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Smoothings of Schemes with Nonisolated Singularities

Nikolaos Tziolas

1. Introduction

The purpose of this paper is to describe the deformation and $\mathbb{Q}$-Gorenstein deformation theory of schemes defined over a field k with nonisolated singularities and to obtain criteria for the existence of smoothings and $\mathbb{Q}$-Gorenstein smoothings. The motivation for doing so comes from many different problems. Two of the most important ones are the compactification of the moduli space of surfaces of general type (and its higher-dimensional analogues) and the minimal model program.

Let $0 \in C$ be the germ of a smooth curve and let $U = C - 0$. It is well known [A; KoSh] that any family $f_U : \mathcal{X}_U \to U$ of smooth surfaces of general type over U can be completed in a unique way to a family $f : \mathcal{X} \to C$ such that $\omega^{[k]}_{\mathcal{X}/C}$ is invertible and ample for some $k > 0$ and the central fiber $X = f^{-1}(0)$ is a stable surface. A *stable* surface is a proper 2-dimensional reduced scheme X such that X has only semi-log-canonical singularities and $\omega_X^{[k]}$ is locally free and ample for some $k > 0$. Hence the moduli space of surfaces of general type can be compactified by adding the stable surfaces. Therefore, we should like to know which stable surfaces are smoothable and which are not. For an overview of recent advances in this area and the higher-dimensional analogues, see [A].

We would like to mention two applications from the minimal model program that are related to the smoothability problem.

1. The outcome of the minimal model program starting with a smooth, n-dimensional projective variety X is a terminal projective variety Y such that either K_Y is nef or Y has a Mori fiber space structure, which means that there is a projective morphism $f : Y \to Z$ with $-K_Y$ f-ample. Suppose that the second case occurs and $\dim Z = 1$. Let $z \in Z$ and $Y_z = f^{-1}(z)$. Then Y_z is a Fano variety of dimension $n-1$ and Y is a $\mathbb{Q}$-Gorenstein smoothing Y_z. In general, Y_z has nonisolated singularities and may not even be normal. Hence the classification of Mori fiber spaces in dimension n is directly related to the classification of smoothable Fano varieties of dimension $n - 1$.

2. One of the two fundamental maps that appear in the context of the 3-dimensional minimal model program is an extremal neighborhood. A 3-fold *terminal extremal* neighborhood [KoMo] is a proper birational map $\Delta \subset Y \xrightarrow{f} X \ni P$ such that Y is the germ of a 3-fold along a proper curve Δ, $\Delta_{\text{red}} = f^{-1}(P)$, Y is terminal,

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and $-K_Y$ is f-ample. An extremal neighborhood is the local analogue of a flipping contraction or a divisorial contraction that contracts a divisor onto a curve. In this setting, then, Y is a 1-parameter $\mathbb{Q}$-Gorenstein smoothing of the general member $H \in |\mathcal{O}_Y|$. The singularities of H are, in general, difficult to understand, and H may even be nonnormal. Of course, there are natural higher-dimensional analogues of the previous construction.

It is therefore of interest to study the deformation theory of schemes with nonisolated singularities and to obtain criteria for a scheme X to be smoothable. The case when X is a reduced scheme with normal crossing singularities has been extensively studied by Friedman [Fr]. In particular, he obtained a condition (called *d-semistability*) in order for X to be smoothable with a smooth total space and he studied the obstruction theory for a d-semistable scheme to be smoothable. As an application of these methods, Friedman showed that any d-semistable $K3$ surface is smoothable. Pinkham and Persson [PiP] have studied the problem of whether a d-semistable scheme is smoothable and derived examples showing that this is not always so. Kawamata and Namikawa [KaNa] have defined and studied the notion of logarithmic deformations of a normal crossing reduced scheme, extending Friedman's result on the smoothability of normal crossing $K3$ surfaces to higher-dimensional normal crossing Calabi–Yau varieties.

Typically, one first studies this problem locally and then globally. The local problem is to study which singularities are smoothable and the global is to find obstructions for the local smoothings to exist globally. If X has isolated singularities only, then it is well known that $H^2(T_X)$ is an obstruction space for the globalization of the local deformations. Hence, if X is locally smoothable and $H^2(T_X) = 0$, then X itself is smoothable. However, if the singular locus of X has dimension greater than 1, then there are examples of locally smoothable varieties whose obstruction in $H^2(T_X)$ is zero that are not globally smoothable [PiP]. The reason behind this is that, if the singularities are not isolated, then there are many local automorphisms of deformations that do not lift to higher order. Another major difference between the cases of isolated and nonisolated singularities is that Schlessinger's cotangent cohomology sheaves $T^i(X)$ no longer have finite support. Instead, they are sheaves supported on the singular locus of X and are, in general, difficult to describe [Tz1].

In this paper we seek to present a systematic study of the deformation theory of schemes with positive-dimensional singular locus and also write a few smoothability and nonsmoothability criteria. Some of the results that we prove are already known, but many others are (to our knowledge) new. We have tried to obtain the most general results with the fewest possible restrictions on the singularities. We hope this paper will be a useful reference to anyone using deformation theory.

The paper is organized as follows. In Section 3 we define the deformation functors $\mathrm{Def}(Y, X)$ and $\mathrm{Def}^{qG}(Y, X)$, where $Y \subset X$ is a closed subscheme of a scheme X defined over a field k. If $Y = X$ then these are the usual deformation and $\mathbb{Q}$-Gorenstein deformation functors of X. If $P \in X$ is an affine isolated singularity, then $\mathrm{Def}(P, X) = \mathrm{Def}(P \in X)$ is the functor of algebraic deformations of isolated singularities defined by Artin [A, Def. 5.1]. More generally, if $Y \neq X$ then these are deformation functors of $\hat{X}$, the formal completion of X along Y with certain algebraizability conditions that are explained in Definition 3.2. They are algebraic analogues of deformations of germs of analytic spaces. We also define the local deformation functors $\mathrm{Def}_{\mathrm{loc}}(Y,X)$ and $\mathrm{Def}^{qG}_{\mathrm{loc}}(Y,X)$, which parameterize the local deformations of $Y \subset X$. In almost all applications—and for the deformation functors to have good properties—we assume that Y contains the singular locus of X.

In Section 4 we describe the tangent spaces $\mathbb{T}^1(Y,X)$ and $\mathbb{T}^1_{qG}(Y,X)$ of $\mathrm{Def}(Y,X)$ and $\mathrm{Def}^{qG}(Y,X)$. Moreover, in Proposition 4.2 we obtain the local-to-global sequence for the functors $\mathrm{Def}(Y,X)$ and $\mathrm{Def}^{qG}(Y,X)$, which is a generalization of the usual local-to-global sequence for $\mathrm{Def}(X)$ [Se, Thm. 2.4.1].

In Section 5 we study the existence of a pro-representable hull for the deformation functors defined in Section 3. It is known that $\mathrm{Def}(Y,X)$ has a pro-representable hull if its tangent space $\mathbb{T}^1(Y,X)$ is finite dimensional [S]. In Theorem 5.4 we show that this also holds for $\mathrm{Def}^{qG}(Y,X)$ and in Theorem 5.5 we show that, under some strong restrictions on the singularities of X, $\mathrm{Def}^{qG}_{\mathrm{loc}}(Y,X)$ and $\mathrm{Def}_{\mathrm{loc}}(Y,X)$ have a hull, too. Finally, in Proposition 5.3 we exhibit some cases where $\mathbb{T}^1(Y,X)$ and $\mathbb{T}^1_{qG}(Y,X)$ are finite dimensional over the base field k.

In Sections 6 and 7 we explain the main technical tool used to study the deformation theory of X, Kawamata's T^1-lifting property [Ka1; Ka2].

In Section 8 we use the T^1-lifting property to study the global deformation theory of $Y \subset X$. In particular, in Theorem 8.1 we show that, if X is a pure and reduced scheme defined over a field of characteristic 0 and if $X - Y$ is smooth, then $\mathrm{Ext}^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$ is an obstruction space to lifting a deformation $X_n \in \mathrm{Def}(Y,X)(A_n)$ to A_{n+1}, where $\hat{X}$ is the formal completion of X along Y and $A_n = k[t]/(t^{n+1})$. Moreover, we exhibit an explicit obstruction element.

In Section 9 we study the problem of when local deformations of $Y \subset X$ exist globally. The main results are as follows.

(1) In Proposition 9.1 we show that, under very strong restrictions on the singularities of X, the global-to-local map

$$\pi : \mathrm{Def}(Y,X) \to \mathrm{Def}_{\mathrm{loc}}(Y,X)$$

is smooth if $H^2(\hat{T}_X) = 0$, where $\hat{T}_X$ is the completion of T_X along Y. However, in general π may fail to be smooth. This is in contrast to the case of isolated singularities, for which it is well known that the global-to-local map is always smooth if $H^2(T_X) = 0$.

(2) To get around the failure of π to be smooth, for any small extension

$$0 \to J \to B \to A \to 0$$

and for any $X_A \in \mathrm{Def}(Y,X)(A)$ we define the spaces $\mathrm{Def}(X_A/A, B)$ and $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$, parameterizing global and local liftings of X_A to B with certain local compatibility conditions that are explained in Definition 9.2. In Theorem 9.4 we describe them and show that there is an exact sequence

$$0 \to H^1(\hat{T}_X \otimes J) \xrightarrow{\alpha} \mathrm{Def}(X_A/A,B) \xrightarrow{\pi} \mathrm{Def}_{\mathrm{loc}}(X_A/A,B) \xrightarrow{\partial} H^2(\hat{T}_X \otimes J)$$

generalizing the first-order global-to-local exact sequence. Moreover, we show that there must be two successive obstructions in $H^0(T^2(X) \otimes J)$ and $H^1(T^1(X) \otimes J)$ in order for $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B) \neq \emptyset$. If these obstructions vanish, then there must be another obstruction in $H^2(\hat{T}_X \otimes J)$ in order for $\mathrm{Def}(X_A/A, B) \neq \emptyset$—that is, for the local deformations to exist globally. These obstruction spaces are well known if $X = Y$ [H3].

In Section 10 we extend all results obtained for the functor $\mathrm{Def}(Y, X)$ to $\mathrm{Def}^{qG}(Y, X)$. We do this by using that, locally, any $\mathbb{Q}$-Gorenstein deformation of X is induced by a deformation of its index-1 cover [KoSh].

Let X be a scheme of finite type over a field k, and let $f : \mathcal{X} \to \mathcal{S}$ be a deformation of X over the spectrum of a discrete valuation ring (R, m). In Section 11 we compare properties of the global deformation f with properties of the associated formal deformation $f_n : X_n \to S_n$, where $S_n = \operatorname{Spec} R/m^{n+1}$ and $X_n = \mathcal{X} \times_S S_n$. In particular, we obtain criteria on the associated formal deformation in order for the global one to be a smoothing. This is important because the deformations obtained with our methods are only formal and are not necessarily algebraic. But when they are algebraic it is of interest to know which properties of the global deformation can be read from properties of the associated formal deformation.

