}), \quad \mathbf{x} \in \Omega, \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = \bar{\theta}, \end{cases} \tag{1.2}$$

where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$, and $\bar{\theta}$ is a constant.

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The Boussinesq system is potentially relevant to the study of atmospheric and oceanographic turbulence, as well as to other astrophysical situations where rotation and stratification play a dominant role (see e.g. [22]). In fluid mechanics, system (1.1) is used in the field of buoyancy-driven flow. It describes the motion of an inviscid incompressible fluid subject to convective heat transfer under the influence of gravitational force (cf. [21]).

In recent years, the 2D Boussinesq equations have attracted significant attention. When $\Omega = \mathbb{R}^2$, the Cauchy problem for 2D Boussinesq equations has been well studied. In [4], Cannon and DiBenedetto studied the Cauchy problem for the Boussinesq equations with "full viscosity":

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = \nu \Delta U + \theta \mathbf{e}_2, \\ \theta_t + U \cdot \nabla \theta = \kappa \Delta \theta, \\ \nabla \cdot U = 0, \\ (U, \theta)(\mathbf{x}, 0) = (U_0, \theta_0)(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \end{cases} \tag{1.3}$$

which describes the motion of a viscous incompressible fluid subject to convective heat transfer under the influence of gravitational force, where $\nu > 0$ and $\kappa > 0$ are constants. They found a unique, global in time, weak solution. Furthermore, they improved the regularity of the solution when initial data is smooth. Recently, the result of global existence of smooth solutions to (1.3) is generalized to the cases of "partial viscosity" (i.e., either $\nu > 0$ and $\kappa = 0$, or $\nu = 0$ and $\kappa > 0$) by Hou-Li [13] and Chae [5] independently. On the other hand, the global regularity/ singularity question for (1.3) with "zero viscosity" (i.e., $\nu = \kappa = 0$) still remains an outstanding open problem in mathematical fluid mechanics, and we refer the readers to [8; 9; 10; 25; 27] for studies in this direction.

In the real world, flows often move in bounded domains with constraints from boundaries, where the initial-boundary value problems appear. The solutions of the initial-boundary value problems usually exhibit different behaviors and much richer phenomena than with the Cauchy problem. In this direction, the case of "full viscosity" has been analyzed in great extent (see e.g. [20] and references therein). The local existence and blow-up criterion of smooth solutions for the case of "zero viscosity" was established in [14]; see also [6]. Concerning the case of "partial viscosity", in [16] the authors proved the global existence of smooth solutions to the IBVP for 2D viscous Boussinesq equations (i.e., $\nu > 0$ and $\kappa = 0$) subject to the no-slip boundary condition (i.e., $U|_{\partial\Omega} = 0$). However, to the author's knowledge, the question of global regularity/finite time singularity for the case of $\nu = 0$ and $\kappa > 0$ is still open. We will give a definite answer to this question in this paper.

Owing to the dissipation in the temperature equation of (1.1) and the boundary effects, the temperature is expected to converge to its boundary value. This suggests that the equilibrium state of the temperature should be $\bar{\theta}$. In this paper, we will prove that there exists a unique global smooth solution to (1.1)–(1.2) for smooth initial data. Moreover, we will show that the temperature converges exponentially to its boundary value as time goes to infinity, and the velocity and vorticity are uniformly bounded in time.

Throughout this paper, $\|\cdot\|_{L^p}$, $\|\cdot\|_{L^\infty}$, and $\|\cdot\|_{W^{s,p}}$ denote the norms of the usual Lebesgue measurable space $L^p(\Omega)$, $L^\infty(\Omega)$, and the usual Sobolev space $W^{s,p}(\Omega)$ respectively. For $p = 2$, we denote the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{W^{s,2}}$ by $\|\cdot\|$ and $\|\cdot\|_{H^s}$ respectively. The function spaces under consideration are:

$$C([0,T]; H^3(\Omega)) \quad \text{and} \quad L^2([0,T]; H^4(\Omega)),$$

equipped with norms

$$\sup_{0\le t\le T} \|\Psi(\cdot,t)\|_{H^3} \quad \text{for } \Psi \in C([0,T]; H^3(\Omega)),$$

$$\left(\int_0^T \|\Psi(\cdot,\tau)\|_{H^4}^2 \, d\tau\right)^{1/2} \quad \text{for } \Psi \in L^2([0,T]; H^4(\Omega)).$$

Unless specified, C_i $(i = 1,2,\dots,9)$ will denote generic constants that are independent of ρ and U but may depend on Ω, κ, initial data, and the time T; while c_i $(i = 1,2,\dots,20)$ will denote generic constants that depend on Ω, κ, and initial data but are independent of ρ, U, and t.

In this paper, we generalize the study of [5] to a bounded domain with typical physical boundary conditions $(1.2)_2$. For global existence of smooth solutions, we require the following compatibility conditions:

$$\begin{aligned} U_0 \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla \cdot U_0 = 0, \\ \theta_0|_{\partial\Omega} = \bar{\theta}, \quad U_0 \cdot \nabla\theta_0 - \kappa\Delta\theta_0|_{\partial\Omega} = 0. \end{aligned} \tag{1.4}$$

Our main result is stated in the following theorem.

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. If $(U_0(\mathbf{x}), \theta_0(\mathbf{x})) \in H^3(\Omega)$ satisfies the compatibility conditions* (1.4), *then there exists a unique solution (U,θ) to* (1.1)–(1.2) *globally in time such that $U \in C([0,T); H^3(\Omega))$ and $\theta \in C([0,T); H^3(\Omega)) \cap L^2([0,T); H^4(\Omega))$ for any $T > 0$. Moreover, there exist positive constants $\gamma, \eta, \bar{c}, c(p), \tilde{c}$ independent of t such that for any fixed $p \in [2,\infty)$, it holds that*

$$\begin{aligned} \|(\theta - \bar{\theta})(\cdot,t)\|_{H^3} \le \gamma e^{-\eta t}, \quad \|\theta_t\|_{L^2([0,t]; H^2(\Omega))} \le \bar{c} \quad \forall t \ge 0, \\ \|U(\cdot,t)\|_{W^{1,p}} \le c(p), \quad \|\omega(\cdot,t)\|_{L^\infty} \le \tilde{c} \quad \forall t \ge 0, \end{aligned} \tag{1.5}$$

where $\omega = v_x - u_y$ is the 2D vorticity.

REMARK 1.1. The constants $\gamma, \eta, \bar{c}, c(p), \tilde{c}$ in (1.5) depend on Ω, κ, U_0, and θ_0. In particular, for fixed domain and given initial data, we have $\eta = O(\kappa^{18} e^{-9\kappa^{-1}})$, $\gamma = O(\kappa^{-50} e^{24\kappa^{-1}})$, $\bar{c} = O(\kappa^{-78} e^{33\kappa^{-1}})$, $c(p) = O(\kappa^{-9} e^{4\kappa^{-1}})$, and $\tilde{c} = O(\kappa^{-43} e^{21\kappa^{-1}})$ as $\kappa \to 0$. Therefore, the smaller the diffusivity is, the slower the solution decays. We refer readers to Section 3 for explicit expressions of these constants. In this paper, since we are interested in global existence and large time behavior of smooth solutions for fixed $\kappa > 0$ instead of the vanishing viscosity limit, all these constants are finite.

We prove Theorem 1.1 by showing the global existence and large time behavior of solutions to the IBVP for the perturbation $(\theta - \bar{\theta}, U)$. The proof begins with the global existence of weak solutions—that is, solutions satisfying the following definition.

DEFINITION 1.1. (U,θ) is said to be a *global weak solution* to (1.1)–(1.2) if for any $T > 0$, $U \in C([0,T); H^1(\Omega))$, $\theta \in C([0,T); L^2(\Omega)) \cap L^2([0,T); H^1(\Omega))$, and it holds that

$$\int_\Omega U_0 \cdot \Phi(\mathbf{x},0)\,d\mathbf{x} + \int_0^T \int_\Omega (U\cdot\Phi_t + U\cdot(U\cdot\nabla\Phi) + \theta\mathbf{e}_2\cdot\Phi)\,d\mathbf{x}\,dt = 0,$$

$$\int_\Omega \theta_0\psi(\mathbf{x},0)\,d\mathbf{x} + \int_0^T \int_\Omega (\theta\psi_t + \theta U\cdot\nabla\psi - \nabla\theta\cdot\nabla\psi)\,d\mathbf{x}\,dt = 0$$

for any $\Phi = (\phi_1,\phi_2) \in C^\infty(\Omega\times[0,T])^2$ satisfying $\Phi(\mathbf{x},T) = 0$, $\nabla\cdot\Phi = 0$, and $\Phi\cdot\mathbf{n}|_{\partial\Omega} = 0$ and for any $\psi \in C_0^\infty(\Omega\times[0,T])$ satisfying $\psi(\mathbf{x},T) = 0$.

We then build up the regularity of the solution by energy estimates under the initial and boundary conditions (1.2). For simplicity of presentation, we prove global regularity and large time behavior of the solution simultaneously. The energy estimate is somewhat delicate mainly because of the coupling between the velocity and temperature equations by convection, gravitational force, and boundary effects. Great efforts have been made to simplify the proof. The current proof involves intensive applications of Sobolev embeddings and Ladyzhenskaya's inequality; see Lemma 2.1. The proof distinguishes itself from the Cauchy problem in [5] mainly by the fact that the problem is set on the bounded domain. Roughly speaking, because of the lack of the spatial derivatives of the solution at the boundary, our energy framework proceeds as follows. We first apply the standard energy estimate on the solution and the temporal derivatives of the solution. We then apply standard results on elliptic equations to recover estimates of the spatial derivatives. Such a process will be repeated up to third order, and then the carefully coupled estimates will be composed into a desired estimate leading to global regularity and exponential decay of the temperature. Then classical results on 2D Euler equations, see Lemma 2.5, will be implemented on the first equation in (1.1) to establish the global regularity of the velocity field. The uniqueness of the solution then follows in a straightforward way. This result suggests that, without viscous dissipation, thermal diffusion is still strong enough to compensate the effects of gravitational force and nonlinear convection in order to prevent the development of singularity of the system.

