Michigan Math. J. 59 (2010), 179–188

Comodules for Some Simple $\mathcal{O}$-forms of $\mathbb{G}_m$

N. E. Csima & R. E. Kottwitz

Tannakian theory allows one to understand an affine group scheme G over a commutative base ring A in terms of the category $\operatorname{Rep}(G)$ of G-modules, by which is meant comodules for the Hopf algebra corresponding to G. The theory is especially well developed [Sa] in the case that A is a field, and some parts of the theory still work well over more general rings A, say discrete valuation rings (see [Sa; W]).

When A is a field of characteristic 0 and G is connected reductive, the category $\operatorname{Rep}(G)$ is very well understood. However, with the exception of groups as simple as the multiplicative and additive groups, little seems to be known about what $\operatorname{Rep}(G)$ looks like concretely when A is no longer assumed to be a field, even in the most favorable case in which A is a discrete valuation ring and G is a flat affine group scheme over A with connected reductive general fiber.

The modest goal of this paper is to give a concrete description of $\operatorname{Rep}(G)$ for certain flat group schemes G over a discrete valuation ring $\mathcal{O}$ such that the general fiber of G is $\mathbb{G}_m$. It should be noted that $\mathcal{O}$-forms of $\mathbb{G}_m$ are natural first examples to consider, as $\mathbb{G}_m/\mathbb{Q}_p$ arises in the Tannakian description [Sa] of the category of isocrystals with integral slopes.

Choose a generator π of the maximal ideal of $\mathcal{O}$ and write F for the field of fractions of $\mathcal{O}$. For any nonnegative integer k, the construction of Section 1.1, when applied to $f = \pi^k$, yields a commutative flat affine group scheme G_k over $\mathcal{O}$ whose general fiber is $\mathbb{G}_m$. The $\mathcal{O}$-points of G_k are given by

$$G_k(\mathcal{O}) = \{t \in \mathcal{O}^\times : t \equiv 1 \bmod \pi^k\},$$

a principal congruence subgroup arising naturally in the much more general context of Moy–Prasad [MoP] subgroups of p-adic reductive groups. These form a projective system

$$\cdots \to G_2 \to G_1 \to G_0 = \mathbb{G}_m$$

in an obvious way, and we may form the projective limit $G_\infty := \operatorname{proj\,lim} G_k$. The Hopf algebra S_k corresponding to G_k can be described explicitly (see Sections 1.1 and 1.2). The Hopf algebra S_∞ corresponding to G_∞ is

$$\operatorname{inj\,lim} S_k = \left\{ \sum_{i\in\mathbb{Z}} x_i T^i \in F[T, T^{-1}] : \sum_{i\in\mathbb{Z}} x_i \in \mathcal{O} \right\}.$$

Received September 29, 2008. Revision received February 26, 2009.
Partially supported by NSF Grant no. DMS-0245639.

The categories $\mathrm{Rep}(G_\infty)$ and $\mathrm{Rep}(G_k)$ can be described very concretely. Indeed, $\mathrm{Rep}(G_\infty)$ consists of the category of $\mathcal{O}$-modules M equipped with a $\mathbb{Z}$-grading on $F \otimes_{\mathcal{O}} M$ (see Section 2.3, where a much more general result is proved). As for $\mathrm{Rep}(G_k)$, we proceed in two steps.

First, the full subcategory of $\mathrm{Rep}(G_k)$ consisting of those G_k-modules that are flat as $\mathcal{O}$-modules is equivalent (see Theorem 1.3.1) to the category of pairs (V, M) consisting of a $\mathbb{Z}$-graded F-vector space V and an *admissible* $\mathcal{O}$-submodule M of V, where admissible means that the canonical map $F \otimes_{\mathcal{O}} M \to V$ is an isomorphism and $C_n M \subset M$ for all $n \geq 0$, where $C_n : V \to V$ is the graded linear map given by multiplication by $\pi^{kn}\binom{i}{n}$ on the ith graded piece of V. The G_k-module corresponding to (V, M) is M, equipped with the obvious comultiplication.

