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Effective Base Point Free Theorem for Log Canonical Pairs, II. Angehrn–Siu Type Theorems

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1. Introduction

The main purpose of this paper is to advertise the power of the new cohomological technique introduced in [Am]. By this new method, we generalize Angehrn–Siu type effective base point freeness and point separation (see [AS] and [Ko, 5.8 and 5.9]) for *log canonical* pairs. Here, we adopt Kollár's formulation in [Ko] because it is suitable for singular varieties. The main ingredients of our proof are the inversion of adjunction on log canonicity (see [Ka]) and the new cohomological technique (see [Am]). For the Kollár type effective freeness for log canonical pairs, see [F5]. In [F4], we give a simple new proof of the base point free theorem for log canonical pairs. It is closely related to the arguments in this paper.

The following theorems are the main theorems of this paper.

THEOREM 1.1 (Effective Freeness; cf. [Ko, Thm. 5.8]). *Let (X, Δ) be a projective log canonical pair and M a line bundle on X. Assume that $M \equiv K_X + \Delta + N$, where N is an ample $\mathbb{Q}$-divisor on X. Let $x \in X$ be a closed point and assume that there are positive numbers $c(k)$ with the following properties.*

(1) *If $x \in Z \subset X$ is an irreducible (positive dimensional) subvariety, then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

(2) *The numbers $c(k)$ satisfy the inequality*

$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \leq 1.$$

Then M has a global section not vanishing at x.

THEOREM 1.2 (Effective Point Separation; cf. [Ko, Thm. 5.9]). *Let (X, Δ) be a projective log canonical pair and M a line bundle on X. Assume that $M \equiv K_X + \Delta + N$, where N is an ample $\mathbb{Q}$-divisor on X. Let $x_1, x_2 \in X$ be closed points and assume that there are positive numbers $c(k)$ with the following properties.*

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(1) *If $Z \subset X$ is an irreducible (positive dimensional) subvariety that contains x_1 or x_2, then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

(2) *The numbers $c(k)$ satisfy the inequality*

$$\sum_{k=1}^{\dim X} 2^{1/k} \frac{k}{c(k)} \leq 1.$$

Then global sections of M separate x_1 and x_2.

To strengthen these two theorems, we introduce the following definition.

DEFINITION 1.3. Let X be a normal variety and B an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $x \in X$ be a closed point. If (X, B) is Kawamata log terminal at x, then we put $\mu(x, X, B) = \dim X$. When (X, B) is log canonical (lc) but not Kawamata log terminal at x, we define

$$\mu(x, X, B) = \min\{\dim W \mid W \text{ is an lc center of } (X, B) \text{ such that } x \in W\}.$$

If (X, B) is not log canonical at x, then we do not define $\mu(x, X, B)$ for such x. For the details of lc centers, see Theorem 3.1.

In Remark 1.4 (resp. Remark 1.5), we discuss a slight generalization of Theorem 1.1 (resp. Theorem 1.2).

REMARK 1.4. In Theorem 1.1, if $\mu = \mu(x, X, \Delta) < \dim X$ and W is the minimal lc center of (X, B) with $x \in W$, then we can weaken the assumptions as follows. If $Z \subset W$ is an irreducible (positive dimensional) subvariety that contains x, then

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}$$

and the numbers $c(k)$ satisfy the inequality

$$\sum_{k=1}^{\mu} \frac{k}{c(k)} \leq 1.$$

In particular, if $\mu(x, X, \Delta) = 0$, then we need no assumptions on $c(k)$.

REMARK 1.5. In Theorem 1.2, we put $\mu_1 = \mu(x_1, X, \Delta)$ and $\mu_2 = \mu(x_2, X, \Delta)$. Possibly after switching x_1 and x_2, we can assume that $\mu_1 \leq \mu_2$. Let W_1 (resp. W_2) be the minimal lc center of (X, Δ) such that $x_1 \in W_1$ (resp. $x_2 \in W_2$) when $\mu_1 < \dim X$ (resp. $\mu_2 < \dim X$). Otherwise, we put $W_1 = X$ (resp. $W_2 = X$).

