Michigan Math. J. 59 (2010), 297–302

The Automorphism Group of the Free Group of Rank 2 Is a CAT(0) Group

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1. Introduction

A *CAT(0) metric space* is a proper complete geodesic metric space in which each geodesic triangle with side lengths a, b, and c is "at least as thin" as the Euclidean triangle with side lengths a, b, and c (see [5] for details). We say that a finitely generated group G is a *CAT(0) group* if G may be realized as a cocompact and properly discontinuous subgroup of the isometry group of a CAT(0) metric space X. Equivalently, G is a CAT(0) group if there exists a CAT(0) metric space X and a faithful geometric action of G on X. It is perhaps not standard to require that the group action be faithful, a point we address in Remark 1.

For each integer $n \geq 2$, we write F_n for the free group of rank n and B_n for the braid group on n strands.

In [3], Brady exhibited a subgroup $H \leq \mathrm{Aut}(F_2)$ of index 24 that acts faithfully and geometrically on a CAT(0) 2-complex. In subsequent work [4], the same author showed that B_4 acts faithfully and geometrically on a CAT(0) 3-complex. It follows that $\mathrm{Inn}(B_4)$ acts faithfully and geometrically on a CAT(0) 2-complex X_0 (this fact is explained explicitly by Crisp and Paoluzzi in [8]). Now, $\mathrm{Inn}(B_n)$ has index 2 in $\mathrm{Aut}(B_n)$ [10], and $\mathrm{Aut}(F_2)$ is isomorphic to $\mathrm{Aut}(B_4)$ [16, 10]; thus the result in the title of this paper is proved if we exhibit an extra isometry of X_0 that extends the faithful geometric action of $\mathrm{Inn}(B_4)$ to a faithful geometric action of $\mathrm{Aut}(B_4)$. We do this in Section 2.

In the language of [14], X_0 is a systolic simplicial complex. By [14, Thm. 13.1], a group that acts simplicially, properly discontinuously, and cocompactly on such a space is biautomatic. Since the action of $\mathrm{Aut}(F_2)$ provided here is of this type, it follows that $\mathrm{Aut}(F_2)$ is biautomatic.

Our results reinforce the striking contrast between those properties enjoyed by $\mathrm{Aut}(F_2)$ and those enjoyed by the automorphism groups of finitely generated free groups of higher ranks. We can now say that $\mathrm{Aut}(F_2)$ is a CAT(0) group, that it is a biautomatic group, and that it has a faithful linear representation [9; 16]; while $\mathrm{Aut}(F_n)$ is neither a CAT(0) group [12] nor a biautomatic group [6], and it does not have a faithful linear representation [11] whenever $n \geq 3$.

Received November 26, 2008. Revision received February 25, 2009.
The third-named author was supported by N.S.F. Grant no. DMS-0805908.

We regard the CAT(0) 2-complex X_0 as a geometric companion to the Auter space (of rank 2) [13], a topological construction equipped with a group action by $\mathrm{Aut}(F_2)$.

Let W_3 denote the universal Coxeter group of rank 3—that is, W_3 is the free product of three copies of the group of order 2. Since $\mathrm{Aut}(F_2)$ is isomorphic to $\mathrm{Aut}(W_3)$ (see Remark 2), we also learn that $\mathrm{Aut}(W_3)$ is a CAT(0) group.

REMARK 1. As pointed out in the opening paragraph, our definition of a CAT(0) group is perhaps not standard because of the requirement that the group action be faithful. We note that such a requirement is redundant when giving an analogous definition of a word hyperbolic group. This follows from the fact that word hyperbolicity is an invariant of the quasi-isometry class of a group. In contrast, the CAT(0) property is not an invariant of the quasi-isometry class of a group. Examples are known of two quasi-isometric groups, one of which is CAT(0) and the other of which is not. Examples of this type may be constructed using the fundamental groups of graph manifolds [15] and the fundamental groups of Seifert fibre spaces [1; 5, p. 258]. So the adjective "faithful" is not so easily discarded in our definition of a CAT(0) group. We do not know of two abstractly commensurable groups, one of which is CAT(0) and the other of which is not. We pose the following question.

QUESTION 1. *Is the property of being a* CAT(0) *group an invariant of the abstract commensurability class of a group?*

Some relevant results in the literature show that two natural approaches to this question do not work in general. If G acts geometrically on a CAT(0) space X and G' is a finite extension with $[G' : G] = n$, then G' acts properly and isometrically on the CAT(0) space X^n with the product metric [7, p. 190; 18]. However, proving that this action is cocompact is either difficult or impossible in general. In [2], the authors give examples of the following type: G is a group acting faithfully and geometrically on a CAT(0) space X, G' is a finite extension of G, yet G' does not act faithfully and geometrically on X. However, G' may act faithfully and geometrically on some other CAT(0) space.