It should be pointed out that, in Theorem 1.1, no smallness restriction is put upon the initial data.

The plan of the rest of this paper is as follows. In Section 2, we first reformulate the original system to get the one for the perturbation $(\theta-\bar\theta, U)$. Then we give some facts that will be used in this paper and prove the global existence of weak solutions. In Section 3, we prove Theorem 1.1 by energy estimates.

2. Preliminaries and Weak Solutions

In this section, we first reformulate the initial-boundary value problem (1.1)–(1.2). Let $\bar P = P - \bar\theta y$ and $\Theta = \theta - \bar\theta$; then we get from the original system:

$$\begin{cases} U_t + U \cdot \nabla U + \nabla \bar{P} = \Theta \mathbf{e}_2, \\ \Theta_t + U \cdot \nabla \Theta = \kappa \Delta \Theta, \\ \nabla \cdot U = 0. \end{cases} \tag{2.1}$$

The initial and boundary conditions become

$$\begin{cases} (U, \Theta)(\mathbf{x}, 0) = (U_0, \Theta_0)(\mathbf{x}), \\ U \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \Theta|_{\partial\Omega} = 0, \end{cases} \tag{2.2}$$

where $\Theta_0 = \theta_0 - \bar{\theta}$. It is clear that, for smooth solutions, (2.1)–(2.2) are equivalent to (1.1)–(1.2). By definition, the same is true for weak solutions. Hence, for the rest of this paper, we shall work on the reformulated problem (2.1)–(2.2).

Now we collect several facts that will be used in the proof of Theorem 1.1. First, we recall some inequalities of Sobolev and Ladyzhenskaya type (cf. [26]).

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Then*

(i) $\|f\|_{L^\infty}^2 \le c_1 \|f\|_{H^2}^2$;
(ii) $\|f\|_{L^\infty}^2 \le c_2 \|f\|_{W^{1,p}}^2$ *for all* $p > 2$;
(iii) $\|f\|_{L^p}^2 \le c_3 \|f\|_{H^1}^2$ *for all* $1 \le p < \infty$;
(iv) $\|f\|_{L^4}^2 \le c_4 (\|f\| \|\nabla f\| + \|f\|^2)$

for some constants c_i, $i = 1, \ldots, 4$ depending only on Ω and p.

Next, we recall some classical result on elliptic equations (cf. [1; 11; 17; 18]).

LEMMA 2.2. *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Consider the Dirichlet problem*:

$$\begin{cases} \kappa \Delta \Theta = f & \text{in } \Omega, \\ \Theta = 0 & \text{on } \partial\Omega. \end{cases}$$

If $f \in W^{m,p}$, then $\Theta \in W^{m+2,p}$ and there exists a constant $c_5 = c_5(p, m, \Omega)$ such that

$$\|\Theta\|_{W^{m+2,p}} \le \frac{c_5}{\kappa} \|f\|_{W^{m,p}}$$

for any $p \in (1, \infty)$ and the integer $m \ge -1$.

The next three lemmas are useful in the estimation of the velocity field.

LEMMA 2.3 (cf. [3]). *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$, and let $U \in W^{s,p}(\Omega)$ be a vector-valued function satisfying $\nabla \cdot U = 0$ and $U \cdot \mathbf{n}|_{\partial\Omega} = 0$, where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$. Then there exists a constant $c_6 = c_6(s, p, \Omega)$ such that*

$$\|U\|_{W^{s,p}} \le c_6 (\|\nabla \times U\|_{W^{s-1,p}} + \|U\|_{L^p})$$

for any $s \ge 1$ and $p \in (1, \infty)$.

LEMMA 2.4 (cf. [21]). *Let $\Omega \subset \mathbb{R}^2$ be any bounded domain with smooth boundary $\partial\Omega$. Then for any multiindex β with order $|\beta| \ge 3$ and any functions $f \in H^{|\beta|}(\Omega)$ and $g \in H^{|\beta|-1}(\Omega)$, it holds that*

$$\|D^\beta(fg) - fD^\beta g\| \le c_7(\|\nabla f\|_{L^\infty}\|g\|_{H^{|\beta|-1}} + \|f\|_{H^{|\beta|}}\|g\|_{L^\infty})$$

for some constant $c_7 = c_7(|\beta|, \Omega)$.

LEMMA 2.5 (cf. [15]). *Let* $\Omega \subset \mathbb{R}^2$ *be any bounded domain with smooth boundary* $\partial\Omega$. *Consider the initial-boundary value problem*:

$$\begin{cases} U_t + U \cdot \nabla U + \nabla P = G, \\ \nabla \cdot U = 0, \\ U(\mathbf{x}, 0) = U_0(\mathbf{x}), \quad U \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{2.3}$$

where $\mathbf{n}$ *is the unit outward normal to* $\partial\Omega$. *Let* $U_0(\mathbf{x}) \in C^{1+\gamma}(\bar\Omega)$ *satisfying* $\nabla \cdot U_0(\mathbf{x}) = 0$, $U_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$. *For any fixed* $T > 0$, *let* $G \in C([0,T]; C^{1+\nu}(\bar\Omega))$ *for some* $0 < \nu < 1$. *Then there exists a solution* (U, P) *to* (2.3) *such that* $(U, P) \in C^1(\bar\Omega \times [0,T])$.

Next, we establish the global existence of weak solutions to (2.1)–(2.2). The result will be proved by a fixed point argument and the method of energy estimates. The proof is standard and we only give an outline of it. We refer readers to [19] for more details.

LEMMA 2.6 (Global existence of weak solutions). *Under the assumptions of Theorem 1.1, there exists a global weak solution* (U, Θ) *to* (2.1)–(2.2), *as defined in Definition 1.1, such that, for any* $T > 0$, $U \in C([0,T); H^1(\Omega))$ *and* $\Theta \in C([0,T); L^2(\Omega)) \cap L^2([0,T); H_0^1(\Omega))$.

Outline of Proof

Step 1. We fix any $T \in [0,\infty)$ and consider the problem (2.1)–(2.2) in $[0,T]$. Let B be the closed convex set in $C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega))$ defined by

$$B = \big\{\sigma \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)) \mid \\ \|\sigma\|^2_{C([0,T]; L^2(\Omega))} + \|\sigma\|^2_{L^2([0,T]; H_0^1(\Omega))} \le R_0\big\},$$

where R_0 is to be determined.

Step 2. For fixed $\varepsilon \in (0,1)$, we define a mapping F_ε from B as follows. For any $\sigma \in B$, we first apply the standard procedure of regularization to get a smooth approximation σ_ε for σ, $\Theta_0^\varepsilon(\mathbf{x})$ for $\Theta_0(\mathbf{x})$, and $U_0^\varepsilon(\mathbf{x})$ for $U_0(\mathbf{x})$ respectively. We then solve the 2D incompressible Euler equations with smooth external forcing term $\sigma_\varepsilon \mathbf{e}_2$ and smooth initial data

$$\begin{cases} U_t + U \cdot \nabla U + \nabla \bar P = \sigma_\varepsilon \mathbf{e}_2, \\ \nabla \cdot U = 0, \\ U(\mathbf{x}, 0) = U_0^\varepsilon(\mathbf{x}), \quad U \cdot \mathbf{n}|_{\partial\Omega} = 0, \end{cases} \tag{2.4}$$

and we denote the solution by U^ε and the corresponding pressure by $\bar P^\varepsilon$. Next, we solve the linear parabolic equation with smooth initial data

$$\begin{cases} \Theta_t + U^\varepsilon \cdot \nabla\Theta = \kappa\Delta\Theta, \\ \Theta(\mathbf{x}, 0) = \Theta_0^\varepsilon(\mathbf{x}), \quad \Theta|_{\partial\Omega} = 0, \end{cases} \tag{2.5}$$

and we denote the solution by Θ^ε. Then we define the mapping $F_\varepsilon(\sigma) = \Theta^\varepsilon$.

Step 3. We apply Schauder fixed point theorem to construct a sequence of approximate solutions to (2.1)–(2.2). For this purpose, we need to show that, for any fixed $\varepsilon \in (0, 1)$, $F_\varepsilon \colon B \to B$ is compact and continuous. Let $\|\cdot\|_B = \|\cdot\|_{C([0,T];\,L^2(\Omega))} + \|\cdot\|_{L^2([0,T];\,H_0^1(\Omega))}$. We shall show that, for any fixed $\varepsilon \in (0, 1)$, the following hold:

- $\|\Theta^\varepsilon\|_B^2 \le C_1$ for some constant $C_1 > 0$ independent of R_0 and ε;
- $\Theta^\varepsilon \in C([0, T];\, H_0^1(\Omega)) \cap L^2([0, T];\, H^2(\Omega))$;
- $\|F_\varepsilon(\sigma_1) - F_\varepsilon(\sigma_2)\|_B^2 \le C_2\|\sigma_1 - \sigma_2\|_B^2$ for any $\sigma_1, \sigma_2 \in B$.

