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Complete Intersection Points on General Surfaces in $\mathbb{P}^3$

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1. Introduction

A recurrent theme in classical projective geometry is the study of special subvarieties of some given family of varieties. For example: How many isolated singular points can a surface of degree d in $\mathbb{P}^3$ have? Under what conditions do members of a certain family of varieties contain rational curves, or contain a linear space of some positive dimension? Other examples of similar questions can easily be provided by the reader.

The study of the special case of complete intersection subvarieties of hypersurfaces in $\mathbb{P}^n$ has been the subject of a great deal of research. It was known to Severi [S] that, for $n \geq 4$, the only complete intersections of codimension 1 on a general hypersurface are obtained by intersecting that hypersurface with another. Indeed, the problem can be seen as a particular instance of the general problem of determining subvarieties of general hypersurfaces in $\mathbb{P}^n$ (see e.g. [GrH; Gro; L]).

In [CChG] we proposed a new approach to the problem of studying complete intersection subvarieties of hypersurfaces. This approach uses a mix of projective geometry and commutative algebra and is more direct than the usual methods for addressing the general problem. With our approach we were able to give a complete description of the situation for complete intersections of codimension r in $\mathbb{P}^n$ that lie on a general hypersurface of degree d whenever $2r \leq n+2$. The main result of [CChG] is as follows.

Theorem 1.1. *Let $X \subset \mathbb{P}^n$ be a generic degree-d hypersurface, where $n, d > 1$. Then X contains a complete intersection of type $(a_1, \dots, a_r)$, with $2r \leq n+2$ and the a_i all less than d, in the following (and only in the following) instances.*

- $n = 2$: *then $r = 2$, d is arbitrary, and a_1 and a_2 can assume any value less than d.*
- $n = 3, r = 2$: *for $d \leq 3$ we have that a_1 and a_2 can assume any value less than d.*
- $n = 4, r = 3$: *for $d \leq 5$ we have that a_1, a_2, and a_3 can assume any value less than d.*
- $n = 6, r = 4$ *or* $n = 8, r = 5$: *for $d \leq 3$ we have that $a_1, \dots, a_r$ can assume any value less than d.*

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- $n = 5, 7$ *or* $n > 8$, $2r = n + 1$ *or* $2r = n + 2$: *we have only linear spaces on quadrics*; *that is,* $d = 2$ *and* $a_1 = \cdots = a_r = 1$.

In this paper we are interested in the first case *not* covered by Theorem 1.1: the case $n = 3$ and $r = 3$ (i.e., complete intersection points on surfaces of $\mathbb{P}^3$). Although this is a very natural question, we are not aware of any reference to the subject in the literature. We can use the methods of [CChG] to prove the following result (where "CI" denotes "complete intersection").

THEOREM 1.2. *For nonnegative integers* a, b, c, d *such that* $a \le b \le c < d$, *the following statements hold.*

- *If* $a \le 4$, *then the generic degree-d surface of* $\mathbb{P}^3$ *contains a* $\mathrm{CI}(a, b, c)$.
- *If* $a = 5$ *and* $b \le 11$, *then the generic degree-d surface of* $\mathbb{P}^3$ *contains a* $\mathrm{CI}(5, b, c)$; *if* $a = 5$, $b = 12$, *and* $c = 12$, *then the generic degree-d surface of* $\mathbb{P}^3$ *contains a* $\mathrm{CI}(5, 12, 12)$; *if* $a = 5$, $b = 12$, *and* $c \ge 13$, *then the generic degree-*$d \ge 2c + 15$ *surface does not contain a* $\mathrm{CI}(5, 12, c)$; *if* $a = 5$ *and* $b \ge 13$, *then the generic surface of degree* $d \ge b + c + 2$ *does not contain a* $\mathrm{CI}(5, b, c)$.
- *If* $a = 6$ *and* $b \le 7$, *then the generic degree-d surface of* $\mathbb{P}^3$ *contains a* $\mathrm{CI}(6, b, c)$; *if* $a = 6$, $b = 8$, *and* $c = 8, 9$, *then the generic degree-d surface of* $\mathbb{P}^3$ *contains a* $\mathrm{CI}(6, 8, c)$; *if* $a = 6$, $b = 8$, *and* $c \ge 10$, *then the generic surface of degree* $d \ge 2c + 12$ *does not contain a* $\mathrm{CI}(6, 8, c)$; *if* $a = 6$ *and* $b \ge 9$, *then the generic surface of degree* $d \ge b + c + 3$ *does not contain a* $\mathrm{CI}(6, b, c)$.
- *If* $a \ge 7$, *then the generic degree-*$d \ge a + b + c - 3$ *surface of* $\mathbb{P}^3$ *does not contain a* $\mathrm{CI}(a, b, c)$.