In Section 12 we apply the theory developed in the previous sections to give some smoothing and nonsmoothing criteria for a pure and reduced scheme of finite type over a field k. The main results are as follows.

(1) Let D be either $\mathrm{Def}(X)$ or $\mathrm{Def}^{qG}(X)$. In Theorem 12.3 we show that if at any generic point of its singular locus X has normal crossing singularities and if

$$H^0(p(T^1_D(X))) = H^1_Z(p(T^1_D(X))) = 0,$$

then X is not smoothable, where $p(T^1_D(X))$ is the quotient of $T^1_D(X)$ by its torsion and Z is the support of the torsion part. As a special case we get that if X has normal crossing singularities and $H^0(T^1(X)) = 0$, then X is not smoothable.

(2) In Theorem 12.5 we show that if X is a locally smoothable $\mathbb{Q}$-Gorenstein scheme such that the index-1 covers of all its singular points have complete intersection singularities, $T^1_{qG}(X)$ is finitely generated by its global sections, and $H^1(T^1_{qG}(X)) = H^2(T_X) = 0$, then X has a formal $\mathbb{Q}$-Gorenstein smoothing. Various other more specialized smoothing criteria are also given.

In Section 13 we apply the theory developed earlier in order to give examples in the context of the moduli of stable surfaces and the 3-dimensional minimal model program. First we give two examples of nonsmoothable stable surfaces. The components of the moduli space of stable surfaces to which these surfaces belong do not contain any smooth surfaces of general type, so these are extra components that appear by compactifying the moduli space of surfaces of general type. Then, by deforming a particular nonnormal surface H, we construct a 3-dimensional divisorial extremal neighborhood $f : Y \to X$ such that H is the general member of $|\mathcal{O}_Y|$.

2. Preliminaries

(1) All schemes in this paper are separated and Noetherian defined over a field k. Additional properties will be stated as needed.
(2) We denote by $\mathrm{Art}(k)$ the category of Artin local k-algebras.
(3) For any coherent sheaf $\mathcal{F}$ on a scheme X, we denote $\mathcal{F}^{[n]} = (\mathcal{F}^{\otimes n})^{**}$.
(4) Let $F\colon \mathrm{Art}(k) \to \mathrm{Sets}$ be a deformation functor. Then, following the notation introduced by Schlessinger [S], its tangent space is the set $F(k[t]/(t^2))$ and is denoted by T^1_F.
(5) A *small* extension of local Artin k-algebras is a square zero extension

$$0 \to J \to B \to A \to 0$$

of local Artin k-algebras (A, m_A) and (B, m_B) such that J is a principal ideal of B and $m_B J = 0$ (and therefore $J \cong k$ as a B-module).
(6) Let $X \to Y$ be a morphism of Noetherian separated schemes and $\mathcal{F}$ a coherent sheaf on X. Then by $T^i(X/Y, \mathcal{F})$ we denote Schlessinger's cotangent cohomology sheaves [LiS].
(7) Let X be a scheme. A *formal deformation* of X is a flat morphism of formal schemes $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$, where $\mathfrak{S} = \mathrm{Specf}\, R$, (R, m_R) is a complete local ring, and $X \cong \mathfrak{X} \times_{\mathfrak{S}} \mathrm{Specf}(R/m_R)$. Equivalently, a formal deformation of X over (R, m_R) is a collection of compatible deformations $f_n\colon X_n \to \mathrm{Spec}\, R_n$ for all $n \in \mathbb{Z}_{>0}$, where $R_n = R/m_R^{n+1}$. Suppose that X is of finite type over a field k. Then the formal deformation is called *effective* if and only if there is a flat morphism of finite type $f\colon \mathcal{X} \to \mathcal{S} = \mathrm{Spec}\, R$ of schemes with $X = \mathcal{X} \times_{\mathcal{S}} \mathrm{Spec}(R/m_R) = X$ and such that $\mathfrak{X} = \hat{\mathcal{X}}$, the formal completion of $\mathcal{X}$ along X. In this case, $\mathfrak{f}$ is called the *associated* formal deformation of f. If in addition f is induced from a deformation $f'\colon \mathcal{X}' \to \mathrm{Spec}\, A$, where (A, m_A) is a localization of a finitely generated k-algebra such that $\hat{A} \cong R$, then the deformation is said to be *algebraic.*
(8) A reduced scheme X is called $\mathbb{Q}$-*Gorenstein* if and only if it is Cohen–Macauley, it is Gorenstein in codimension 1, and there is an $n \in \mathbb{Z}_{>0}$ such that $\omega_X^{[n]}$ is invertible.
(9) A *smoothing* of a scheme X is a flat morphism $f\colon \mathcal{X} \to T = \mathrm{Spec}\, R$, where (R, m) is a discrete valuation ring such that $\mathcal{X} \times_T \mathrm{Spec}(R/m) \cong X$ and the generic fiber $\mathcal{X} \times_T \mathrm{Spec}\, K(R)$ is smooth over $K(R)$. If in addition X is $\mathbb{Q}$-Gorenstein and there is an $n \in \mathbb{Z}_{>0}$ such that $\omega_{\mathcal{X}/T}^{[n]}$ is invertible, then the smoothing is called $\mathbb{Q}$-Gorenstein. To avoid degenerate situations we will assume *either* that X is a local scheme and f is a morphism of local schemes *or* that X and f are proper and of finite type.

3. The Deformation Functors

First we recall the definition of an étale neighborhood of a closed subscheme Y of a scheme X [C].

DEFINITION 3.1. Let X be a Noetherian scheme defined over a field k, and let $Y \subset X$ be a closed subscheme of X. An *étale neighborhood* of Y in X is an étale morphism $Z \to X$ such that $Z \times_X Y \cong Y$.

Next we define the deformation functors that we shall study in this paper.

DEFINITION 3.2. Let X be a Noetherian scheme defined over a field k, and let $Y \subset X$ be a closed subscheme of X. Let $\hat{X}$ be the formal completion of X along Y. Then $\mathrm{Def}(Y, X)\colon \mathrm{Art}(k) \to \mathrm{Sets}$ is the functor such that, for any finite local Artin k-algebra A, $\mathrm{Def}(Y, X)(A)$ is the set of isomorphism classes of flat morphisms of formal schemes $f : \mathcal{X} \to \operatorname{Specf} A$ such that

(1) $\mathcal{X} \times_{\operatorname{Specf} A} \operatorname{Specf} k \cong \hat{X}$ and
(2) there exist an open cover $\mathcal{U}_i$ of $\mathcal{X}$ and flat morphisms of schemes $f_i : U_i \to \operatorname{Spec} A$ such that:
    (a) $U_i \times_{\operatorname{Spec} A} \operatorname{Spec} k$ is a local étale neighborhood of Y in X; and
    (b) $\mathcal{U}_i \to \operatorname{Specf} A$ is the formal completion of $U_i \to \operatorname{Spec} A$ along Y.

Next we define the notion of $\mathbb{Q}$-Gorenstein deformations and the corresponding deformation functor $\mathrm{Def}^{qG}(Y, X)$. In order for this to make sense, it is necessary to define the notion of relative dualizing sheaves for a formal family as in Definition 3.2.

DEFINITION 3.3. Let X be a Cohen–Macauley scheme that is Gorenstein in codimension 1 and defined over a field k, and let $Y \subset X$ be a closed subscheme of X. Let $f : \mathcal{X} \to \mathcal{S} = \operatorname{Specf} A$ be an element of $\mathrm{Def}(Y, X)(A)$, where $A \in \mathrm{Art}(k)$. Let $\mathcal{U}_i$ be an open cover of $\mathcal{X}$ as in Definition 3.2. Then the sheaves $(\omega_{U_i/A}^{[n]})^\wedge$ glue together to form a coherent sheaf on $\mathcal{X}$, which we denote by $\omega_{\mathcal{X}/\mathcal{S}}^{[n]}$. Note that the construction is independent of the cover chosen.

DEFINITION 3.4. Let X be a $\mathbb{Q}$-Gorenstein scheme defined over a field k, and let $Y \subset X$ be a closed subscheme of X. The functor of $\mathbb{Q}$-Gorenstein deformations is the functor $\mathrm{Def}^{qG}(Y, X)\colon \mathrm{Art}(k) \to \mathrm{Sets}$ such that, for any finite local Artin k-algebra A, $\mathrm{Def}^{qG}(Y, X)(A)$ is the set of isomorphism classes of flat morphisms $\mathcal{X} \to \mathcal{S} = \operatorname{Specf} A$ in $\mathrm{Def}(Y, X)$ such that the sheaf $\omega_{\mathcal{X}/\mathcal{S}}^{[n]}$ is invertible for some $n \in \mathbb{Z}_{>0}$.

It is not immediately clear whether $\mathrm{Def}^{qG}(Y, X)$ as just defined is a functor. This would be true if $\omega_{\mathcal{X}/\mathcal{S}}$ being $\mathbb{Q}$-Gorenstein is a stable property under base extension, which is known to be true [HasK, Lemma 2.6].

REMARK 3.5.

(1) If $Y = X$, then the functors $\mathrm{Def}(X, X)$ and $\mathrm{Def}^{qG}(X, X)$ are just the familiar deformation functors $\mathrm{Def}(X)$ and $\mathrm{Def}^{qG}(X)$.
(2) Let $P \in X$ be an affine isolated singularity. Then it follows from the definitions and from Theorem 11.1 [Ar1, Cor. 2.6] that $\mathrm{Def}(P, X)$ is the functor of algebraic deformations of an isolated singularity [Ar2, Def. 5.1]. This functor is usually denoted by $\mathrm{Def}(P \in X)$, and we will frequently use this notation.

More generally, if X has isolated singularities and $Y = X^{\text{sing}} = \{P_1, \dots, P_k\}$, then $\text{Def}(Y, X) = \prod_{i=1}^{k} \text{Def}(P_i \in X)$.

Moreover, as we shall see later, in order to obtain reasonable results about $\text{Def}(Y, X)$ or $\text{Def}^{qG}(Y, X)$ (in particular, existence of pro-representable hulls), we will assume that Y is proper and that $X - Y$ is smooth.