REMARK 2. The fact that $\mathrm{Aut}(F_2)$ is isomorphic to $\mathrm{Aut}(W_3)$ appears to be well known in certain mathematical circles, but it is rarely recorded explicitly. We now outline a proof: The subgroup $E \leq W_3$ of even-length elements is isomorphic to F_2 and characteristic in W_3, and $C_{W_3}(E) = \{1\}$; it follows from [17, Lemma 1.1] that the induced homomorphism $\pi : \mathrm{Aut}(W_3) \to \mathrm{Aut}(E)$ is injective. One easily confirms that the image of π contains a set of generators for $\mathrm{Aut}(E)$, and hence π is an isomorphism. A topological proof may also be constructed using the fact that the subgroup E of even-length words in W_3 corresponds to the 2-fold orbifold cover of the the orbifold $S^2(2, 2, 2, \infty)$ by the once-punctured torus.

The authors would like to thank Jason Behrstock and Martin Bridson for pointing out the examples in [1; 5, p. 258; 15] and Luisa Paoluzzi for discussions regarding [8].

2. Aut(B_4) Is a CAT(0) Group

We shall describe an apt presentation of B_4, give a concise combinatorial description of Brady's space X_0, describe the faithful geometric action of $\mathrm{Inn}(B_4)$ on X_0, and, finally, introduce an isometry of X_0 to extend the action of $\mathrm{Inn}(B_4)$ to a faithful geometric action of $\mathrm{Aut}(B_4)$.

The interested reader will find an informative, and rather more geometric, account of X_0 and the associated action of $\mathrm{Inn}(B_4)$ in [8].

AN APT PRESENTATION OF B_4. A standard presentation of the group B_4 is

$$\langle a, b, c \mid aba = bab,\ bcb = cbc,\ ac = ca \rangle. \tag{1}$$

Introducing generators $d = (ac)^{-1}b(ac)$, $e = a^{-1}ba$, and $f = c^{-1}bc$, one may verify that B_4 is also presented by

$$\langle a, b, c, d, e, f \mid ba = ae = eb,\ de = ec = cd,\ bc = cf = fb,\\ df = fa = ad,\ ca = ac,\ ef = fe \rangle. \tag{2}$$

We set $x = bac$ and write $\langle x \rangle \subset B_4$ for the infinite cyclic subgroup generated by x. The center of B_4 is the infinite cyclic subgroup generated by x^4.

THE SPACE X_0. Consider the 2-dimensional piecewise Euclidean CW-complex X_0 constructed as follows:

(0-S) the vertices of X_0 are in one-to-one correspondence with the left cosets of $\langle x \rangle$ in B_4—we write $v_{g\langle x \rangle}$ for the vertex corresponding to the coset $g\langle x \rangle$;

(1-S) distinct vertices $v_{g_1\langle x \rangle}$ and $v_{g_2\langle x \rangle}$ are connected by an edge of unit length if and only if there exists an element $\ell \in \{a, b, c, d, e, f\}^{\pm 1}$ such that $g_2^{-1} g_1 \ell \in \langle x \rangle$;

(2-S) three vertices $v_{g_1\langle x \rangle}$, $v_{g_2\langle x \rangle}$, and $v_{g_3\langle x \rangle}$ are the vertices of a Euclidean (equilateral) triangle if and only if the vertices are pairwise adjacent.

The link of the vertex $v_{\langle x \rangle}$ in X_0, just like the link of each vertex in X_0, consists of twelve vertices (one for each of the cosets represented by elements in $\{a, b, c, d, e, f\}^{\pm 1}$) and sixteen edges (one for each of the distinct ways to spell x as a word of length 3 in the alphabet $\{a, b, c, d, e, f\}$—see [8] for more details). It can be viewed as the 1-skeleton of a Möbius strip. In Figure 1 we depict the infinite cyclic cover of the link of $v_{\langle x \rangle}$. Each vertex with label g in the figure lies above the vertex $v_{g\langle x \rangle}$ in the link of $v_{\langle x \rangle}$. The link is formed by identifying identically labeled vertices and identifying edges with the same start and end points.