Observe that Theorem 1.2 gives a complete asymptotic solution to the existence problem for $\mathrm{CI}(a, b, c)$ on a general surface of degree d in $\mathbb{P}^3$. More precisely, we have the following result.

COROLLARY 1.3. *Let* $a \le b \le c < d$ *be integers. Then, the generic degree-d surface contains a* $\mathrm{CI}(a, b, c)$ *if*:

- $a \le 4$;
- $a = 5$ *and* $b \le 11$;
- $a = 5$, $b = 12$, *and* $c = 12$;
- $a = 6$ *and* $b \le 7$;
- $a = 6$, $b = 8$, *and* $c = 8, 9$.

For any other a, b, c *and* $d \gg 0$ (*depending on* a, b, c), *the generic degree-d surface does not contain a* $\mathrm{CI}(a, b, c)$.

REMARK 1.4. We also observe that the kind of asymptotic problem just solved can be considered only for points. More precisely, if we choose a family $\mathcal{F}$ of subschemes of $\mathbb{P}^n$ then we can ask: Is it true that, for $d \gg 0$, the generic degree-d hypersurface of $\mathbb{P}^n$ contains an object of the family $\mathcal{F}$?

Using a standard incidence correspondence argument, it is easy to see that a positive answer can be given only if

$$\dim \mathcal{F} + 1 - h_{\mathcal{F}}(d) \geq 0,$$

where $h_{\mathcal{F}}(d)$ is the Hilbert polynomial of the objects in $\mathcal{F}$. Clearly this can be the case only if $h_{\mathcal{F}}(d)$ is bounded and hence constant. This implies that $\mathcal{F}$ is a family parameterizing 0-dimensional schemes.

The balance of the paper is structured as follows. In Section 2, we formalize the question we wish to study and treat the first simple instances; in Section 3, we recall the results we need from [CChG]. In Sections 4–6 we apply our method to produce the intermediate results necessary to prove Theorem 1.2. Finally, in Section 7 we prove Theorem 1.2 and state a conjecture for the expected behavior in the cases that remain open.

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2. The Question

In this paper we study complete intersection points in projective 3-space. We say that $\mathbb{X} \subset \mathbb{P}^3$ is a complete intersection 0-dimensional scheme if its ideal $I_{\mathbb{X}} = (F, G, H)$, where the forms F, G, and H are a regular sequence in the ring $R = \mathbb{C}[x_0, \dots, x_3]$. Moreover, if $\deg F = a$, $\deg G = b$, and $\deg H = c$ then we say that $\mathbb{X}$ is a *complete intersection of type* (a, b, c). We will always assume that $a \leq b \leq c$ and will write $\mathrm{CI}(a, b, c)$ to describe a complete intersection of type (a, b, c).

Our basic question is: *For which integers a, b, c and d does the general degree-d surface of* $\mathbb{P}^3$ *contain a* $\mathrm{CI}(a, b, c)$?

There are cases where the answer is straightforward. If $d = c$, then the answer is clearly affirmative because we are cutting a complete intersection curve of type (a, b) with a surface of degree d (and similarly for $d = a$ or $d = b$). If $d < a$, then the answer is negative because no form of degree less than a belongs to the ideal of a $\mathrm{CI}(a, b, c)$; and similarly for $a < d < b$, since a generic form is irreducible. If $b < d < c$ then we are actually looking for a complete intersection of type (a, b) on the generic degree-d surface, and this is covered by Theorem 1.1. Hence, it is enough to focus on the following refinement of our question: *For which integers a, b, c and d, $a \leq b \leq c < d$, does the general degree-d surface of* $\mathbb{P}^3$ *contain a* $\mathrm{CI}(a, b, c)$?