(1) If $W_1 \not\subset W_2$, then the assumptions in Theorem 1.2 can be replaced as follows. If $x_1 \in Z \subset W_1$ is an irreducible (positive dimensional) subvariety, then

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}$$

and the numbers $c(k)$ satisfy the inequality

$$\sum_{k=1}^{\mu_1} \frac{k}{c(k)} \leq 1.$$

(2) If $W_1 \subsetneq W_2$, then the assumptions in Theorem 1.2 can be replaced as follows. If $x_2 \in Z \subset W_2$ is an irreducible (positive dimensional) subvariety, then

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}$$

and the numbers $c(k)$ satisfy the inequality

$$\sum_{k=1}^{\mu_2} \frac{k}{c(k)} \leq 1.$$

(3) If $W_1 = W_2$, then we can weaken the assumptions in Theorem 1.2 as follows. If $Z \subset W_1 = W_2$ is an irreducible (positive dimensional) subvariety that contains x_1 or x_2, then

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}$$

and the numbers $c(k)$ satisfy the inequality

$$\sum_{k=1}^{\mu} 2^{1/k} \frac{k}{c(k)} \leq 1,$$

where $\mu = \mu_1 = \mu_2$.

1.6. Let us quickly review the usual technique for base point free theorems via multiplier ideal sheaves (cf. [AS; Ko]). Let (X, Δ) be a projective log canonical pair and M a line bundle on X. Assume that $M \equiv K_X + \Delta + N$, where N is an ample $\mathbb{Q}$-divisor on X. Let $x \in X$ be a closed point. Assume that (X, Δ) is Kawamata log terminal (klt) around x. In this case, it is sufficient to find an effective $\mathbb{Q}$-divisor E on X such that $E \equiv cN$ for $0 < c < 1$, $(X, \Delta + E)$ is log canonical around x, and x is an isolated non-klt locus of $(X, \Delta + E)$. Once we obtain E, we have $H^1(X, M \otimes \mathcal{J}(X, \Delta + E)) = 0$ by the Kawamata–Viehweg–Nadel vanishing theorem and $M - (K_X + \Delta + E) \equiv (1 - c)N$, where $\mathcal{J}(X, \Delta + E)$ is the multiplier ideal sheaf associated to $(X, \Delta + E)$. Therefore, the restriction map $H^0(X, M) \to H^0(X, M \otimes \mathcal{O}_X/\mathcal{J}(X, \Delta + E))$ is surjective. By the choice of E, x is isolated in $\mathrm{Supp}(\mathcal{O}_X/\mathcal{J}(X, \Delta + E))$. So, we can obtain a section of M that does not vanish at x. To construct E, we need the Ohsawa–Takegoshi L^2-extension theorem or the inversion of adjunction. For the details, see [AS] and [Ko].

From now on, we assume that (X, Δ) is log canonical but not Kawamata log terminal at x. When x is an isolated non-klt locus of (X, Δ), we can apply the preceding arguments. However, in general, x is not isolated in $\mathrm{Supp}(\mathcal{O}_X/\mathcal{J}(X, \Delta))$. So, we cannot directly use the techniques in [AS] and [Ko]. Fortunately, by using the new framework introduced in [Am], we know that it is sufficient to find an effective $\mathbb{Q}$-divisor E on X such that $E \equiv cN$ for $0 < c < 1$, $(X, \Delta + E)$ is log canonical around x, and x is an lc center of $(X, \Delta + E)$. This is because we can prove that the restriction map $H^0(X, M) \to \mathbb{C}(x) \otimes M$ is surjective once we obtain such E (see Theorem 3.2). We note that the inversion of adjunction on log canonicity plays a crucial role when we construct E.

We summarize the contents of this paper. In Section 2, we will explain the proof of Theorem 1.1 and 1.2. It is essentially the same as Kollár's proof in [Ko, Sec. 6] if we adopt the new cohomological technique and the inversion of adjunction on log canonicity. So, we will omit some details in Section 2. In Section 3, which is an appendix, we collect some basic properties of lc centers and the new cohomological technique for the reader's convenience since they are not popular yet.

I hope that this paper and [F5] will motivate the reader to study the new cohomological technique. For a systematic and thorough treatment on this topic, that is, the new cohomological technique and the theory of quasi-log varieties, we recommend that the reader see [F3].