REMARK 3.6. The functors $\text{Def}(Y, X)$ and $\text{Def}^{qG}(Y, X)$ are an attempt to establish an algebraic analogue of deformations of germs of analytic spaces. A candidate for an algebraic germ is the formal neighborhood. However, completion along a subscheme is not an algebraic construction. The algebraic analogues of local analytic neighborhoods are étale neighborhoods. Ideally we would like to define the notion of an algebraic germ in such a way such that (i) if two are isomorphic then they are at least locally étale equivalent and (ii) any morphism between two algebraic germs comes, at least locally, from a morphism between étale neighborhoods. It is known [C, Thm. 4] that if $Y \subset X_1$, $Y \subset X_2$ is an embedding of a scheme Y into two schemes X_1 and X_2, and $X_1^h \cong X_2^h$, then—under relatively mild hypotheses—the isomorphism is induced by a common étale neighborhood of Y in X_1 and X_2. However, it is possible that $\hat{X}_1 \cong \hat{X}_2$ but $X_1^h \ncong X_2^h$, in which case X_1 and X_2 are not étale equivalent around Y [C, Ex. 1]. For these reasons, the correct definition of the algebraic germ of $Y \subset X$ would be that of the henselization X^h of X along Y instead of the completion $\hat{X}$. However, owing to technical difficulties of working with henselization, we work with the formal neighborhood and impose a local algebraizability condition in order not to stray too far from the geometry of $Y \subset X$. Moreover, in many cases the results of Artin [Ar1] allow us to move between the formal and the algebraic case.

NOTATION 3.7. For the rest of this paper, whenever we speak of $\text{Def}(Y, X)$ or $\text{Def}^{qG}(Y, X)$, X is assumed to satisfy all the relevant properties stated in Definitions 3.2 and 3.4.

One of the fundamental problems in deformation theory is to determine when a given scheme X admits a smoothing. The natural approach is first to study the problem locally (i.e., to determine which singularities are smoothable) and then to globalize the local smoothings. If X has isolated singularities only, say $P_1, \dots, P_k$, then the globalization of the local deformations is achieved by studying the natural transformation of functors

$$D(X) \to \prod_{i=1}^{k} D(P_i, X), \tag{3.1}$$

where $D(X)$ is either $\text{Def}(X)$ or $\text{Def}^{qG}(X)$ and $D(P_i, X)$ is either $\text{Def}(P_i, X)$ or $\text{Def}^{qG}(P_i, X)$. If the singularities of X are not isolated, then the map (3.1) does not exist. A kind of "sheafification" of the local deformation functors is more appropriate in this case.

DEFINITION 3.8. Let $D(Y, X)$ be either $\text{Def}(Y, X)$ or $\text{Def}^{qG}(Y, X)$. The functor $\underline{D}(Y, X)$ is the functor

$$\underline{D}(Y,X)\colon \mathrm{Art}(k)\to \mathrm{Sh}(X)$$

defined as follows. For any finite local k-algebra A, $\underline{D}(Y,X)(A)$ is the sheaf associated to the presheaf F defined by $F(V)=D(Y\cap V,V)(A)$ for any open set V.

DEFINITION 3.9. Let $D(Y,X)$ be either $\mathrm{Def}(Y,X)$ or $\mathrm{Def}^{qG}(Y,X)$. The *functor of local deformations* of $D(Y,X)$ is the functor $D_{\mathrm{loc}}(Y,X)\colon \mathrm{Art}(k)\to$ Sets defined by

$$D_{\mathrm{loc}}(Y,X)(A)=H^0(\underline{D}(Y,X)(A)).$$

For $D(Y,X)$ as just defined, there is a natural transformation of functors

$$\pi\colon D(Y,X)\to D_{\mathrm{loc}}(Y,X). \tag{3.2}$$

We call this map the *local-to-global* map. If X has isolated singularities and if $Y=X$, then π extends (3.1).

REMARK 3.10. If X has isolated singularities and $H^2(T_X)=0$, then it is well known that π is smooth. But π is not smooth in general because of its inability to lift local automorphisms of deformations to higher order. Under some strong conditions on the singularities of X, however, π is still smooth (Proposition 9.1).

4. The Tangent Space of $\mathrm{Def}(Y,X)$ and $\mathrm{Def}^{qG}(Y,X)$

Let $Y\subset X$ be a closed subscheme of a scheme X. In this section we describe the tangent spaces of the functors $\mathrm{Def}(Y,X)$ and $\mathrm{Def}^{qG}(Y,X)$ as well as the local-to-global map π (3.2) at the level of tangent spaces.

DEFINITION 4.1. We denote by $\mathbb{T}^1(Y,X)$, $T^1(Y,X)$, $\mathbb{T}^1_{qG}(Y,X)$, and $T^1_{qG}(Y,X)$ the tangent spaces of the functors

$$\mathrm{Def}(Y,X),\quad \underline{\mathrm{Def}}(Y,X),\quad \mathrm{Def}^{qG}(Y,X),\quad \text{and}\quad \underline{\mathrm{Def}}^{qG}(Y,X),$$

respectively.

It easily follows from the definitions of the deformation functors involved that $H^0(T^1(Y,X))$ and $H^0(T^1_{qG}(Y,X))$ are the respective tangent spaces of $\mathrm{Def}_{\mathrm{loc}}(Y,X)$ and $\mathrm{Def}^{qG}_{\mathrm{loc}}(Y,X)$. If $X-Y$ is smooth, then $T^1(Y,X)$ is just Schlessinger's $T^1(X)$ sheaf and $T^1_{qG}(Y,X)$ is the subsheaf $T^1_{qG}(X)$ of $T^1(X)$ defined as follows. For any affine open subset $U\subset X$, $T^1_{qG}(X)(U)$ is the $\mathcal{O}_X(U)$-module of isomorphism classes of first-order $\mathbb{Q}$-Gorenstein deformations of U.

The next proposition describes the global-to-local map at the level of tangent spaces. If $X=Y$ and $D=\mathrm{Def}(X)$, then this is just the familiar global-to-local sequence of the functor $\mathrm{Def}(X)$ [Se, Thm. 2.4.1].

PROPOSITION 4.2. *Suppose that X is a reduced scheme and that $Y\subset X$ a closed subscheme. Then the following statements hold.*

(1) *There is a canonical injection*

$$\phi\colon \mathbb{T}^1(Y,X)\to \mathrm{Ext}_{\hat{X}}(\hat{\Omega}_X,\mathcal{O}_{\hat{X}})$$

that is an isomorphism if $X-Y$ is smooth.

(2) *Let D be either* $\mathrm{Def}(Y,X)$ *or* $\mathrm{Def}^{qG}(Y,X)$. *Then there is an exact sequence*

$$0 \to H^1(\hat{T}_X) \to \mathbb{T}^1_D(Y,X) \to H^0(\hat{T}^1_D(X)).$$

If in addition $X - Y$ is smooth, then $\mathbb{T}^1_D(Y,X) = \mathbb{T}^1_D(X)$, $T^1_D(Y,X) = T^1_D(X)$, and there is an extended exact sequence

$$0 \to H^1(\hat{T}_X) \to \mathbb{T}^1_D(X) \to H^0(T^1_D(X)) \to H^2(\hat{T}_X),$$

where $\hat{X}$ is the formal completion of X along Y and where $\hat{\Omega}_X$, $\hat{T}_X$, and $\hat{T}^1_D(X)$ are the corresponding completions of Ω_X, T_X, and $T^1_D(X)$ along Y.

Proof. We first deal with the case $D = \mathrm{Def}(Y,X)$. The proof is based on the one for ordinary schemes [Se, Thm. 2.4.1]. Let $\mathcal{X}_1 \to \mathrm{Specf}\, A_1$ be a first-order deformation of $\hat{X}$. Then by definition there is an open cover $\mathcal{U}_i$ of $\mathcal{X}_1$ such that $\mathcal{U}_i \cong \hat{U}_i$, where U_i is a first-order deformation of a local étale neighborhood V_i of Y in X. Then the extension

$$0 \to k \to A_1 \to k \to 0$$

gives the extension

$$0 \to \mathcal{O}_{V_i} \to \mathcal{O}_{U_i} \to \mathcal{O}_{V_i} \to 0,$$

and since X is assumed to be reduced, there is an exact sequence

$$0 \to \mathcal{O}_{V_i} \to \Omega_{U_i} \otimes \mathcal{O}_{V_i} \to \Omega_{V_i} \to 0$$

and consequently

$$0 \to \mathcal{O}_{\hat{V}_i} \to \hat{\Omega}_{U_i} \otimes \mathcal{O}_{\hat{V}_i} \to \hat{\Omega}_{V_i} \to 0.$$

Patching these all together yields the exact sequence

$$0 \to \mathcal{O}_{\hat{X}} \to \hat{\Omega}_{\mathcal{X}} \otimes \mathcal{O}_{\hat{X}} \to \hat{\Omega}_X \to 0.$$

Hence we get a map

$$\mathbb{T}^1(Y,X) \to \mathrm{Ext}_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}),$$

which is injective (as in the usual scheme case). Conversely, let

$$0 \to \mathcal{O}_{\hat{X}} \to \mathcal{E} \to \hat{\Omega}_X \to 0$$

be any extension in $\mathrm{Ext}_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$. Let $\hat{d}\colon \mathcal{O}_{\hat{X}} \to \hat{\Omega}_X$ be the completion of the universal derivation of X (for detailed definitions and properties of $\hat{d}$ and $\hat{\Omega}_{\mathcal{X}}$ for any formal scheme $\mathcal{X}$, see [TLóR]). Then, exactly as in the scheme case, this gives a first-order deformation $\mathcal{X}$ of $\hat{X}$. However, in general it may not be locally the completion of a deformation of a local étale neighborhood of Y in X.

The standard local-to-global spectral sequence gives

$$0 \to H^1(\hat{T}_X) \to \mathrm{Ext}^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}) \to H^0(\mathcal{E}xt^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})) \to H^2(\hat{T}_X). \quad (4.1)$$

Claim:

$$\mathcal{E}xt^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}) \cong \mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X)^\wedge.$$

In fact, we will show that

$$\mathcal{E}xt^1_{\hat{X}}(\hat{\mathcal{F}}, \hat{\mathcal{P}}) \cong \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{P})^\wedge,$$

where $\mathcal{F}$ and $\mathcal{P}$ are coherent $\mathcal{O}_X$-modules. This is a local result, so we may assume that $X = \operatorname{Spec} A$ and $Y = V(I)$, where $I \subset A$ is an ideal. Then, since $\mathcal{F}$ is coherent, there is an exact sequence

$$0 \to \mathcal{O}_X^k \to \mathcal{O}_X^m \to \mathcal{F} \to 0.$$

Applying $\mathcal{H}om_X(\cdot, \mathcal{P})$ and taking completions, we obtain the exact sequence

$$\hat{\mathcal{P}}^m \to \hat{\mathcal{P}}^k \to \mathcal{E}xt^1_X(\mathcal{F}, \mathcal{P})^\wedge \to 0.$$

Taking completions first and then applying $\mathcal{H}om_{\hat{X}}(\cdot, \hat{P})$, we get the exact sequence

$$\hat{\mathcal{P}}^m \to \hat{\mathcal{P}}^k \to \mathcal{E}xt^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}) \to 0.$$

The claim now follows immediately from the last two exact sequences.