3. Technical Facts

We will treat this question using the method introduced in [CChG], which proceeds as follows. Translate the problem of finding a $\mathrm{CI}(a, b, c)$ on a general surface of degree d, say $M = 0$, as the problem of writing M as

$$M = FF' + GG' + HH',$$

where F, G, H (resp. F', G', H') are forms of degree a, b, c (resp. $d-a$, $d-b$, and $d-c$). Since M is generic, this decomposition problem is actually a problem about joins of varieties of splitting forms, so we use Terracini's lemma to translate the computation of the dimension of the join into a Hilbert function computation. Namely, as first observed in [M], the tangent space to the variety of splitting forms at the point $[FF']$ corresponds to the degree-d homogeneous piece of the ideal (F, F'). Thus, the tangent space at M to the join corresponds to the degree-d homogeneous piece of the ideal spanned by F, F', G, G', H, H'. For more details we refer the reader to [CChG].

In particular, we will need the following result [CChG, Lemma 4.3].

LEMMA 3.1. *For given integers a, b, c and d such that $a \le b \le c < d$, the following two facts are equivalent.*

(1) *The general degree-d surface of $\mathbb{P}^3$ contains a* $\mathrm{CI}(a, b, c)$.
(2) *For a generic choice of forms $F, G, H, H', G', F' \in R$ of respective degrees a, b, c, $d-c$, $d-b$, and $d-a$,*
$$H\left(\frac{R}{(F, G, H, H', G', F')}, d\right) = 0;$$
here $H(\cdot, d)$ denotes the Hilbert function in degree d.

Using Lemma 3.1, we translate our geometric question into a purely algebraic one. In particular, we can take advantage of results about the Lefschetz property [A; St] to deal with our question.

As F, G, H, and H' constitute a regular sequence in R, we have a good understanding of the ring
$$W = \frac{R}{(F, G, H, H')};$$
we will use this to study the Hilbert function of the ring
$$\frac{R}{(F, G, H, H', G', F')} \simeq \frac{W}{([F'], [G'])},$$
where $[\cdot]$ denotes the class in W.

Via the Koszul complex we compute the minimal free resolution of W:
$$0 \leftarrow W \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow M_3 \leftarrow M_4 \leftarrow 0. \qquad (*)$$
Here
$$M_0 = R,$$
$$M_1 = R(-a) \oplus R(-b) \oplus R(-c) \oplus R(-d+c),$$
$$M_2 = R(-a-b) \oplus R(-a-c) \oplus R(-a-d+c) \oplus R(-b-c) \oplus R(-b-d+c) \oplus R(-d),$$
$$M_3 = R(-a-b-c) \oplus R(-a-b-d+c) \oplus R(-a-d) \oplus R(-b-d),$$
$$M_4 = R(-a-b-d).$$
We also make note of the following (see [CChG, Lemma 4.1 and Rem. 4.2]).

LEMMA 3.2. *The following statements are equivalent.*

- *For integers* $a \le b \le c < d$, *a* $\mathrm{CI}(a, b, c)$ *exists on the generic degree-d surface of* $\mathbb{P}^3$.
- *For integers* $a' \le b' \le c' \le d$, *a* $\mathrm{CI}(a', b', c')$ *exists on the generic degree-d surface of* $\mathbb{P}^3$, *where* $a = a'$ *or* $a + a' = d$, *where* $b = b'$ *or* $b + b' = d$, *and where* $c = c'$ *or* $c + c' = d$.

REMARK 3.3. We can use Lemma 3.2 to study our question for integers $a \le b \le c < d/2$ and produce a complete answer for the general case. In fact, either $a \le d/2$ or $a' \le d/2$.

4. The Case $a \le 4$

Here we use Stanley's result [St] showing that the quotient of $R = \mathbb{C}[x_0, \dots, x_3]$ by four generic forms has the strong Lefschetz property. More precisely, given generic forms $F, G, H, F', G' \in R$ of respective degrees $a, b, c, d - c$, and $d - b$, we consider $W = R/(F, G, H, H')$. Then the multiplication by the class of G' has maximal rank, so the sequence

$$W(-d + b) \to W \to \frac{W}{([G'])} \to 0$$

produces $H(W/([G']), d) = \max\{H(W, d) - H(W, b), 0\}$.