NOTATION. We will work over the complex number field $\mathbb{C}$ throughout this paper. *Numerical equivalence* of line bundles and $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors is denoted by $\equiv$. *Linear equivalence* of Cartier divisors is denoted by $\sim$. Let X be a normal variety and B an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Then we can define the discrepancy $a(E, X, B) \in \mathbb{Q}$ for every prime divisor E over X. If $a(E, X, B) \geq -1$ (resp. > -1) for every E, then (X, B) is called *log canonical* (resp. *Kawamata log terminal*). Note that there always exists the maximal Zariski open set U of X such that (X, B) is log canonical on U. If E is a prime divisor over X such that $a(E, X, B) = -1$ and the closure of the image of E on X, which is denoted by $c_X(E)$ and called the *center* of E on X, is not contained in $X \setminus U$, then $c_X(E)$ is called a *center of log canonical singularities* or *log canonical center* of (X, B). Let $x \in X$ be a closed point. Assume that (X, B) is log canonical at x but not Kawamata log terminal. Then there is a unique minimal log canonical center W passing through x, and W is normal at x (see Theorem 3.1 in the Appendix).

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2. Proof of the Main Theorem

The main results of this section are the following propositions.

PROPOSITION 2.1 (cf. [Ko, Thm. 6.4]). *Let (X, Δ) be a projective log canonical pair and N an ample $\mathbb{Q}$-divisor on X. Let $x \in X$ be a closed point and $c(k)$ positive numbers such that if $x \in Z \subset X$ is an irreducible* (*positive dimensional*) *subvariety then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

Assume that

$$\sum_{k=1}^{\dim X} \frac{k}{c(k)} \leq 1.$$

Then there is an effective $\mathbb{Q}$-divisor $D \equiv cN$ with $0 < c < 1$ and an open neighborhood $x \in X^0 \subset X$ such that

(1) $(X^0, \Delta + D)$ *is log canonical and*
(2) x *is a center of log canonical singularities for the pair* $(X, \Delta + D)$.

REMARK 2.2. In Proposition 2.1, the assumptions on Z and $c(k)$ can be replaced as in Remark 1.4.

PROPOSITION 2.3 (cf. [Ko, Thm. 6.5]). *Let* (X, Δ) *be a projective log canonical pair and* N *an ample* $\mathbb{Q}$*-divisor on* X. *Let* $x_1, x_2 \in X$ *be closed points and* $c(k)$ *positive numbers such that if* $Z \subset X$ *is an irreducible (positive dimensional) subvariety such that* $x_1 \in Z$ *or* $x_2 \in Z$ *then*

$$(N^{\dim Z} \cdot Z) > c(\dim Z)^{\dim Z}.$$

Assume also that

$$\sum_{k=1}^{\dim X} 2^{1/k} \frac{k}{c(k)} \leq 1.$$

Then, possibly after switching x_1 *and* x_2, *one can take an effective* $\mathbb{Q}$*-divisor* $D \equiv cN$ *with* $0 < c < 1$ *and an open neighborhood* $x_1 \in X^0 \subset X$ *such that*
(1) $(X^0, \Delta + D)$ *is log canonical,*
(2) x_1 *is a center of log canonical singularities for the pair* $(X, \Delta + D)$, *and*
(3) $(X, \Delta + D)$ *is not log canonical at* x_2.

REMARK 2.4. In Proposition 2.3, the assumptions on Z and $c(k)$ can be replaced as in Remark 1.5.

First, we give a proof of Theorem 1.1 by using Proposition 2.1.

Proof of Theorem 1.1. We consider the pair $(X, \Delta + D)$ constructed in Proposition 2.1. It is not necessarily log canonical but has a natural quasi-log structure (see Theorem 3.2). Let $X \setminus X_{-\infty}$ be the maximal Zariski open set where $(X, \Delta + D)$ is log canonical. Since $(X, \Delta + D)$ is log canonical around x and x is an lc center of $(X, \Delta + D)$, the union of x and $X_{-\infty}$ has a natural quasi-log structure X' induced by the quasi-log structure of $(X, \Delta + D)$ (see Theorem 3.2). We consider the following short exact sequence:

$$0 \to \mathcal{I}_{X'} \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0.$$

Since $M - (K_X + \Delta + D) \equiv (1 - c)N$ is ample, $H^1(X, \mathcal{I}_{X'} \otimes M) = 0$ by the vanishing theorem (see Theorem 3.2). Therefore, $H^0(X, M) \to H^0(X', M)$ is surjective. We note that x is isolated in X' because $x \notin X_{-\infty}$. Thus, the evaluation map $H^0(X, M) \to M \otimes \mathbb{C}(x)$ is surjective. This is what we wanted. □

Next, we give a proof of Theorem 1.2 by Proposition 2.3.