Since X is reduced, it follows that $T^1(X) = \mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X)$. Thus from (4.1) we obtain the exact sequence

$$0 \to H^1(\hat{T}_X) \to \operatorname{Ext}^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}) \to H^0(\hat{T}^1(X)) \to H^2(\hat{T}_X). \tag{4.2}$$

The space $H^1(\hat{T}_X)$ classifies the first-order locally trivial deformations of $\hat{X}$ [Hal], and $\mathbb{T}^1(Y, X) \subset \operatorname{Ext}^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$. Hence there is an exact sequence

$$0 \to H^1(\hat{T}_X) \to \mathbb{T}^1(Y, X) \to H^0(\hat{T}^1(X))$$

as claimed. If in addition $X - Y$ is smooth, then $T^1(X)$ is supported on Y and so $\hat{T}^1(X) = T^1(X)$. Therefore, every first-order deformation $\mathcal{X}$ of $\hat{X}$ arising from an element of $\operatorname{Ext}^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$ is locally the completion of a local deformation of X, and hence in this case $\mathbb{T}^1(Y, X) = \operatorname{Ext}^1_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$. This, together with the exact sequence (4.2), gives the exact sequence claimed in the second part of the proposition.

It remains to consider the $\mathbb{Q}$-Gorenstein functor. Let

$$\psi\colon \mathbb{T}^1(Y, X) \to H^0(T^1(Y, X))$$

be the global-to-local map defined previously. Then

$$H^0(T^1_{qG}(Y, X)) \subset H^0(T^1(Y, X)) \quad \text{and} \quad \mathbb{T}^1_{qG}(Y, X) = \psi^{-1}(H^0(T^1_{qG}(Y, X))).$$

This, together with the results just proven for the usual deformations case, yields the corresponding results for the $\mathbb{Q}$-Gorenstein case. □

REMARK 4.3. From Proposition 4.2 it follows that, in order to obtain reasonable results concerning the tangent space of $\operatorname{Def}(Y, X)$ or $\operatorname{Def}^{qG}(Y, X)$, $X - Y$ must be smooth. From now on we will always assume this.

5. Existence of Pro-representable Hulls

In this section we investigate the existence of pro-representable hulls [S] for all the deformation functors defined in Section 3. To do so, we use the following result of Schlessinger.

THEOREM 5.1 [S]. *Let $F\colon \mathrm{Art}(k) \to \mathrm{Sets}$ be a functor such that $F(k)$ is a single point. Let $A' \to A$ and $A'' \to A$ be morphisms in $\mathrm{Art}(k)$, and consider the map*

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{5.1}$$

Then the following statements hold.

(1) *F has a pro-representable hull if and only if F has the following properties*:
 (H_1) (5.1) *is a surjection whenever $A'' \to A$ is a small extension*;
 (H_2) (5.1) *is a bijection when $A = k$ and $A'' = k[t]/(t^2)$*;
 (H_3) $\dim_k T_F^1 < \infty$.

(2) *F is pro-representable if and only if F has the following additional property*:
 (H_4) *$F(A' \times_A A') \to F(A') \times_{F(A)} F(A')$ is an isomorphism for any small extension $A' \to A$.*

By using the criteria of the previous theorem, Schlessinger showed the following.

PROPOSITION 5.2 [S]. *Let X be a scheme defined over a field k. Then $\mathrm{Def}(X)$ has a pro-representable hull if and only if $\dim \mathbb{T}^1(X) < \infty$.*

The proof given by Schlessinger applies directly to $\mathrm{Def}(Y, X)$, so it follows that $\mathrm{Def}(Y, X)$ has a pro-representable hull if and only if $\dim_k \mathbb{T}^1(Y, X) < \infty$.

Next we present some cases where $\mathbb{T}^1(Y, X)$ and $\mathbb{T}^1_{qG}(Y, X)$ have finite dimension over k. Then we show that $\mathrm{Def}^{qG}(Y, X)$ has a pro-representable hull if and only if $\dim_k \mathbb{T}^1_{qG}(Y, X) < \infty$; and finally we show that, under some strong restrictions on the singularities of X, $\mathrm{Def}_{\mathrm{loc}}(Y, X)$ and $\mathrm{Def}^{qG}_{\mathrm{loc}}(Y, X)$ also have a pro-representable hull.

PROPOSITION 5.3. *Let X be a reduced scheme, and let $Y \subset X$ be a proper subscheme of X. Then $\mathbb{T}^1(Y, X)$ and $\mathbb{T}^1_{qG}(Y, X)$ have finite dimension over the base field k in any of the following cases.*

(1) *$X = Y$.*
(2) *Both X and Y are proper and smooth, and the normal bundle $N_{Y/X}$ of Y in X is ample.*
(3) *Y is contractible to an isolated singularity—in other words, there is a proper morphism $f\colon X \to Z$ such that $f(Y)$ is a point, $X - Y \cong Z - f(Y)$, $Z - f(Y)$ is smooth, and $R^i f_* \mathcal{O}_X = 0$ for all $i \geq 1$.*
(4) *$\dim Y = 1$, $X - Y$ is smooth, and $I_Y/I_Y^{(2)}$ is ample, where $I_Y^{(2)}$ is the second symbolic power of the ideal sheaf I_Y of Y in X.*

Proof. We use Proposition 4.2. Then the first part is immediate and the second part was proved by Hartshorne [H2]. The third part is well known in the analytic category, but owing to the lack of a reference we present a proof here. The result is local around Y, so we may assume that $Z = \mathrm{Spec}\, A$, where (A, m) is the localization of a finitely generated k-algebra. Let $f\colon X \to Z$ be the birational map in the assumption. Now, since f is proper and birational, $H^1(T_X)$ is a finitely generated torsion A-module and hence $H^1(T_X)^\wedge = H^1(T_X)$, where $H^1(T_X)^\wedge$ is the m-adic completion of $H^1(T_X)$. Then, according to the formal functions theorem,

$$H^1(T_X) \cong H^1(\hat{T}_X).$$

Dualizing the standard exact sequence

$$f^*\Omega_Z \to \Omega_X \to \Omega_{X/Z} \to 0$$

and taking into consideration that f is birational, we obtain the exact sequence

$$0 \to T_X \to (f^*\Omega_Z)^* \to M \to 0,$$

where M is a coherent $\mathcal{O}_X$-module supported on Y. Hence $\dim_k H^1(T_X) < \infty$ if and only if $\dim_k H^1((f^*\Omega_Z)^*) < \infty$. Moreover, there is a natural map ψ: $f^*T_Z \to (f^*\Omega_Z)^*$, and the supports of both $\mathrm{Ker}(\psi)$ and $\mathrm{Coker}(\psi)$ are contained in Y. It therefore suffices to show that $\dim_k H^1(f^*T_Z) < \infty$. Since Z is affine, there is an exact sequence

$$0 \to N \to \mathcal{O}_Z^m \to T_Z \to 0$$

and hence an exact sequence

$$0 \to Q \to f^*N \to \mathcal{O}_X^m \to f^*T_Z \to 0,$$

where Q is supported on Y. This breaks into two short exact sequences:

$$0 \to Q \to f^*N \to M \to 0;$$
$$0 \to M \to \mathcal{O}_X^m \to f^*T_Z \to 0.$$

Thus, since $R^1f_*\mathcal{O}_X = 0$, it now follows that

$$\dim_k H^1(f^*T_Z) < \infty \iff \dim_k H^2(f^*N) < \infty.$$

If we repeat the above argument then the result follows by induction.

It remains to show the last part. So, assume that $\dim Y = 1$, that $I_Y/I_Y^{(2)}$ is ample, and that $X - Y$ is smooth. Then, by Proposition 4.2, it suffices to show that $\dim_k H^1(\hat{T}_X) < \infty$. The completion $\hat{X}$ of X along Y can be calculated via the ideal sheaves $I_Y^{(n)}$, so

$$H^1(\hat{T}_X) = \varprojlim H^1(T_X \otimes \mathcal{O}_X/I_Y^{(n)}).$$

The short exact sequence

$$0 \to I_Y^{(n)}/I_Y^{(n+1)} \to \mathcal{O}_X/I_Y^{(n+1)} \to \mathcal{O}_X/I_Y^{(n)} \to 0$$

gives the exact sequence

$$0 \longrightarrow K_n \longrightarrow I_Y^{(n)}/I_Y^{(n+1)} \otimes T_X \longrightarrow \mathcal{O}_X/I_Y^{(n+1)} \otimes T_X \xrightarrow{\alpha_n} \mathcal{O}_X/I_Y^{(n)} \otimes T_X \longrightarrow 0.$$

We will show that $H^1(\mathrm{Ker}(\alpha_n)) = 0$ for n sufficiently large and hence, since Y is proper, $\dim_k H^1(\hat{T}_X) < \infty$. Since $\dim Y = 1$, it follows that $H^2(K_n) = 0$ and so it suffices to show $H^1(I_Y^{(n)}/I_Y^{(n+1)} \otimes T_X) = 0$ for n sufficiently large. The natural map

$$S^n(I_Y/I_Y^{(2)}) \to I_Y^{(n)}/I_Y^{(n+1)}$$

is generically surjective along Y; hence there exists an exact sequence

$$S^n(I_Y/I_Y^{(2)}) \otimes T_X \to I_Y^{(n)}/I_Y^{(n+1)} \otimes T_X \to T_n \to 0,$$

where T_n has 0-dimensional support. Because $I_Y/I_Y^{(2)}$ is ample, there must exist an $n_0 \in \mathbb{Z}$ such that $H^1(S^n(I_Y/I_Y^{(2)}) \otimes T_X) = 0$ for all $n \geq n_0$. Therefore, $H^1(I_Y^{(n)}/I_Y^{(n+1)} \otimes T_X) = 0$ for all $n \geq n_0$ and hence $\dim_k H^1(\hat{T}_X) < \infty$ as claimed. □

THEOREM 5.4. *Let X be a $\mathbb{Q}$-Gorenstein scheme, and let $Y \subset X$ be a closed subscheme of X. Assume also that* $\dim_k \mathbb{T}^1_{qG}(Y, X) < \infty$ (*this occurs, for example, when $Y \subset X$ satisfy the conditions of Proposition 5.3*). *Then the functor* $\mathrm{Def}^{qG}(Y, X)$ *has a pro-representable hull.*