Second, any G_k-module (see Section 1.4) is obtained as the cokernel of some injective homomorphism $M_1 \to M_0$ coming from a morphism $(V_1, M_1) \to (V_0, M_0)$ of pairs of the type just described.

When $\mathcal{O}$ is a $\mathbb{Q}$-algebra, the situation is even simpler: M is an admissible $\mathcal{O}$-submodule of the graded vector space V if and only if $C_1 M \subset M$ and $F \otimes_{\mathcal{O}} M \cong V$. Moreover, in case $\mathcal{O}$ is the formal power series ring $\mathbb{C}[[\varepsilon]]$, there is an interesting connection with affine Springer fibers (see Section 1.5).

1. A Description of $\mathrm{Rep}(G)_f$ for Certain Group Schemes G

Throughout this section we consider a commutative ring A and a nonzerodivisor $f \in A$. Thus the canonical homomorphism $A \to A_f$ is injective, where A_f denotes the localization of A with respect to the multiplicative subset $\{f^n : n \geq 0\}$. For the rest of this section we denote A_f by B and use the canonical injection $A \hookrightarrow B$ to identify A with a subring of B.

1.1. The Group Scheme G over A

We are now going to define a commutative affine group scheme G, flat and finitely presented over A. There will be a canonical homomorphism $G \to \mathbb{G}_m$ that becomes an isomorphism after extending scalars from A to B.

We begin by specifying the functor of points for G. For any commutative A-algebra R we put

$$\begin{aligned} G(R) &:= \{(t, x) \in R^\times \times R : t - 1 = fx\} \\ &= \{x \in R : 1 + fx \in R^\times\}. \end{aligned}$$

Then G is represented by the A-algebra

$$\begin{aligned} S &:= A[T, T^{-1}, X]/(T - 1 - fX) \\ &= A[X]_{1+fX}, \end{aligned} \tag{1.1.1}$$

which is clearly flat and finitely presented.

The multiplication on $G(R)$ is defined as $(t, x)(t', x') = (tt', x + x' + fxx')$. The identity element is $(1, 0)$ and the inverse of (t, x) is $(t^{-1}, -t^{-1}x)$.

There is a canonical homomorphism $\lambda\colon G \to \mathbb{G}_m$ given by $(t,x) \mapsto t$. When f is a nonzerodivisor in R, the homomorphism $\lambda\colon G(R) \to R^\times$ identifies $G(R)$ with $\ker[R^\times \to (R/fR)^\times]$, and when f is a unit in R, then $G(R) = R^\times$, showing that the homomorphism $\lambda\colon G \to \mathbb{G}_m$ becomes an isomorphism after extending scalars from A to B. Thus there is a canonical isomorphism $B \otimes_A S \cong B[T,T^{-1}]$.

Lemma 1.1.1. *Let M be an A-module on which f is a nonzerodivisor. Let F be any flat A-module. Then f is also a nonzerodivisor on $F \otimes_A M$.*

Proof. Tensor the injection $M \xrightarrow{f} M$ over A with F. □

Corollary 1.1.2. *The canonical homomorphism $S \to B \otimes_A S = B[T,T^{-1}]$ is injective, so that we may identify S with a subring of $B[T,T^{-1}]$.*

Proof. Just note that S is flat over A and that f is a nonzerodivisor on A. Therefore f is a nonzerodivisor on $S \otimes_A A = S$, and this means that $S \to B \otimes_A S$ is injective. □

1.2. Description of S as a Subring of $B[T,T^{-1}]$

We have just identified S with a subring of $B[T,T^{-1}]$. It is obvious from (1.1.1) that S is the A-subalgebra of $B[T,T^{-1}]$ generated by $T, T^{-1}, (T-1)/f$. However there is a more useful description of S in terms of B-module maps

$$L_n\colon B[T,T^{-1}] \to B,$$

one for each nonnegative integer n, defined by the formula

$$L_n\left(\sum_{i\in\mathbb{Z}} b_i T^i\right) = \sum_{i\in\mathbb{Z}} f^n \binom{i}{n} b_i.$$

Here $\binom{i}{n}$ is the binomial coefficient $i(i-1)\cdots(i-n+1)/n!$ defined for all $i \in \mathbb{Z}$. When $n = 0$, we have $\binom{i}{n} = 1$ for all $i \in \mathbb{Z}$.