Proof of Theorem 1.2. We use the same notation as in the proof of Theorem 1.1. In this case, x_2 is on $X_{-\infty}$. In particular, x_2 is a point of X'. Since $H^0(X, M) \to H^0(X', M)$ is surjective and x_1 is an isolated point of X', we can take a section of M that separates x_1 and x_2. □

Therefore, all we have to prove are Propositions 2.1 and 2.3. Let us recall the following easy but important result. Here, we need the inversion of adjunction on log canonicity.

PROPOSITION 2.5 (cf. [Ko, Thm. 6.7.1]). *Let (X, Δ) be a projective log canonical pair with* $\dim X = n$ *and $x \in X$ a closed point. Let H be an ample $\mathbb{Q}$-divisor on X such that $(H^n) > n^n$. Then there is an effective $\mathbb{Q}$-divisor $B_x \equiv H$ such that $(X, \Delta + B_x)$ is not log canonical at x.*

Proof. If we adopt Lemma 2.6, then the proof of [Ko, Thm. 6.7.1] works without any changes. □

LEMMA 2.6 (cf. [Ko, Cor. 7.8]). *Let (X, Δ) be a log canonical pair and $B_c : c \in C$ an algebraic family of $\mathbb{Q}$-divisors on X parameterized by a smooth pointed curve $0 \in C$. Assume that $(X, \Delta + B_0)$ is log canonical at $x \in X$. Then there is a Euclidean open neighborhood $x \in W \subset X$ such that $(X, \Delta + B_c)$ is log canonical on W for $c \in C$ near zero.*

Lemma 2.6 is a direct consequence of Kawakita's inversion of adjunction on log canonicity (see [Ka, Thm.]). The next proposition is a reformulation of [Ko, Thm. 6.8.1]. In Kollár's proof in [Ko, Sec. 6], he cuts down the non-klt locus. On the other hand, we cut down the minimal lc center passing through x. The advantage of our method is in the fact that the minimal lc center is always *irreducible* by its definition. So, we do not need to use *tie breaking technique* (see [Ko, 6.9, Step 3]) to make the non-klt locus irreducible even when (X, Δ) is Kawamata log terminal.

PROPOSITION 2.7 (cf. [Ko, Thm. 6.8.1]). *Let (X, Δ) be a projective log canonical pair and $x \in X$ a closed point. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $(X, \Delta + D)$ is log canonical in a neighborhood of x. Assume that Z is the minimal lc center of $(X, \Delta + D)$ such that $x \in Z$ with $k =$* $\dim Z > 0$. *Let H be an ample $\mathbb{Q}$-divisor on X such that $(H^k \cdot Z) > k^k$. Then there are an effective $\mathbb{Q}$-divisor $B \equiv H$ and a rational number* $0 < c < 1$ *such that*

(1) *$(X, \Delta + D + cB)$ is log canonical in a neighborhood of x and*
(2) *there is a minimal lc center Z_1 of $(X, \Delta + D + cB)$ such that $x \in Z_1$ and* $\dim Z_1 < \dim Z$.

Proof. By the assumption, there are a projective birational morphism $f : Y \to X$ and a divisor $E \subset Y$ such that $a(E, X, \Delta + D) = -1$ and $f(E) = Z$. We write $K_Y = f^*(K_X + \Delta + D) + \sum e_i E_i$ where $E = E_1$ and so $e_1 = -1$. Let $Z^0 \subset Z$ be an open subset such that:

(1) Z^0 is smooth and $f|_E : E \to Z$ is smooth over Z^0; and
(2) if $z \in Z^0$, then $(f|_E)^{-1}(z) \not\subset E_i$ for $i \neq 1$.