These are achieved by the method of energy estimates. Since we will perform energy estimates for the nonlinear system (2.1) in Section 3, our energy framework also works for the linear equation (2.5). Therefore, we omit the details here.

Step 4. To ensure compactness of the sequence of approximate solutions, we show that $\|U^\varepsilon\|_{C([0,T];\,H^1(\Omega))}^2 \le C_3$ for some constant C_3 independent of ε. This is also achieved by the energy estimates given in Section 3.

Step 5. With the uniform (ε-independent) estimates established in Step 3 and Step 4, it is standard to check that the limiting function (Θ, U) of $(\Theta^\varepsilon, U^\varepsilon)$ as $\varepsilon \to 0_+$ is a weak solution to (2.1)–(2.2) in $\Omega \times [0, T]$. We conclude the argument by noticing that T is arbitrary.

3. Global Regularity and Large Time Behavior

In this section, we shall establish the regularity, uniqueness, and large time behavior of the solution obtained in Lemma 2.6 and thereby give a proof of our main result, Theorem 1.1. The following theorem gives the key estimates.

THEOREM 3.1. *Under the assumptions of Theorem 1.1, the solution obtained in Lemma 2.6 satisfies*

$$U \in C([0, T);\, H^3(\Omega)), \qquad \Theta \in C([0, T);\, H^3(\Omega)) \cap L^2([0, T);\, H^4(\Omega))$$

for any $T > 0$. Moreover, there exist positive constants $\gamma, \eta, \bar{c}, c(p), \tilde{c}$ independent of t such that for any fixed $p \in [2, \infty)$, it holds that

$$\|\Theta(\cdot, t)\|_{H^3} \le \gamma e^{-\eta t}, \quad \|\Theta_t\|_{L^2([0,t];\,H^2(\Omega))} \le \bar{c} \quad \forall t \ge 0,$$
$$\|U(\cdot, t)\|_{W^{1,p}} \le c(p), \quad \|\omega(\cdot, t)\|_{L^\infty} \le \tilde{c} \quad \forall t \ge 0.$$

The proof of Theorem 3.1 is based on several steps of careful energy estimates which are stated as a sequence of lemmas.

3.1. L^2 Estimate of Θ

First, we give the decay estimate of $\|\Theta\|$.

LEMMA 3.1. *Under the assumptions of Theorem 1.1, it holds that*

$$\|\Theta(\cdot, t)\|^2 \le \|\Theta_0\|^2 e^{-2\beta_0 t}, \qquad \int_0^t e^{\beta_0 \tau}\|\nabla\Theta(\cdot, \tau)\|^2\, d\tau \le \alpha_0\|\Theta_0\|^2 \quad \forall t \ge 0,$$

where $\beta_0 = \kappa/c_0$ *for* c_0 *the constant in Poincaré's inequality on the domain* Ω, *and* $\alpha_0 = 1/\kappa$.

Proof. Taking the L^2 inner product of $(2.1)_2$ with Θ, we have

$$\frac{d}{dt}\|\Theta\|^2 + 2\kappa\|\nabla\Theta\|^2 = 0. \tag{3.1}$$

Since $\Theta|_{\partial\Omega} = 0$, Poincaré's inequality implies that

$$\|\Theta\|^2 \leq c_0\|\nabla\Theta\|^2 \tag{3.2}$$

for some constant c_0 depending only on Ω. Replacing $\|\nabla\Theta\|^2$ in (3.1) by $\|\Theta\|^2$ we have

$$\frac{d}{dt}\|\Theta\|^2 + \frac{2\kappa}{c_0}\|\Theta\|^2 \leq 0,$$

which yields immediately that

$$\|\Theta(\cdot,t)\|^2 \leq \|\Theta_0\|^2 e^{-2\beta_0 t} \quad \forall t \geq 0, \tag{3.3}$$

where $\beta_0 = \kappa/c_0$.

Next, we multiply (3.1) by $e^{\beta_0 t}$ and use (3.3) to get

$$\frac{d}{dt}(e^{\beta_0 t}\|\Theta\|^2) + 2\kappa e^{\beta_0 t}\|\nabla\Theta\|^2 = \beta_0 e^{\beta_0 t}\|\Theta\|^2 \leq \beta_0 e^{-\beta_0 t}\|\Theta_0\|^2. \tag{3.4}$$

For any $t \geq 0$, upon integrating (3.4) in time over $[0,t]$ we have

$$e^{\beta_0 t}\|\Theta(\cdot,t)\|^2 - \|\Theta_0\|^2 + 2\kappa\int_0^t e^{\beta_0\tau}\|\nabla\Theta(\cdot,\tau)\|^2\,d\tau \leq (1 - e^{-\beta_0 t})\|\Theta_0\|^2,$$

which implies that (by dropping $e^{\beta_0 t}\|\Theta(\cdot,t)\|^2$ from the LHS)

$$\int_0^t e^{\beta_0\tau}\|\nabla\Theta(\cdot,\tau)\|^2\,d\tau \leq \alpha_0\|\Theta_0\|^2 \quad \forall t \geq 0, \tag{3.5}$$

where $\alpha_0 = 1/\kappa$. This completes the proof. □

3.2. H^1 Estimate of (Θ, U)

To improve the decay estimate of Θ, we proceed to find the uniform estimate of $\|U\|_{H^1}$. This will be achieved with the help of (3.5).

LEMMA 3.2. *Under the assumptions of Theorem 1.1, it holds that*

$$\|U(\cdot,t)\|_{H^1}^2 \leq d_1 \quad \forall t \geq 0,$$

where $d_1 = c_6 e^{1/\beta_0}(\|U_0\|^2 + \|\omega_0\|^2 + (\alpha_0 + 1/\beta_0)\|\Theta_0\|^2) = O(\kappa^{-1}e^{\kappa^{-1}})$ *as* $\kappa \to 0$, *and* c_6 *is the constant in Lemma 2.3.*

Proof. Taking the L^2 inner product of $(2.1)_1$ with U we have

$$\frac{d}{dt}\|U\|^2 = 2\int_\Omega \Theta\mathbf{e}_2 \cdot U\,d\mathbf{x}.$$

The Cauchy–Schwartz inequality and (3.3) then imply that

$$
\begin{aligned}
\frac{d}{dt}\|U\|^2 &\le e^{-\beta_0 t}\|U\|^2 + e^{\beta_0 t}\|\Theta\|^2 \\
&\le e^{-\beta_0 t}\|U\|^2 + e^{-\beta_0 t}\|\Theta_0\|^2.
\end{aligned} \tag{3.6}
$$

Applying Gronwall's inequality to (3.6) we have

$$
\begin{aligned}
\|U(\cdot,t)\|^2 &\le \exp\left\{\int_0^t e^{-\beta_0\tau}\,d\tau\right\}\left(\|U_0\|^2 + \int_0^t e^{-\beta_0\tau}\|\Theta_0\|^2\,d\tau\right) \\
&\le e^{1/\beta_0}\left(\|U_0\|^2 + \frac{\|\Theta_0\|^2}{\beta_0}\right) \quad \forall t \ge 0.
\end{aligned} \tag{3.7}
$$

Taking the curl of $(2.1)_1$ we have

$$
\omega_t + U\cdot\nabla\omega = \Theta_x, \tag{3.8}
$$

where $\omega = v_x - u_y$ is the 2D vorticity. Taking the L^2 inner product of (3.8) with ω and using Cauchy–Schwartz inequality we have

$$
\frac{d}{dt}\|\omega\|^2 \le e^{-\beta_0 t}\|\omega\|^2 + e^{\beta_0 t}\|\nabla\Theta\|^2.
$$

Gronwall's inequality and (3.5) then yield

$$
\begin{aligned}
\|\omega(\cdot,t)\|^2 &\le \exp\left\{\int_0^t e^{-\beta_0\tau}\,d\tau\right\}\left(\|\omega_0\|^2 + \int_0^t e^{\beta_0\tau}\|\nabla\Theta(\cdot,\tau)\|^2\,d\tau\right) \\
&\le e^{1/\beta_0}(\|\omega_0\|^2 + \alpha_0\|\Theta_0\|^2) \quad \forall t \ge 0.
\end{aligned} \tag{3.9}
$$

We conclude the proof by combining (3.7), (3.9), and Lemma 2.3 with $s = 1$, $p = 2$. □

Now we prove the exponential decay of $\|\Theta\|_{H^1}$. The important role played by the uniform estimate of $\|U\|_{H^1}^2$ will be revealed in the proof.

LEMMA 3.3. *Under the assumptions of Theorem 1.1, there exist constants $\alpha_1 > 0$ and $\beta_1 > 0$ independent of t such that*

$$
\|\Theta(\cdot,t)\|_{H^1}^2 \le \alpha_1\|\Theta_0\|_{H^1}^2 e^{-\beta_1 t} \quad \forall t \ge 0,
$$

where $\alpha_1 = O(\kappa^{-5}e^{2\kappa^{-1}})$ and $\beta_1 = O(\kappa)$ as $\kappa \to 0$.