The following remarks may help in understanding the maps L_n. For any nonnegative integer n, we have the divided-power differential operator

$$D^{[n]}\colon B[T,T^{-1}] \to B[T,T^{-1}]$$

defined by

$$D^{[n]}\left(\sum_{i\in\mathbb{Z}} b_i T^i\right) = \sum_{i\in\mathbb{Z}} \binom{i}{n} b_i T^{i-n}. \tag{1.2.1}$$

The Leibniz formula says that

$$D^{[n]}(gh) = \sum_{r=0}^{n} D^{[r]}(g) D^{[n-r]}(h). \tag{1.2.2}$$

For any $g \in B[T] \subset B[T,T^{-1}]$ the Taylor expansion of g at $T = 1$ reads

$$g = \sum_{n=0}^{\infty} (D^{[n]}g)(1) \cdot (T-1)^n, \tag{1.2.3}$$

the sum having only finitely many nonzero terms.

For any $g \in B[T, T^{-1}]$ we have $L_n(g) = f^n(D^{[n]}g)(1)$. It follows from (1.2.2) that for all $g, h \in B[T, T^{-1}]$

$$L_n(gh) = \sum_{r=0}^{n} L_r(g)L_{n-r}(h), \tag{1.2.4}$$

and for all $h \in B[T] \subset B[T, T^{-1}]$ it follows from (1.2.3) that

$$h = \sum_{n=0}^{\infty} L_n(h)\left(\frac{T-1}{f}\right)^n. \tag{1.2.5}$$

Now we are in a position to prove the following statement.

PROPOSITION 1.2.1. *The subring S of $B[T, T^{-1}]$ is equal to*

$$\{g \in B[T, T^{-1}] : L_n(g) \in A \ \forall n \geq 0\}.$$

Proof. Write S' for $\{g \in B[T, T^{-1}] : L_n(g) \in A \ \forall n \geq 0\}$. Obviously S' is an A-submodule of $B[T, T^{-1}]$, and it follows from (1.2.4) that S' is a subring of $B[T, T^{-1}]$. A simple calculation shows that $T, T^{-1}, (T-1)/f$ lie in S', and as these three elements generate S as A-algebra, we conclude that $S \subset S'$.

Now let $g \in S'$. There exists an integer n large enough that $h := T^m g$ lies in the subring $B[T]$. Note that $h \in S'$. Equation (1.2.5) shows that $h \in S$, since $(T-1)/f \in S$ and $L_n(h) \in A$. Therefore $g = T^{-m}h \in S$. □

Now let M be an A-module on which f is a nonzerodivisor, so that we may use the canonical A-module map $M \to B \otimes_A M$ (sending m to $1 \otimes m$) to identify M with an A-submodule of $N := B \otimes_A M$.

It follows from Lemma 1.1.1 that the canonical A-module map

$$S \otimes_A M \to B \otimes_A (S \otimes_A M) = B[T, T^{-1}] \otimes_B N$$

identifies $S \otimes_A M$ with an A-submodule of $B[T, T^{-1}] \otimes_B N$. We will now derive from Proposition 1.2.1 a description of $S \otimes_A M$ inside $B[T, T^{-1}] \otimes_B N$. For this we will need the B-module maps $\mathbf{L}_n \colon B[T, T^{-1}] \otimes_B N \to N$ defined by

$$\mathbf{L}_n\left(\sum_{i \in \mathbb{Z}} T^i \otimes x_i\right) = \sum_{i \in \mathbb{Z}} f^n \binom{i}{n} x_i.$$

Here $x_i \in N$, all but finitely many being 0.