LEMMA 2.8 (cf. [Ko, Claim 6.8.3]). *With notation as before, choose $m \gg 1$ such that mH is Cartier. Let U be the Zariski open subset of X where $(X, \Delta + D)$ is log canonical. Then, for every $z \in Z^0$, the following assertions hold.*

(1) *There is a divisor* $F_z \sim mH|_Z$ *such that* $\operatorname{mult}_z F_z > mk$.
(2) $\mathcal{O}_X(mH) \otimes I_Z$ *is generated by global sections and*

$$H^1(X, \mathcal{O}_X(mH) \otimes I_Z) = 0.$$

In particular, $H^0(X, \mathcal{O}_X(mH)) \to H^0(Z, \mathcal{O}_Z(mH|_Z))$ *is surjective.*
(3) *For any* $F \sim mH|_Z$*, there is an* $F^X \sim mH$ *such that* $F^X|_Z = F$ *and* $(X, \Delta + D + (1/m)F^X)$ *is log canonical on* $U \setminus Z$.
(4) *Let* $F_z^X \sim mH$ *be such that* $F_z^X|_Z = F_z$*. Then* $(X, \Delta + D + (1/m)F_z^X)$ *is not log canonical at* z.

The proof of Lemma 2.8 is the same as the proof of [Ko, Claim 6.8.3]. So, we omit it here. Pick $z_0 \in Z$ arbitrary. Let $0 \in C$ be a smooth affine curve and $g\colon C \to Z$ a morphism such that $z_0 = g(0)$ and $g(c) \in Z^0$ for general $c \in C$. For general $c \in C$, pick $F_c := F_{g(c)}$ as in Lemma 2.8(1). Let $F_0 = \lim_{c \to 0} F_c$. Then we obtain the following lemma. For the precise meaning of $\lim_{c \to 0} F_c$, see the proof of [Ko, Thm. 6.7.1].

LEMMA 2.9 (cf. [Ko, Claim 6.8.4]). *With notation as before, there exists a divisor* $F_0^X \in |mH|$ *such that*

(1) $F_0^X|_Z = F_0$,
(2) $(X, \Delta + D + (1/m)F_0^X)$ *is log canonical on* $U \setminus Z$,
(3) $(X, \Delta + D + (1/m)F_0^X)$ *is log canonical at the generic point of* Z, *and*
(4) $(X, \Delta + D + (1/m)F_0^X)$ *is not log canonical at* z_0.

The proof of Lemma 2.9 is the same as the proof of [Ko, Claim 6.8.4] if we adopt Lemma 2.6. To finish the proof of Proposition 2.7, we set $B = (1/m)F_0^X$. Let c be the maximal value such that $(X, \Delta + D + cB)$ is log canonical at x. Then we have the desired properties. □

Proof of Proposition 2.1. Without loss of generality, we can assume that $c(k) \in \mathbb{Q}$ for every k. If (X, Δ) is Kawamata log terminal around x, then we put $Z_1 = X$. Otherwise, let Z_1 be the minimal lc center of (X, Δ) such that $x \in Z_1$. If $\dim Z_1 = k_1 > 0$, then we can find $x \in D_1 \equiv \frac{k_1}{c(k_1)} N$ and $0 < c_1 < 1$ such that $(X, \Delta + c_1 D_1)$ is log canonical around x and $k_2 = \dim Z_2 < k_1$, where Z_2 is the minimal lc center of $(X, \Delta + c_1 D_1)$ with $x \in Z_2$ (see Proposition 2.7). Repeat this process. Then we can find $n \geq \mu_1 = k_1 > k_2 > \cdots > k_l > 0$, where $k_i \in \mathbb{Z}$, with the following properties:

(1) there exists an effective $\mathbb{Q}$-divisor D_i such that $D_i \equiv \frac{k_i}{c(k_i)} N$ for every i,
(2) there exists a rational number c_i with $0 < c_i < 1$ for every i,
(3) $(X, \Delta + \sum_{i=1}^{l} c_i D_i)$ is log canonical around x, and
(4) x is an lc center of the pair $(X, \Delta + \sum_{i=1}^{l} c_i D_i)$.

We put $D = \sum_{i=1}^{l} c_i D_i$. Then D has the desired properties. We note that $0 < c = \sum_{i=1}^{l} c_i \frac{k_i}{c(k_i)} < 1$ and $D \equiv cN$. □

From now on, we consider the proof of Proposition 2.3. We can prove Proposition 2.10 by the same way as Proposition 2.5.