Proof. We only show the case $X = Y$; the general case is similar. For convenience, set $D = \mathrm{Def}^{qG}(Y, X)$. We follow the general lines of the proof given by Schlessinger for the usual deformation functor $\mathrm{Def}(X)$ [S, Prop. 3.10]. It suffices to show that D satisfies Schlessinger's conditions (H_1), (H_2), and (H_3) (see Theorem 5.1). Condition (H_3) is satisfied by assumption, and (H_2) will follow from (H_1) because it is satisfied for the usual deformation functor $\mathrm{Def}(Y, X)$. Let $A'' \to A$ and $A' \to A$ be homomorphisms between Artin local k-algebras such that $A'' \to A$ is a small extension; that is, there exists a square zero extension

$$0 \to k \to A'' \to A \to 0.$$

We will show that the natural map

$$D(A'' \times_A A') \to D(A'') \times_{D(A)} D(A')$$

is surjective (this is condition (H_1)). Let $X_{A''} \in D(A'')$, $X_{A'} \in D(A')$, and $X_A \in D(A)$ such that $X_{A'} \otimes_{A''} A = X_{A'} \otimes_{A'} A = X_A$. Then there are natural maps $\mathcal{O}_{X_{A''}} \to \mathcal{O}_{X_A}$ and $\mathcal{O}_{X_{A'}} \to \mathcal{O}_{X_A}$. Let $R = A'' \times_A A'$ and let X_R be the scheme $(|X|, \mathcal{O}_{X_R})$, where $|X|$ is the underlying topological space of X and $\mathcal{O}_{X_R} = \mathcal{O}_{X_{A''}} \times_{\mathcal{O}_{X_A}} \mathcal{O}_{X_{A'}}$. Then $\mathcal{O}_{X_R}$ is a flat R-algebra, $\mathcal{O}_{X_R} \otimes_R A'' = \mathcal{O}_{X_{A''}}$, and $\mathcal{O}_{X_R} \otimes_R A' = \mathcal{O}_{X_{A'}}$ [S]. To conclude the proof we must show that X_R is $\mathbb{Q}$-Gorenstein (i.e., that it is Cohen–Macauley), that X_R is Gorenstein in codimension 1, and that there is an $n \in \mathbb{Z}$ such that $\omega_{X_R/R}^{[n]}$ is invertible. Because X_R is a deformation of X over an Artin local ring R, it is Cohen–Macauley and Gorenstein in codimension 1. Let n be the index of X. Then there is a natural map

$$\phi\colon \omega_{X_R/R}^{[n]} \to \omega_{X_{A''}/A''}^{[n]} \times_{\omega_{X_A/A}^{[n]}} \omega_{X_{A'}/A'}^{[n]}.$$

We will show that this map is an isomorphism. First observe that, since X is $\mathbb{Q}$-Gorenstein of index n and $X_A, X_{A'}, X_{A''}$ are also $\mathbb{Q}$-Gorenstein, they also have index n [KoSh] and hence the right-hand side is invertible. Since $\omega_{X_R/R}^{[n]}$ is reflexive and X_R is Cohen–Macauley, it suffices to show that ϕ is an isomorphism over the Gorenstein locus. Let $X^0 \subset X$ be the Gorenstein locus of X. Then there is a commutative diagram

$$\begin{array}{ccc} \omega^{[n]}_{X^0_R/R} & \longrightarrow & \omega^{[n]}_{X^0_{A'}/A'} \\ \downarrow & & \downarrow \\ \omega^{[n]}_{X^0_{A''}/A''} & \longrightarrow & \omega^{[n]}_{X^0_A/A} \end{array}$$

and, moreover, since $\omega^{[n]}_{X^0_R/R} = \omega^{\otimes n}_{X^0_R/R}$ and $\omega^{[n]}_{X^0_{A''}/A''} = \omega^{\otimes n}_{X^0_{A''}/A''}$ are invertible,

$$\omega^{[n]}_{X^0_R/R} \otimes_R A'' = \omega^{[n]}_{X^0_{A''}/A''}.$$

Hence [S, Cor. 3.6]

$$\omega^{[n]}_{X^0_R/R} \cong \omega^{[n]}_{X^0_{A''}/A''} \times_{\omega^{[n]}_{X^0_A/A}} \omega^{[n]}_{X^0_{A'}/A'}$$

as claimed and therefore X_R is $\mathbb{Q}$-Gorenstein. □

The next proposition shows that, under some strong restrictions on the singularities of X, the local deformation functors $\mathrm{Def}_{\mathrm{loc}}(Y, X)$ and $\mathrm{Def}^{qG}_{\mathrm{loc}}(Y, X)$ have a hull, too. This is useful in the cases when $\mathrm{Def}(Y, X)$ and $\mathrm{Def}^{qG}(Y, X)$ do not have a hull, a deficiency that arises because they may not have finite-dimensional tangent spaces. However, the tangent spaces of the local functors are $H^0(T^1(Y, X))$ and $H^0(T^1_{qG}(Y, X))$, and since $T^{(}Y, X)$, $T^1_{qG}(Y, X)$ are coherent sheaves supported on the singular locus of X, it follows that $H^0(T^1(Y, X))$ and $H^0(T^1_{qG}(Y, X))$ will be finite-dimensional if the singular locus of X is proper and is contained in Y.

THEOREM 5.5. *Let X be a scheme, and let $Y \subset X$ be a subscheme of X. Assume that the singular locus Z of X is proper and that $Z \subset Y$. Let D be either* $\mathrm{Def}(Y, X)$ *or* $\mathrm{Def}^{qG}(Y, X)$. *Suppose that one of the following conditions are satisfied*:

(1) *with the exception of finitely many singular points, D locally satisfies Schlessinger's condition* (H_4);
(2) *the codimension of Z in X is at least 3 and* $\mathrm{depth}_P(\mathcal{O}_{X,P}) \geq 3$ *for any point $P \in Z$ (closed or not).*

Then the local deformation functor D_{loc} has a pro-representable hull.

Proof. We prove the theorem only for $Y = X$. The proof of the general case is similar.

It suffices to verify Schlessinger's conditions (H_1), (H_2), and (H_3). The tangent space of D_{loc} is $H^0(T^1_D(X))$. Since $T^1_D(X)$ is a coherent sheaf supported on the singular locus of X, it follows that $H^0(T^1_D(X))$ is finite dimensional over the base field k. So (H_3) is satisfied.

Assume now that either one of the conditions in the statement is satisfied. If the second one holds, then $\mathrm{Def}(X) = \mathrm{Def}(X - Z)$ [Ar2] and, since $X - Z$ is smooth, it locally satisfies (H_4). Hence we need only assume that the first condition is satisfied.

Let $A'' \to A$ and $A' \to A$ be homomorphisms between Artin local k-algebras such that $A'' \to A$ is a small extension. We will show (H_1)—in other words, that the natural map

$$D_{\mathrm{loc}}(A'' \times_A A') \to D_{\mathrm{loc}}(A'') \times_{D_{\mathrm{loc}}(A)} D_{\mathrm{loc}}(A')$$

is surjective. By definition, $D_{\mathrm{loc}}(B) = H^0(\underline{D}(B))$ for any local finite k-algebra B. Let $s' \in H^0(\underline{D}(A'))$ and $s'' \in H^0(\underline{D}(A''))$ be such that they map to $s \in H^0(\underline{D}(A))$ under the natural maps $\lambda'\colon H^0(\underline{D}(A')) \to H^0(\underline{D}(A))$ and $\lambda''\colon H^0(\underline{D}(A'')) \to H^0(\underline{D}(A))$. Let $\{U_i\}$ be an affine open cover of X, and let $U_{ij} = U_i \cap U_j$. Let $\mathcal{X}_i$ be any deformation of U_i over a ring B. In what follows we will use $\mathcal{X}_{ij}$ to denote the restriction of $\mathcal{X}_{ij}$ on U_{ij}.

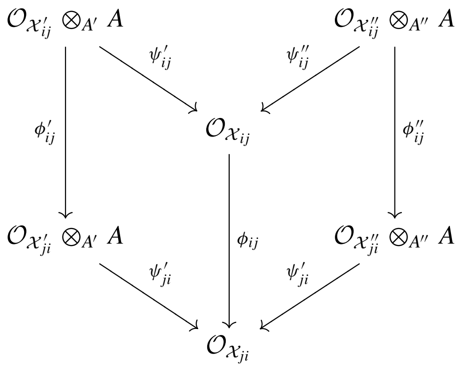
The section s is equivalent to a collection of deformations $\mathcal{X}_i$ of U_i over A and A-isomorphisms $\phi_{ij}\colon \mathcal{X}_{ij} \to \mathcal{X}_{ji}$. Similarly, s' is equivalent to a collection of deformations $\mathcal{X}'_i$ of U_i over A' and A'-isomorphisms $\phi'_{ij}\colon \mathcal{X}'_{ij} \to \mathcal{X}'_{ji}$ and s'' is equivalent to a collection of deformations $\mathcal{X}''_i$ of U_i over A'' and A''-isomorphisms $\phi''_{ij}\colon \mathcal{X}''_{ij} \to \mathcal{X}''_{ji}$. Since $\lambda'(s') = \lambda''(s'') = s$, there must exist A-isomorphisms $\psi'_i\colon \mathcal{X}'_i \otimes_{A'} A \to \mathcal{X}_i$ and $\psi''_i\colon \mathcal{X}''_i \otimes_{A''} A \to \mathcal{X}_i$. Then $\mathcal{O}_{\mathcal{X}'_i} \times_{\mathcal{O}_{\mathcal{X}_i}} \mathcal{O}_{\mathcal{X}''_i}$ is a deformation of U_i over $R = A'' \times_A A'$. The collection $\{\mathcal{O}_{\mathcal{X}'_i} \times_{\mathcal{O}_{\mathcal{X}_i}} \mathcal{O}_{\mathcal{X}''_i}\}$ forms a section in $H^0(\underline{D}(R))$ if and only if there are R-isomorphisms

$$\lambda_{ij}\colon \mathcal{O}_{\mathcal{X}'_{ij}} \times_{\mathcal{O}_{\mathcal{X}_{ij}}} \mathcal{O}_{\mathcal{X}''_{ij}} \to \mathcal{O}_{\mathcal{X}'_{ji}} \times_{\mathcal{O}_{\mathcal{X}_{ji}}} \mathcal{O}_{\mathcal{X}''_{ji}}.$$

The natural candidate for such an isomorphism is

$$\phi'_{ij} \times \phi''_{ij}\colon \mathcal{O}_{\mathcal{X}'_{ij}} \times \mathcal{O}_{\mathcal{X}''_{ij}} \to \mathcal{O}_{\mathcal{X}'_{ji}} \times \mathcal{O}_{\mathcal{X}''_{ji}}.$$

This isomorphism induces an isomorphism of $\mathcal{O}_{\mathcal{X}'_{ij}} \times_{\mathcal{O}_{\mathcal{X}_{ij}}} \mathcal{O}_{\mathcal{X}''_{ij}}$ if and only if there is a commutative diagram



By our assumption, we can refine the open cover in such a way that the U_{ij} satisfy (H_4). We can now modify the ϕ_{ij} so that the left-hand side of the diagram commutes and then, since the U_{ij} satisfy (H_4), we lift them to X''_{ij}. Hence we get a section and therefore D_{loc} satisfies (H_1). Similarly, it also satisfies (H_2) (note that (H_2) is satisfied without any restrictions on the singularities of X) and hence D_{loc} has a hull. □

Next we present a simple case when D_{loc} has a hull.