LEMMA 1.2.2. *The A-submodule $S \otimes_A M$ of $B[T, T^{-1}] \otimes_B N$ is equal to*

$$\{x \in B[T, T^{-1}] \otimes_B N : \mathbf{L}_n(x) \in M \ \forall n \geq 0\}.$$

Proof. From Proposition 1.2.1 we see that there is an exact sequence

$$0 \to S \to B[T, T^{-1}] \xrightarrow{L} \prod_{n \geq 0} B/A,$$

the nth component of the map L being the composition

$$B[T, T^{-1}] \xrightarrow{L_n} B \twoheadrightarrow B/A.$$

In fact the map L takes values in $\bigoplus_{n\geq 0} B/A$. Indeed, for any $g \in B[T, T^{-1}]$ there exists an integer m large enough that $f^m g \in A[T, T^{-1}]$, and then $L_n(g) \in A$ for all $n \geq m$. Moreover L maps $B[T, T^{-1}]$ *onto* $\bigoplus_{n\geq 0} B/A$. Indeed, a simple calculation shows that for $b \in B$ and $m \geq 0$

$$L_n(bf^{-m}(T-1)^m) = \begin{cases} b & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

(First check that $D^{[n]}((T-1)^m) = \binom{m}{n}(T-1)^{m-n}$, say by induction on m; note that this formula is valid even if $n > m$, since $\binom{m}{n} = 0$ when $0 \leq m < n$.)

We now have a short exact sequence

$$0 \to S \to B[T, T^{-1}] \xrightarrow{L} \bigoplus_{n\geq 0} B/A \to 0$$

of A-modules. Tensoring with the A-module M, we obtain an exact sequence

$$S \otimes_A M \to B[T, T^{-1}] \otimes_A M \xrightarrow{L\otimes \mathrm{id}_M} \Big(\bigoplus_{n\geq 0} B/A\Big) \otimes_A M \to 0. \tag{1.2.6}$$

Now

$$B[T, T^{-1}] \otimes_A M = B[T, T^{-1}] \otimes_B B \otimes_A M = B[T, T^{-1}] \otimes_B N$$

and

$$\Big(\bigoplus_{n\geq 0} B/A\Big) \otimes_A M = \bigoplus_{n\geq 0} N/M.$$

With these identifications (and recalling that $S \otimes_A M \to B[T, T^{-1}] \otimes_B N$ is injective), we see that (1.2.6) describes $S \otimes_A M$ as the subset of $B[T, T^{-1}] \otimes_B N$ consisting of elements x such that $\mathbf{L}_n(x) \in M$ for all $n \geq 0$, and this completes the proof. □

1.3. Comodules for S

Since G is an affine group scheme over A, the A-algebra S is actually a commutative Hopf algebra, and we can consider $\mathrm{Rep}(G)$, the category of S-comodules. We denote by $\mathrm{Rep}(G)_f$ the full subcategory of $\mathrm{Rep}(G)$ consisting of S-comodules M such that f is a nonzerodivisor on the A-module underlying M. Our next goal is to give a concrete description of $\mathrm{Rep}(G)_f$.

In order to do so, we need one more construction. Let $N = \bigoplus_{i\in\mathbb{Z}} N_i$ be a $\mathbb{Z}$-graded B-module. For each nonnegative integer n we define an endomorphism $C_n : N \to N$ of the graded B-module N by requiring that C_n be given by multiplication by $f^n\binom{i}{n}$ on N_i. Thus

$$C_n\Big(\sum_{i\in\mathbb{Z}} x_i\Big) = \sum_{i\in\mathbb{Z}} f^n \binom{i}{n} x_i.$$

Here $x_i \in N_i$, all but finitely many being 0.

Let $\mathcal{C}$ be the category whose objects are pairs (N, M), N being a $\mathbb{Z}$-graded B-module, and M being an A-submodule of N such that the natural map $B \otimes_A M \to N$ is an isomorphism and such that $C_n M \subset M$ for all $n \geq 0$. A morphism

$(N, M) \to (N', M')$ is a homomorphism $\phi\colon N \to N'$ of graded B-modules such that $\phi M \subset M'$.

