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Singular Loci of Grassmann–Hibi Toric Varieties

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Introduction

Let K denote the base field, which we assume to be algebraically closed and of arbitrary characteristic. Given a distributive lattice $\mathcal{L}$, let $X(\mathcal{L})$ denote the affine variety in $\mathbb{A}^{\#\mathcal{L}}$ whose vanishing ideal is generated by the binomials $X_\tau X_\varphi - X_{\tau\vee\varphi}X_{\tau\wedge\varphi}$ in the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$ (here, $\tau \vee \varphi$ (resp. $\tau \wedge \varphi$) denotes the *join*—the smallest element of $\mathcal{L}$ greater than both τ and φ (resp. the *meet*—the largest element of $\mathcal{L}$ smaller than both τ and φ)). These varieties were extensively studied by Hibi in [10], where it is proved that $X(\mathcal{L})$ is a normal variety. On the other hand, Eisenbud and Sturmfels [6] showed that a binomial prime ideal is toric (here, "toric ideal" is in the sense of [17]). Thus one obtains that $X(\mathcal{L})$ is a normal toric variety. We shall refer to such an $X(\mathcal{L})$ as a *Hibi toric variety*.

For $\mathcal{L}$ the Bruhat poset of Schubert varieties in a minuscule G/P, it is shown in [8] that $X(\mathcal{L})$ flatly deforms to $\widehat{G/P}$ (the cone over G/P); in other words, there exists a flat family over $\mathbb{A}^1$ with $\widehat{G/P}$ as the generic fiber and $X(\mathcal{L})$ as the special fiber. More generally, for a Schubert variety $X(w)$ in a minuscule G/P, it is shown in [8] that $X(\mathcal{L}_w)$ flatly deforms to $\widehat{X(w)}$, the cone over $X(w)$ (here, $\mathcal{L}_w$ is the Bruhat poset of Schubert subvarieties of $X(w)$). In a subsequent paper [9], the authors studied the singularities of $X(\mathcal{L})$ for $\mathcal{L}$ the Bruhat poset of Schubert varieties in the Grassmannian; they also gave a conjecture (see [9, Sec. 11]; see also Remark 9.1 of this paper) giving a necessary and sufficient condition for a point on $X(\mathcal{L})$ to be smooth and proved the sufficiency part of the conjecture. Subsequently, the necessary part of the conjecture was proved in [2] by Batyrev and colleagues. The toric varieties $X(\mathcal{L})$ for $\mathcal{L}$ the Bruhat poset of Schubert varieties in the Grassmannian play an important role in the area of mirror symmetry; for more details, see [1; 2]. We refer to such an $X(\mathcal{L})$ as a *Grassmann–Hibi toric variety* (or G-H toric variety).

The proof (in [9]) of the sufficiency part of the conjecture in [9] uses the Jacobian criterion for smoothness, whereas the proof (in [2]) of the necessary part of the conjecture in [9] uses certain desingularization of $X(\mathcal{L})$.

It should be remarked that neither [9] nor [2] discusses the relationship between the singularities of $X(\mathcal{L})$ and the combinatorics of the polyhedral cone associated

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to $X(\mathcal{L})$; the main goal of this paper is to bring out this relationship. Our first main result is a description of the singular locus of a G-H toric variety $X(\mathcal{L})$ in terms of the faces of the associated polyhedral cone; in particular, we give a proof of the conjecture of [9] using only the combinatorics of the polyhedral cone associated to the toric variety $X(\mathcal{L})$. Furthermore, we prove (Theorem 6.19) that the singular locus of $X(\mathcal{L})$ is pure and of codimension 3 in $X(\mathcal{L})$ and that the generic singularities are of cone type (more precisely, the singularity type is the same as that at the vertex of the cone over the quadric surface $x_1x_4 - x_2x_3 = 0$ in $\mathbb{P}^3$); we also determine the tangent cone to $X(\mathcal{L})$ at a generic singularity, which turns out to be a toric variety (by a "generic singularity" we mean a point $P \in X(\mathcal{L})$ such that the closure of the torus orbit through P is an irreducible component of Sing $X(\mathcal{L})$, the singular locus of $X(\mathcal{L})$). We further obtain (see Corollary 7.3 and Theorem 7.11) an interpretation of the multiplicities at some of the singularities as certain Catalan numbers when $\mathcal{L}$ is the Bruhat poset of Schubert varieties in the Grassmannian of 2-planes in K^n. We also present a product formula (Theorem 7.17).

It turns out that the conjecture of [9] does not extend to a general $X(\mathcal{L})$ (see Section 9.2 for a counterexample). However, in [4] we proved the conjecture of [9] for other minuscule posets.

This paper contains more results for the Grassmann–Hibi toric varieties that cannot be deduced from the results of [4]; for example, the multiplicity formulas provided in Sections 7 and 8 of this paper are not discussed in [4]. The singularities of the Hibi toric variety were also studied by Wagner in [18], where it was shown that all Hibi toric varieties have a singular locus of codimension ≥ 3. In this paper, we go into much more detail about the singularities of a G-H toric variety.

The balance of the paper is organized as follows. In Sections 1 and 2 we recall generalities on toric varieties and distributive lattices, respectively. In Section 3, we introduce the Hibi toric variety $X(\mathcal{L})$ and recall some results from [9; 14] on $X(\mathcal{L})$. In Section 4, we recall results from [14] on the polyhedral cone associated to $X(\mathcal{L})$; in Section 5, we introduce the Grassmann–Hibi toric variety. In Section 6 we prove our first main result, giving the description of Sing $X(\mathcal{L})$ in terms of faces of the cone associated to $X(\mathcal{L})$; we also present our results on the tangent cones and deduce the multiplicities at the associated points. In Section 7 we present certain product formulas for $X(\mathcal{L})$, where $\mathcal{L}$ is the Bruhat poset of Schubert varieties in the Grassmannian of 2-planes in K^n. In Section 8, we present a formula for the multiplicity at the unique T-fixed point of $X(\mathcal{L})$ for $\mathcal{L}$ the Bruhat poset of Schubert varieties in the Grassmannian of d-planes in K^n. In Section 9 we present a counterexample to show that the conjecture of [9] does not extend to a general $X(\mathcal{L})$; in this section, we also present two conjectures that concern extending the multiplicity formulas of Sections 7 and 8.

1. Generalities on Toric Varieties

Our main object of study is a certain affine toric variety, so in this section we recall some basic definitions on affine toric varieties. Let $T = (K^*)^m$ be an

m-dimensional torus. Let $\mathbb{A}^l$ be the affine l-space (i.e., l-tuples of elements of the field K).

DEFINITION 1.1 [7; 12]. An *equivariant affine embedding* of a torus T is an affine variety $X \subseteq \mathbb{A}^l$ containing T as an open subset and equipped with a T-action $T \times X \to X$ extending the action $T \times T \to T$ given by multiplication. If in addition X is normal, then X is called an *affine toric variety.*

1.2. THE CONE ASSOCIATED TO A TORIC VARIETY. Let M be the character group of T, and let N be the $\mathbb{Z}$-dual of M. Let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, and recall [7; 12] that there exists a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$ such that

$$K[X] = K[S_\sigma],$$

where S_σ is the subsemigroup $\sigma^\vee \cap M$ for $\sigma^\vee$ the cone in $M_{\mathbb{R}}$ dual to σ. Note that S_σ is a finitely generated subsemigroup in M.

1.3. ORBIT DECOMPOSITION IN AFFINE TORIC VARIETIES. We shall denote X also by X_σ. We may suppose, without loss of generality, that σ spans $N_{\mathbb{R}}$ so that the dimension of σ equals $\dim N_{\mathbb{R}} = \dim T$. (By "dimension of σ" we mean the vector space dimension of the span of σ.)

DEFINITION 1.4. A *face* τ of σ is a convex polyhedral subcone of σ of the form $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$, and it is denoted $\tau < \sigma$. Note that σ itself is considered a face.

We have that X_τ is a principal open subset of X_σ; namely,

$$X_\tau = (X_\sigma)_u.$$

Each face τ determines a (closed) point P_τ in X_σ: the point corresponding to the maximal ideal in $K[X] = K[S_\sigma]$ given by the kernel of $e_\tau \colon K[S_\sigma] \to K$, where for $u \in S_\sigma$ we have

$$e_\tau(u) = \begin{cases} 1 & \text{if } u \in \tau^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 1.5. As a point in $\mathbb{A}^l$, P_τ may be identified with the l-tuple with 1 at the ith place if χ_i is in $\tau^\perp$ and with 0 otherwise. (Here, χ_i denotes the weight of the T-weight vector y_i—the class of x_i in $K[X_\sigma]$.)

1.6. ORBIT DECOMPOSITION. Let O_τ denote the T-orbit in X_σ through P_τ. We have the following orbit decomposition in X_σ:

$$X_\sigma = \bigcup_{\theta \le \sigma} O_\theta, \quad \overline{O_\tau} = \bigcup_{\theta \ge \tau} O_\theta, \quad \dim \tau + \dim O_\tau = \dim X_\sigma.$$

See [7; 12] for details.

Thus $\tau \mapsto \overline{O_\tau}$ defines an order-reversing bijection between {faces of σ} and {T-orbit closures in X_σ}.

LEMMA 1.7 [7, Sec. 3.1]. *For a face $\tau < \sigma$, $K[\overline{O_\tau}] = K[S_\sigma \cap \tau^\perp]$.*

2. Finite Distributive Lattices

We shall study a special class of toric varieties—namely, the toric varieties associated to distributive lattices. We shall first collect some definitions on finite partially ordered sets. A partially ordered set is also called a *poset*.

Definition 2.1. A finite poset P is called *bounded* if it has both a unique maximal and a unique minimal element, denoted $\hat{1}$ and $\hat{0}$ respectively. A totally ordered subset C of P is called a *chain*, and the number $\#C - 1$ is called the *length* of the chain. A bounded poset P is said to be *graded* (or *ranked*) if all maximal chains have the same length. If P is graded, then the length of a maximal chain in P is called the *rank* of P.

Definition 2.2. Let P be a graded poset. For $\lambda, \mu \in P$ with $\lambda \geq \mu$, the graded poset $\{\tau \in P \mid \mu \leq \tau \leq \lambda\}$ is called the *interval from* μ *to* λ and is denoted by $[\mu, \lambda]$.

Definition 2.3. Let P be a graded poset, and let $\lambda, \mu \in P$ with $\lambda \geq \mu$. The ordered pair (λ, μ) is called a *cover* (and we also say that λ *covers* μ) if $[\mu, \lambda] = \{\mu, \lambda\}$.