Proof. Taking the L^2 inner product of $(2.1)_2$ with Θ_t we have

$$
\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \|\Theta_t\|^2 = -\int_\Omega (U\cdot\nabla\Theta)\Theta_t\,d\mathbf{x}. \tag{3.10}
$$

We estimate the right-hand side of (3.10) as follows. First, using Cauchy–Schwartz inequality we have

$$
\left|\int_\Omega (U\cdot\nabla\Theta)\Theta_t\,d\mathbf{x}\right| \le \|U\cdot\nabla\Theta\|^2 + \frac{1}{4}\|\Theta_t\|^2. \tag{3.11}
$$

For the first term on the RHS of (3.11), using Lemma 2.1(iii) and Lemma 3.2 we have

$$\begin{aligned}\|U\cdot\nabla\Theta\|^2 &\le \|U\|_{L^4}^2\|\nabla\Theta\|_{L^4}^2\\ &\le c_3\|U\|_{H^1}^2\|\nabla\Theta\|_{L^4}^2\\ &\le c_3d_1\|\nabla\Theta\|_{L^4}^2.\end{aligned}$$

So we update (3.10) as

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2+\frac{3}{4}\|\Theta_t\|^2\le c_3d_1\|\nabla\Theta\|_{L^4}^2. \tag{3.12}$$

For the RHS of (3.12), applying Lemma 2.1(iv) to $\nabla\Theta$ we have

$$\begin{aligned}c_3d_1\|\nabla\Theta\|_{L^4}^2 &\le c_3d_1c_4(\|\nabla\Theta\|\|D^2\Theta\|+\|\nabla\Theta\|^2)\\ &\le \delta\|D^2\Theta\|^2+\frac{c_3d_1c_4(4\delta+c_3d_1c_4)}{4\delta}\|\nabla\Theta\|^2,\end{aligned} \tag{3.13}$$

where δ is a number to be determined. To estimate $\|D^2\Theta\|$, we rewrite $(2.1)_2$ as

$$\kappa\Delta\Theta=f\equiv\Theta_t+U\cdot\nabla\Theta. \tag{3.14}$$

Since $\Theta|_{\partial\Omega}=0$, Lemma 2.2 with $m=0$ and $p=2$ implies that

$$\|\Theta\|_{H^2}^2\le\frac{2c_5}{\kappa}(\|\Theta_t\|^2+\|U\cdot\nabla\Theta\|^2). \tag{3.15}$$

For the second term on the RHS of (3.15), similarly to (3.13) we have

$$\|U\cdot\nabla\Theta\|^2\le c_3d_1c_4(\|\nabla\Theta\|\|D^2\Theta\|+\|\nabla\Theta\|^2). \tag{3.16}$$

Then, using Cauchy–Schwartz inequality and (3.16) we update (3.15) as

$$\begin{aligned}\|\Theta\|_{H^2}^2 &\le \frac{2c_5}{\kappa}\|\Theta_t\|^2+\frac{2c_3d_1c_4c_5}{\kappa}(\|\nabla\Theta\|\|D^2\Theta\|+\|\nabla\Theta\|^2)\\ &\le \frac{1}{2}\|\Theta\|_{H^2}^2+\frac{2c_5}{\kappa}\|\Theta_t\|^2+\frac{2c_3d_1c_4c_5(c_3d_1c_4c_5+\kappa)}{\kappa^2}\|\nabla\Theta\|^2.\end{aligned} \tag{3.17}$$

Let

$$c_8=\frac{4c_3d_1c_4c_5(c_3d_1c_4c_5+\kappa)}{\kappa^2}. \tag{3.18}$$

Using Lemma 3.2 we have

$$c_8=O(\kappa^{-4}e^{2\kappa^{-1}})\quad\text{as }\ \kappa\to0. \tag{3.19}$$

Then, by (3.17)–(3.18) we have

$$\|\Theta\|_{H^2}^2\le\frac{4c_5}{\kappa}\|\Theta_t\|^2+c_8\|\nabla\Theta\|^2. \tag{3.20}$$

Using (3.20) we update (3.13) as

$$c_3d_1\|\nabla\Theta\|_{L^4}^2\le\delta\frac{4c_5}{\kappa}\|\Theta_t\|^2+\left(\frac{c_3d_1c_4(4\delta+c_3d_1c_4)}{4\delta}+\delta c_8\right)\|\nabla\Theta\|^2. \tag{3.21}$$

Choosing $\delta=\kappa/(16c_5)$ in (3.21) we have

$$c_3d_1\|\nabla\Theta\|_{L^4}^2\le\frac{1}{4}\|\Theta_t\|^2+c_9\|\nabla\Theta\|^2, \tag{3.22}$$

where

$$c_9 = c_3 d_1 c_4 + \frac{4c_5(c_3 d_1 c_4)^2}{\kappa} + \frac{c_8 \kappa}{16 c_5}. \tag{3.23}$$

By coupling (3.12) and (3.22) we have

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2 \le c_9\|\nabla\Theta\|^2. \tag{3.24}$$

By (3.23), Lemma 3.2, and (3.19) we have

$$c_9 = O(\kappa^{-3} e^{2\kappa^{-1}}) \quad \text{as } \kappa \to 0. \tag{3.25}$$

To prove the decay of $\|\Theta\|_{H^1}$, we consider the operation $(c_9/\kappa) \times (3.1) + (3.24)$, which gives

$$\frac{d}{dt}\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) + c_9\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2 \le 0. \tag{3.26}$$

Using (3.4) one easily checks that

$$\beta_1\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) \le c_9\|\nabla\Theta\|^2, \tag{3.27}$$

where

$$\beta_1 = \left(\frac{c_0}{\kappa} + \frac{\kappa}{2c_9}\right)^{-1} = O(\kappa) \quad \text{as } \kappa \to 0. \tag{3.28}$$

Replacing $c_9\|\nabla\Theta\|^2$ on the LHS of (3.26) by the LHS of (3.27) we have

$$\frac{d}{dt}\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) + \beta_1\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) + \frac{1}{2}\|\Theta_t\|^2 \le 0,$$

which implies (by dropping $\frac{1}{2}\|\Theta_t\|^2$ from the left-hand side) that

$$\left(\frac{c_9}{\kappa}\|\Theta(\cdot,t)\|^2 + \frac{\kappa}{2}\|\nabla\Theta(\cdot,t)\|^2\right) \le \left(\frac{c_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\right)e^{-\beta_1 t}. \tag{3.29}$$

Therefore, we have

$$\|\Theta(\cdot,t)\|_{H^1}^2 \le \alpha_1\|\Theta_0\|_{H^1}^2 e^{-\beta_1 t} \quad \forall t \ge 0,$$

where (by using (3.23))

$$\alpha_1 = \frac{\max\left\{\frac{c_9}{\kappa}, \frac{\kappa}{2}\right\}}{\min\left\{\frac{c_9}{\kappa}, \frac{\kappa}{2}\right\}} = O(\kappa^{-5} e^{2\kappa^{-1}}) \quad \text{as } \kappa \to 0. \tag{3.30}$$

This completes the proof. □

REMARK 3.1. The coupling of energy estimates used in the proof of Lemma 3.3 will be repeated twice to establish the exponential decay of $\|\Theta\|_{H^2}$ and $\|\Theta\|_{H^3}$.

3.3. $W^{1,p}$ Estimate of U and H^2 Estimate of Θ

In order to improve the decay estimate of Θ, we need to establish a higher-order uniform estimate of U.

LEMMA 3.4. *Under the assumptions of Theorem 1.1, for any fixed* $p \in [2,\infty)$, *there exists a constant* $d_2 = d_2(p) > 0$ *independent of* t *such that*

$$\|U(\cdot,t)\|_{W^{1,p}}^2 \le d_2 \quad \forall t \ge 0,$$

where $d_2 = O(\kappa^{-9}e^{4\kappa^{-1}})$ *as* $\kappa \to 0$.

Proof. By virtue of Lemma 2.1(iii), Lemma 3.2, and Lemma 2.3, it suffices to prove the uniform estimate of $\|\omega\|_{L^p}$ in order to prove the lemma. For any fixed $p \in [2,\infty)$, taking the L^2 inner product of (3.8) with $|\omega|^{p-2}\omega$ we have

$$\frac{1}{p}\frac{d}{dt}\|\omega\|_{L^p}^p = -\int_\Omega \Theta_x |\omega|^{p-2}\omega \, d\mathbf{x}. \tag{3.31}$$

Using Hölder's inequality, we estimate the RHS of (3.31) as

$$\left|\int_\Omega \Theta_x |\omega|^{p-2}\omega \, d\mathbf{x}\right| \le \|\nabla\Theta\|_{L^p}\|\omega\|_{L^p}^{p-1}. \tag{3.32}$$

Combining (3.31) with (3.32) and using Lemma 2.1(iii) we have

$$\frac{d}{dt}\|\omega\|_{L^p} \le \|\nabla\Theta\|_{L^p} \le c_3\|\Theta\|_{H^2}. \tag{3.33}$$

Upon integrating in time and using Hölder's inequality we have

$$\begin{aligned}
&\|\omega(\cdot,t)\|_{L^p} \\
&\le \|\omega(\cdot,0)\|_{L^p} + c_3\int_0^t \|\Theta(\cdot,\tau)\|_{H^2}\, d\tau \\
&\le \|\omega(\cdot,0)\|_{L^p} + c_3\left(\int_0^t e^{\beta_1\tau/2}\|\Theta(\cdot,\tau)\|_{H^2}^2\, d\tau\right)^{1/2}\left(\int_0^t e^{-\beta_1\tau/2}\, d\tau\right)^{1/2}.
\end{aligned}\tag{3.34}$$

To estimate the middle term on the RHS of (3.34), we establish a result similar to (3.5) for $\|\Theta\|_{H^2}$. Multiplying (3.26) by $e^{\beta_1 t/2}$ and applying (3.29) we have

$$\begin{aligned}
\frac{d}{dt}\left[e^{\beta_1 t/2}\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right)\right] + e^{\beta_1 t/2}\left(c_9\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2\right)& \\
\le \frac{\beta_1}{2}e^{-\beta_1 t/2}\left(\frac{c_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\right).&
\end{aligned}\tag{3.35}$$