PROPOSITION 4.1. *For any choice of* a, b, c *and* d *positive integers such that* $a \le 4 \le b \le c$ *and* $d \ge a + b + c - 3$, *the general degree-d surface in* $\mathbb{P}^3$ *contains a* $\mathrm{CI}(a, b, c)$.

Proof. By Lemma 3.3, it is enough to consider the case when $a \le b \le c \le d/2$. According to Proposition 3.1(2), we need only show that

$$H(W/([G']), d) = \max\{H(W, d) - H(W, b), 0\} = 0.$$

By the resolution of W given in $(*)$, we immediately obtain:

- if $b < c$, then

$$\begin{aligned} H(W, b) &= \binom{b+3}{3} - \binom{b-a+3}{3} - 1 \\ &= 1/6a^3 - 1/2a^2b + 1/2ab^2 - a^2 + 2ab + 11/6a - 1; \end{aligned}$$

- if $b = c$, then

$$\begin{aligned} H(W, b) &= \binom{b+3}{3} - \binom{b-a+3}{3} - 2 \\ &= 1/6a^3 - 1/2a^2b + 1/2ab^2 - a^2 + 2ab + 11/6a - 2. \end{aligned}$$

Led by the resolution of W, we also consider the following polynomial:

$$
\begin{aligned}
&h(W,d)\\
&\quad=\binom{d+3}{3}-\left[\binom{d-a+3}{3}+\binom{d-b+3}{3}+\binom{d-c+3}{3}+\binom{c+3}{3}\right]\\
&\qquad+\binom{d-a-b+3}{3}+\binom{d-a-c+3}{3}+\binom{c-a+3}{3}\\
&\qquad+\binom{d-b-c+3}{3}+\binom{c-b+3}{3}+1\\
&\qquad-\left[\binom{d-a-b-c+3}{3}+\binom{c-a-b+3}{3}\right],
\end{aligned}
$$

where $\binom{x}{3}$ is the polynomial $\frac{1}{6}x(x-1)(x-2)$. Performing the computation yields

$$h(W,d) = 1/2a^2b + 1/2ab^2 - 2ab + 1.$$

Notice that, for given a, b, c such that $c-a-b \geq -3$ and $d \geq a+b+c-3$, the evaluation of $h(W,d)$ coincides with the Hilbert function of W in degree d; in other words, $h(W,d) = H(W,d)$. Moreover, the inequalities

$$a \leq 4 \quad \text{and} \quad c-a-b \leq -4$$

hold only when $a = 4$ and $b = c$ (recall that $a \leq b \leq c$), and in this case

$$H(W,d) = h(W,d) + \binom{c-a-b+3}{3} = h(W,d) - 1.$$

Finally, we compute $H(W,d) - H(W,b)$ by distinguishing two cases.

Case 1: $a < 4$ or $b < c$. If $b < c$ then we use the value of $H(W,b)$ and the equality $H(W,d) = h(W,d)$ previously determined to get

$$H(W,d) - H(W,b) = -1/6a^3 + a^2b + a^2 - 4ab - 11/6a + 2.$$

This polynomial is linear in b and does not involve d, and it is easy to see that for $a \leq 4$ we have

$$H(W,d) - H(W,b) \leq 0.$$

If $a < 4$ and $b = c$ then a completely analogous argument can be used.

Case 2: $a = 4$ and $b = c$. Mutatis mutandis, we compute again and obtain

$$H(W,d) - H(W,b) = -1/6a^3 + a^2b + a^2 - 4ab - 11/6a + 2,$$

the same polynomial as before. This finishes the proof. □

Proposition 4.1 gives an asymptotic result yielding that, when one of the degrees of the complete intersection is at most 4, then for d big enough a complete intersection of the given type exists on a generic surface of degree d. With a slightly more careful analysis, this can be improved and the condition on d dropped.