Definition 2.4. A *lattice* is a partially ordered set $(\mathcal{L}, \leq)$ such that, for every pair of elements $x, y \in \mathcal{L}$, there exist elements $x \vee y$ and $x \wedge y$, called (respectively) the *join* and the *meet* of x and y, defined by:

$$x \vee y \geq x, \quad x \vee y \geq y, \quad \text{and} \quad \text{if } z \geq x \text{ and } z \geq y \text{ then } z \geq x \vee y;$$
$$x \wedge y \leq x, \quad x \wedge y \leq y, \quad \text{and} \quad \text{if } z \leq x \text{ and } z \leq y \text{ then } z \leq x \wedge y.$$

It is easy to check that the operations $\vee$ and $\wedge$ are commutative and associative.

Definition 2.5. Given a lattice $\mathcal{L}$, a subset $\mathcal{L}' \subset \mathcal{L}$ is called a *sublattice* of $\mathcal{L}$ if $x, y \in \mathcal{L}'$ implies that $x \wedge y \in \mathcal{L}'$ and $x \vee y \in \mathcal{L}'$.

Definition 2.6. A lattice is called *distributive* if the following identities hold:
(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2.7. An element z of a lattice $\mathcal{L}$ is called *join-irreducible* (resp. *meet-irreducible*) if $z = x \vee y$ (resp. $z = x \wedge y$) implies either $z = x$ or $z = y$. The set of join-irreducible elements of $\mathcal{L}$ is denoted by $J(\mathcal{L})$.

The following lemma is easily checked.

Lemma 2.8. *Let $\mathcal{L}$ be a finite distributive lattice. Then*

$$J(\mathcal{L}) = \{\tau \in \mathcal{L} \mid \textit{there exists at most one cover of the form } (\tau, \lambda)\}.$$

Definition 2.9. A subset I of a poset P is called an *ideal* of P if, for all $x, y \in P$,

$$x \in I \text{ and } y \leq x \implies y \in I.$$

THEOREM 2.10 [3, Chap. III, Sec. 3]. *Let $\mathcal{L}$ be a finite distributive lattice with minimal element $\hat{0}$, and let $P = J(\mathcal{L}) \setminus \{\hat{0}\}$ with the induced partial order of $\mathcal{L}$. Then $\mathcal{L}$ is isomorphic to the lattice of ideals of P by means of the lattice isomorphism*

$$\alpha \mapsto I_\alpha = \{\tau \in P \mid \tau \le \alpha\}, \quad \alpha \in \mathcal{L}.$$

For $\alpha \in \mathcal{L}$, let I_α denote the ideal corresponding to α under the isomorphism in Theorem 2.10.

REMARK 2.11. As a consequence of Theorem 2.10, we have that every finite distributive lattice is graded.

3. The Variety $X(\mathcal{L})$

Throughout the following sections, let $\mathcal{L}$ be a finite distributive lattice.

Consider the polynomial algebra $K[X_\alpha, \alpha \in \mathcal{L}]$, and let $I(\mathcal{L})$ be the ideal generated by $\{X_\alpha X_\beta - X_{\alpha\vee\beta}X_{\alpha\wedge\beta}, \alpha, \beta \in \mathcal{L}\}$. Then one knows [10] that $K[X_\alpha, \alpha \in \mathcal{L}]/I(\mathcal{L})$ is a normal domain; in particular, we have that $I(\mathcal{L})$ is a prime ideal. Let $X(\mathcal{L})$ be the affine variety of the zeroes in K^l of $I(\mathcal{L})$ (here $l = \#\mathcal{L}$). Then $X(\mathcal{L})$ is an affine normal variety defined by binomials; on the other hand, by [6], a binomial prime ideal is a toric ideal (here, "toric ideal" is in the sense of [17]). Hence $X(\mathcal{L})$ is a toric variety for the action by a suitable torus T.

In the sequel, we shall denote $R(\mathcal{L}) = K[X_\alpha, \alpha \in \mathcal{L}]/I(\mathcal{L})$. Also, for $\alpha \in \mathcal{L}$, we shall denote the image of X_α in $R(\mathcal{L})$ by x_α.

DEFINITION 3.1. The variety $X(\mathcal{L})$ will be called a *Hibi toric variety.*

REMARK 3.2. An extensive study of $X(\mathcal{L})$ appeared first in [10].

We have that $\dim X(\mathcal{L}) = \dim T$.

THEOREM 3.3 [14]. *The dimension of $X(\mathcal{L})$ is equal to $\#J(\mathcal{L})$, which is also equal to the cardinality of the set of elements in a maximal chain of $\mathcal{L}$.*

DEFINITION 3.4. For a finite distributive lattice $\mathcal{L}$, we call the cardinality of $J(\mathcal{L})$ the *dimension* of $\mathcal{L}$, denoted $\dim \mathcal{L}$. If $\mathcal{L}'$ is a sublattice of $\mathcal{L}$, then the *codimension* of $\mathcal{L}'$ in $\mathcal{L}$ is defined as $\dim \mathcal{L} - \dim \mathcal{L}'$.

DEFINITION 3.5 [18]. A sublattice $\mathcal{L}'$ of $\mathcal{L}$ is called an *embedded sublattice of* $\mathcal{L}$ if

$$\tau, \phi \in \mathcal{L},\ \tau \vee \phi \in \mathcal{L}',\ \tau \wedge \phi \in \mathcal{L}' \implies \tau, \phi \in \mathcal{L}'.$$

Given a sublattice $\mathcal{L}'$ of $\mathcal{L}$, consider the variety $X(\mathcal{L}')$ and the canonical embedding $X(\mathcal{L}') \hookrightarrow \mathbb{A}^{\#\mathcal{L}'} \hookrightarrow \mathbb{A}^{\#\mathcal{L}}$.

PROPOSITION 3.6 [9, Prop. 5.16]. *$X(\mathcal{L}')$ is a subvariety of $X(\mathcal{L})$ if and only if $\mathcal{L}'$ is an embedded sublattice of $\mathcal{L}$.*

3.7. MULTIPLICITY OF $X(\mathcal{L})$ AT THE ORIGIN. Let B be a $\mathbb{Z}_+$-graded and finitely generated K-algebra, $B = \bigoplus B_m$. Let $\phi_m(B)$ denote the Hilbert function,

$$\phi_m(B) = \dim_K B_m,$$

and let $P_B(x)$ denote the Hilbert polynomial of B. Recall that:

- $P_B(x) \in \mathbb{Q}[x]$;
- $\deg P_B(x) = \dim \operatorname{Proj} B = s$, say; and
- the leading coefficient of $P_B(x)$ is of the form $e_B/s!$.

DEFINITION 3.8. The number e_B is called the *degree* of the graded ring B, or the degree of Proj B.

THEOREM 3.9. *The degree of* $K[X(\mathcal{L})]$ *is equal to the number of maximal chains in* $\mathcal{L}$.

Proof. Let $I(\mathcal{L})$ be as before. We begin by putting a monomial order on $K[X_\alpha, \alpha \in \mathcal{L}]$. Consider the reverse partial order on $\mathcal{L}$ and extend it to a total order, denoted $\leq_{\text{tot}}$, on the variables $\{X_\alpha, \alpha \in \mathcal{L}\}$. We now take the monomial order defined as follows. For $\alpha_1 \leq_{\text{tot}} \cdots \leq_{\text{tot}} \alpha_r$ and $\beta_1 \leq_{\text{tot}} \cdots \leq_{\text{tot}} \beta_s$, we say that $X_{\alpha_1} \cdots X_{\alpha_r} \prec X_{\beta_1} \cdots X_{\beta_s}$ if and only if either $r < s$ or $r = s$ and there exists a $t < r$ such that $\alpha_1 = \beta_1, \ldots, \alpha_t = \beta_t$ with $\alpha_{t+1} <_{\text{tot}} \beta_{t+1}$. From [9] we have that $\{X_\alpha X_\beta - X_{\alpha\wedge\beta} X_{\alpha\vee\beta} \mid \alpha, \beta \in \mathcal{L} \text{ non-comparable}\}$ is a Gröbner basis for $I(\mathcal{L})$ for this monomial order. Hence, letting I be the ideal generated by initial terms of elements of $I(\mathcal{L})$, we have that $\{X_\alpha X_\beta \mid \alpha, \beta \text{ noncomparable}\}$ is a generating set for I. Let us denote $K[X(\mathcal{L})]$ by S and $K[X_\alpha, \alpha \in \mathcal{L}]/I$ by R. By [5, Sec. 15.8] we have a flat degeneration of $\operatorname{Spec}(S)$ to $\operatorname{Spec}(R)$. Hence, the degree of S equals the degree of R.

Let $J = \{j_1, \ldots, j_s\}$ be a subset of $\mathcal{L}$ such that $X_{j_1} \cdots X_{j_s} \notin I$. Note that J is thus a chain of length $s - 1$ in $\mathcal{L}$. We have

$$R = K \oplus \bigoplus_{J=\{j_1,\ldots,j_s\}} (X_{j_1} \cdots X_{j_s}) K[X_{j_1}, \ldots, X_{j_s}],$$

where J runs over all chains of any length in $\mathcal{L}$. Therefore,

$$\phi_m(R) = \dim R_m = \sum_{J=\{j_1,\ldots,j_s\}} \binom{s + (m-s) - 1}{m - s} = \sum_{J=\{j_1,\ldots,j_s\}} \binom{m-1}{s-1}.$$

Note that for m sufficiently large, the leading term appears in the summation only for J of maximal cardinality s. The result follows from this. □

Next we recall $\operatorname{mult}_P X$, the multiplicity of an algebraic variety at a point $P \in X$. Let $\mathcal{O}_{X,P} = (A, \mathfrak{m})$. Let C_P be the tangent cone at P; that is, $C_P = \operatorname{Spec} A(P)$, where $A(P) = \operatorname{gr}(A, \mathfrak{m})$. Then the multiplicity of X at P is defined to be

$$\operatorname{mult}_P X = \deg \operatorname{Proj} A(P) \ (= \deg A(P)).$$

Thus, using the notation from Section 3.7, we obtain $e_B = \operatorname{mult}_0 \operatorname{Spec}(B)$, the multiplicity of $\operatorname{Spec}(B)$ at the origin.

The following result is a direct consequence of Theorem 3.9.

THEOREM 3.10. *The multiplicity of $X(\mathcal{L})$ at the origin is equal to the number of maximal chains in $\mathcal{L}$.*

4. Cone and Dual Cone of $X(\mathcal{L})$

Let $M = \mathbb{Z}^d$ for $d = \#J(\mathcal{L})$, with basis $\{f_z, z \in J(\mathcal{L})\}$. Let N be the $\mathbb{Z}$-dual of M, with basis $\{e_z, z \in J(\mathcal{L})\}$ dual to $\{f_z, z \in J(\mathcal{L})\}$. We denote the torus acting on the toric variety $X(\mathcal{L})$ by T, and we identify M with the character group $X(T)$. Thus, for $t = (t_y)_{y \in J(\mathcal{L})} \in T$ (under the identification of T with $(K^*)^d$), we let $f_z(t) = t_z$ for $z \in J(\mathcal{L})$.