We now define a functor $F\colon \mathrm{Rep}(G)_f \to \mathcal{C}$. Let M be an object of $\mathrm{Rep}(G)_f$. Then $N := B \otimes_A M$ is a comodule for $B \otimes_A S = B[T, T^{-1}]$. It is known (see [DGr], Exp. 1) that the category of $B[T, T^{-1}]$-comodules is equivalent to the category of $\mathbb{Z}$-graded B-modules. Thus N has a $\mathbb{Z}$-grading $N = \bigoplus_{i\in\mathbb{Z}} N_i$, and the comultiplication $\Delta_N\colon N \to B[T, T^{-1}] \otimes_B N$ is given by $\sum_{i\in\mathbb{Z}} x_i \mapsto \sum_{i\in\mathbb{Z}} T^i \otimes x_i$. Since f is a nonzerodivisor on M, the canonical map $M \to B \otimes_A M = N$ identifies M with an A-submodule of N.

We define our functor F by $FM := (N, M)$. For this to make sense we must check that $C_n M \subset M$ for all $n \geq 0$. Let $m \in M$, and write $m = \sum_{i\in\mathbb{Z}} x_i$ in $\bigoplus_{i\in\mathbb{Z}} N_i = N$. Since the comodule N was obtained from M by extension of scalars, the element $x = \Delta_N m = \sum_{i\in\mathbb{Z}} T^i \otimes x_i \in B[T, T^{-1}] \otimes_B N$ lies in the image of $S \otimes_A M \to B[T, T^{-1}] \otimes_B N$. Lemma 1.2.2 then implies that $\mathbf{L}_n(x) = \sum_{i\in\mathbb{Z}} f^n \binom{i}{n} x_i = C_n(m)$ lies in M, as desired.

THEOREM 1.3.1. *The functor* $F\colon \mathrm{Rep}(G)_f \to \mathcal{C}$ *is an equivalence of categories.*

Proof. Let us first show that F is essentially surjective. Let (N, M) be an object in $\mathcal{C}$. We are going to use the comultiplication $\Delta_N\colon N \to B[T, T^{-1}] \otimes_B N$ to turn M into an S-comodule.

Since M is an A-submodule of N, it is clear that f is a nonzerodivisor on M. As we have seen before, it follows that f is a nonzerodivisor on $S \otimes_A M$ and hence that the natural map $S \otimes_A M \to B \otimes_A (S \otimes_A M) = B[T, T^{-1}] \otimes_B N$ identifies $S \otimes_A M$ with an A-submodule of $B[T, T^{-1}] \otimes_B N$.

Using Lemma 1.2.2, we see that our assumption that $C_n M \subset M$ for all $n \geq 0$ is simply the statement that $\Delta_N M \subset S \otimes_A M$. In other words, there exists a unique A-module map $\Delta_M\colon M \to S \otimes_A M$ such that Δ_M yields Δ_N after extending scalars from A to B.

We claim that Δ_M makes M into an S-comodule. For this we must check the commutativity of two diagrams, and this follows from the commutativity of these diagrams after extending scalars from A to B, once one notes that for any two A-modules M_1, M_2 on which f is a nonzerodivisor

$$\mathrm{Hom}_A(M_1, M_2) = \{\phi \in \mathrm{Hom}_B(B \otimes_A M_1, B \otimes_A M_2) : \phi(M_1) \subset M_2\}. \tag{1.3.1}$$

Here of course we are identifying M_1 and M_2 with A-submodules of $B \otimes_A M_1$ and $B \otimes_A M_2$, respectively. (At one point we need that f is a nonzerodivisor on $S \otimes_A S \otimes_A M$, which is true since $S \otimes_A S$ is flat over A.)

As F takes M to (N, M), we are done with essential surjectivity. It is easy to see that F is fully faithful; this too uses (1.3.1). □

1.4. Principal Ideal Domains A

One defect of the theorem we have just proved is that it only describes those G-modules on which f is a nonzerodivisor. When A is a principal ideal domain, as we assume for the rest of this subsection, we can do better. Now f is simply any nonzero element of A. As a consequence of Theorem 1.3.1 we obtain an equivalence of categories between the category $\mathrm{Rep}(G)_{\mathrm{flat}}$ of G-modules M such that M is flat as A-module and the full subcategory of $\mathcal{C}$ consisting of pairs (N, M) for which M is a flat A-module (in which case $N \cong B \otimes_A M$ is necessarily a flat B-module).