THEOREM 4.2. *Let a, b, c and d be integers such that $a \leq b \leq c < d$. If $a \leq 4$, then a $\mathrm{CI}(a,b,c)$ exists on the generic degree-d surface in $\mathbb{P}^3$.*

Proof. By Proposition 4.1, we need only check values of d in the range $c < d \leq a+b+c-4$. The idea is to use Lemma 3.2 to reduce the degree of the complete intersection without changing d, so that Proposition 4.1 can be applied.

For $a = 2$, we consider the existence of a $\mathrm{CI}(2,b,c)$ on the generic degree-d surface for $c < d \leq b+c-2$. Such a complete intersection exists if the same occurs for a $\mathrm{CI}(2, d-c, d-b)$. Yet by Proposition 4.1 this happens as soon as

$$d \geq 2 + (d-c) + (d-b) - 3;$$

this is equivalent to $d \leq b+c-1$, which is actually the case. We argue similarly for $a = 3$.

The case $a = 4$ is treated in analogy with the previous ones, except for $d = b+c$. According to Lemma 3.2, in this situation we must study $\mathrm{CI}(4,b,b)$s on a generic surface of degree $d \geq b$. Repeating the same argument as before, we have to treat values of d in the range $b \leq d \leq 2b$. Proposition 6.1 gives the existence for all d where $d = 2b$. By Proposition 3.1, we must consider the coordinate ring W of a complete intersection of type (b,b,b,b) and its Hilbert function $H(\cdot)$. Given that multiplication by one form has maximal rank in W, we have only to compare $H(2b)$ and $H(2b-4)$. However, these values are the same (since W is a Gorenstein ring with socle degree $4b-4$), and this finishes the proof. □

5. The Case $a > 4$: Nonexistence Results

In this section we will prove asymptotic nonexistence results when $a > 4$. For nonnegative integers a,b,c,d with $a \leq b \leq c < d$, we consider generic forms $F, G, H, H', G', F' \in R$ of respective degrees a, b, c, $d-c$, $d-b$, and $d-a$. Consider the ring $W = R/(F,G,H,H')$ and observe that, by a straightforward dimensional argument, if

$$H(W,a) + H(W,b) - H(W,d) < 0$$

then

$$H\left(\frac{W}{([G'],[F'])}, d\right) \neq 0.$$

Hence, by Lemma 3.1(2), if $H(W,d) - H(W,a) - H(W,b) < 0$ then the generic degree-d surface of $\mathbb{P}^3$ does not contain a $\mathrm{CI}(a,b,c)$. We can use this idea to help prove the following result.

THEOREM 5.1. *Let $a \leq b \leq c$ and d be nonnegative integers such that*

$$a = 5 \quad \textit{and} \quad b \geq 13$$

or

$$a = 6 \quad \textit{and} \quad b \geq 9$$

or

$$a \geq 7.$$

Then, for $d \geq a+b+c-3$, the generic degree-d surface of $\mathbb{P}^3$ does not contain a $\mathrm{CI}(a,b,c)$.

In order to prove this theorem, we need the following technical fact.

LEMMA 5.2. *Let a, b, c be nonnegative integers with $a > 4$. Assume that, for integers c_0 and d such that*

$$c_0 \geq b \quad \textit{and} \quad d > a + b + c_0 - 4,$$

one has the Hilbert function inequality

$$H(W, a) + H(W, b) - H(W, d) < 0,$$

where W is the ring

$$W = \frac{R}{(F, G, H, H')}$$

and where the forms F, G, H, and H' are generic and have the respective degrees a, b, c_0, and $d - c_0$.

If A, B, C, and D are forms of respective degrees $a, b, c \geq c_0$, and $d - c$ and if

$$W' = \frac{R}{(A, B, C, D)},$$

then

$$H(W', a) + H(W', b) - H(W', d) < 0$$

for $d > a + b + c - 4$.