Denote by $\mathcal{I}$ the lattice of ideals of $J(\mathcal{L})$. For $A \in \mathcal{I}$, set

$$f_A := \sum_{z \in A} f_z.$$

Let $V = N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\sigma \subset V$ be the cone such that $X(\mathcal{L}) = X_\sigma$, and let $\sigma^\vee \subset V^*$ be the cone that is dual to σ. Let $S_\sigma = \sigma^\vee \cap M$, so that $K[X(\mathcal{L})]$ equals the semigroup algebra $K[S_\sigma]$.

From [10; 14, Prop. 4.6] we have the following statement.

PROPOSITION 4.1. *The semigroup S_σ is generated by f_A for $A \in \mathcal{I}$.*

Let $M(J(\mathcal{L}))$ be the set of maximal elements in the poset $J(\mathcal{L})$. Let $Z(J(\mathcal{L}))$ denote the set of all covers in the poset $J(\mathcal{L})$ (i.e., (z, z') with $z > z'$ in the poset $J(\mathcal{L})$, and there is no other element $y \in J(\mathcal{L})$ such that $z > y > z'$). For a cover $(y, y') \in Z(J(\mathcal{L}))$, denote

$$v_{y,y'} := e_{y'} - e_y.$$

PROPOSITION 4.2 [14, Prop. 4.7]. *The cone σ is generated by*

$$\{e_z,\ z \in M(J(\mathcal{L}));\ v_{y,y'},\ (y, y') \in Z(J(\mathcal{L}))\}.$$

4.3. ANALYSIS OF FACES OF σ. We shall concern ourselves just with the closed points in $X(\mathcal{L})$. So in the sequel, by a point in $X(\mathcal{L})$ we shall mean a closed point. Let τ be a face of σ. Let P_τ be the distinguished point of O_τ with the associated maximal ideal being the kernel of the map

$$K[S_\sigma] \to K, \qquad u \in S_\sigma, \quad u \mapsto \begin{cases} 1 & \text{if } u \in \tau^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for a point $P \in X(\mathcal{L})$ (identified with a point in $\mathbb{A}^l$, $l = \#\mathcal{L}$) and denoting by $P(\alpha)$ the αth coordinate of P, we have

$$P_\tau(\alpha) = \begin{cases} 1 & \text{if } f_{I_\alpha} \in \tau^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Now let

$$D_\tau = \{\alpha \in \mathcal{L} \mid P_\tau(\alpha) \neq 0\}.$$

4.4. THE BIJECTION $\mathcal{D}$ (cf. [14]). We have a bijection

$$\mathcal{D}\colon \{\text{faces of } \sigma\} \overset{\text{bij}}{\longleftrightarrow} \{\text{embedded sublattices of } \mathcal{L}\}, \quad \mathcal{D}(\tau) = D_\tau$$

PROPOSITION 4.5 [14, Prop. 4.11]. *Let τ be a face of σ. Then $\overline{O_\tau} = X(D_\tau)$.*

5. The Distributive Lattice $I_{d,n}$ and the Grassmann–Hibi Toric Variety

We now turn our focus to a particular distributive lattice—namely,

$$\mathcal{L} = I_{d,n} = \{x = (i_1, \dots, i_d) \mid 1 \le i_1 < \dots < i_d \le n\}.$$

The partial order $\ge$ on $I_{d,n}$ is given by

$$(i_1, \dots, i_d) \ge (j_1, \dots, j_d) \iff i_1 \ge j_1, \dots, i_d \ge j_d.$$

For $x \in I_{d,n}$, we denote the jth entry in x by $x(j)$, $1 \le j \le d$.

REMARK 5.1. It is a well-known fact (see e.g. [13]) that the partially ordered set $I_{d,n}$ is isomorphic to the poset determined by the set of Schubert varieties in $G_{d,n}$, the Grassmannian of d-dimensional subspaces in an n-dimensional space, where the Schubert varieties are partially ordered by inclusion. From [15, Sec. 3] we get that $I_{d,n}$ is a distributive lattice.

REMARK 5.2. Some readers may prefer to work with the lattice of Young diagrams that fit into a rectangle with d rows and $n - d$ columns, which we will denote by $\Lambda_{d,n-d}$. In this case one may go from $I_{d,n}$ to $\Lambda_{d,n-d}$ using the following bijection:

$$(i_1, \dots, i_d) \mapsto \lambda = (\lambda_1, \dots, \lambda_d),\ \lambda_1 = i_d - d,\ \lambda_2 = i_{d-1} - (d-1),\ \dots,\ \lambda_d = i_1 - 1.$$

In the next lemma, by a *segment* we shall mean a set consisting of consecutive integers.

LEMMA 5.3 [9]. *For $\mathcal{L} = I_{d,n}$, the following statements hold.*

(i) *The element $\tau = (i_1, \dots, i_d)$ is join-irreducible if and only if either τ is a segment (we shall call these elements Type I) or τ consists of two disjoint segments (μ, ν), with μ starting with 1 (Type II).*

(ii) *The element $\tau = (i_1, \dots, i_d)$ is meet-irreducible if and only if either τ is a segment or τ consists of two disjoint segments (μ, ν), with ν ending with n.*

(iii) *The element $\tau = (i_1, \dots, i_d)$ is join-irreducible and meet-irreducible if and only if either τ is a segment or τ consists of two disjoint segments (μ, ν), with μ starting with 1 and ν ending with n.*

REMARK 5.4. The join irreducible elements of $\Lambda_{d,n-d}$ are those Young diagrams that are rectangles (i.e., the nonzero rows all have the same length).

DEFINITION 5.5. We shall denote $X(I_{d,n})$ by just $X_{d,n}$ and will refer to it as a *Grassmann–Hibi toric variety,* or a G-H toric variety for short.

6. Singular Faces of the G-H Toric Variety $X_{d,n}$

Let $\mathcal{L}$ represent the distributive lattice $I_{d,n}$. From Lemma 5.3 we have that the elements of $J(\mathcal{L})$ are of two types, Type I and Type II.

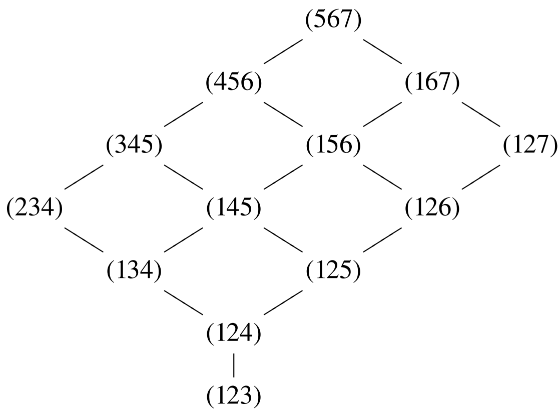
Since the generators of the cone σ are determined by $J(\mathcal{L})$ (Proposition 4.2), we will often consider $J(\mathcal{L})$ as a partially ordered set with the partial order induced from $\mathcal{L}$. Notice that $J(\mathcal{L})$ has one maximal element, which is also the maximal element of $\mathcal{L}$: $\hat{1} = (n-d+1, \dots, n)$; and $J(\mathcal{L})$ has one minimal element, which is also the minimal element of $\mathcal{L}$: $\hat{0} = (1, \dots, d)$. For each element x of $J(\mathcal{L})$, there are at most two covers of the form (y, x) in $J(\mathcal{L})$.

For example, if $x = (1, \dots, k, l+1, \dots, l+d-k) \in J(\mathcal{L})$ then we have $y = (1, \dots, k, l+2, \dots, l+d-k+1)$ and $y' = (1, \dots, k-1, l, \dots, l+d-k)$, forming the two covers of x in $J(\mathcal{L})$ (if $k = 1$, then $y' = (l, \dots, l+d-1)$). If $l = n-d+k$ or if x is of Type I, then x has only one cover.

The following lemma is a corollary of [15, Prop. 3.2].

LEMMA 6.1. *The partially ordered set $J(I_{d,n})$ is a distributive lattice.*

REMARK 6.2. As a lattice, $J(\mathcal{L})$ looks like a tessellation of diamonds in the shape of a rectangle with sides of length $d-1$ and $n-d-1$. For example, let $d = 3$ and $n = 7$. Then $J(\mathcal{L})$ is the following lattice.



As in Section 4, let σ be the cone associated to $X(\mathcal{L})$.

DEFINITION 6.3. For $1 \le i \le n-d-1$ and $1 \le j \le d-1$, let

$$\mu_{ij} = (1, \dots, j, i+j+1, \dots, i+d),$$
$$\lambda_{ij} = (i+1, \dots, i+j, n+1+j-d, \dots, n).$$

Define

$$\mathcal{L}_{ij} = \mathcal{L} \setminus [\mu_{ij}, \lambda_{ij}].$$

REMARK 6.4. (i) By [9, Lemma 11.5] we have that $\mathcal{L}_{ij}$ is an embedded sublattice.

(ii) For $\alpha, \beta \in J(\mathcal{L})$ noncomparable, $\alpha \wedge \beta = \mu_{ij}$ for some $1 \leq i \leq n-d-1$ and $1 \leq j \leq d-1$; thus every diamond in $J(\mathcal{L})$ has a μ_{ij} as its minimal element.

DEFINITION 6.5. Let σ_{ij} be the face of σ for which $D_{\sigma_{ij}} = \mathcal{L}_{ij}$.

DEFINITION 6.6. A face τ of σ is a *singular* (resp. *nonsingular*) face if P_τ is a singular (resp. nonsingular) point of X_σ.

Our first result is that σ_{ij} is a singular face. To prove this, we start by determining a set of generators for σ_{ij}.

DEFINITION 6.7. Let us denote by $W(\sigma)$ (or simply W) the set of generators for σ, as described in Proposition 4.2. For a face τ of σ, define

$$W(\tau) = \{v \in W \mid f_{I_\alpha}(v) = 0 \ \forall \alpha \in D_\tau\}.$$

(Here, D_τ is as in Section 4.4.) Then $W(\tau)$ gives a set of generators for τ.