The next lemma is a variant of [Se, Prop. 3].

LEMMA 1.4.1. *Let A be a principal ideal domain, let C be a flat A-coalgebra, and let E be a C-comodule. Then there exists a short exact sequence of C-comodules*

$$0 \to F_1 \to F_0 \to E \to 0$$

in which F_0 and F_1 are flat as A-modules.

Proof. We imitate Serre's proof. Recall [Se, 1.2] that for any A-module M the map $\Delta \otimes \mathrm{id}_M : C \otimes_A M \to C \otimes_A C \otimes_A M$ (Δ being the comultiplication for C) gives $C \otimes_A M$ the structure of C-comodule, and [Se, 1.4] that the comultiplication map $\Delta_E : E \to C \otimes_A E$ is an injective comodule map when $C \otimes_A E$ is given the comodule structure just described. We use Δ_E to identify E with a subcomodule of $C \otimes_A E$.

Now choose a surjective A-linear map $p : F \to E$, where F is a free A-module. Let F_0 denote the preimage of E under the surjective comodule map $\mathrm{id} \otimes p : C \otimes_A F \twoheadrightarrow C \otimes_A E$. Since F_0 is the kernel of

$$C \otimes_A F \to C \otimes_A E \to (C \otimes_A E)/E,$$

it is a subcomodule of $C \otimes_A F$. Moreover $\mathrm{id} \otimes p$ restricts to a surjective comodule map $F_0 \to E$, whose kernel we denote by F_1. Since C and F are flat, so too are $C \otimes_A F$, F_0, and F_1, and we are done. We used that for principal ideal domains, a module is flat if and only if it is torsion-free, and the property of being torsion-free is inherited by submodules. □

Returning to our Hopf algebra S, we see that any G-module E has a resolution $0 \to F_1 \to F_0 \to E \to 0$ in which F_1 and F_0 are objects of $\mathrm{Rep}(G)_{\mathrm{flat}}$ and hence are described by our theorem. We conclude that E has the following form. There exist an injective homomorphism $\phi : N \to N'$ of graded B-modules and flat A-submodules M, M' of N, N' respectively such that $\phi M \subset M'$ and $(N, M), (N', M') \in \mathcal{C}$, having the property that E is isomorphic to $M'/\phi M$ as a G-module.

1.5. A Special Case

When A is a $\mathbb{Q}$-algebra, the category $\mathcal{C}$ is very simple. Indeed, there is a polynomial $P_n \in \mathbb{Q}[U]$ of degree n such that $\binom{i}{n} = P_n(i)$, and therefore $C_n = Q_n(C)$, where $C = C_1$ and $Q_n := f^n P_n(f^{-1}U) \in A[U]$. Therefore $\mathcal{C}$ is the category of pairs (N, M) consisting of a $\mathbb{Z}$-graded B-module N and an A-submodule M of N such that the natural map $B \otimes_A M \to N$ is an isomorphism and such that $CM \subset M$, where C is the endomorphism of the graded module $N = \bigoplus_{i \in \mathbb{Z}} N_i$ given by multiplication by fi on N_i.

When A is the formal power series ring $\mathcal{O} := \mathbb{C}[[\varepsilon]]$, and $f = \varepsilon^k$ (for some nonnegative integer k) our constructions yield a group scheme G over $\mathcal{O}$ such that $G(\mathcal{O}) = \{t \in \mathcal{O}^\times : t \equiv 1 \bmod \varepsilon^k\}$, and the category of representations of G on free $\mathcal{O}$-modules of finite rank is equivalent to the category of pairs (V, M), where V is a finite-dimensional graded vector space over $F := \mathbb{C}((\varepsilon))$ and M is an $\mathcal{O}$-lattice in V such that $CM \subset M$, where C is given by multiplication by $i\varepsilon^k$ on the ith graded piece of V. It is amusing to note that for fixed V, the space of all M satisfying $CM \subset M$ is an affine Springer fiber, which, when all the nonzero graded pieces of V are one-dimensional, is actually one of the affine Springer fibers studied at some length in [GKM], where it was shown to be paved by affine spaces. Finally, since $\mathcal{O}$ is a principal ideal domain, the results in Section 1.4 give a concrete description of all G-modules.