COROLLARY 5.6. *Assume that, with the exception of finitely many singular points, the index-1 cover of any singular point of X is smooth and that the singular locus of X is proper. Then* $\mathrm{Def}^{qG}_{\mathrm{loc}}(Y, X)$ *has a hull.*

Proof. By Theorem 5.5 we need only show that, with the exception of finitely many singular points, property (H_4) is satisfied. This is equivalent to showing that local automorphisms of deformations lift to higher order. Since the result is local, we may assume that X is affine. Then let $\pi: \tilde{X} \to X$ be the index-1 cover of X. Let X_A be a $\mathbb{Q}$-Gorenstein deformation of X over A. Let $A \to B$ be a finite local A-algebra and let $X_B = X_A \otimes_A B$. Let θ be a B-automorphism of X_B. Let $\tilde{X}_A \to X_A$ be the index-1 cover of X_A. Then $\tilde{X}_A$ is a deformation of $\tilde{X}$ [KoSh] over A and $\tilde{X}_A \otimes_A B$ is the index-1 cover of X_B. From the construction of the index-1 cover, θ lifts to an automorphism of $\tilde{X}_B$ that is smooth by assumption. This now lifts to an automorphism of $\tilde{X}_A$ and hence to an automorphism of X_A. □

REMARK 5.7. From the proof of Theorem 5.5, it is clear that the obstruction to the local deformation functors having a hull is the presence of automorphisms. In fact, the only time we were able to show existence of a hull is when there are no automorphisms. In view of this, perhaps it would be better to consider the stack of deformations instead.

6. The T^1-lifting Property

The main technical tool that we will use to study the deformation theory of a scheme X is Kawamata's T^1-lifting property [Ka1; Ka2]. We recall the basic definitions and properties.

Let $D: \mathrm{Art}(k) \to \mathrm{Sets}$ be a deformation functor of some scheme X defined over a field of characteristic 0—that is, a covariant functor that satisfies Schlessinger's conditions (H_1) and (H_2). Assume moreover that D has an obstruction space T^2_D. For $A \in \mathrm{Art}(k)$, $D(A)$ is the set of isomorphism classes of pairs (X_A, ϕ_0) consisting of deformations X_A of X and marking isomorphisms $\phi_0: X_A \otimes_A k \to X$. The class of (X_A, ϕ_0) will be denoted by $[X_A, \phi_0]$.

Let $B_n = k[x, y]/(x^{n+1}, y^2)$ and $C_n = k[x, y]/(x^{n+1}, y^2, x^n y)$. There are natural maps $\alpha_n: A_{n+1} \to A_n$, $\beta_n: B_n \to A_n$, $\gamma_n: B_n \to C_n$, $\delta_n: C_n \to B_{n-1}$, $\zeta_n: A_n \to C_n$, and $\varepsilon_n: A_{n+1} \to B_n$ with $\beta_n(x) = t$, $\beta_n(y) = 0$, $\varepsilon_n(t) = x + y$, and $\zeta_n(t) = x + y$.

DEFINITION 6.1. Let $[X_n, \phi_0] \in D(A_n)$. Then we define:

(1) $\mathbb{T}^1_D(X_n/A_n)$ to be the set of isomorphism classes of pairs (Y_n, ψ_n) consisting of deformations Y_n of X over B_n and marking isomorphisms $\psi_n: Y_n \otimes_{B_n} A_n \to X_n$; and

(2) $T^1_D(X_n/A_n)$ to be the sheaf of sets on X associated to the presheaf $\mathcal{F}$ such that $\mathcal{F}(U) = \mathbb{T}^1_D(U_n/A_n)$ for any open $U \subset X$, where $U_n = \mathcal{X}_n|_U$.

If D is $\mathrm{Def}(Y, X)$ or $\mathrm{Def}^{qG}(Y, X)$ then we use the notation $\mathbb{T}^1(X_n/A_n)$, $T^1(X_n/A_n)$, $\mathbb{T}^1_{qG}(X_n/A_n)$, and $T^1_{qG}(X_n/A_n)$, respectively.

DEFINITION 6.2 [Ka1; Ka2]. We say that the deformation functor D satisfies the *T^1-lifting property* if and only if, for any $X_n \in D(A_n)$, the natural map

$$\phi_n\colon \mathbb{T}^1_D(X_n/A_n) \to \mathbb{T}^1_D(X_{n-1}/A_{n-1})$$

is surjective, where $X_{n-1} = D(\alpha_{n-1})(X_n)$.

THEOREM 6.3 [Ka1, Thm. 1]. *Let D be a deformation functor that satisfies the T^1-lifting property. Then D is smooth. In particular, if D has a hull, then its hull is smooth.*

In fact, the proof of the previous theorem shows the following.

THEOREM 6.4. *Let D be a deformation functor, $X_n \in D(A_n)$, $X_{n-1} = D(\alpha_n)(X_n)$, and $Y_{n-1} = D(\varepsilon_{n-1})(X_n) \in \mathbb{T}^1_D(X_{n-1}/A_{n-1})$. Then X_n lifts to A_{n+1} (i.e., is in the image of $D(A_{n+1}) \to D(A_n)$) if and only if Y_{n-1} is in the image of the natural map*

$$\phi_n\colon \mathbb{T}^1_D(X_n/A_n) \to \mathbb{T}^1_D(X_{n-1}/A_{n-1}).$$

The advantage of Theorem 6.4 is that it allows us to exhibit in the next section a very explicit obstruction element to the lifting of X_n to A_{n+1}. The following result is also useful.

PROPOSITION 6.5. *With assumptions as in Theorem 6.4, let $Y_n \in \mathbb{T}^1(X_n/A_n)$ be a lifting of Y_{n-1}; that is, $\phi_n(Y_n) = Y_{n-1}$. Then there is a lifting X_{n+1} of X_n over A_{n+1} such that $Y_n = D(\varepsilon_n)(X_{n+1})$.*

The proof of the proposition depends on the following result of Schlessinger.

THEOREM 6.6 [S]. *Let $D\colon \mathrm{Art}(k) \to \mathrm{Sets}$ be a functor that satisfies* (H_2). *Let*

$$0 \to J \to B \xrightarrow{\alpha} A \to 0$$

be a small extension of local Artin k-algebras, and let $D(\alpha)\colon D(B) \to D(A)$ be the natural map. Then, for any $\xi_A \in D(A)$, there is a natural action of the tangent space t_D of D on the set $D(\alpha)^{-1}(\xi_A)$. Moreover, if D satisfies (H_1), *then the action is transitive.*

A careful look at the proof of the previous theorem reveals that the action described satisfies the following functorial property.

COROLLARY 6.7. *With assumptions as in Theorem 6.6, let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B & \xrightarrow{\alpha} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & J' & \longrightarrow & B' & \xrightarrow{\alpha'} & A' & \longrightarrow & 0 \end{array}$$

be a commutative diagram of small extensions of local Artin k-algebras such that f is a k-isomorphism. Let $\xi_A \in D(A)$ and $\xi_{A'} = D(h)(\xi_A) \in D(A')$. Then the natural map $D(\alpha)^{-1}(\xi_A) \to D(\alpha')^{-1}(\xi_{A'})$ is t_D-equivariant.

If f is not an isomorphism, then the previous result is not true.

Proof of Proposition 6.5. Let $\zeta_n \colon A_n \to C_n$ be defined by $\zeta_n(t) = x + y$, and let $\delta_n \colon C_n \to B_{n-1}$ be the natural map. Then $\delta_n \zeta_n = \varepsilon_{n-1}$. Consider the commutative diagram of small extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & A_{n+1} & \longrightarrow & A_n & \longrightarrow & 0 \\ & & \big\downarrow f & & \big\downarrow \varepsilon_n & & \big\downarrow \zeta_n & & \\ 0 & \longrightarrow & J' & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \end{array}$$

where $J = (t^{n+1})$, $J' = (xy^n)$, and f is the isomorphism given by sending t^{n+1} to xy^n. This diagram induces the commutative diagram

$$\begin{array}{ccccc} D(A_{n+1}) & \xrightarrow{D(\alpha_n)} & D(A_n) & \longrightarrow & T_D^2 \otimes J \\ \big\downarrow D(\varepsilon_n) & & \big\downarrow D(\zeta_n) & & \big\| \\ D(B_n) & \xrightarrow{D(\gamma_n)} & D(C_n) & \longrightarrow & T_D^2 \otimes J' \end{array}$$

where T_D^2 is an obstruction space for D. Let $Z_n = D(\zeta_n)(X_n)$. Then the T^1-lifting property implies that $D(\gamma_n)(Y_n) = Z_n$ [Ka1; Ka2]. Let X'_{n+1} be a lifting of X_n, which exists by the T^1-lifting property, and let $Y'_n = D(\varepsilon)(X'_{n+1})$. Then $Y_n, Y'_n \in D(\gamma_n)^{-1}(Z_n)$, which is a homogeneous t_D-space by Theorem 6.6. Hence there is a $\theta \in t_D$ such that $\theta \cdot Y'_n = Y_n$. Moreover, by Corollary 6.7, the natural map $D(\alpha_n)^{-1}(X_n) \to D(\zeta_n)^{-1}(Z_n)$ is t_D-equivariant. Hence $D(\varepsilon_n)(X_{n+1}) = Y_n$, where $X_{n+1} = \theta \cdot X'_{n+1}$. □

REMARK 6.8. The T^1-lifting property was originally introduced by Ran [Ra] in order to study infinitesimal deformations of a complex manifold; it was later generalized by Kawamata [Ka1; Ka2] to the case of an arbitrary deformation functor D. Later, a stronger version of the T^1-lifting property was introduced by Fantechi and Manetti [FM2]. According to their definition, a deformation functor D has the T^1-lifting property if, for any $n \in \mathbb{N}$, the natural map

$$D(B_{n+1}) \to D(B_n) \times_{D(A_n)} D(A_{n+1})$$

is surjective; they show that if D has the T^1-lifting property and k has characteristic 0, then D is smooth. Then, naturally, for any $X_n \in D(A_n)$ one can define $T_D^1(X_n/A_n) = \{Y_n \in D(B_n), D(\beta_n)(Y_n) = X_n\}$. Hence D has the new T^1-lifting property if and only if the natural map $T^1(X_n/A_n) \to T^1(X_{n-1}/A_{n-1})$ is surjective for any $X_n \in D(A_n)$. This is a stronger condition because it depends only on D and does not take into consideration any automorphisms of X_n. However,

$T^1(X_n/A_n)$ does not have any natural k-vector space structures even when $D = \mathrm{Def}(X)$. For this reason we consider the weaker definition given by Kawamata: it has the advantage that $\mathbb{T}^1_D(X_n/A_n)$ has a natural k-vector space structure if D is either $\mathrm{Def}(Y, X)$ or $\mathrm{Def}^{qG}(Y, X)$, which are the cases of interest in this paper.