Proof. The key observation is that

$$H(W, d) = H(W, a + b - 4).$$

In fact, since W is a Gorenstein ring, its Hilbert function is symmetric and

$$H(W, x) = H(W, y) \quad \text{if } x + y = d + a + b - 4.$$

So we compute

$$H(W, a) + H(W, b) - H(W, a + b - 4)$$

using the formulas in the proof of Proposition 4.1, for which we need the assumption on d. It is clear that the final expression does not involve either c or d, and the proof follows. For example, in the case $a < b$ one gets $H(W, a) + H(W, b) - H(W, a + b - 4) < 0$ if and only if

$$b \geq \frac{\frac{1}{2}a^3 - a^2 + \frac{11}{2}a - 4}{(a^2 - 4a)}. \qquad \square$$

We can now prove Theorem 5.1.

Proof of Theorem 5.1. We let $b = c$ and show that, for the required values of a and b, we have the inequality $H(W, a) + H(W, b) - H(W, d) < 0$. Then we apply Lemma 5.2 to get the result when $c \geq b$.

We remark once again that $H(W, d) = H(W, a + b - 4)$. The proof is divided into two cases depending on whether $a = b$ or $a < b$.

Case 1: $a < b$. Using the resolution of the ring W, the inequality

$$H(W,a) + H(W,b) - H(W,a+b-4) \geq 0$$

is readily seen to be equivalent to

$$b \leq \frac{\frac{1}{2}a^3 - a^2 + \frac{11}{2}a - 4}{(a^2 - 4a)}.$$

Recalling that $a < b$, we have

$$H(W,a) + H(W,b) - H(W,d) \geq 0$$

only if

$$-\frac{1}{2}a^3 + 3a^2 + \frac{11}{2}a - 4 \geq 0,$$

and this inequality holds if and only if

$$a = 5 \quad \text{or} \quad a = 6.$$

Therefore,

$$H(W,a) + H(W,b) - H(W,d) \geq 0$$

implies that

$$a = 5 \quad \text{and} \quad b \leq 12$$

or

$$a = 6 \quad \text{and} \quad b \leq 8.$$

Case 2: $a = b$. Computations show that

$$H(W,a) + H(W,b) - H(W,d) = -\frac{2}{3}a^3 + 4a^2 + \frac{11}{3}a - 3 \geq 0$$

only if $a < 7$, and this finishes the proof. □

To prove some more nonexistence results, we need the following statement.

PROPOSITION 5.3. *Let* $a \leq b \leq c < d$ *and* $d > 2c + b + a - 3$. *If no* $\mathrm{CI}(a,b,c)$ *exists on the generic degree-d hypersurface, then none exists on the generic hypersurface of degree* $d' > d$, *either.*

Proof. It is enough to treat the case $d' = d + 1$. For generic forms F, G, H of respective degrees a, b, c, let

$$A = \frac{\mathbb{C}[x_0, \ldots, x_3]}{(F, G, H)}.$$

By hypothesis, for the generic choice of F', G', and H' of respective degrees $d-a$, $d-b$, and $d-c$ in A, we know that the degree-d part of

$$\frac{A}{(F', G', H')}$$

is not zero. Now consider elements F'', G'', and H'' of respective degrees $d+1-a$, $d+1-b$, and $d+1-c$. Observe that

$$d - a \geq d - b \geq d - c > c + b + a - 3$$

and recall that $A_i \simeq A_j$ as $\mathbb{C}$ vector spaces if i and j are each greater than $2c+b+a-3$. Hence for a general linear form L we have

$$F'' = LF^*, \quad G'' = LG^*, \quad H'' = LH^*,$$

and the forms F^*, G^*, and H^* have respective degrees $d-a$, $d-b$, and $d-c$. Thus we have an isomorphism

$$(F'', G'', H'')_{d+1} \simeq (F^*, G^*, H^*)_d,$$

and this is enough to conclude that the degree-$(d+1)$ part of

$$\frac{A}{(F'', G'', H'')}$$

is not zero. The result follows. □

LEMMA 5.4. *If $c \geq 13$ then, for $d \geq 2c+15$, the generic degree-d surface of $\mathbb{P}^3$ does not contain a* $\mathrm{CI}(5,12,c)$.

If $c \geq 10$ then, for $d \geq 2c+12$, the generic degree-d surface of $\mathbb{P}^3$ does not contain a $\mathrm{CI}(6,8,c)$.