PROPOSITION 2.10. *Let* (X, Δ) *be a projective log canonical pair with* $\dim X = n$ *and* $x, x' \in X$ *closed points. Let* H *be an ample* $\mathbb{Q}$*-divisor on* X *such that* $(H^n) > 2n^n$*. Then there is an effective* $\mathbb{Q}$*-divisor* $B_{x,x'} \equiv H$ *such that* $(X, \Delta + B_{x,x'})$ *is not log canonical at* x *and* x'*.*

By Proposition 2.10, we can modify Proposition 2.7 as follows. We leave the details as an exercise for the reader.

PROPOSITION 2.11. *Let* (X, Δ) *be a projective log canonical pair and* $x, x' \in X$ *closed points. Let* D *be an effective* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* X *such that* $(X, \Delta + D)$ *is log canonical in a neighborhood of* x *and* x'*. Assume that there exists a minimal lc center* Z *of* $(X, \Delta + D)$ *such that* $x, x' \in Z$ *with* $k = \dim Z > 0$*. Let* H *be an ample* $\mathbb{Q}$*-divisor on* X *such that* $(H^k \cdot Z) > 2k^k$*. Then there are an effective* $\mathbb{Q}$*-divisor* $B \equiv H$ *and a rational number* $0 < c < 1$ *such that*:

(1) $(X, \Delta + D + cB)$ *is not Kawamata log terminal at the points* x *and* x' *and is log canonical at one of them, say at* x; *and*
(2) *there is a minimal lc center* Z_1 *of* $(X, \Delta + D + cB)$ *such that* $x \in Z_1$ *and* $\dim Z_1 < \dim Z$.

The next proposition is very easy.

PROPOSITION 2.12. *Let* (X, Δ) *be a projective log canonical pair and* $x, x' \in X$ *closed points. Let* D *be an effective* $\mathbb{Q}$*-Cartier* $\mathbb{Q}$*-divisor on* X *such that* $(X, \Delta + D)$ *is log canonical in a neighborhood of* x *and* x'*. Assume that the minimal lc center* $Z \ni x'$ *of* $(X, \Delta + D)$ *does not contain* x*, that is,* $x \notin Z$*. Let* H *be an ample* $\mathbb{Q}$*-divisor on* X*. Then there is an effective* $\mathbb{Q}$*-divisor* $B \equiv H$ *such that* $x \notin B$ *and* $(X, \Delta + \varepsilon B)$ *is not log canonical at* x' *for every* $\varepsilon > 0$.

Proof. We take a general member A of $H^0(X, \mathcal{O}_X(lH) \otimes I_Z)$, where l is sufficiently large and divisible. Note that H is an ample $\mathbb{Q}$-divisor. We put $B = (1/l)A$. Then $x \notin B$ and $(X, \Delta + D + \varepsilon B)$ is not log canonical at x' for every $\varepsilon > 0$. □

Let us start the proof of Proposition 2.3.

Proof of Proposition 2.3. We use the notation in Remark 1.5. Let W_1 (resp. W_2) be the minimal lc center of (X, Δ) such that $x_1 \in W_1$ (resp. $x_2 \in W_2$) when $\mu_1 = \mu(x_1, X, \Delta) < \dim X$ (resp. $\mu_2 = \mu(x_2, X, \Delta) < \dim X$). Otherwise, we put $W_1 = X$ (resp. $W_2 = X$). Possibly after switching x_1 and x_2, we can assume that $\mu_1 \leq \mu_2$. Without loss of generality, we can assume that $c(k) \in \mathbb{Q}$ for every k.

Case 1. If $W_1 \not\subset W_2$, then we can see that $x_1 \notin W_2$ (see Theorem 3.1(2)). By Proposition 2.12, we can find an effective $\mathbb{Q}$-divisor $B \equiv N$ such that $x_1 \notin B$ and $(X, \Delta + \varepsilon B)$ is not log canonical at x_2 for every $\varepsilon > 0$. In this case, we cut down the minimal lc center passing through x_1 as in the proof of Proposition 2.1 by using Proposition 2.7. Then we obtain a $\mathbb{Q}$-divisor D on X satisfying conditions (1), (2), and (3) in Proposition 2.3.