7. Description of $\mathbb{T}^1(X_n/A_n)$ and $T^1(X_n/A_n)$

Let X be a pure and reduced scheme defined over a field of characteristic 0, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. Let $\mathcal{X}_n \in \mathrm{Def}(Y, X)(A_n)$. In this section we describe the spaces $\mathbb{T}^1(\mathcal{X}_n/A_n)$ and the sheaves $T^1(\mathcal{X}_n/A_n)$.

First we state a simple technical result that will be needed later.

LEMMA 7.1. *Let X be a pure scheme, and let X_R be a deformation of X over a local Artin k-algebra R. Then X_R is also pure.*

Proof. The proof will be by induction on the length $l(R)$ of R. If $l(R) = 1$ then $X_R = X$, which by assumption is pure. Now, for any Artin ring R, the maximal ideal m has a composition sequence $(0) = I_0 \subset I_1 \subset \cdots \subset I_{k-1} \subset I_k = m$ such that $I_k/I_{k+1} \cong R/m$. Since $I_1 = A/m$ and since $I_1 \to I_1/I_1^2$ is surjective, it follows that $I_1^2 = 0$. Hence there is a square zero extension

$$0 \to k \to R \to B \to 0,$$

which gives the square zero extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_R} \xrightarrow{p} \mathcal{O}_{X_B} \longrightarrow 0.$$

Let $J \subset \mathcal{O}_{X_R}$ be an ideal sheaf such that $\dim \mathrm{Supp}(J) < \dim X$. Then, by induction, $p(J) = 0$; hence $J \subset \mathcal{O}_X$ and so $J = 0$ since X is pure. □

PROPOSITION 7.2. *Suppose that X is a pure and reduced scheme, that $Y \subset X$ is a closed subscheme, and that $X - Y$ is smooth. Let $\mathcal{X}_n \in \mathrm{Def}(Y, X)(A_n)$. Then*

$$\mathbb{T}^1(\mathcal{X}_n/A_n) \cong \mathrm{Ext}^1_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\mathcal{X}_n})$$

and

$$T^1(\mathcal{X}_n/A_n) \cong \mathcal{E}xt^1_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\mathcal{X}_n}).$$

Proof. The proof is similar to our proof of Proposition 4.2. We will show only the first isomorphism; the proof of second is identical. Let $\{\mathcal{U}_n^i\}$ be an open cover of $\mathcal{X}_n$ such that $\mathcal{U}_n^i = \hat{U}_n^i$, where U_n^i is a deformation over A_n of a local étale neighborhood V^i of Y in X. Let also $\mathcal{Y}_n \in D(B_n)$ and let $\{\mathcal{W}_n^i\}$ be the corresponding open cover such that $\mathcal{W}_n^i = \hat{W}_n^i$, where W_n^i is a deformation over B_n of a local étale neighborhood Z^i of Y in X. By Lemma 7.1, U_n^i, V^i, W_n^i, and Z^i are also pure.

We know that B_n is the trivial square zero extension of A_n by A_n. Therefore, the trivial extension

$$0 \to A_n \to B_n \to A_n \to 0$$

gives the extension (not necessarily trivial) of A_n-algebras

$$0 \to \mathcal{O}_{W_n^i} \otimes_{B_n} A_n \to \mathcal{O}_{W_n^i} \to \mathcal{O}_{W_n^i} \otimes_{B_n} A_n \to 0.$$

There is a right exact sequence

$$\mathcal{O}_{W_n^i} \otimes_{B_n} A_n \xrightarrow{\alpha_n} \Omega_{W_n^i/A_n} \otimes_{B_n} A_n \to \Omega_{W_n^i \otimes_{B_n} A_n/A_n} \to 0.$$

Since X is pure and reduced, it follows that $W_n^i \otimes_{B_n} A_n$ is pure and hence α_n is injective. Taking completions, we obtain the exact sequence

$$0 \to \mathcal{O}_{\mathcal{U}_i} \to (\Omega_{W_n^i/A_n} \otimes_{B_n} A_n)^\wedge \to (\Omega_{W_n^i \otimes_{B_n} A_n/A_n})^\wedge \to 0.$$

Now if (A, m) is a local k-algebra, then $\hat{\Omega}_{A/k} \cong \hat{\Omega}_{\hat{A}/k}$, where $\hat{A}$ is the m-adic completion of A [TLóR]. Therefore, and patching the preceding sequences together, it follows that there is an exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}_n} \to \hat{\Omega}_{\mathcal{Y}_n/A_n} \otimes_{B_n} A_n \to \hat{\Omega}_{\mathcal{X}_n} \to 0.$$

Hence we have a map

$$\mathbb{T}_D^1(\mathcal{X}_n/A_n) \to \operatorname{Ext}^1_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\mathcal{X}_n}),$$

which (as in the usual scheme case) is injective. We will show that it is also surjective.

Let

$$0 \to \mathcal{O}_{\mathcal{X}_n} \to \mathcal{E} \to \hat{\Omega}_{X_n} \to 0$$

be an element of $\operatorname{Ext}^1_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\mathcal{X}_n})$. Let

$$\hat{d} \colon \mathcal{O}_{\mathcal{X}_n} \to \hat{\Omega}_{\mathcal{X}_n/A_n}$$

be the completion of the universal derivation [TLóR]. Then, again as in the usual scheme case, we get a square zero extension of A_n-algebras

$$0 \to \mathcal{O}_{\mathcal{X}_n} \xrightarrow{\sigma} \mathcal{O}_{\mathcal{Y}_n} \to \mathcal{O}_{\mathcal{X}_n} \to 0. \tag{7.1}$$

Moreover, if we argue exactly as in the proof of Proposition 4.2, it follows that the extension (7.1) is locally the completion of an extension of U_n^i by U_n^i. To complete the proof we need to show that $\mathcal{O}_{\mathcal{Y}_n}$ admits the structure of a flat B_n-algebra and that $\mathcal{Y}_n \otimes_{B_n} A_n = \mathcal{X}_n$. The algebra $\mathcal{O}_{\mathcal{Y}_n}$ is already an A_n-algebra, and it can be made into an A_1-algebra via $\lambda \colon k[t]/(t^2) \to \mathcal{O}_{\mathcal{Y}_n}$ by setting $\lambda(t) = \sigma(1)$. In this way, $\mathcal{O}_{\mathcal{Y}_n}$ becomes a $B_n = (A_1 \otimes A_n)$-algebra. The flatness is a consequence of the following straightforward generalization of [Se, Lemma A.9].

LEMMA 7.3. *Let (B, m_B) be a local ring, A a B-algebra, and M a finitely generated A-module. Let*

$$0 \to M \to A' \to A \to 0 \tag{7.2}$$

be a square zero extension of A by M. Let R be an A'-algebra. Then R is a flat A'-algebra if and only if the sequence (7.2) $\otimes_{A'} R$ *is exact and $R \otimes_{A'} A$ is a flat A-algebra.*

From the construction of the B_n-algebra structure on $\mathcal{O}_{\mathcal{Y}_n}$ we have $\mathcal{O}_{\mathcal{Y}_n} \otimes_{B_n} A_n = \mathcal{O}_{\mathcal{X}_n}$. Furthermore, since $X - Y$ is smooth, (7.1) $\otimes_{B_n} A_n$ is exact on $X - Y$ and, since X is pure, it follows that (7.1) $\otimes_{B_n} A_n$ is, in fact, exact. Hence $\mathcal{O}_{\mathcal{Y}_n}$ is flat over B_n. □

REMARK 7.4. If $X = Y$ then Proposition 7.2 says simply that

$$\mathbb{T}^1(X_n/A_n) \cong \mathrm{Ext}^1_{X_n}(\Omega_{X_n/A_n}, \mathcal{O}_{X_n})$$

and

$$T^1(X_n/A_n) \cong \mathcal{E}xt^1_{X_n}(\Omega_{X_n/A_n}, \mathcal{O}_{X_n}),$$

where $X_n \in \mathrm{Def}(X)(A_n)$.

REMARK 7.5. Proposition 7.2 was proved by Namikawa [Na] for the case $X = Y$.

As a corollary of Proposition 7.2, the spectral sequence relating the functors Ext and $\mathcal{E}xt$ gives the local-to-global sequence for T^1.

COROLLARY 7.6. *Given the assumptions in Proposition 7.2, there exists an exact sequence*

$$0 \to H^1(\hat{T}_{\mathcal{X}_n/A_n}) \to \mathbb{T}^1(\mathcal{X}_n/A_n) \to H^0(T^1(\mathcal{X}_n/A_n)) \to H^2(\hat{T}_{\mathcal{X}_n/A_n}).$$

The next technical lemma will be needed in the sequel.

LEMMA 7.7. *Let X be a pure and reduced scheme, and let X_A be a deformation of X over a local Artin k-algebra A. Let F_A be a coherent sheaf on X_A for which there is a nonempty open subset $U_A \subset X_A$ such that the restriction $F_A|_{U_A}$ is flat over A. Let $A \to B$ be a homomorphism of finite Artin local k-algebras, and let $X_B = X_A \otimes_A B$. Let $i\colon X_B \to X_A$ be the inclusion, and let G_B be a coherent $\mathcal{O}_{X_B}$-module. Then, for all $k \geq 0$,*

$$\mathrm{Ext}^k_{X_A}(F_A, i_*G_B) \cong \mathrm{Ext}^k_{X_B}(i^*F_A, G_B)$$

and

$$\mathcal{E}xt^k_{X_A}(F_A, i_*G_B) \cong \mathcal{E}xt^k_{X_B}(i^*F_A, G_B).$$

Proof. For any k there are natural maps

$$\phi^k_F\colon \mathrm{Ext}^k_{X_B}(i^*F_A, G_B) \to \mathrm{Ext}^k_{X_A}(F_A, i_*G_B),$$
$$\psi^k_F\colon \mathcal{E}xt^k_{X_B}(i^*F_A, G_B) \to \mathcal{E}xt^k_{X_A}(F_A, i_*G_B)$$

defined as follows. Let $[E_B]$ be an element of $\mathrm{Ext}^k_{X_B}(i^*F_A, G_B)$. This is represented by an extension

$$0 \to G_B \to E_1 \to E_2 \to \cdots \to E_k \to i^*F_A \to 0.$$

Moreover, there is a natural map $\lambda_A\colon F_A \to i_*i^*F_A$. We define $\phi^k_F([E_A]) \in \mathrm{Ext}^k_{X_A}(F_A, i_*G_B)$ to be the extension obtained by pulling back $[E_A]$ with λ, and similarly for ψ^k_F.