6.8. DETERMINATION OF $W(\sigma_{ij})$. It will aid our proof below to observe a few facts about the generators of σ_{ij}. First of all, $e_{\hat{1}}$ is not a generator for any σ_{ij}, since $\hat{1} \in \mathcal{L}_{ij}$ for all $1 \leq i \leq n-d-1$ and $1 \leq j \leq d-1$ and since $e_{\hat{1}}$ is nonzero on $f_{I_{\hat{1}}}$. Similarly, for the cover $(y', \hat{0})$ where $y' = (1, \ldots, d-1, d+1)$, $e_{\hat{0}} - e_{y'}$ is not a generator for any σ_{ij}.

Second, for any cover (y', y) in $J(\mathcal{L})$, if $y \in \mathcal{L}_{ij}$ then $e_y - e_{y'}$ is not a generator of σ_{ij} because $f_{I_y}(e_y - e_{y'}) \neq 0$. Thus, in determining elements of $W(\sigma_{ij})$, we need only be concerned with elements $e_y - e_{y'}$ such that $y \in J(\mathcal{L}) \cap [\mu_{ij}, \lambda_{ij}]$. The elements of $J(\mathcal{L}) \cap [\mu_{ij}, \lambda_{ij}]$ are

$$y_t = (1, \ldots, j, i+j+1+t, \ldots, i+d+t) \quad \text{for } 0 \leq t \leq n-d-i,$$
$$z_t = (1, \ldots, j-t, i+j+1-t, \ldots, i+d) \quad \text{for } 0 \leq t \leq j.$$

Note that $y_0 = z_0 = \mu_{ij}$ and $z_j = (i+1, \ldots, i+d)$. In the next theorem we prove that $W(\sigma_{ij})$ consists of precisely four elements, forming a diamond in the distributive lattice $J(\mathcal{L})$ with μ_{ij} as the smallest element.

THEOREM 6.9. *$W(\sigma_{ij}) = \{e_{\mu_{ij}} - e_A, e_{\mu_{ij}} - e_B, e_A - e_C, e_B - e_C\}$, where A, B, and C are defined in the proof.*

Proof. We divide the proof into two cases, $j = 1$ and $j > 1$.

Case 1. Let $j = 1$ and $1 \leq i \leq n-d-1$. Here we have

$$\mu_{ij} = (1, i+2, \ldots, i+d) \quad \text{and} \quad \lambda_{ij} = (i+1, n-d+2, \ldots, n).$$

As discussed previously, we find that μ_{ij} is covered in $J(\mathcal{L})$ by $A = (i+1, \dots, i+d)$ and $B = (1, i+3, \dots, i+d+1)$. We have that both A and B are in the interval $[\mu_{ij}, \lambda_{ij}]$. Let C be the join of A and B in the lattice $J(\mathcal{L})$:

$$C = (i+2, \dots, i+d+1).$$

Note that (C, A) and (C, B) are covers in $J(\mathcal{L})$.

We first observe that,

$$\text{for } x = (x_1, \dots, x_d) \in \mathcal{L}_{ij}, \quad \text{if } x \geq \mu_{ij} \text{ then } x \geq C. \tag{6.1}$$

(This follows because $x \geq \mu_{ij}$ and $x \in \mathcal{L}_{ij}$ imply that $x \not\leq \lambda_{ij}$ and hence $x_1 \geq i+2$.)

Claim (i): $e_{\mu_{ij}} - e_A$ and $e_{\mu_{ij}} - e_B$ are both in $W(\sigma_{ij})$. We shall prove the claim for $e_{\mu_{ij}} - e_A$ (the proof for $e_{\mu_{ij}-e_B}$ is similar). To prove that $e_{\mu_{ij}} - e_A$ is in $W(\sigma_{ij})$, we need to show that there does not exist an $x = (x_1, \dots, x_d) \in \mathcal{L}_{ij}$ such that $x \geq \mu_{ij}$ and $x \not\geq A$. But this follows from (6.1) (which implies that, for $x = (x_1, \dots, x_d) \in \mathcal{L}_{ij}$, if $x \geq \mu_{ij}$ then $x \geq A$).

Claim (ii): $e_A - e_C$ and $e_B - e_C$ are in $W(\sigma_{ij})$. The proof is similar to that of Claim (i). Again we show the result for $e_A - e_C$ (the proof for $e_B - e_C$ is similar). We must demonstrate that there does not exist an $x = (x_1, \dots, x_d) \in \mathcal{L}_{ij}$ such that $x \geq A$ but $x \not\geq C$. Again this follows from (6.1) (note that $x \geq A$ implies in particular that $x \geq \mu_{ij}$).

Claim (iii): $W(\sigma_{ij}) = \{e_{\mu_{ij}} - e_A, e_{\mu_{ij}} - e_B, e_A - e_C, e_B - e_C\}$. In the case under consideration, since $j = 1$ it follows that the only elements of $J(\mathcal{L}) \cap [\mu_{ij}, \lambda_{ij}]$ are of the following forms:

$$y_t = (1, i+t+2, \dots, i+d+t) \quad \text{for } 0 \leq t \leq n-d-i;$$
$$z_1 = (i+1, \dots, i+d).$$

Let $y'_t = (i+t+1, \dots, i+d+t)$ for $1 \leq t \leq n-d-i$; thus we have covers of type (y'_t, y_t) for $0 \leq t \leq n-d-i$ and of type (y_{t+1}, y_t) for $0 \leq t \leq n-d-1-i$. Observe that $y_0 = \mu_{ij}$, $y_1 = B$, $z_1 = y'_0 = A$, and $y'_1 = C$. In Claims (i) and (ii) we have shown that the covers (y_1, y_0), (y'_0, y_0), (y'_1, y_1), and (y'_1, z_1) yield elements of $W(\sigma_{ij})$. Also note that C is the only cover of A. Hence, it only remains to show that $e_{y_t} - e_{y'_t} \notin W(\sigma_{ij})$ for $2 \leq t \leq n-d-i$ and that $e_{y_t} - e_{y_{t+1}} \notin W(\sigma_{ij})$ for $1 \leq t \leq n-d-1-i$. For each of these covers, we shall exhibit an $x \in \mathcal{L}_{ij}$ such that f_{I_x} is nonzero on the cover under consideration.

Define $x_t = (i+t, i+t+2, \dots, i+d+t)$; then $x_t \in \mathcal{L}_{ij}$ for $2 \leq t \leq n-d-i$. Furthermore, $f_{I_{x_t}}$ is nonzero on $e_{y_t} - e_{y_{t+1}}$ for $2 \leq t \leq n-d-i-1$ and on $e_{y_t} - e_{y'_t}$ for $2 \leq t \leq n-d-i$. For (y_2, y_1), note that $C \in \mathcal{L}_{ij}$ and f_{I_C} is nonzero on $e_{y_1} - e_{y_2}$. This completes the proof of Case 1.

Case 2. Now let $2 \leq j \leq d-1$ and $1 \leq i \leq n-d-1$. We have

$$\mu_{ij} = (1, \dots, j, i+j+1, \dots, i+d),$$
$$\lambda_{ij} = (i+1, \dots, i+j, n+1+j-d, \dots, n).$$

As in Case 1, we look for covers of μ_{ij} in $J(\mathcal{L})$. They are $A = (1,\dots,j-1, i+j,\dots,i+d)$ and $B = (1,\dots,j,i+j+2,\dots,i+d+1)$. Define C to be the join of A and B in the lattice $J(\mathcal{L})$; thus,

$$C = (1,\dots,j-1,i+j+1,\dots,i+d+1).$$

Claim (iv): $\{e_{\mu_{ij}} - e_A, e_{\mu_{ij}} - e_B, e_A - e_C, e_B - e_C\}$ *are in* $W(\sigma_{ij})$. We first observe that,

$$\text{for } x = (x_1,\dots,x_d) \in \mathcal{L}_{ij}, \quad \text{if } x \ge \mu_{ij} \text{ then } x \ge C. \tag{6.2}$$

For suppose that $x \not\ge C$; now $x \ge \mu_{ij}$ and $x \in \mathcal{L}_{ij}$ together imply that $x \not\le \lambda_{ij}$ and thus $x_l > i+l$ for some $1 \le l \le j$. Also, $x \not\ge C$; hence $x_k < i+k+1$ for some $j \le k \le d$. Therefore,

$$x = (x_1,\dots,x_{l-1},x_l > i+l,\, x_{l+1} > i+l+1,\,\dots,\, x_{k-1} > i+k-1,\\ i+k+1 > x_k > i+k,\dots).$$

Clearly, no such x_k exists and thus (6.2) follows.

By (6.2) we have that, if $x \in \mathcal{L}_{ij}$ is such that $x \ge \mu_{ij}$, then $x \ge A, B, C$. Hence Claim (iv) follows.

Claim (v): $W(\sigma_{ij}) = \{e_{\mu_{ij}} - e_A, e_{\mu_{ij}} - e_B, e_A - e_C, e_B - e_C\}$. As in Claim (iii), we will show that all other covers in $J(\mathcal{L})$ of the form (y', y), $y \in J(\mathcal{L}) \cap [\mu_{ij}, \lambda_{ij}]$, are not in $W(\sigma_{ij})$. As in Section 6.8, all of the elements of $J(\mathcal{L}) \cap [\mu_{ij}, \lambda_{ij}]$ are

$$y_t = (1,\dots,j,i+j+1+t,\dots,i+d+t) \quad \text{for } 0 \le t \le n-d-i,$$
$$z_t = (1,\dots,j-t,i+j+1-t,\dots,i+d) \quad \text{for } 0 \le t \le j$$

(note that $z_j = (i+1,\dots,i+d)$). We will examine covers of these elements; notice that $y_0 = z_0 = \mu_{ij}$, $z_1 = A$, and $y_1 = B$.

Let $z'_t = (1,\dots,j-t,i+j+2-t,\dots,i+d+1)$ for $1 \le t \le n-d-i$, and let $z'_j = (i+2,\dots,i+d+1)$. First we want to show that the covers $(z_{t+1}, z_t)_{1 \le t \le j-1}$ and $(z'_t, z_t)_{2 \le t \le j}$ do not yield elements in $W(\sigma_{ij})$. Observe that $(z'_1, z_1) = (C, A)$ and $e_A - e_C \in W(\sigma_{ij})$. Also, $C \in \mathcal{L}_{ij}$ and $f_{I_C}(e_{z_1} - e_{z_2})$ is nonzero; thus $e_{z_1} - e_{z_2} \notin W(\sigma_{ij})$, and we may restrict our attention to $t \ge 2$. Let

$$x_t = (1,\dots,j-t,i+j+1-t,n-d+j-t+2,\dots,n) \quad \text{for } 2 \le t \le j-1,$$
$$x_j = (i+1,n-d+2,\dots,n).$$

Now, on the interval $2 \le t \le j-1$, we have the following facts:

(1) $x_t \ge z_t$,
(2) $x_t \not\ge z_{t+1}$,
(3) $x_t \not\ge z'_t$,
(4) $x_t \not\le \lambda_{ij}$.