Integrating (3.35) in time over $[0,t]$ we have

$$\begin{aligned}
e^{\beta_1 t/2}\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) + \int_0^t e^{\beta_1\tau/2}\left(c_9\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2\right) d\tau& \\
\le 2\left(\frac{c_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\right).&
\end{aligned}$$

In particular, we have

$$\int_0^t e^{\beta_1\tau/2}\left(c_9\|\nabla\Theta\|^2 + \frac{1}{2}\|\Theta_t\|^2\right) d\tau \le 2\left(\frac{c_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\right). \tag{3.36}$$

Letting $c_{10} = 2(\min\{c_9, 1/2\})^{-1}$ we get from (3.36) that

$$\int_0^t e^{\beta_1\tau/2}(\|\nabla\Theta(\cdot,\tau)\|^2 + \|\Theta_t(\cdot,\tau)\|^2)\,d\tau \le c_{10}\left(\frac{c_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\right). \quad (3.37)$$

Using (3.20) and (3.37) we obtain the following estimate on $\|\Theta\|_{H^2}$:

$$\int_0^t e^{\beta_1\tau/2}\|\Theta(\cdot,\tau)\|_{H^2}^2\,d\tau \le c_{11} \quad \forall t \ge 0, \quad (3.38)$$

where

$$c_{11} = c_{10}\left(\max\left\{\frac{4c_5}{\kappa}, c_8\right\}\right)\left(\frac{c_9}{\kappa}\|\Theta_0\|^2 + \frac{\kappa}{2}\|\nabla\Theta_0\|^2\right).$$

By definition of c_{10}, (3.19), and (3.25) we have

$$c_{11} = O(\kappa^{-8}e^{4\kappa^{-1}}) \quad \text{as } \kappa \to 0. \quad (3.39)$$

Substituting (3.38) into (3.34) we have

$$\|\omega(\cdot,t)\|_{L^p} \le \|\omega(\cdot,0)\|_{L^p} + c_3\sqrt{2c_{11}/\beta_1} \quad \forall t \ge 0.$$

Therefore, Lemmas 2.1, 2.3, and 3.2 together with (3.31) imply that for any fixed $p \in [2,\infty)$,

$$\begin{aligned}\|U(\cdot,t)\|_{W^{1,p}} &\le c_6(\|\omega(\cdot,t)\|_{L^p} + \|U(\cdot,t)\|_{L^p})\\ &\le c_6(\|\omega(\cdot,t)\|_{L^p} + c_3\|U(\cdot,t)\|_{H^1})\\ &\le c_6\big(\|\omega(\cdot,0)\|_{L^p} + c_3\sqrt{2c_{11}/\beta_1} + c_3\sqrt{d_1}\big) \equiv \sqrt{d_2} \quad \forall t \ge 0.\end{aligned}$$

Using (3.28), (3.39), and Lemma 3.2 we have

$$d_2 = O(\kappa^{-9}e^{4\kappa^{-1}}) \quad \text{as } \kappa \to 0. \quad (3.40)$$

This completes the proof. □

With the help of Lemma 3.4 we are now ready to show the exponential decay of $\|\Theta\|_{H^2}$.

LEMMA 3.5. *Under the assumptions of Theorem 1.1, there exist constants $\alpha_2 > 0$ and $\beta_2 > 0$ independent of t such that*

$$\|\Theta(\cdot,t)\|_{H^2}^2 \le \alpha_2 e^{-\beta_2 t} \quad \forall t \ge 0,$$

where $\alpha_2 = O(\kappa^{-23}e^{11\kappa^{-1}})$ and $\beta_2 = O(\kappa^4 e^{-2\kappa^{-1}})$ as $\kappa \to 0$.

Proof. Taking the temporal derivative of $(2.1)_2$ we have

$$\Theta_{tt} + U_t\cdot\nabla\Theta + U\cdot\nabla\Theta_t = \kappa\Delta\Theta_t. \quad (3.41)$$

Since $\Theta|_{\partial\Omega} = 0$ and $\Omega \subset \mathbb{R}^2$ is fixed, we have $\Theta_t|_{\partial\Omega} = 0$. Taking the L^2 inner product of (3.41) with Θ_t we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta_t\|^2 + \kappa\|\nabla\Theta_t\|^2 &= -\int_\Omega (U_t\cdot\nabla\Theta)\Theta_t\,d\mathbf{x}\\ &= \int_\Omega \Theta(U_t\cdot\nabla\Theta_t)\,d\mathbf{x}. \quad (3.42)\end{aligned}$$

Using the Cauchy–Schwartz inequality and Lemma 2.1(i) we estimate the RHS of (3.42) as:

$$\begin{aligned}\left|\int_\Omega \Theta(U_t\cdot\nabla\Theta_t)\,d\mathbf{x}\right| &\le \frac{1}{2\kappa}\|\Theta U_t\|^2+\frac{\kappa}{2}\|\nabla\Theta_t\|^2\\ &\le \frac{1}{2\kappa}\|\Theta\|_{L^\infty}^2\|U_t\|^2+\frac{\kappa}{2}\|\nabla\Theta_t\|^2\\ &\le \frac{c_1}{2\kappa}\|\Theta\|_{H^2}^2\|U_t\|^2+\frac{\kappa}{2}\|\nabla\Theta_t\|^2. \end{aligned}\tag{3.43}$$

To estimate $\|U_t\|^2$, we take the L^2 inner product of $(2.1)_1$ with U_t to get

$$\|U_t\|^2=-\int_\Omega U_t\cdot(U\cdot\nabla U)\,d\mathbf{x}+\int_\Omega \Theta v_t\,d\mathbf{x},$$

from which we derive, using Lemma 2.1(ii), Lemma 3.1, Lemma 3.2, and Lemma 3.4, that

$$\begin{aligned}\|U_t\|^2 &\le \frac14\|U_t\|^2+\|U\cdot\nabla U\|^2+\|\Theta\|^2+\frac14\|U_t\|^2\\ &\le \frac12\|U_t\|^2+\|U\|_{L^\infty}^2\|\nabla U\|^2+\|\Theta\|^2\\ &\le \frac12\|U_t\|^2+c_2\|U\|_{W^{1,4}}^2\|\nabla U\|^2+\|\Theta\|^2\\ &\le \frac12\|U_t\|^2+c_2d_2d_1+\|\Theta_0\|^2.\end{aligned}$$

Hence,

$$\|U_t(\cdot,t)\|^2\le 2(c_2d_2d_1+\|\Theta_0\|^2)\equiv c_{12}\quad \forall t\ge 0.\tag{3.44}$$

By Lemma 3.2 and Lemma 3.4 we have

$$c_{12}=O(\kappa^{-10}e^{5\kappa^{-1}})\quad\text{as }\ \kappa\to 0.\tag{3.45}$$

Using (3.44) we update (3.43) as

$$\left|\int_\Omega \Theta(U_t\cdot\nabla\Theta_t)\,d\mathbf{x}\right|\le\frac{c_1c_{12}}{2\kappa}\|\Theta\|_{H^2}^2+\frac{\kappa}{2}\|\nabla\Theta_t\|^2,$$

which, together with (3.42), implies that

$$\frac{d}{dt}\|\Theta_t\|^2+\kappa\|\nabla\Theta_t\|^2\le c_{13}\|\Theta\|_{H^2}^2,\tag{3.46}$$

where (by using (3.45))

$$c_{13}=\frac{c_1c_{12}}{\kappa}=O(\kappa^{-11}e^{5\kappa^{-1}})\quad\text{as }\ \kappa\to 0.\tag{3.47}$$

For the RHS of (3.46), using (3.20) we have

$$\frac{d}{dt}\|\Theta_t\|^2+\kappa\|\nabla\Theta_t\|^2\le c_{14}(\|\nabla\Theta\|^2+\|\Theta_t\|^2),\tag{3.48}$$

where (by using (3.19) and (3.47))

$$c_{14} = c_{13} \max\left\{\frac{2c_5}{\kappa}, c_8\right\} = O(\kappa^{-15} e^{7\kappa^{-1}}) \quad \text{as } \kappa \to 0. \tag{3.49}$$

In order to control the RHS of (3.48), we consider the estimate (3.26). Letting $c_{15} = \min\{c_9, 1/2\}$ we get from (3.26) that

$$\frac{d}{dt}\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) + c_{15}(\|\nabla\Theta\|^2 + \|\Theta_t\|^2) \le 0. \tag{3.50}$$

The operation (3.50) $\times$ $(2c_{14}/c_{15})$ + (3.48) then yields

$$\frac{d}{dt}(E(t)) + c_{14}(\|\nabla\Theta\|^2 + \|\Theta_t\|^2) + \kappa\|\nabla\Theta_t\|^2 \le 0, \tag{3.51}$$

where

$$E(t) = \frac{2c_{14}}{c_{15}}\left(\frac{c_9}{\kappa}\|\Theta\|^2 + \frac{\kappa}{2}\|\nabla\Theta\|^2\right) + \|\Theta_t\|^2. \tag{3.52}$$

With the help of Poincaré's inequality it is easy to check that

$$\beta_2 E(t) \le c_{14}(\|\nabla\Theta\|^2 + \|\Theta_t\|^2), \tag{3.53}$$

where (by using definition of c_{15}, (3.25), and (3.49))

$$\beta_2 = \left(\max\left\{\frac{2}{c_{15}}\left(\frac{c_0 c_9}{\kappa} + \frac{\kappa}{2}\right), \frac{1}{c_{14}}\right\}\right)^{-1} = O(\kappa^4 e^{-2\kappa^{-1}}) \quad \text{as } \kappa \to 0. \tag{3.54}$$

Using (3.53) we update (3.51) as

$$\frac{d}{dt}(E(t)) + \beta_2 E(t) + \kappa\|\nabla\Theta_t\|^2 \le 0,$$

which implies that

$$E(t) \le E(0)e^{-\beta_2 t}.$$

By virtue of (3.20) we have

$$\|\Theta(\cdot,t)\|_{H^2}^2 \le \alpha_2 e^{-\beta_2 t} \quad \forall t \ge 0,$$

where (by using (3.19), (3.25), (3.49), and (3.52))

$$\begin{aligned} \alpha_2 &= E(0)\max\left\{\frac{2c_5}{\kappa}, c_8\right\}\left(\min\left\{\frac{\kappa c_{14}}{c_{15}}, 1\right\}\right)^{-1} \\ &= O(\kappa^{-23} e^{11\kappa^{-1}}) \quad \text{as } \kappa \to 0. \end{aligned} \tag{3.55}$$

This completes the proof. □

3.4. H^3 Estimate of Θ

The next lemma is concerned with the decay of $\|\nabla\Theta_t\|^2$, based on which we can prove the decay of $\|\Theta\|_{H^3}^2$.