Proof. We begin with the study of $\mathrm{CI}(5,12,c)$. Let $c = 13+x$ and $d = 2c+a+b-2 = 41+2x$, and consider the ring

$$W = \frac{R}{(F, G, H, H')};$$

here the forms F, G, H, and H' have respective degrees 5, 12, $13+x$, and $28+x$. The generic degree-d surface does not contain a $\mathrm{CI}(5,12,c)$ if $H(W,5) + H(W,12) - H(W,d) < 0$, where $H(W,d) = H(W, 41+2x) = H(W,14)$. Now we compute

$$H(5) = \binom{8}{3} - 1 = 55;$$

$$H(12) = \binom{15}{3} - \binom{10}{3} - 1 = 334;$$

$$H(14) = \begin{cases} \binom{17}{3} - \binom{12}{3} - \binom{5}{3} = 450 & \text{if } x > 1, \\ 449 & \text{if } x = 1, \\ 446 & \text{if } x = 0. \end{cases}$$

Thus $H(W,5) + H(W,12) - H(W,d) < 0$, and by Proposition 5.3 we conclude that the generic degree-d' surface does not contain a $\mathrm{CI}(5,12,c)$ for $d' \geq d = 41+2x = 15+2c$.

The case of $\mathrm{CI}(6,8,c)$ is solved by completely analogous computations. □

6. The Case $a > 4$: Existence Results

Theorem 5.1 does not cover small values of a and b, so in this section we derive a result analogous to Theorem 4.2 in these cases. We begin by proving two technical facts.

PROPOSITION 6.1. *Let $a \le b \le c \le d$ and $d \ge a+b+c-3$. If a $\mathrm{CI}(a,b,c)$ exists on the generic degree-d surface, then it also exists on the generic surface of degree $d' > d$.*

Proof. Let $d' = d+1$ and notice that it is enough to treat this case. The hypothesis reads as follows: the degree-d part of the ring

$$\frac{A}{(F', G', H')}$$

is zero for generic forms F', G', and H' of respective degrees $d-a$, $d-b$, and $d-c$, where $A = R/(F,G,H)$. If $L \in A$ is a generic linear form, then by [St] we know that multiplication by L is an isomorphism in degree $\ge a+b+c-4$. Hence, the degree-$(d+1)$ piece of

$$\frac{A}{(LF', LG', LH')}$$

is zero. This is enough to complete the proof because, if three special forms (viz., LF', LG', and LH') have maximal span, then the same property holds for a generic choice. □

LEMMA 6.2. *Let a, b, and d be nonnegative integers such that $4 < a \le b$ and $d = 2a+2b-6$. If the generic degree-d surface in $\mathbb{P}^3$ contains a $\mathrm{CI}(a,b,a+b-3)$, then the generic degree-d' surface contains a $\mathrm{CI}(a,b,c)$ for any $d' \ge a+b+c-3$ and any $c \ge a+b-3$.*

Proof. By Proposition 6.1, the generic degree-$(d+\varepsilon)$ surface in $\mathbb{P}^3$ contains a $\mathrm{CI}(a,b,a+b-3)$ for all $\varepsilon \ge 0$. Hence, by Lemma 3.2, the same holds for $\mathrm{CI}(a,b,a+b-3+\varepsilon)$ and surfaces of degree $d+\varepsilon$. Making ε vary and again applying Proposition 6.1 then yields the result. □

THEOREM 6.3. *Let a, b, c and d be nonnegative integers such that $a \le b \le c < d$. If $a = 5$ and $b \le 11$ or if $a = 6$ and $b \le 7$, then a $\mathrm{CI}(a,b,c)$ exists on the generic degree-d surface of $\mathbb{P}^3$. If $a = 5$, $b = 12$, and $c = 12$ or if $a = 6$, $b = 8$, and $c = 8, 9$, then a $\mathrm{CI}(a,b,c)$ exists on the generic degree-d surface of $\mathbb{P}^3$.*

Proof. To prove the thesis we combine all the previous results and technical facts. Other crucial ingredients include some explicit computations performed using the computer algebra system CoCoA [Co].

To determine whether a $\mathrm{CI}(a,b,c)$ exists on the generic surface of degree d in $\mathbb{P}^3$, we proceed as follows.