Facts (1), (3), and (4) hold for the case $t = j$; it is just a separate check. Hence, for $2 \le t \le j$ (resp. $2 \le t \le j-1$) we have that $x_t \in \mathcal{L}_{ij}$ and $f_{I_{x_t}}$ is nonzero on $e_{z_t} - e_{z'_t}$ (resp. $e_{z_t} - e_{z_{t+1}}$).

Next, we must concern ourselves with covers involving y_t. Define

$$y'_t = (1, \dots, j-1, i+j+t, \dots, i+d+t) \quad \text{for } 1 \le t \le n-d-i.$$

To complete Claim (v), we must show that the covers

$$(y_{t+1}, y_t)_{1 \le t \le n-d-i-1} \quad \text{and} \quad (y'_t, y_t)_{2 \le t \le n-d-i}$$

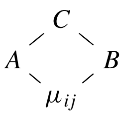
do not yield elements of $W(\sigma_{ij})$. Note that $(y'_1, y_1) = (C, B)$ and thus does yield an element of $W(\sigma_{ij})$. Also, $f_{I_C}(e_{y_1} - e_{y_2})$ is nonzero; we can therefore restrict our attention to $t \ge 2$. Let $x'_t = (1, \dots, j-1, i+j+1, i+j+t+1, \dots, i+d+t)$. On the interval $2 \le t \le n-d-i$, we have the following facts:

(1′) $x'_t \ge y_t$,
(2′) $x'_t \not\ge y'_t$,
(3′) $x'_t \not\ge y_{t+1}$ for $t \le n-d-i-1$,
(4′) $x'_t \not\le \lambda_{ij}$.

Therefore, on the interval $2 \le t \le n-d-i$ (resp. $2 \le t \le n-d-i-1$), we have that $x'_t \in \mathcal{L}_{ij}$ and $f_{I_{x'_t}}$ is nonzero on $e_{y_t} - e_{y'_t}$ (resp. $e_{y_t} - e_{y_{t+1}}$).

This completes Claim (v), Case 2, and the proof of Theorem 6.9. □

REMARK 6.10. The face σ_{ij} corresponds to the following diamond in $J(\mathcal{L})$.



This diamond is a poset of rank 2.

LEMMA 6.11. *The face σ_{ij} has dimension 3.*

Proof. We have $\{e_{\mu_{ij}} - e_A, e_{\mu_{ij}} - e_B, e_A - e_C, e_B - e_C\}$, a set of generators for σ_{ij}. We can see that a subset of three of these generators is linearly independent. Thus, if the fourth generator can be put in terms of the first three, the result follows. Notice that

$$(e_{\mu_{ij}} - e_A) - (e_{\mu_{ij}} - e_B) + (e_A - e_C) = e_B - e_C. \qquad \square$$

Our next theorem is an immediate consequence of Theorem 6.9 and Lemma 6.11.

THEOREM 6.12. *We have an identification of the (open) affine piece in $X(\mathcal{L})$ corresponding to the face $\sigma_{i,j}$ with the product $Z \times (K^*)^{\#J(\mathcal{L})-3}$, where Z is the cone over the quadric surface $x_1x_4 - x_2x_3 = 0$ in $\mathbb{P}^3$.*

We now prove two lemmas that hold for a general toric variety.

LEMMA 6.13. *Let X_τ be an affine toric variety with τ as the associated cone. Then X_τ is a nonsingular variety if and only if it is nonsingular at the distinguished point P_τ.*

Proof. Only the $\Leftarrow$ implication requires a proof. Let then P_τ be a smooth point. Let us assume (if possible) that Sing $X_\tau \ne \varnothing$. We have the following facts.

- Sing X_τ is a closed T-stable subset of X_τ.
- $P_\tau \in \overline{O_\theta}$ for every face θ of τ (see Section 1.6); in particular, $P_\tau \in \overline{O_\theta}$ for some face θ such that P_θ is a singular point (such a θ exists because, by our assumption, Sing X_τ is nonempty).

We thus obtain that $P_\tau \in$ Sing X_τ, a contradiction. Hence our assumption is wrong and the result follows. □

LEMMA 6.14. *Let τ be a face of σ (for σ a convex polyhedral cone). Then P_τ is a smooth point of X_σ if and only if P_τ is a smooth point of X_τ—that is, if and only if τ is generated by a part of a basis of N (where N is the $\mathbb{Z}$-dual of the character group of the torus).*

Proof. We have that X_τ is a principal open subset of X_σ. Hence X_σ is nonsingular at P_τ if and only if X_τ is nonsingular at P_τ. By Lemma 6.13, X_τ is nonsingular at P_τ if and only if X_τ is a nonsingular variety; but by [7, Sec. 2.1], this is true if and only if τ is generated by a part of a basis of N. □

We now return to the case where σ is the convex polyhedral cone associated to $X(I_{d,n})$.

THEOREM 6.15. *Let $\tau = \sigma_{i,j}$. Then the following statements hold.*

(i) $P_\tau \in$ Sing X_σ.
(ii) *We have an identification of $TC_{P_\tau}X_\sigma$ with $Z \times (K^*)^{\#J(\mathcal{L})-3}$, with Z as in Theorem 6.12; furthermore, $TC_{P_\tau}X_\sigma$ is a toric variety.*
(iii) *The singularity at P_τ is of the same type as that at the vertex of the cone over the quadric surface $x_1x_4 - x_2x_3 = 0$ in $\mathbb{P}^3$. In particular,* $\operatorname{mult}_{P_\tau} X_\sigma = 2$.

Proof. Assertion (i) follows from Lemma 6.13, Lemma 6.14, and Theorem 6.12. Because X_τ is open in X_σ, we may identify $TC_{P_\tau}X_\sigma$ with $TC_{P_\tau}X_\tau$, which in turn coincides with X_τ (since X_τ is of cone type, where P_τ is identified with the origin). Assertion (ii) follows from this in view of Theorem 6.12 and given that X_τ is a toric variety. Assertion (iii) is immediate from (ii). □

Next, we will show that the faces containing some σ_{ij} are the only singular faces. We first prove some preparatory lemmas.

LEMMA 6.16. *Let $A \neq \hat{0}$. If $e_A - e_C$ is in W (the set of generators of σ as described in Proposition 4.2), then $e_A - e_C$ is in $W(\sigma_{ij})$ (cf. Definition 6.7) for some (i,j), where $1 \le i \le n-d-1$ and $1 \le j \le d-1$.*

Proof. If A is equal to some μ_{ij}, then C must be one of the two covers of $\mu_{ij} = (1,\dots,j,i+j+1,\dots,i+d)$ in $J(\mathcal{L})$ and we are done by Theorem 6.9. So we will assume that $A \neq \mu_{ij}$. Hence A is a join irreducible of one of the following two forms.

Case 1: $A = (1,\dots,k,n-d+k+1,\dots,n)$ for some k. Then $\mu_{n-d-1,k} = (1,\dots,k,n-d+k,\dots,n-1)$, and $(A,\mu_{n-d-1,k})$ is a cover in $J(\mathcal{L})$. Also, A has only one cover in $J(\mathcal{L})$, which must be C; thus $e_A - e_C$ is an element of $W(\sigma_{n-d-1,k})$, as shown in Theorem 6.9.

Case 2: $A = (k+1, \dots, k+d)$, $1 \leq k \leq n-d-1$ (*note that* $k < n-d$, *since* $C > A$ *because* $e_A - e_C \in W$). Then we have $\mu_{k,1} = (1, k+2, \dots, k+d)$, and $(A, \mu_{k,1})$ is a cover in $J(\mathcal{L})$. Also, we must have $C = (k+2, \dots, k+d+1)$, and $e_A - e_C$ is an element of $W(\sigma_{k,1})$ by Case 1 of Theorem 6.9. □

We now return to the case of a Grassmann–Hibi toric variety.

THEOREM 6.17. *Let* τ *be a face such that* D_τ *is not contained in any* $\mathcal{L}_{ij}$ *for* $1 \leq i \leq n-d-1$ *and* $1 \leq j \leq d-1$. *Then the associated face* τ *is nonsingular* (*i.e., if a face* τ *does not contain any one* σ_{ij}, *then* τ *is nonsingular*).

Proof. By Lemma 6.14, for τ to be nonsingular it must be generated by part of a basis for N. Since τ is generated by a subset $W(\tau)$ of W, for τ to be singular its generators would have to be linearly dependent. (Generally this is not enough to prove that a face is singular or nonsingular, but since all generators in W have coefficients equal to ± 1, any linearly independent set will serve as part of a basis for N.) Suppose τ is singular; then there is some subset of the elements of $W(\tau)$ equal to $\{e_1 - e_2, \dots\}$ such that $\sum a_{ij}(e_i - e_j) = 0$, with coefficients a_{ij} nonzero for at least one (i, j).

Recall that the elements of W can be represented as all the line segments in the lattice $J(\mathcal{L})$ with the exception of $e_{\hat{1}}$ (see diagram in Remark 6.2). Therefore, the linearly dependent generators of τ must represent a "loop" of line segments in $J(\mathcal{L})$. This loop will have at least one bottom corner, left corner, top corner, and right corner.

Choose some particular $\mathcal{L}_{ij}$. By Theorem 6.9, $W(\sigma_{ij}) = \{e_{\mu_{ij}} - e_A, e_{\mu_{ij}} - e_B, e_A - e_C, e_B - e_C\}$. These four generators are represented by the four sides of a diamond in $J(\mathcal{L})$. Thus, by hypothesis, the generators of τ represent a loop in $J(\mathcal{L})$ that does not traverse all four sides of the diamond representing all four generators of σ_{ij}.