LEMMA 3.6. *Under the assumptions of Theorem 1.1, there exist constants $\alpha_3 > 0$ and $\beta_3 > 0$ independent of t such that*

$$\|\Theta_t(\cdot,t)\|_{H^1}^2 \le \alpha_3 e^{-\beta_3 t} \quad \forall t \ge 0,$$

where $\alpha_3 = O(\kappa^{-36} e^{17\kappa^{-1}})$ and $\beta_3 = O(\kappa^{18} e^{-9\kappa^{-1}})$ as $\kappa \to 0$.

Proof. Taking the L^2 inner product of (3.41) with Θ_{tt} we have

$$\frac{\kappa}{2}\frac{d}{dt}\|\nabla\Theta_t\|^2+\|\Theta_{tt}\|^2=-\int_\Omega \Theta_{tt}(U_t\cdot\nabla\Theta)\,d\mathbf{x}-\int_\Omega \Theta_{tt}(U\cdot\nabla\Theta_t)\,d\mathbf{x}. \tag{3.56}$$

Using (3.44) and Lemma 2.1(ii), we estimate the first term on the RHS of (3.56) as:

$$\begin{aligned}\left|\int_\Omega \Theta_{tt}(U_t\cdot\nabla\Theta)\,d\mathbf{x}\right| &\le \frac{1}{4}\|\Theta_{tt}\|^2+\|U_t\|^2\|\nabla\Theta\|^2_{L^\infty}\\ &\le \frac{1}{4}\|\Theta_{tt}\|^2+c_{12}c_2\|\Theta\|^2_{W^{2,3}}.\end{aligned} \tag{3.57}$$

To estimate $\|\Theta\|^2_{W^{2,3}}$, we use (3.14) and Lemma 2.2 with $m=0$ and $p=3$ to get

$$\|\Theta\|^2_{W^{2,3}}\le 2c_5(\|\Theta_t\|^2_{L^3}+\|U\cdot\nabla\Theta\|^2_{L^3}). \tag{3.58}$$

Using Lemma 2.1(iii), Lemma 3.2, and (3.20) we have

$$\begin{aligned}\|\Theta_t\|^2_{L^3}+\|U\cdot\nabla\Theta\|^2_{L^3} &\le c_3\|\Theta_t\|^2_{H^1}+\|U\|^2_{L^6}\|\nabla\Theta\|^2_{L^6}\\ &\le c_3\|\Theta_t\|^2_{H^1}+c_3^2\|U\|^2_{H^1}\|\Theta\|^2_{H^2}\\ &\le c_3\|\Theta_t\|^2_{H^1}+c_3^2d_1\|\Theta\|^2_{H^2}\\ &\le c_3\|\Theta_t\|^2_{H^1}+c_3^2d_1\left(\frac{4c_5}{\kappa}\|\Theta_t\|^2+c_8\|\nabla\Theta\|^2\right).\end{aligned}$$

So we update (3.57) as

$$\left|\int_\Omega \Theta_{tt}(U_t\cdot\nabla\Theta)\,d\mathbf{x}\right|\le\frac{1}{4}\|\Theta_{tt}\|^2+c_{16}(\|\Theta_t\|^2_{H^1}+\|\nabla\Theta\|^2), \tag{3.59}$$

where (by using (3.19), (3.45), and Lemma 3.2)

$$c_{16}=2c_2c_3c_5c_{12}\left(1+\frac{4d_1c_3c_5}{\kappa}+d_1c_3c_8\right)=O(\kappa^{-15}e^{8\kappa^{-1}})\quad\text{as }\kappa\to 0. \tag{3.60}$$

For the second term on the RHS of (3.56), by Lemma 3.4 and Lemma 2.1(ii) we have

$$\begin{aligned}\left|\int_\Omega \Theta_{tt}(U\cdot\nabla\Theta_t)\,d\mathbf{x}\right| &\le \frac{1}{4}\|\Theta_{tt}\|^2+\|U\|^2_{L^\infty}\|\nabla\Theta_t\|^2\\ &\le \frac{1}{4}\|\Theta_{tt}\|^2+c_2d_2\|\nabla\Theta_t\|^2.\end{aligned} \tag{3.61}$$

Combining (3.56), (3.59), and (3.61) we have

$$\kappa\frac{d}{dt}\|\nabla\Theta_t\|^2+\|\Theta_{tt}\|^2\le c_{17}(\|\nabla\Theta\|^2+\|\Theta_t\|^2+\|\nabla\Theta_t\|^2), \tag{3.62}$$

where (by using (3.40) and (3.60))

$$c_{17}=2(c_{16}+c_2d_2)=O(\kappa^{-15}e^{8\kappa^{-1}})\quad\text{as }\kappa\to 0. \tag{3.63}$$

Now we consider the estimate (3.51). Letting $c_{18}\equiv\min\{c_{14},\kappa\}$ we have

$$\frac{d}{dt}(E(t))+c_{18}(\|\nabla\Theta\|^2+\|\Theta_t\|^2+\|\nabla\Theta_t\|^2)\le 0. \tag{3.64}$$

The operation (3.64) $\times$ $(2c_{17}/c_{18})$ + (3.62) then yields

$$\frac{d}{dt}\left(\frac{2c_{17}}{c_{18}}E(t)+\kappa\|\nabla\Theta_t\|^2\right)+c_{17}(\|\nabla\Theta\|^2+\|\Theta_t\|^2+\|\nabla\Theta_t\|^2)+\|\Theta_{tt}\|^2\le 0. \tag{3.65}$$

Using (3.52) one easily checks that

$$\beta_3\left(\frac{2c_{17}}{c_{18}}E(t)+\kappa\|\nabla\Theta_t\|^2\right)\le c_{17}(\|\nabla\Theta\|^2+\|\Theta_t\|^2+\|\nabla\Theta_t\|^2), \tag{3.66}$$

where (by using (3.49), (3.54), and (3.63))

$$\beta_3=c_{17}\left(\max\left\{\frac{2c_{14}c_{17}}{\beta_2c_{18}},\kappa\right\}\right)^{-1}=O(\kappa^{18}e^{-9\kappa^{-1}})\quad\text{as }\kappa\to 0. \tag{3.67}$$

Therefore, coupling (3.65) and (3.66) we have

$$\left(\frac{2c_{17}}{c_{18}}E(t)+\kappa\|\nabla\Theta_t\|^2\right)\le\left(\frac{2c_{17}}{c_{18}}E(0)+\kappa\|\nabla\Theta_t(0)\|^2\right)e^{-\beta_3t}.$$

By definition of $E(t)$ we have

$$\|\Theta_t(\cdot,t)\|_{H^1}^2\le\alpha_3e^{-\beta_3t},$$

where

$$\begin{aligned}\alpha_3&=\left(\frac{2c_{17}}{c_{18}}E(0)+\kappa\|\nabla\Theta_t(0)\|^2\right)\left(\min\left\{\frac{2c_{14}c_{17}}{c_{18}\beta_2},\kappa\right\}\right)^{-1}\\&=O(\kappa^{-36}e^{17\kappa^{-1}})\quad\text{as }\kappa\to 0.\end{aligned} \tag{3.68}$$

This completes the proof. □

With the help of Lemmas 3.2–3.6, we are now ready to prove the exponential decay of $\|\Theta\|_{H^3}^2$.

LEMMA 3.7. *Under the assumptions of Theorem 1.1, there exist constants $\gamma>0$ and $\eta>0$ independent of t such that*

$$\|\Theta(\cdot,t)\|_{H^3}^2\le\gamma e^{-\eta t}\quad\forall t\ge 0,$$

where $\gamma=O(\kappa^{-50}e^{24\kappa^{-1}})$ and $\eta=O(\kappa^{18}e^{-9\kappa^{-1}})$ as $\kappa\to 0$.