Case 2. If $W_1 \subsetneq W_2$, then $x_2 \notin W_1$. Thus, the proof in Case 1 works after switching x_1 and x_2.

Case 3. We assume $W_1 = W_2$. If $W_1 = W_2 = X$, then we apply Proposition 2.10. Otherwise, we use Proposition 2.11 and cut down $W_1 = W_2$. Then, we obtain $(X, \Delta + G)$ such that:

(1) G is an effective $\mathbb{Q}$-divisor on X and a rational number d such that $G \equiv d\frac{2^{1/k}k}{c(k)}N$ with $0 < d < 1$, where $k = \dim W_1$;
(2) $(X, \Delta + G)$ is not Kawamata log terminal at the points x_1 and x_2 and is log canonical at one of them, say at x_1; and
(3) there is a minimal lc center W_1' of $(X, \Delta + G)$ such that $x \in W_1'$ and $\dim W_1' < \dim W_1$.

If $(X, \Delta + G)$ is log canonical at both the points x_1 and x_2, and if x_1 and x_2 stay on the same new minimal lc center of $(X, \Delta + G)$, then we apply Proposition 2.11 again. By repeating this process, we obtain the situation where there is a suitable effective $\mathbb{Q}$-divisor G' on X such that $(X, \Delta + G')$ is not log canonical at one of x_1 and x_2, or x_1 and x_2 are on different minimal lc centers of the pair $(X, \Delta + G')$. Then, we can apply the same arguments as in Case 1 and Case 2.

Thus, we finish the proof. □

3. Appendix

In this appendix, we collect some basic properties of lc centers and the new cohomologial technique (cf. [Am]). Here, we do not explain the definition of quasi-log varieties because it is very difficult to grasp. We think that Theorem 3.2 is sufficient for our purpose in this paper. We recommend the reader interested in the theory of quasi-log varieties to see [F1].

Throughout this appendix, X is a normal variety and B is an effective $\mathbb{Q}$-divisor on X such that $K_X + B$ is $\mathbb{Q}$-Cartier.

Theorem 3.1 (cf. [Am, Prop. 4.8]). *Assume that (X, B) is log canonical. Then we have the following properties.*

(1) *(X, B) has at most finitely many lc centers.*
(2) *An intersection of two lc centers is a union of lc centers.*
(3) *Any union of lc centers of (X, B) is semi-normal.*
(4) *Let $x \in X$ be a closed point such that (X, B) is log canonical but not Kawamata log terminal at x. Then there is a unique minimal lc center W_x passing through x, and W_x is normal at x.*

The next theorem is one of the most important results in the theory of quasi-log varieties (see [Am, Thm. 4.4]).

Theorem 3.2. *The pair (X, B) has a natural quasi-log structure. Let $X \setminus X_{-\infty}$ be the maximal Zariski open set where (X, B) is log canonical. Let X' be the union of $X_{-\infty}$ and some lc centers of (X, B). Then X' has a natural quasi-log* *structure, which is induced by the quasi-log structure of* (X, B). *We consider the short exact sequence*

$$0 \to \mathcal{I}_{X'} \to \mathcal{O}_X \to \mathcal{O}_{X'} \to 0,$$

where $\mathcal{I}_{X'}$ *is the defining ideal sheaf of* X' *on* X. *Note that* X' *is reduced on* $X \setminus X_{-\infty}$. *Assume that* X *is projective. Let* L *be a line bundle on* X *such that* $L - (K_X + B)$ *is ample. Then* $H^i(X, \mathcal{I}_{X'} \otimes L) = 0$ *for all* $i > 0$. *In particular, the restriction map*

$$H^0(X, L) \to H^0(X', L|_{X'})$$

is surjective.

For the proofs of Theorem 3.1 and Theorem 3.2, see [Am] and [F3, Sec. 3.2]. We close this appendix with an important remark.

REMARK 3.3. The vanishing and torsion-free theorems required for the proofs of Theorem 3.1 and Theorem 3.2 can be proved easily by investigating mixed Hodge structures on compact support cohomology groups of smooth varieties. Therefore, [F2] is sufficient for our purposes. We do not need [F3, Chap. 2], which is very technical and seems to be inaccessible to non-experts. See also [F6, Secs. 5, 6].

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