By hypothesis, D_τ is not contained in any $\mathcal{L}_{ij}$ for $1 \leq i \leq n-d-1$ and $1 \leq j \leq d-1$; hence there must be at least one element of D_τ in the interval $[\mu_{ij}, \lambda_{ij}]$, say $\alpha \in [\mu_{ij}, \lambda_{ij}]$. We have $\alpha \geq \mu_{ij}$ and $\alpha \not\geq C$ for C as defined in the proof of Theorem 6.9. Based on how α compares with both A and B, we can eliminate certain elements of W from $W(\tau)$. There are four possibilities; we list all four, as well as the corresponding generators in $W(\sigma_{ij})$ that are not in $W(\tau)$ (i.e., those generators v in $W(\sigma_{ij})$ such that $f_{I_\alpha}(v) \neq 0$):

$$\alpha \not\geq A, \alpha \not\geq B \implies e_{\mu_{ij}} - e_A, e_{\mu_{ij}} - e_B \notin W(\tau);$$
$$\alpha \geq A, \alpha \not\geq B \implies e_A - e_C, e_{\mu_{ij}} - e_B \notin W(\tau);$$
$$\alpha \not\geq A, \alpha \geq B \implies e_{\mu_{ij}} - e_A, e_B - e_C \notin W(\tau);$$
$$\alpha \geq A, \alpha \geq B \implies e_A - e_C, e_B - e_C \notin W(\tau).$$

Therefore, it is impossible to have $\{e_{\mu_{ij}} - e_A, e_A - e_C\}$ or $\{e_{\mu_{ij}} - e_B, e_B - e_C\}$ contained in $W(\tau)$. This is true for any (i, j) and so, in view of Lemma 6.16, our "loop" in $J(\mathcal{L})$ that represented the generators of τ cannot have a left corner or a right corner. Thus it is really not possible to have a loop at all; hence the generators of τ are linearly independent, and the result follows. □

Corollary 6.18. *The G-H toric variety $X_{d,n}$ is smooth along the orbit O_τ if and only if the face τ does not contain any σ_{ij}.*

Combining this corollary with Theorem 6.15 and Lemma 6.11 yields our first main theorem as follows.

Theorem 6.19. *Let $\mathcal{L} = I_{d,n}$. Then the following statements hold.*

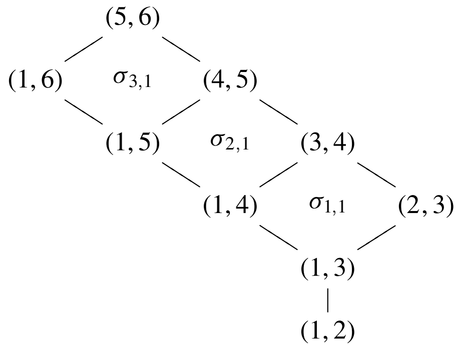
(i) Sing $X(\mathcal{L}) = \bigcup_{\sigma_{i,j}} \bar{O}_{\sigma_{i,j}}$, *where the union is taken over all the* $\sigma_{i,j}$ (*as in Theorem 6.9*).
(ii) Sing $X(\mathcal{L})$ *is pure and of codimension 3 in* $X(\mathcal{L})$, *and the generic singularities are of cone type* (*more precisely, the singularity type is the same as that at the vertex of the cone over the quadric surface* $x_1x_4 - x_2x_3 = 0$ *in* $\mathbb{P}^3$).
(iii) *For* $\tau = \sigma_{i,j}$, $TC_{P_\tau}X(\mathcal{L})$ *is a toric variety and* $\mathrm{mult}_{P_\tau} X(\mathcal{L}) = 2$.

Remark 6.20. Theorem 6.19 thus proves the conjecture of [9] using just the combinatorics of the cone associated to the toric variety $X_{d,n}$ (for a statement of the conjecture of [9], see Remark 9.1). Further, it gives a description of Sing $X_{d,n}$ purely in terms of the faces of the cone associated to $X_{d,n}$.

7. Multiplicities of Singular Faces of $X_{2,n}$

In this section we take $\mathcal{L} = I_{2,n}$, determine the multiplicity of $X_{2,n}$ $(= X(I_{2,n}))$ at P_τ for certain of the singular faces of $X_{2,n}$, and deduce a product formula. For $I_{d,n}$ we have defined $\mathcal{L}_{ij}$ and the corresponding face $\sigma_{i,j}$ for $1 \le j \le d-1$ and $1 \le i \le n-d-1$; hence, for $I_{2,n}$ we need only consider $\mathcal{L}_{i,1}$ for $1 \le i \le n-3$.

For example, the following diagram is the poset of join irreducibles for $I_{2,6}$. We write $\sigma_{i,1}$ inside each diamond because the four segments surrounding it represent the four generators of the face.



In order to go from the join irreducibles of $I_{2,6}$ to $I_{2,7}$, we simply add $(1,7)$ and $(6,7)$ to the poset above, forming $\sigma_{4,1}$. We will see that this makes the calculation of the multiplicities of singular faces of $I_{2,n}$ much easier.

In the sequel, we shall denote the set of join irreducibles of $I_{2,n}$ by $J_{2,n}$; also, as in the previous sections, σ will denote the polyhedral cone corresponding to $X_{2,n}$.

7.1. $\text{mult}_{P_\sigma} X_{2,n}$. Because $X_{d,n}$ is now of cone type (i.e., the vanishing ideal is homogeneous), we have a canonical identification of $T_{P_\sigma} X_{d,n}$ (the tangent cone to $X_{d,n}$ at P_σ) with $X_{d,n}$. Hence, by Theorem 3.10, $\text{mult}_{P_\sigma} X_{d,n}$ equals the number of maximal chains in $I_{d,n}$. So we begin by counting the number of maximal chains in $I_{2,n}$.

As we move through a chain from $(1,2)$, at any point (i, j) we have at most two possibilities for the next point, $(i+1, j)$ or $(i, j+1)$. For each cover in our chain, we assign a value: for a cover of type $((i, j+1), (i, j))$, we assign +1; for a cover of type $((i+1, j), (i, j))$, we assign −1.

A maximal chain C in $I_{2,n}$ contains $2n-3$ lattice points, so every chain can be uniquely represented by a $(2n-4)$-tuple of 1s and −1s; let us denote this $(2n-4)$-tuple by $n_C = \langle a_1, \dots, a_{2n-4} \rangle$.

For any such n_C, it is clear that 1 and −1 occur precisely $n-2$ times. Also, we can see that $a_1 = +1$ and that, for any $1 \le k \le 2n-4$, if $\{a_1, \dots, a_k\}$ contains more −1s than +1s then we have arrived at a point (i, j) with $i > j$, which is not a lattice point. Thus, we must have $a_1 + \dots + a_k \ge 0$ for every $1 \le k \le 2n-4$.

THEOREM 7.2 [16, Cor. 6.2.3]. *The Catalan number*

$$\text{Cat}_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \ge 0,$$

counts the number of sequences $a_1, \dots, a_{2n}$ *of 1s and −1s with*

$$a_1 + \dots + a_k \ge 0 \quad (k = 1, 2, \dots, 2n),$$

and $a_1 + \dots + a_{2n} = 0$.

COROLLARY 7.3. *The multiplicity of* $X_{2,n}$ *at* P_σ *is equal to the Catalan number*

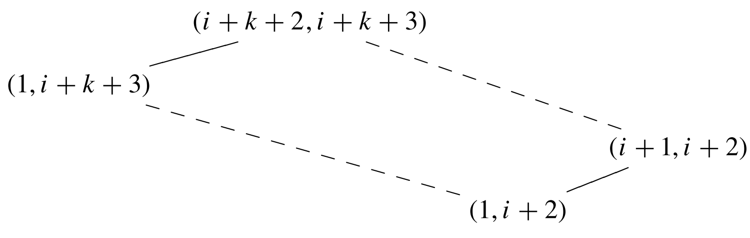
$$\text{Cat}_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

7.4. $\text{mult}_{P_\tau} X_{2,n}$. Next we shall determine $\text{mult}_{P_\tau} X_{2,n}$ for τ of block type (see Definition 7.7 to follow). Let τ be a face of σ such that the associated (embedded sublattice) D_τ is of the form

$$\begin{aligned} D_\tau &= [(1,2),(i,i+1)] \cup [(i+k+2, i+k+3),(n-1,n)] \\ &= I_1 \cup I_2 \quad \text{(say)}, \end{aligned}$$

where $I_1 = [(1,2),(i,i+1)]$ and $I_2 = [(i+k+2, i+k+3),(n-1,n)]$ for $1 \le i \le n-3$ and $0 \le k \le n-i-3$.

We shall now determine $W(\tau)$ (cf. Section 6.7). Let A_τ denote the interval $[(1, i+2), (i+k+2, i+k+3)]$ in $J_{2,n}$:



LEMMA 7.5. *With τ as just described, we have that $W(\tau) = \{e_{y'} - e_y \mid (y, y')$ is a cover in $A_\tau\}$.*

Proof. Clearly, $e_{(n-1,n)}$ (the element in $W(\sigma)$ corresponding to the unique maximal element $(n-1, n)$ in $J_{2,n}$) is not in $W(\tau)$, since $(n-1, n) \in D_\tau$. Let us denote

$$\theta = (i + k + 2, i + k + 3) \quad \text{and} \quad \delta = (i, i + 1).$$

Claim 1: For a cover (y, y') in A_τ, $f_{I_\alpha}(e_{y'} - e_y) = 0$ for all $\alpha \in D_\tau$. The claim follows in view of the following facts for a cover (y, y') in A_τ:

- $y, y' \in I_\theta$ and hence $y, y' \in I_\alpha$ for all $\alpha \in I_2$;
- $y, y' \notin I_\delta$ and hence $y, y' \notin I_\alpha$ for all $\alpha \in I_1$.

Claim 2: For a cover (y, y') in $J_{2,n}$ not contained in A_τ, there exists an $\alpha \in D_\tau$ such that $f_{I_\alpha}(e_{y'} - e_y) \neq 0$. Note that a cover in $J_{2,n}$ is one of the following three types.

Type I: $((1, j), (1, j-1)), 3 \le j \le n.$
Type II: $((j-1, j), (j-2, j-1)), 4 \le j \le n.$
Type III: $((j-1, j), (1, j)), 3 \le j \le n.$

Let now (y, y') be a cover not contained in A_τ.

If (y, y') is of Type I, then $(y, y') = ((1, j), (1, j-1))$, where either $j \le i + 2$ or $j \ge i + k + 4$. Letting

$$\alpha = \begin{cases} (1, j-1) & \text{if } j \le i + 2, \\ (j-2, j-1) & \text{if } j \ge i + k + 4, \end{cases}$$

we have $\alpha \in D_\tau$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

If (y, y') is of Type II, then $(y, y') = ((j-1, j), (j-2, j-1))$, where either $j \le i + 2$ or $j \ge i + k + 4$. Letting $\alpha = (j-2, j-1)$, we have $\alpha \in D_\tau$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

If (y, y') is of Type III, then $(y, y') = ((j-1, j), (1, j))$, where either $j \le i + 1$ or $j \ge i + k + 4$. Letting

$$\alpha = \begin{cases} (1, j) & \text{if } j \le i + 1, \\ (j-2, j) & \text{if } j \ge i + k + 4, \end{cases}$$

we have $\alpha \in D_\tau$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

The required result now follows from Claims 1 and 2. □

COROLLARY 7.6. *With τ as in Lemma 7.5, we have*

$$\tau = \sigma_{i,1} \cup \sigma_{i+1,1} \cup \cdots \cup \sigma_{i+k,1}.$$

DEFINITION 7.7. We define a face τ as in Lemma 7.5 as a *J-block* (i.e., τ is a union of consecutive $\sigma_{i,1}$).