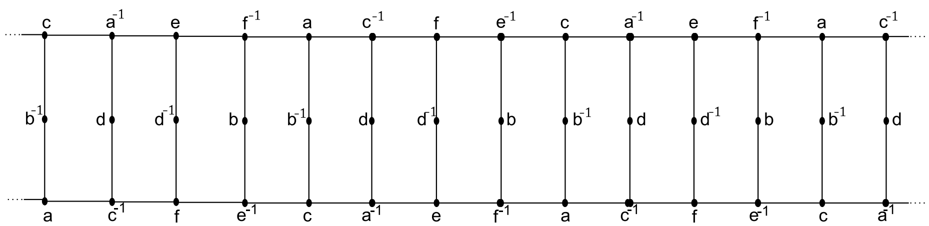

Figure 1 A covering of the link of $v_{\langle x \rangle}$ in X_0

That X_0 is CAT(0) follows most naturally from the alternative construction of X_0 described in detail in [8]. Alternatively, a complex constructed from isometric Euclidean triangles is CAT(0) if and only if it is simply connected and satisfies the "link condition" [5, Thm. II.5.4, p. 206]. For a 2-dimensional complex, the link condition requires that each injective loop in the link of a vertex have length at least 2π, where edges in a link are assigned the length of the angle they subtend [5, Lemma II.5.6, p. 207]. It is easily seen that X_0 satisfies the link condition because each injective loop in Figure 1 crosses at least six edges and each edge has length $\pi/3$. Thus one might show that X_0 is CAT(0) by showing that it is simply connected. We shall not digress from the task at hand to provide such an argument.

BRADY'S FAITHFUL GEOMETRIC ACTION OF $\mathrm{Inn}(B_4)$ ON X_0. We shall describe Brady's faithful geometric action of $\mathrm{Inn}(B_4)$ on X_0. We shall do so by describing an isometric action $\rho\colon B_4 \to \mathrm{Isom}(X_0)$ such that the image of ρ is a properly discontinuous and cocompact subgroup of $\mathrm{Isom}(X_0)$ that is isomorphic to $\mathrm{Inn}(B_4)$.

It follows immediately from (1-S) that, for each $g \in B_4$, the "left-multiplication by g" map on the 0-skeleton of X_0, $g_1\langle x\rangle \mapsto gg_1\langle x\rangle$, extends to a simplicial isometry of the 1-skeleton of X_0. It follows immediately from (2-S) that any simplicial isometry of the 1-skeleton of X_0 extends to a simplicial isometry of X_0. We write ϕ_g for the isometry of X_0 determined by g in this way, and we write $\rho\colon B_4 \to \mathrm{Isom}(X_0)$ for the map $g \mapsto \phi_g$. We compute that $\rho(g_1g_2)(v_{g\langle x\rangle}) = v_{g_1g_2g\langle x\rangle} = \rho(g_1)\rho(g_2)(v_{g\langle x\rangle})$ for each $g_1, g_2, g \in B_4$, so ρ is a homomorphism. Further, $\phi_g(v_{\langle x\rangle}) = v_{g\langle x\rangle}$ for each $g \in B_4$, so the vertices of X_0 are contained in a single ρ-orbit. It follows that ρ is a cocompact isometric action of B_4 on X_0.

To show that the image of ρ is isomorphic to $\mathrm{Inn}(B_4)$, it suffices to show that the kernel of ρ is exactly the center of B_4. One easily computes that $\rho(x^4)$ is the identity isometry of X_0. Thus the kernel of ρ contains the center of B_4. It is also clear that the stabilizer of $v_{\langle x\rangle}$, which contains the kernel of ρ, is the infinite subgroup $\langle x\rangle$. So to establish that the kernel of ρ is exactly the center of B_4, it suffices to show that ϕ_x, ϕ_{x^2}, and ϕ_{x^3} are nontrivial and distinct isometries of X_0. We achieve this by showing that these elements act nontrivially and distinctly on the link of $v_{\langle x\rangle}$ in X_0. We compute that x acts as follows on the cosets corresponding to vertices in the link of $v_{\langle x\rangle}$, where $\delta = \pm 1$:

$$a^\delta\langle x\rangle \mapsto e^\delta\langle x\rangle \mapsto c^\delta\langle x\rangle \mapsto f^\delta\langle x\rangle \mapsto a^\delta\langle x\rangle \quad \text{and} \quad b^\delta\langle x\rangle \leftrightarrow d^\delta\langle x\rangle.$$

Thus the restriction of ϕ_x to the link of $v_{\langle x\rangle}$ may be understood, with reference to Figure 2, as translation two units to the right followed by reflection across the horizontal dotted line. It follows that $\phi_x, \phi_{x^2}, \phi_{x^3}$ are nontrivial and distinct isometries of X_0, as required.