2. Certain Hopf Algebras and Their Comodules

Throughout this section A is a commutative ring and B is a commutative algebra such that the canonical homomorphism $B \otimes_A B \to B$ (given by $b_1 \otimes b_2 \mapsto b_1 b_2$) is an isomorphism. For example B might be of the form $S^{-1}A/I$ for some multiplicative subset S of A and some ideal I in $S^{-1}A$.

Let N be a B-module. Then the canonical B-module map $B \otimes_A N \to N$ (given by $b \otimes n \mapsto bn$) is an isomorphism. It follows that the canonical A-module homomorphism $N \to B \otimes_A N$ (given by $n \mapsto 1 \otimes n$) is actually an isomorphism of B-modules (since $N \to B \otimes_A N \to N$ is the identity).

Moreover, for any two B-modules N_1 and N_2, we have isomorphisms

$$\operatorname{Hom}_B(N_1, N_2) \cong \operatorname{Hom}_A(N_1, N_2) \tag{2.0.1}$$

and

$$N_1 \otimes_A N_2 \cong N_1 \otimes_B N_2. \tag{2.0.2}$$

2.1. General Remarks on Hopf Algebras and Their Comodules

Let S be a Hopf algebra over A. The composition $A \to S \to A$ of the unit and counit is the identity, and therefore there is a direct sum decomposition $S = A \oplus S_0$ of A-modules, where S_0 is by definition the kernel of the counit $S \to A$. In this subsection all tensor products will be taken over A and the subscript A will be omitted.

We denote by $\Delta\colon S \to S \otimes S$ the comultiplication for S. The counit axioms imply that Δ takes the form $\Delta(a + s_0) = a + s_0 \otimes 1 + 1 \otimes s_0 + \bar{\Delta}(s_0)$ when we identify S with $A \oplus S_0$ and $S \otimes S$ with $A \oplus (S_0 \otimes A) \oplus (A \otimes S_0) \oplus (S_0 \otimes S_0)$. Here $\bar{\Delta}$ is a uniquely determined A-module map $S_0 \to S_0 \otimes S_0$.

For any S-comodule M with comultiplication $\Delta_M\colon M \to S \otimes M$ the counit axiom for M implies that $\Delta_M(m) = 1 \otimes m + \bar{\Delta}_M(m)$ for a uniquely determined A-module map

$$\bar{\Delta}_M\colon M \to S_0 \otimes M.$$

In this way we obtain an equivalence of categories between S-comodules and A-modules M equipped with an A-linear map $\bar{\Delta}_M : M \to S_0 \otimes M$ such that the diagram

$$\begin{CD} M @>{\bar{\Delta}_M}>> S_0 \otimes M \\ @V{\bar{\Delta}_M}VV @VV{\bar{\Delta}\otimes \mathrm{id}}V \\ S_0 \otimes M @>{\mathrm{id}\otimes\bar{\Delta}_M}>> S_0 \otimes S_0 \otimes M \end{CD} \tag{2.1.1}$$

commutes.