- If $c \le a+b-3$, then we make explicit computations for all $d \le a+b+c-3$; a positive answer for $d = a+b+c-3$ solves the cases for bigger d by Proposition 6.1.
- If $c = a+b-3$ and $d \ge 2a+2b-6$, we verify each statement with an explicit computation for $d = 2a+2b-6$; if the answer is positive then, by Lemma 6.2, we conclude the same for $c \ge a+b-3$ and $d \ge a+b+c-3$.

We sketch this procedure for $a = 6$ (the case $a = 5$ is completely analogous but lengthier). We need to perform explicit computations in the following cases:

- $\mathrm{CI}(6, 6, c_1)$ for $c_1 \leq 9$ and $d_1 \leq 9 + c_1$;
- $\mathrm{CI}(6, 7, c_2)$ for $c_2 \leq 10$ and $d_3 \leq 10 + c_2$.

The computations (see Example 6.4) show that the complete intersections exist on the generic surfaces of the required degrees. Hence we conclude that the generic surface of degree d contains a $\mathrm{CI}(6, b, c)$ for all $b \leq 7$ and any c, d such that $d > c$. We conclude the proof for $a = 6$ by verifying existence in the cases $\mathrm{CI}(6, 8, 8)$ for $d = 19$ and $\mathrm{CI}(6, 8, 9)$ for $d = 20$. $\square$

Example 6.4. We begin with verifying that the generic surface of degree $7 \leq d \leq 15$ contains a $\mathrm{CI}(6, 6, 6)$. Given Proposition 3.1, it is enough to show that the ring

$$S = \frac{\mathbb{C}[x_0, \dots, x_3]}{(F, G, H, H', G', F')}$$

is zero in degree d, where the forms F, G, H, H', G', F' are generic and have respective degrees 6, 6, 6, $d - 6$, $d - 6$, and $d - 6$. Hence, for each d, we choose random forms with rational coefficients of the required degrees. Then we ask CoCoA [Co] to compute the Hilbert function of S in degree d. Since for all d we get $H(S, d) = 0$, we conclude (by semicontinuity) that this is the case for a generic choice of forms of the appropriate degrees. In particular, since $15 = 6 + 6 + 6 - 3$ and $H(S, 15) = 0$, Proposition 6.1 yields that a $\mathrm{CI}(6, 6, 6)$ exists on the generic degree-$d \geq 15$ surface of $\mathbb{P}^3$.

The same argument works in complete analogy for $c \leq 8$. For $c = 9$ we make an explicit computation for $d = 18$; then Lemma 6.2 allows us to show existence of a $\mathrm{CI}(6, 6, c)$ on the generic degree-d surface for $c \geq 9$ and $d \geq c + 9$. The cases for $c < d < c + 9$ are solved by using Lemma 3.1 together with the results for $c \leq 8$ and $a \leq 4$.

7. Main Theorem and Final Remarks

We can now prove our main theorem.

Proof of Theorem 1.2. The existence part for the case $a \leq 4$ is Theorem 4.2, and existence for the remaining cases is Theorem 6.3. The asymptotic nonexistences are given by Lemma 5.4 and Theorem 5.1. $\square$

Theorem 1.2 produces a complete asymptotic answer to our original question. We also derive many existence and nonexistence results for small values of d. However, there remain infinitely many cases that have not been solved—for example, $a = 7$ with any b, c, and d such that $7 \leq b \leq c$ and $c + 5 \leq d \leq a + b + c - 4$.

We now state a conjecture completing Theorem 1.2.

Conjecture. *Given nonnegative integers a, b, c and d such that $a \leq b \leq c < d$, there exists a function $d(a, b, c)$, possibly assuming the value $+\infty$, such that the generic degree-d surface in $\mathbb{P}^3$ contains a $\mathrm{CI}(a, b, c)$ if and only if $d < d(a, b, c)$.*

As support for this conjecture, we remark that it fits with the asymptotic statement and with the other results of Theorem 1.2. For example, $d(a,b,c) = +\infty$ for $a \leq 4$ and $d(a,b,c) < a+b+c-3$ for $7 \leq a$.

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