REMARK 7.8. Note that a union of faces need not be a face.

7.9. THE HIBI VARIETY $Z_{2,r}$. For an integer $r \ge 3$, let $\widetilde{I_{2,r}}$ denote the distributive lattice $I_{2,r} \setminus \{(1, 2), (r-1, r)\}$. We define $Z_{2,r}$ to be the Hibi variety associated to $\widetilde{I_{2,r}}$. Note (cf. Proposition 4.2) that the cone associated to $Z_{2,r}$ has a set of generators consisting of $\{e_{y'} - e_y\}$, where (y, y') is a cover in the sublattice $[(1,3),(r-1,r)]$ of $J_{2,r}$ (the set of join irreducibles of $I_{2,r}$). In view of Theorem 3.10 we have

$$\text{mult}_{\mathbf{0}}\, Z_{2,r} = \text{mult}_{P_\sigma}\, X_{2,r} = \text{Cat}_{r-2},$$

where $\mathbf{0}$ denotes the origin.

THEOREM 7.10. *Let τ be a face of σ that is a "J-block" of $k+1$ consecutive $\sigma_{i,1}$ (as in Definition 7.7). We have an identification of X_τ (the open affine piece of X_σ corresponding to τ) with $Z_{2,k+4} \times (K^*)^m$, where $m = \text{codim}_\sigma\, \tau = 2(n-k) - 6$.*

Proof. In view of Section 1.6 and Proposition 4.5, we have

$$\text{codim}_\sigma\, \tau = \dim X(D_\tau) = \#\{\text{elements in a maximal chain in } D_\tau\}.$$

From this it is clear that $\text{codim}_\sigma\, \tau = 2(n-k) - 6$. Next, in view of Lemma 7.5 and Section 7.9, we obtain an identification of X_τ with $Z_{2,k+4} \times (K^*)^m$ (for m as in the theorem). □

THEOREM 7.11. *Let τ be as in Theorem 7.10.*

(i) *We have an identification of $TC_{P_\tau} X_\sigma$ with $Z_{2,k+4} \times (K^*)^m$, where $m =$ $\text{codim}_\sigma\, \tau = 2(n-k) - 6$; also, $TC_{P_\tau} X_\sigma$ is a toric variety.*
(ii) $\text{mult}_{P_\tau}\, X_{2,n} = \text{Cat}_{k+2} = \frac{1}{k+3}\binom{2k+4}{k+2}$.

Proof. Since X_τ is open in X_σ, we may identify $TC_{P_\tau} X_\sigma$ with $TC_{P_\tau} X_\tau$, which in turn coincides with X_τ (because X_τ is of cone type, where P_τ is identified with the origin). Assertion (i) follows from this in view of Theorem 7.10 (and the fact that X_τ is a toric variety).

Assertion (ii) follows from (i) and Corollary 7.3. □

7.12. A PRODUCT FORMULA. Here we give a product formula for $\text{mult}_{P_\tau}\, X_{2,n}$, where τ is a union of pairwise nonintersecting and nonconsecutive J-blocks (see Remark 7.15).

Let τ be a face of σ such that the associated (embedded sublattice) D_τ is of the form

$$\begin{aligned} D_\tau &= [(1,2),(i_1,i_1+1)] \cup [(i_1+k_1+2, i_1+k_1+3),(i_2,i_2+1)] \\ &\quad \cup [(i_2+k_2+2, i_2+k_2+3),(n-1,n)] \\ &= J_1 \cup J_2 \cup J_3 \quad \text{(say)}, \end{aligned}$$

where $i_1 + k_1 + 1 < i_2$ and where

$$\begin{aligned} J_1 &= [(1,2),(i_1,i_1+1)], \\ J_2 &= [(i_1+k_1+2, i_1+k_1+3),(i_2,i_2+1)], \\ J_3 &= [(i_2+k_2+2, i_2+k_2+3),(n-1,n)]. \end{aligned}$$

Consider the following sublattices in $J_{2,n}$ (the set of join irreducibles in $I_{2,n}$):

$$\begin{aligned} A &= [(1, i_1+2),(i_1+k_1+2, i_1+k_1+3)], \\ B &= [(1, i_2+2),(i_2+k_2+2, i_2+k_2+3)]. \end{aligned}$$

Lemma 7.13. *With τ as before, we have $W(\tau) = \{e_{y'} - e_y \mid (y, y')$ is a cover in $A \cup B\}$.*

Proof. We proceed as in the proof of Lemma 7.5, where $e_{(n-1,n)}$ is not in $W(\tau)$ (since $(n-1, n) \in D_\tau$). Let us denote:

$$\theta_1 = (i_1 + k_1 + 2, i_1 + k_1 + 3), \qquad \theta_2 = (i_2 + k_2 + 2, i_2 + k_2 + 3);$$
$$\delta_1 = (i_1, i_1 + 1), \qquad \delta_2 = (i_2, i_2 + 1).$$

For any cover (y, y') in $A \cup B$, we clearly have $y, y' \in I_{\theta_2}$ and hence $y, y' \in I_\alpha$ for all $\alpha \in J_3$; also, $y, y' \notin I_{\delta_1}$ and hence $y, y' \notin I_\alpha$ for all $\alpha \in J_1$. Thus we obtain that

$$f_{I_\alpha}(e_{y'} - e_y) = 0 \quad \text{for all } \alpha \in J_1 \cup J_3. \tag{7.1}$$

Next, if (y, y') is a cover in A, then $y, y' \in I_{\theta_1}$ and hence $y, y' \in I_\alpha$ for all $\alpha \in J_2$. If (y, y') is a cover in B, then $y, y' \notin I_{\delta_2}$ and hence $y, y' \notin I_\alpha$ for all $\alpha \in J_2$. Note that θ_1 (resp. δ_2) is the smallest (resp. largest) element in J_2. Therefore,

$$f_{I_\alpha}(e_{y'} - e_y) = 0 \quad \text{for all } \alpha \in J_2 \tag{7.2}$$

Together, (7.1) and (7.2) imply the inclusion "$\supseteq$". We shall prove the inclusion "$\subseteq$" by showing that, if a cover (y, y') is not contained in $A \cup B$, then there exists an $\alpha \in D_\tau$ such that $f_{I_\alpha}(e_{y'} - e_y) \neq 0$. This proof runs along lines similar to the proof of Lemma 7.5. Let then (y, y') be a cover in $J_{2,n}$ not contained in $A \cup B$. It is convenient to introduce the following sublattices in $J_{2,n}$:

$$P = [(1, 2), (i_1 + 1, i_1 + 2)],$$
$$Q = [(1, i_1 + k_1 + 3), (i_2 + 1, i_2 + 2)],$$
$$R = [(1, i_2 + k_2 + 3), (n - 1, n)].$$

We distinguish three cases as follows.

Case 1: (y, y') is of type I (cf. proof of Lemma 7.5)—say, $((1, j), (1, j - 1))$.

(i) If (y, y') is contained in P, then $j \le i_1 + 2$. We let $\alpha = (1, j - 1)$. Note that $\alpha \in J_1$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

(ii) If (y, y') is contained in Q (resp. R), then $i_1 + k_1 + 4 \le j \le i_2 + 2$ (resp. $i_2 + k_2 + 4 \le j \le n$). We let $\alpha = (j - 2, j - 1)$. Note that $\alpha \in J_2$ (resp. J_3) and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

Case 2: (y, y') is of type II—say, $((j - 1, j), (j - 2, j - 1))$. Then $3 \le j \le i_1 + 2$, $i_1 + k_1 + 4 \le j \le i_2 + 2$, or $i_2 + k_2 + 4 \le j \le n$ accordingly as (y, y') is contained in P, Q, or R. We let $\alpha = (j - 2, j - 1)$. Note that $\alpha \in J_1, J_2, J_3$ accordingly as (y, y') is contained in P, Q, R, and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

Case 3: (y, y') is of type III—say, $((j - 1, j), (1, j))$.

(i) If (y, y') is contained in P, then $j \le i_1 + 1$. We let $\alpha = (1, j)$. Note that $\alpha \in J_1$ and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$.

(ii) If (y, y') is contained in Q (resp. R), then $i_1 + k_1 + 4 \le j \le i_2 + 1$ (resp. $i_2 + k_2 + 4 \le j \le n$). We let $\alpha = (j - 2, j)$. Note that $\alpha \in J_2, J_3$ accordingly as (y, y') is contained in Q, R, and $f_{I_\alpha}(e_{y'} - e_y) \neq 0$. □

As an immediate consequence of Lemma 7.13 and Corollary 7.6, we have the following result.

COROLLARY 7.14. *Let τ be as in Lemma 7.13. Then $\tau = \tau_1 \cup \tau_2$, where*

$$\tau_1 = \sigma_{i_1,1} \cup \cdots \cup \sigma_{i_1+k_1,1},$$
$$\tau_2 = \sigma_{i_2,1} \cup \cdots \cup \sigma_{i_2+k_2,1}.$$

REMARK 7.15. We refer to a pair (τ_1, τ_2) of faces as in Corollary 7.14 as *nonintersecting J-blocks.*

THEOREM 7.16. *Let $\tau = \tau_1 \cup \tau_2$, where τ_1 and τ_2 are two nonintersecting (and nonconsecutive) J-blocks (see Corollary 7.14). We have an identification of X_τ (the open affine piece of X_σ corresponding to τ) with $Z_{2,k_1+4} \times Z_{2,k_2+4} \times (K^*)^m$, where $m =$* $\operatorname{codim}_\sigma \tau = 2(n - k_1 - k_2) - 9$.

The proof is similar to that of Theorem 7.11 (using Lemma 7.13).

Our next theorem follows as an immediate consequence.

THEOREM 7.17. *Let $\tau = \tau_1 \cup \tau_2$, where τ_1 and τ_2 are two nonintersecting (and nonconsecutive) J-blocks.*

(i) *We have an identification of $TC_{P_\tau} X_\sigma$ with $Z_{2,k_1+4} \times Z_{2,k_2+4} \times (K^*)^m$, where $m =$* $\operatorname{codim}_\sigma \tau = 2(n-k_1-k_2)-9$; *in particular, $TC_{P_\tau} X_\sigma$ is a toric variety.*
(ii) $\operatorname{mult}_{P_\tau} X_{2,n} = (\operatorname{mult}_{P_{\tau_1}} X_{2,n}) \cdot (\operatorname{mult}_{P_{\tau_2}} X_{2,n})$.

The proof is similar to that of Theorem 7.11 (using Theorem 7.16).