Proof. First, since $\Theta|_{\partial\Omega}=0$, using (3.14) and Lemma 2.2 with $m=1$ and $p=2$ we have

$$\|\Theta\|_{H^3}^2\le\frac{2c_5}{\kappa}(\|\Theta_t\|_{H^1}^2+\|U\cdot\nabla\Theta\|_{H^1}^2). \tag{3.69}$$

By virtue of Lemma 3.6, it suffices to estimate $\|U\cdot\nabla\Theta\|_{H^1}^2$ in order to prove the lemma. For this purpose, we observe, by Lemma 3.4 and Lemma 2.1(ii), that

$$\begin{aligned}\|(U\cdot\nabla\Theta)(\cdot,t)\|_{H^1}^2&\le\|U\|_{L^\infty}^2\|\Theta\|_{H^2}^2+\|\nabla U\|^2\|\nabla\Theta\|_{L^\infty}^2\\&\le c_2d_2(\|\Theta\|_{H^2}^2+\|\Theta\|_{W^{2,3}}^2).\end{aligned} \tag{3.70}$$

From the derivations of (3.58)–(3.59) we have

$$\|\Theta\|_{W^{2,3}}^2 \le \frac{c_{16}}{c_2 c_{12}}(\|\Theta_t\|_{H^1}^2 + \|\nabla\Theta\|^2). \tag{3.71}$$

Substituting (3.71) into (3.70) we have

$$\|(U\cdot\nabla\Theta)(\cdot,t)\|_{H^1}^2 \le c_{19}(\|\Theta_t\|_{H^1}^2 + \|\Theta\|_{H^2}^2), \tag{3.72}$$

where (by using (3.40), (3.45), and (3.60))

$$c_{19} = c_2 d_2 + \frac{c_{16} d_2}{c_{12}} = O(\kappa^{-14} e^{7\kappa^{-1}}) \quad \text{as } \kappa \to 0. \tag{3.73}$$

Plugging (3.72) into (3.69) we have

$$\|\Theta\|_{H^3}^2 \le \left(\frac{2c_5}{\kappa} + c_{19}\right)(\|\Theta\|_{H^2}^2 + \|\Theta_t\|_{H^1}^2). \tag{3.74}$$

Lemma 3.5 and Lemma 3.6 then imply that

$$\|\Theta\|_{H^3}^2 \le \left(\frac{2c_5}{\kappa} + c_{19}\right)(\alpha_2 e^{-\beta_2 t} + \alpha_3 e^{-\beta_3 t}) \le \gamma e^{-\eta t}, \tag{3.75}$$

where (by using (3.73), Lemma 3.5, and Lemma 3.6)

$$\begin{aligned} \gamma &= \left(\frac{2c_5}{\kappa} + c_{19}\right)\max\{\alpha_2,\alpha_3\} = O(\kappa^{-50} e^{24\kappa^{-1}}), \\ \eta &= \min\{\beta_2,\beta_3\} = O(\kappa^{18} e^{-9\kappa^{-1}}) \quad \text{as } \kappa \to 0. \end{aligned} \tag{3.76}$$

This completes the proof. □

3.5. L^∞ *Estimate of* ω

As a consequence of Lemma 3.7, we show the uniform estimate of $\|\omega\|_{L^\infty}$.

LEMMA 3.8. *Under the assumptions of Theorem 1.1, there exists a constant* $\tilde{c} > 0$ *independent of* t *such that*

$$\|\omega(\cdot,t)\|_{L^\infty} \le \tilde{c} \quad \forall t \ge 0,$$

where $\tilde{c} = O(\kappa^{-43} e^{21\kappa^{-1}})$ *as* $\kappa \to 0$.

Proof. We note from (3.33) that for any $p \ge 2$, it holds that

$$\frac{d}{dt}\|\omega\|_{L^p} \le \|\nabla\Theta\|_{L^p} \le \|\nabla\Theta\|_{L^\infty}|\Omega|^{1/p} \le \|\nabla\Theta\|_{L^\infty}\max\{1,|\Omega|\}. \tag{3.77}$$

By Lemma 2.1(i) and Lemma 3.7 we have

$$\|\nabla\Theta\|_{L^\infty} \le c_1\|\Theta\|_{H^3} \le c_1\gamma^{1/2} e^{-\eta t/2}. \tag{3.78}$$

Plugging (3.78) into (3.77) we have

$$\frac{d}{dt}\|\omega\|_{L^p} \le c_{20} e^{-\eta t/2}, \tag{3.79}$$

where (by using (3.76))

$$c_{20} = \max\{1,|\Omega|\}c_1\gamma^{1/2} = O(\kappa^{-25} e^{12\kappa^{-1}}) \quad \text{as } \kappa \to 0. \tag{3.80}$$

Upon integrating (3.79) in time we have

$$\|\omega(\cdot,t)\|_{L^p} \le \|\omega(\cdot,0)\|_{L^p} + 2c_{20}/\eta. \tag{3.81}$$

We note that the constant $2c_{20}/\eta$ is independent of t and $p \ge 2$. Therefore, letting $p \to \infty$ in (3.81) we have

$$\|\omega(\cdot,t)\|_{L^\infty} \le \tilde{c}, \tag{3.82}$$

where (by using (3.76) and (3.80))

$$\tilde{c} = \|\omega(\cdot,0)\|_{L^\infty} + 2c_{20}/\eta = O(\kappa^{-43}e^{21\kappa^{-1}}) \quad \text{as } \kappa \to 0. \tag{3.83}$$

This completes the proof. □

3.6. H^3 Estimate of U

Now we turn to the regularity of the velocity field.

LEMMA 3.9. *Under the assumptions of Theorem 1.1, there exists a constant $M > 0$ depending on T and other constants indicated in previous lemmas such that*

$$\|U\|^2_{C([0,T];H^3(\Omega))} \le M(T) < \infty \quad \textit{for all } 0 < T < \infty.$$

Proof. First we note that, by Lemma 3.7 and Sobolev embedding,

$$\|\Theta\|^2_{C([0,T];C^{1+\nu}(\bar{\Omega}))} \le c(\Omega)\gamma e^{-\eta t}$$

for some $\nu \in (0,1)$. Therefore, $(2.1)_1$ and Lemma 2.5 with $G = \Theta\mathbf{e}_2$ imply that for any fixed $T > 0$,

$$\|U\|^2_{C([0,T];C^1(\bar{\Omega}))} \le C_4 < \infty. \tag{3.84}$$

By virtue of Lemma 2.3 and Lemma 3.4, it suffices to show the estimate of $\|\omega\|^2_{H^2}$ in order to prove the lemma. We consider the vorticity equation (3.8). For any mixed spatial derivative D^α with $0 \le |\alpha| \le 2$, taking the L^2 inner product of D^α(3.8) with $D^\alpha\omega$ we have

$$\frac{1}{2}\frac{d}{dt}\|D^\alpha\omega\|^2 = -\int_\Omega D^\alpha(U\cdot\nabla\omega)D^\alpha\omega\,d\mathbf{x} - \int_\Omega D^\alpha\Theta_x D^\alpha\omega\,d\mathbf{x}. \tag{3.85}$$

Since $\nabla\cdot U = 0$ and $U\cdot\mathbf{n}|_{\partial\Omega} = 0$, we rewrite the first term on the RHS of (3.85) as

$$\begin{aligned}-\int_\Omega D^\alpha(U\cdot\nabla\omega)D^\alpha\omega\,d\mathbf{x} &= -\int_\Omega D^\alpha\nabla\cdot(U\omega)D^\alpha\omega\,d\mathbf{x}\\ &= -\int_\Omega (D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega)D^\alpha\omega\,d\mathbf{x}. \end{aligned}\tag{3.86}$$

Plugging (3.86) into (3.85) and using Cauchy–Schwartz inequality we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|D^\alpha\omega\|^2\\ &\quad \le \frac{1}{2}\|(D^\alpha\nabla\cdot(U\omega) - U\cdot\nabla D^\alpha\omega)\|^2 + \frac{1}{2}\|D^\alpha\Theta_x\|^2 + \|D^\alpha\omega\|^2. \end{aligned}\tag{3.87}$$

Now, it is easy to see that

$$\|(D^\alpha \nabla \cdot (U\omega) - U \cdot \nabla D^\alpha \omega)\|^2 \le \|\nabla U\|_{L^\infty}^2 \|\omega\|^2 \quad \text{for } |\alpha| = 0 \tag{3.88}$$

and

$$\|(D^\alpha \nabla \cdot (U\omega) - U \cdot \nabla D^\alpha \omega)\|^2 \le \|\nabla U\|_{L^\infty}^2 \|\nabla\omega\|^2 \quad \text{for } |\alpha| = 1. \tag{3.89}$$

For $|\alpha| = 2$, with the help of Lemma 2.4 with $f = U$, $g = \omega$, and $|\beta| = 3$ we obtain

$$\|(D^\alpha \nabla \cdot (U\omega) - U \cdot \nabla D^\alpha \omega)\|^2 \le c_7(\|\nabla U\|_{L^\infty}^2 \|\omega\|_{H^2}^2 + \|U\|_{H^3}^2 \|\omega\|_{L^\infty}^2). \tag{3.90}$$

Combining (3.88)–(3.90) we see that for any multiindex α with $0 \le |\alpha| \le 2$ it holds that

$$\begin{aligned} &\|(D^\alpha \nabla \cdot (U\omega) - U \cdot \nabla D^\alpha \omega)\|^2 \\ &\qquad \le (c_7 + 1)(\|\nabla U\|_{L^\infty}^2 \|\omega\|_{H^2}^2 + \|U\|_{H^3}^2 \|\omega\|_{L^\infty}^2). \end{aligned} \tag{3.91}$$

Plugging (3.91) into (3.87) and using Lemma 2.3 with $s = 3$ and $p = 2$ we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|D^\alpha \omega\|^2 &\le \frac{c_7+1}{2}(\|\nabla U\|_{L^\infty}^2 \|\omega\|_{H^2}^2 + \|U\|_{H^3}^2 \|\omega\|_{L^\infty}^2) \\ &\quad + \|D^\alpha\omega\|^2 + \frac{1}{2}\|D^\alpha \Theta_x\|^2 \\ &\le \frac{c_7+1}{2}(c_6+1)\|\nabla U\|_{L^\infty}^2 (\|\omega\|_{H^2}^2 + \|U\|^2) \\ &\quad + \|D^\alpha\omega\|^2 + \frac{1}{2}\|D^\alpha \Theta_x\|^2. \end{aligned} \tag{3.92}$$

Summing (3.92) over all α with $0 \le |\alpha| \le 2$ and using (3.84) and Lemma 3.7 we have

$$\begin{aligned} \frac{d}{dt}\|\omega\|_{H^2}^2 &\le 2(c_7+1)(c_6+1)\|\nabla U\|_{L^\infty}^2 (\|\omega\|_{H^2}^2 + \|U\|^2) + 2\|\omega\|_{H^2}^2 + \|\Theta\|_{H^3}^2 \\ &\le C_5 \|\omega\|_{H^2}^2 + C_6. \end{aligned}$$

Then Gronwall's inequality implies that

$$\sup_{t\in[0,T]} \|\omega(\cdot,t)\|_{H^2}^2 \le C_7.$$

This completes the proof. □

3.7. H^4 Estimate of Θ

To complete the regularity stated in Theorem 3.1, it remains only to estimate $\|\Theta\|^2_{L^2([0,T];\,H^4(\Omega))}$. The proof is straightforward by using the results obtained in previous lemmas.