We next show that the image of ρ is a properly discontinuous subgroup of $\mathrm{Isom}(X_0)$. Now, the action ρ is not properly discontinuous because, as noted above, the ρ-stabilizer of $v_{\langle x\rangle}$ is the infinite subgroup $\langle x\rangle$ (so infinitely many elements of B_4 fail to move the unit ball about $v_{\langle x\rangle}$ off itself). But the image of $\langle x\rangle$ under the map $B_4 \to \mathrm{Inn}(B_4)$ has order 4. It follows that the image of ρ is a properly discontinuous subgroup of $\mathrm{Isom}(X)$.

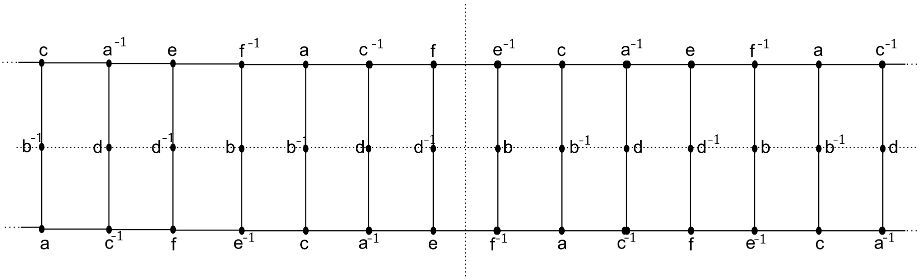

Figure 2 A covering of the link of the vertex $v_{\langle x\rangle}$ and the fixed point sets of some reflections

Thus we have that the image of ρ is a properly discontinuous and cocompact subgroup of $\mathrm{Isom}(X_0)$ that is isomorphic to $\mathrm{Inn}(B_4)$.

EXTENDING ρ BY FINDING ONE MORE ISOMETRY. It was shown in [10] that the unique nontrivial outer automorphism of B_n is represented by the automorphism that inverts each of the generators in presentation (1). Consider the automorphism $\tau \in \mathrm{Aut}(B_4)$ determined by

$$a \mapsto a^{-1},\ b \mapsto d^{-1},\ c \mapsto c^{-1},\ d \mapsto b^{-1},\ e \mapsto f^{-1},\ f \mapsto e^{-1}.$$

Note that τ is achieved by first applying the automorphism that inverts each of the generators a, b, and c and then applying the inner automorphism $w \mapsto (ac)^{-1}w(ac)$ for each $w \in B_4$. It follows that τ is an involution that represents the unique nontrivial outer automorphism of B_4. Writing $J := B_4 \rtimes_\tau \mathbb{Z}_2$, we have $\mathrm{Aut}(B_4) \cong J/\langle x^4\rangle$. We identify B_4 with its image in J.

The automorphism $\tau \in \mathrm{Aut}(B_4)$ permutes the elements of $\{a,b,c,d,e,f\}^{\pm 1}$ and maps the subgroup $\langle x\rangle$ to itself (in fact, $\tau(x) = x^{-1}$). It follows from (1-S) that the map $v_{g_1\langle x\rangle} \mapsto v_{\tau(g_1)\langle x\rangle}$ on the 0-skeleton of X_0 extends to a simplicial isometry of the 1-skeleton of X_0 and hence also to a simplicial isometry θ of X_0. We compute that $\theta\phi_g\theta = \phi_{\tau(g)}$ for each $g \in B_4$. Thus we have an isometric action $\rho'\colon J \to \mathrm{Isom}(X_0)$ given by

$$g \mapsto \phi_g \ \text{ for each } g \in B_4 \quad \text{and} \quad \tau \mapsto \theta.$$

We also compute that the restriction of θ to the link of $v_{\langle x\rangle}$ may be understood as reflection across the vertical dotted line shown in Figure 2. It follows that θ is a nontrivial isometry of X_0 that is distinct from ϕ_x, ϕ_{x^2}, and ϕ_{x^3}. Thus the kernel of ρ' is still the center of B_4, and the image of ρ' is a properly discontinuous and cocompact subgroup of $\mathrm{Isom}(X_0)$ that is isomorphic to $\mathrm{Aut}(B_4)$. Hence we have a faithful geometric action of $\mathrm{Aut}(B_4)$ on X_0, as required.

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