REMARK 7.18. It is clear that we can extend this multiplicative property to $\tau = \tau_1 \cup \cdots \cup \tau_s$, a union of s pairwise nonintersecting, nonconsecutive J-blocks.

8. A Multiplicity Formula for $X_{d,n}$

In this section we give a formula for $\operatorname{mult}_{P_\sigma} X_{d,n}$. By Theorem 3.10, $\operatorname{mult}_{P_\sigma} X_{d,n}$ equals the number of maximal chains in $I_{d,n}$. We shall provide an explicit formula for the number of maximal chains in $I_{d,n}$. Observe that the number of chains in $I_{d,n}$ from $(1, 2, \ldots, d)$ to $(n - d + 1, \ldots, n)$ is the same as the number of chains from $(0, 0, \ldots, 0)$ to $(n - d, n - d, \ldots, n - d)$; hence, for any $(i_1, \ldots, i_d)$ in the chain, $i_1 \geq i_2 \geq \cdots \geq i_d \geq 0$. Now set

$$\mu = (\mu_1, \mu_2, \ldots, \mu_d) = (n - d, n - d, \ldots, n - d). \tag{8.1}$$

For any $\lambda \vdash m$, let $f^\lambda = K_{\lambda,1^m}$—that is, the number of standard Young tableaux of shape λ (cf. [16]).

PROPOSITION 8.1 [16, Prop. 7.10.3]. *Let λ be a partition of m. Then the number f^λ counts the lattice paths $0 = v_0, v_1, \ldots, v_m$ in $\mathbb{R}^l$ (where $l = l(\lambda)$) from the origin v_0 to $v_m = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, with each step a coordinate vector, and staying within the region (or cone) $x_1 \geq x_2 \geq \cdots \geq x_l \geq 0$.*

Thus, for μ as described in (8.1), the number of maximal chains in $I_{d,n}$ is equal to f^{μ}.

An explicit description of f^{λ} is given in [16, Cor. 7.21.5].

Proposition 8.2. *Let $\lambda \vdash m$. Then*

$$f^{\lambda} = \frac{m!}{\prod_{u\in\lambda} h(u)}.$$

The statement of the proposition refers to $u \in \lambda$ as a box in the Young diagram of λ and to $h(u)$ as the "hook length" of u. The hook length is easily defined as the number of boxes to the right and below of u, including u once.

Let us take, for example, $I_{3,6}$. Then $\mu = (3,3,3)$, and the Young diagram of shape μ with hook lengths given in their corresponding boxes is as follows.

5	4	3
4	3	2
3	2	1

Therefore,

$$f^{\mu} = \frac{9!}{5 \cdot 4^2 \cdot 3^3 \cdot 2^2 \cdot 1} = 42.$$

In fact, in the $I_{d,n}$ scenario our derived partition μ (given by (8.1)) will always be a rectangle, and we can deduce a formula for f^{μ} that does not require the Young tableau. The top left box of μ will always have hook length $(n-d)+d-1 = n-1$; the box directly below it (and the box directly to the right of it) will have length $n-2$. For any box of μ, the box below and the box to the right will have hook length 1 less than that of the box with which we started.

Since the posets $I_{d,n}$, and $I_{n-d,n}$ are isomorphic, we may assume that $d \le n-d$. Then

$$\prod_{u\in\mu} h(u) = (n-1)(n-2)^2 \cdots (n-d)^d (n-d-1)^d \cdots (d)^d (d-1)^{d-1} \cdots (2)^2(1).$$

Thus we arrive at the following statement.

Theorem 8.3. *The multiplicity of $X_{d,n}$ at P_{σ} is equal to*

$$\frac{(d(n-d))!}{(n-1)(n-2)^2 \cdots (n-d)^d (n-d-1)^d \cdots (d)^d (d-1)^{d-1} \cdots (2)^2(1)}.$$

9. Conjectures

In this section, we give two conjectures on the multiplicity at a singular point. We also mention a result relevant to this paper on Sing $X(\mathcal{L})$ for $\mathcal{L}$ the Bruhat poset of Schubert varieties in any minuscule G/P.

The generating set $W(\tau)$ of a face τ consists of $\{e_{y'} - e_y\}$ for certain covers (y, y') in $J(\mathcal{L})$ (assuming that $\hat{1} \in D_{\tau}$, so that $e_{\hat{1}}$ is not in $W(\tau)$). Thus $W(\tau)$ determines a subset $H(\tau) := \dot{\bigcup} H(\tau)_i$ of $J(\mathcal{L})$ such that $W(\tau)$ consists of all the covers in the $H(\tau)_i$. Thus, $H(\tau)$ for $\tau = \sigma_{ij}$ would be the diamond given in Remark 6.10. In Section 7.12, if $\tau = \tau_1 \cup \tau_2$ for τ_1, τ_2 a pair of nonconsecutive and nonintersecting J-blocks, then $H(\tau) = H(\tau_1) \dot{\cup} H(\tau_2)$.

Conjecture 1. The multiplicity formula for $X_{2,n}$ in Theorem 7.17 extends to $X_{d,n}$. Namely, let σ be the convex polyhedral cone associated to $X_{d,n}$ and let τ, τ_1, τ_2 be faces of σ such that $\tau = \tau_1 \cup \tau_2$. Then, if $H(\tau_1) \cap H(\tau_2)$ is empty, we have $\mathrm{mult}_{P_\tau} X_{d,n} = (\mathrm{mult}_{P_{\tau_1}} X_{d,n}) \cdot (\mathrm{mult}_{P_{\tau_2}} X_{d,n})$.

Theorem 7.11 implies that $\mathrm{mult}_{P_{\tau_1}} X_{2,n} = \mathrm{mult}_{P_{\tau_2}} X_{2,n}$ if both τ_1 and τ_2 are J-blocks of the same length; in particular, $H(\tau_1)$ and $H(\tau_2)$ are isomorphic. Guided by this phenomenon, we make the following conjecture.

Conjecture 2. For a face τ of any Hibi toric variety $X(\mathcal{L})$, $\mathrm{mult}_{P_\tau} X(\mathcal{L})$ is determined by the poset $H(\tau)$. By this we mean that if τ, τ' are such that $H(\tau), H(\tau')$ are isomorphic posets, then the multiplicities of $X(\mathcal{L})$ at the points $P_\tau, P_{\tau'}$ are the same.

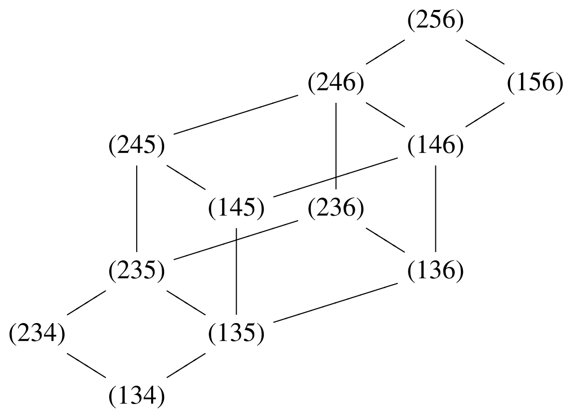
Remark 9.1. Toward generalizing Theorem 6.19 to other Hibi varieties, we will first explain how the lattice points μ_{ij} and λ_{ij} were chosen. Let δ and θ be two incomparable meet and join irreducibles in $I_{d,n}$; say, $\delta = (i+1, \dots, i+d)$ and $\theta = (1, \dots, j, n+j+1-d, \dots, n)$. Then $\theta \wedge \delta = \mu_{ij}$ and $\theta \vee \delta = \lambda_{ij}$. In view of Theorems 6.15 and 6.17, we have the following statement.

> *In $X_{d,n}$, P_τ is a smooth point if and only if, for every pair (θ, δ) of join and meet irreducibles, there is an $\alpha \in [\theta \wedge \delta, \theta \vee \delta]$ such that $P_\tau(\alpha)$, the αth coordinate of P_τ, is nonzero.*

In fact, this is the content of the conjecture of [9, Sec. 11].

These results suggest that we look at such pairs of join–meet irreducibles in other distributive lattices and expect the components of the singular locus of the associated Hibi toric variety to be given by Theorem 6.19(i) for the case of $I_{d,n}$. However, this is not true in general, as the following counterexample shows.

9.2. Counterexample. Let $\mathcal{L}$ be the interval $[(1,3,4),(2,5,6)]$, a sublattice of $I_{3,6}$.



Notice that $\mathcal{L}$ has only one pair of join–meet irreducibles, $(2,3,4)$ and $(1,5,6)$, and thus the corresponding interval $[\theta \wedge \delta, \theta \vee \delta]$ is the entire lattice. Therefore, if our result (Theorem 6.19(i)) on the singular locus of G-H toric varieties were to generalize to other Hibi toric varieties, then any proper face would be nonsingular. This follows because any face τ must correspond to an embedded sublattice D_τ, and naturally this sublattice will intersect the interval, which is just $\mathcal{L}$.

But this is not true! For example, let τ be the face of σ such that $D_\tau = \{(1,5,6)\}$. Then

$$\tau = C\langle e_{145} - e_{156}, e_{136} - e_{156}, e_{135} - e_{145}, e_{135} - e_{136}, e_{134} - e_{135}\rangle$$

is a set of generators for τ. Clearly, τ is not generated by the subset of a basis, so τ is a singular face (see Lemma 6.14).

Nevertheless, Theorem 6.19 holds for minuscule lattices as described next. Let G be semisimple, and let P be a maximal parabolic subgroup with ω as the associated fundamental weight. Let W (resp. W_P) be the Weyl group of G (resp. P). Then the Schubert varieties in G/P are indexed by W/W_P. Let P be *minuscule,* by which we mean that the weights in the fundamental representation associated to ω form one orbit under the Weyl group. It is known that the Bruhat poset W/W_P of the Schubert varieties in G/P is a distributive lattice; see [11] for details.

DEFINITION 9.3. We call $\mathcal{L} := W/W_P$ a *minuscule lattice* and $X(\mathcal{L})$ a *Bruhat–Hibi toric variety.*

REMARK 9.4. Any Grassmann–Hibi toric variety $X_{d,n}$ is also a Bruhat–Hibi toric variety.

Now, for $\mathcal{L}$ a minuscule lattice as in Definition 9.3, consider a pair (α, β) of incomparable join–meet irreducible elements. It has recently been shown [4] that a Bruhat–Hibi toric variety $X(\mathcal{L})$ is smooth at P_τ (for τ a face of σ) if and only if, for each incomparable pair (α, β) of join–meet irreducibles in $\mathcal{L}$, there exists at least one $\gamma \in [(\alpha \wedge \beta), (\alpha \vee \beta)]$ such that $P_\tau(\gamma)$ is nonzero.

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