LEMMA 3.10. *Under the assumptions of Theorem 1.1, there exists a constant $N > 0$ depending on T and other constants indicated in previous lemmas such that*

$$\|\Theta\|^2_{L^2([0,T];\,H^4(\Omega))} \le N(T) < \infty \quad \textit{for all } 0 < T < \infty.$$

Proof. First, we rewrite equation (3.41) in terms of Θ_t as

$$\kappa\Delta(\Theta_t) = \Theta_{tt} + U_t \cdot \nabla\Theta + U \cdot \nabla\Theta_t. \tag{3.93}$$

Since $\Theta_t|_{\partial\Omega} = 0$, applying Lemma 2.2 to (3.93) we have

$$\|\Theta_t\|_{H^2}^2 \leq \frac{4c_5}{\kappa}(\|\Theta_{tt}\|^2 + \|U_t \cdot \nabla\Theta\|^2 + \|U \cdot \nabla\Theta_t\|^2). \tag{3.94}$$

Using Lemma 2.1, (3.44), Lemma 3.4, and Lemmas 3.6–3.7 we estimate the RHS of (3.94) as follows:

$$\begin{aligned}
&\|\Theta_{tt}\|^2 + \|U_t \cdot \nabla\Theta\|^2 + \|U \cdot \nabla\Theta_t\|^2 \\
&\quad \leq \|\Theta_{tt}\|^2 + \|U_t\|^2\|\nabla\Theta\|_{L^\infty}^2 + \|U\|_{L^\infty}^2\|\nabla\Theta_t\|^2 \\
&\quad \leq \|\Theta_{tt}\|^2 + c_{12}c_1\|\Theta\|_{H^3}^2 + c_2 d_2\|\nabla\Theta_t\|^2 \\
&\quad \leq \|\Theta_{tt}\|^2 + c_{12}c_1\gamma e^{-\eta t} + c_2 d_2\alpha_3 e^{-\beta_3 t}.
\end{aligned}$$

Then (3.94) is updated as

$$\|\Theta_t\|_{H^2}^2 \leq \frac{4c_5}{\kappa}(\|\Theta_{tt}\|^2 + c_{12}c_1\gamma e^{-\eta t} + c_2 d_2\alpha_3 e^{-\beta_3 t}). \tag{3.95}$$

To estimate $\|\Theta_{tt}\|^2$, we recall (3.65). For any $t > 0$, upon integrating (3.65) in time over $[0, t]$ we have

$$\int_0^t \|\Theta_{tt}\|^2\, d\tau \leq \frac{2c_{17}}{c_{18}}E(0) + \kappa\|\nabla\Theta_t(\cdot, 0)\|^2,$$

which, together with (3.95), implies that

$$\|\Theta_t\|_{L^2([0,t];\, H^2(\Omega))}^2 \leq \bar{c}, \tag{3.96}$$

where (by using (3.45), (3.50), (3.74), and Lemmas 3.6–3.7)

$$\begin{aligned}
\bar{c} &= \frac{4c_5}{\kappa}\left(\frac{2c_{17}}{c_{18}}E(0) + \kappa\|\nabla\Theta_t(\cdot, 0)\|^2 + \frac{c_{12}c_1\gamma}{\eta} + \frac{c_2 d_2\alpha_3}{\beta_3}\right) \\
&= O(\kappa^{-78}e^{33\kappa^{-1}}) \quad \text{as } \kappa \to 0.
\end{aligned} \tag{3.97}$$

We note that (3.96) completes the uniform estimates stated in Theorem 3.1. For the H^4 norm of Θ, since $\Theta|_{\partial\Omega} = 0$, Lemma 2.2 with $m = 2$, $p = 2$, and previous estimates imply

$$\begin{aligned}
\|\Theta\|_{H^4}^2 &\leq \frac{2c_5}{\kappa}(\|\Theta_t\|_{H^2}^2 + \|U \cdot \nabla\Theta\|_{H^2}^2) \\
&\leq \frac{2c_5}{\kappa}(\|\Theta_t\|_{H^2}^2 + \|U\|_{L^\infty}^2\|\Theta\|_{H^3}^2 + \|\nabla U\|_{L^\infty}^2\|\Theta\|_{H^2}^2 + \|U\|_{H^2}^2\|\nabla\Theta\|_{L^\infty}^2) \\
&\leq \frac{2c_5}{\kappa}(\|\Theta_t\|_{H^2}^2 + C_8).
\end{aligned} \tag{3.98}$$

Therefore, we conclude the proof by combining (3.96) and (3.98). □

3.8. Uniqueness

Lemmas 3.7–3.10 conclude Theorem 3.1. It remains to show uniqueness of the solution in order to complete the proof of Theorem 1.1.

THEOREM 3.2. *Under the assumptions of Theorem 1.1, the solution of* (2.1)–(2.2) *is unique.*

Proof. For any fixed $T > 0$, suppose there are two solutions $(\Theta_1, U_1, \bar{P}_1)$ and $(\Theta_2, U_2, \bar{P}_2)$ to (2.1)–(2.2). Setting $\tilde{\Theta} = \Theta_1 - \Theta_2$, $\tilde{U} = U_1 - U_2$, and $\tilde{P} = \bar{P}_1 - \bar{P}_2$, then $(\tilde{\Theta}, \tilde{U}, \tilde{P})$ satisfy

$$\begin{cases} \tilde{U}_t + U_1 \cdot \nabla \tilde{U} + \tilde{U} \cdot \nabla U_2 + \nabla \tilde{P} = \tilde{\Theta}(0,1)^{\mathrm{T}}, \\ \tilde{\Theta}_t + U_1 \cdot \nabla \tilde{\Theta} + \tilde{U} \cdot \nabla \Theta_2 = \kappa \Delta \tilde{\Theta}, \\ \nabla \cdot \tilde{U} = 0, \\ \tilde{U}(\mathbf{x}, 0) = 0, \quad \tilde{\Theta}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in \Omega, \\ \tilde{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \tilde{\Theta}|_{\partial\Omega} = 0. \end{cases} \tag{3.99}$$

Taking the L^2 inner products of $(3.99)_1$ with $\tilde{U}$ and $(3.99)_2$ with $\tilde{\Theta}$ respectively we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2) + \kappa \|\nabla \tilde{\Theta}\|^2 \\ &\qquad = -\int_\Omega \tilde{\Theta}(\tilde{U} \cdot \nabla \Theta_2)\, d\mathbf{x} - \int_\Omega \tilde{U} \cdot (\tilde{U} \cdot \nabla U_2)\, d\mathbf{x} + \int_\Omega \tilde{\Theta} \tilde{v}\, d\mathbf{x}. \end{aligned}$$

Using the estimates for Θ_2 and U_2, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2) + \kappa \|\nabla \tilde{\Theta}\|^2 \\ &\quad \le \|\nabla \Theta_2\|_{L^\infty}(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2) + \|\nabla U_2\|_{L^\infty}\|\tilde{U}\|^2 + \frac{1}{2}(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2) \\ &\quad \le C_9(\|\tilde{\Theta}\|^2 + \|\tilde{U}\|^2) \quad \forall t \in [0, T], \end{aligned}$$

which implies that

$$(\|\tilde{\Theta}(\cdot, t)\|^2 + \|\tilde{U}(\cdot, t)\|^2) \le e^{-2C_9 t}(\|\tilde{\Theta}(0)\|^2 + \|\tilde{U}(0)\|^2) = 0$$

for any $t \in [0, T]$. We conclude the theorem by noticing that $T > 0$ is arbitrary. □

REMARK 3.2. Using the ideas of this paper, one can study the IBVP for (1.1) with the Neumann boundary condition on θ $\big($i.e., $\frac{\partial \theta}{\partial \mathbf{n}}\big|_{\partial\Omega} = 0\big)$. In this case, owing to the conservation of total mass, the asymptotic state of θ is $\hat{\theta} = \frac{1}{|\Omega|}\int_\Omega \theta_0(\mathbf{x})\, d\mathbf{x}$. Similar results as in Theorem 1.1 hold in this case. We omit the details here.

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Mathematical Biosciences Institute
Ohio State University
Columbus, OH 43210

kzhao@mbi.osu.edu