2.2. *Hopf Algebras for B Give Hopf Algebras for A*

Let S be a Hopf algebra over B. As in Section 2.1, we decompose S as $B \oplus S_0$. It is easy to see that there is a unique Hopf algebra structure on $R := A \oplus S_0$ such that the unit and counit for R are the obvious maps $A \hookrightarrow R$ and $R \twoheadrightarrow A$ and such that the Hopf algebra structure on $B \otimes_A R$ agrees with the given one on S under the natural B-module isomorphism $B \otimes_A R \cong S$. What makes this work is (2.0.2), a consequence of our assumption that $B \otimes_A B \to B$ is an isomorphism, so that, for example, $S_0 \otimes_B S_0 \cong S_0 \otimes_A S_0$. The comultiplications Δ_R, Δ_S on R, S respectively are given by

$$\Delta_R(a + s_0) = a + s_0 \otimes 1 + 1 \otimes s_0 + \bar{\Delta}(s_0), \tag{2.2.1}$$

$$\Delta_S(b + s_0) = b + s_0 \otimes 1 + 1 \otimes s_0 + \bar{\Delta}(s_0), \tag{2.2.2}$$

and similar considerations apply to the multiplication maps $R \otimes_A R \to R$ and $S \otimes_B S \to S$ and the antipodes $R \to R$ and $S \to S$.

PROPOSITION 2.2.1. *The category of R-comodules is equivalent to the category of A-modules M equipped with an S-comodule structure on $N := B \otimes_A M$.*

Proof. We have already observed that giving an R-comodule is the same as giving an A-module M equipped with an A-module map $\bar{\Delta}_M : M \to S_0 \otimes_A M$ such that (2.1.1) commutes. Since S_0 is a B-module and $B \otimes_A B \cong B$, giving $\bar{\Delta}_M$ such that (2.1.1) commutes is the same as giving a B-module map $\bar{\Delta}_N : N \to S_0 \otimes_B N$ such that

$$\begin{CD} N @>{\bar{\Delta}_N}>> S_0 \otimes_B N \\ @V{\bar{\Delta}_N}VV @VV{\bar{\Delta}\otimes \mathrm{id}}V \\ S_0 \otimes_B N @>{\mathrm{id}\otimes\bar{\Delta}_N}>> S_0 \otimes_B S_0 \otimes_B N \end{CD}$$

commutes, or, in other words, giving an S-comodule structure on N. □

2.3. *Special Case*

Let $\mathcal{O}$ be a valuation ring and F its field of fractions. Let G be an affine group scheme over F and let S be the corresponding commutative Hopf algebra over F.

Decompose S as $F \oplus S_0$ and define a commutative Hopf algebra R over $\mathcal{O}$ by $R := \mathcal{O} \oplus S_0$. Corresponding to R is an affine group scheme $\tilde{G}$ over $\mathcal{O}$, and giving a representation of $\tilde{G}$ (i.e., an R-comodule) is the same as giving an $\mathcal{O}$-module M together with an S-comodule structure on $F \otimes_{\mathcal{O}} M$.

For example, when G is the multiplicative group $\mathbb{G}_m$, the Hopf algebra R is $\left\{\sum_{i \in \mathbb{Z}} a_i T^i \in F[T, T^{-1}] : \sum_{i \in \mathbb{Z}} a_i \in \mathcal{O}\right\}$, which is easily seen to be the union of the Hopf subalgebras S_k discussed in the Introduction.

References

[DGr] M. Demazure and A. Grothendieck, *SGA3, Schémas en groupes, tome I,* Lecture Notes in Math., 151, Springer-Verlag, Heidelberg, 1970.

[GKM] M. Goresky, R. Kottwitz, and R. MacPherson, *Purity of equivalued affine Springer fibers,* Represent. Theory 10 (2006), 130–146.

[MoP] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups,* Invent. Math. 116 (1994), 393–408.

[Sa] N. Saavedra Rivano, *Catégories Tannakiennes,* Lecture Notes in Math., 265, Springer-Verlag, Berlin, 1972.

[Se] J.-P. Serre, *Groupes de Grothendieck des schémas en groupes réductifs déployés,* Inst. Hautes Études Sci. Publ. Math. 34 (1968), 37–52.

[W] T. Wedhorn, *On Tannakian duality over valuation rings,* J. Algebra 282 (2004), 575–609.

N. E. Csima
Department of Mathematics
University of Chicago
Chicago, IL 60637

ecsima@math.uchicago.edu

R. E. Kottwitz
Department of Mathematics
University of Chicago
Chicago, IL 60637

kottwitz@math.uchicago.edu