Let $i_*\colon \mathrm{Coh}(X_B) \to \mathrm{Coh}(X_A)$ be the induced map between the corresponding categories of coherent sheaves. Let G be either the $\mathcal{H}om_{X_A}(F_A, \cdot)$ or $\mathrm{Hom}_{X_A}(F_A, \cdot)$ functor. Since i_* is exact, to prove the lemma it suffices to show that i_* sends injectives to G-acyclics. First we show this in the case when $G = \mathcal{H}om_{X_A}(F_A, \cdot)$. Let I_B be an injective $\mathcal{O}_{X_B}$-module. We will show that

$$\mathcal{E}xt^k_{X_A}(F_A, I_B) = 0;$$

this is local, so we may assume that X (and hence X_A) is affine. Then X_A has enough locally free sheaves. So we may write

$$0 \to P_A \to E_A \to F_A \to 0,$$

where E_A is locally free. Hence

$$\mathcal{E}xt^k_{X_A}(F_A, I_B) = \mathcal{E}xt^{k-1}_{X_A}(P_A, I_B).$$

Furthermore, since X is pure it follows that X_A is pure as well. Therefore, E_A is pure and hence P_A is also pure and its restriction on U_A is flat over A. Continuing similarly, we find that

$$\mathcal{E}xt^k_{X_A}(F_A, I_B) = \mathcal{E}xt^1_{X_A}(N_A, I_B),$$

where N_A is also pure and its restriction on U_A is flat over A. Now consider the exact sequence

$$0 \to Q_A \to M_A \to N_A \to 0,$$

where M_A is locally free. Then, as before, Q_A is pure and thus, since N_A is flat over U_A, it follows that

$$0 \to i^*Q_A \to i^*M_A \to i^*N_A \to 0$$

is exact, too. Hence there is a commutative diagram

$$\begin{array}{ccccccccc}
\mathcal{H}om_{X_A}(N_A, i_*I_B) & \longrightarrow & \mathcal{H}om_{X_A}(M_A, i_*I_B) & \longrightarrow & \mathcal{H}om_{X_A}(Q_A, i_*I_B) & \longrightarrow & \mathcal{E}xt^1_{X_A}(N_A, i_*I_B) & \longrightarrow & 0 \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \\
\mathcal{H}om_{X_B}(i^*N_A, I_B) & \longrightarrow & \mathcal{H}om_{X_B}(i^*M_A, I_B) & \longrightarrow & \mathcal{H}om_{X_B}(i^*Q_A, I_B) & \longrightarrow & \mathcal{E}xt^1_{X_A}(i^*N_A, I_B) & \longrightarrow & 0
\end{array}$$

where f_1, f_2, and f_3 are clearly isomorphisms. Consequently, f_4 is also an isomorphism. But since I_B is an injective $\mathcal{O}_{X_B}$-module, we have

$$\mathcal{E}xt^1_{X_A}(G_A, i_*I_B) = \mathcal{E}xt^1_{X_B}(i^*G_A, I_B) = 0$$

and hence

$$\mathcal{E}xt^k_{X_A}(F_A, i_*I_B) = 0$$

for all $k \geq 1$, as claimed. Next we show the corresponding statement for the global Ext. The spectral sequence relating the local and global Ext functors show that

$$\begin{aligned}
\mathrm{Ext}^k_{X_A}(F_A, i_*I_B) &= H^k(\mathcal{H}om_{X_A}(F_A, i_*I_B)) \\
&= H^k(\mathcal{H}om_{X_B}(i^*F_A, I_B)) = \mathrm{Ext}^k_{X_B}(i^*F_A, I_B) = 0.
\end{aligned}$$

The argument about the $\mathcal{E}xt$ sheaves cannot be directly applied to the global Ext functor because there may not be enough locally free sheaves on X_A. □

Next we give a version of the previous results in the case of formal schemes.

COROLLARY 7.8. *With assumptions as in Lemma 7.7, let* $\mathcal{X}_A \in \mathrm{Def}(Y,X)(A)$ *and* $\mathcal{X}_B = \mathcal{X}_A \otimes_A B$. *Let* $\mathcal{F}_A$ *be a coherent sheaf on* $\mathcal{X}_A$ *for which there is an open* $\mathcal{U}_A \subset \mathcal{X}_A$ *such that* $\mathcal{F}_A|_{\mathcal{U}_A}$ *is flat over* A. *Let* $\mathcal{G}_B$ *be a coherent sheaf on* $\mathcal{X}_B$ *and let* $i\colon \mathcal{X}_B \to \mathcal{X}_A$ *be the inclusion. Then*

$$\mathcal{E}xt^i_{\mathcal{X}_A}(\mathcal{F}_A, i_*\mathcal{G}_B) \cong \mathcal{E}xt^i_{\mathcal{X}_B}(i^*\mathcal{F}_A, \mathcal{G}_B)$$

and

$$\mathrm{Ext}^i_{\mathcal{X}_A}(\mathcal{F}_A, i_*\mathcal{G}_B) \cong \mathrm{Ext}^i_{\mathcal{X}_B}(i^*\mathcal{F}_A, \mathcal{G}_B).$$

Proof. The natural map $\phi^i_{\mathcal{F}}$ defined in Lemma 7.7 exists in this case, too. Then the proof proceeds similarly and it is local. Locally $\mathcal{X}_A \cong \hat{V}_A$, where V_A is a deformation over A of a local étale neighborhood V of Y in X. So we may assume that $\mathcal{F}_A = \hat{F}_A$ and $\mathcal{G}_B = \hat{G}_B$, where F_A, G_B are coherent sheaves on V_A, V_B. But then, as we have already seen in Proposition 4.2,

$$\mathcal{E}xt^i_{\hat{V}_A}(\hat{F}_A, \hat{G}_B) = (\mathcal{E}xt^i_{V_A}(F_A, G_B))^{\wedge}.$$

Moreover, if $\mathcal{I}_B$ is an injective $\mathcal{O}_{\mathcal{X}_B}$-module then $\mathcal{I}_B = \hat{I}_B$, where I_B is an injective $\mathcal{O}_{V_B}$-module. Now the proof proceeds exactly as the proof of Lemma 7.7. □

We next state the key result that will enable us to obtain obstructions to lift a deformation $X_n \in \mathrm{Def}(Y,X)(A_n)$ to A_{n+1}.

PROPOSITION 7.9. *Let X be a pure and reduced scheme defined over a field k of characteristic 0, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. Let* $\mathcal{X}_n \in \mathrm{Def}(Y,X)(A_n)$. *Then there are exact sequences*

$$0 \to \hat{T}_X \to \hat{T}_{\mathcal{X}_n/A_n} \to \hat{T}_{\mathcal{X}_{n-1}/A_{n-1}} \to T^1(Y,X) \to T^1(\mathcal{X}_n/A_n) \to T^1(\mathcal{X}_{n-1}/A_{n-1}) \xrightarrow{\theta} \mathcal{E}xt^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$$

and

$$0 \to H^0(\hat{T}_X) \to H^0(\hat{T}_{\mathcal{X}_n/A_n}) \to H^0(\hat{T}_{\mathcal{X}_{n-1}/A_{n-1}}) \to \mathbb{T}^1(Y,X) \to \mathbb{T}^1(\mathcal{X}_n/A_n) \to \mathbb{T}^1(\mathcal{X}_{n-1}/A_{n-1}) \xrightarrow{\Theta} \mathrm{Ext}^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}).$$

Note that, since $X - Y$ is assumed to be smooth, it follows from Proposition 4.2 that $T^1(Y,X) = T^1(X)$.

Proof of Proposition 7.9. Apply $\mathrm{Hom}_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \cdot)$ and $\mathcal{H}om_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \cdot)$ on the square zero extension

$$0 \to \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\mathcal{X}_n} \to \mathcal{O}_{\mathcal{X}_{n-1}} \to 0$$

and then use Proposition 7.2 and Lemma 7.7. □

8. Global Lifting of Deformations

Let $\mathcal{X}_n \in \mathrm{Def}(Y,X)(A_n)$. In this section we obtain obstructions to the lifting of $\mathcal{X}_n$ to A_{n+1}. Let $\mathcal{Y}_{n-1} = \mathrm{Def}(Y,X)(\varepsilon_{n-1})(\mathcal{X}_n) \in \mathbb{T}^1(\mathcal{X}_{n-1}/A_{n-1})$. According to Theorem 6.4, $\mathcal{X}_n$ lifts to A_{n+1} if and only if $\mathcal{Y}_{n-1}$ is in the image of the natural map

$$\mathbb{T}^1(\mathcal{X}_n/A_n) \xrightarrow{\tau_n} \mathbb{T}^1(\mathcal{X}_{n-1}/A_{n-1}).$$

According to Proposition 7.9, there is an exact sequence

$$\mathbb{T}^1(\mathcal{X}_n/A_n) \xrightarrow{\tau_n} \mathbb{T}^1(\mathcal{X}_{n-1}/A_{n-1}) \xrightarrow{\Theta} \mathrm{Ext}^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}).$$

Identifying $\mathbb{T}^1(\mathcal{X}_{n-1}/A_{n-1})$ with

$$\mathrm{Ext}^1_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\mathcal{X}_{n-1}}) = \mathrm{Ext}^1_{\mathcal{X}_{n-1}}(\hat{\Omega}_{\mathcal{X}_{n-1}/A_{n-1}}, \mathcal{O}_{\mathcal{X}_{n-1}})$$

and identifying $\mathbb{T}^1(\mathcal{X}_n/A_n)$ with $\mathrm{Ext}^1_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\mathcal{X}_n})$, we see that $\mathcal{Y}_{n-1}$ is represented by the extension

$$0 \to \mathcal{O}_{\mathcal{X}_{n-1}} \to E \to \hat{\Omega}_{\mathcal{X}_n/A_n} \to 0,$$

which is the pullback of the extension

$$0 \to \mathcal{O}_{\mathcal{X}_{n-1}} \to \hat{\Omega}_{\mathcal{Y}_{n-1}/A_{n-1}} \otimes_{B_{n-1}} A_{n-1} \to \hat{\Omega}_{\mathcal{X}_{n-1}/A_{n-1}} \to 0$$

under the natural map $\hat{\Omega}_{\mathcal{X}_n/A_n} \to \hat{\Omega}_{\mathcal{X}_{n-1}/A_{n-1}}$. Hence

$$E = (\hat{\Omega}_{\mathcal{Y}_{n-1}/A_{n-1}} \otimes_{B_{n-1}} A_{n-1}) \times_{\hat{\Omega}_{\mathcal{X}_{n-1}/A_{n-1}}} \hat{\Omega}_{\mathcal{X}_n/A_n}.$$

Then $\Theta(\mathcal{Y}_{n-1}) \in \mathrm{Ext}^2_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\hat{X}}) = \mathrm{Ext}^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$ is represented by the two-term extension

$$0 \to \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\mathcal{X}_n} \to E \to \hat{\Omega}_{\mathcal{X}_n/A_n} \to 0.$$

We can therefore use Theorem 6.4 to obtain the following result.

THEOREM 8.1. *With assumptions as in Proposition 7.9, let*

$$\mathcal{Y}_{n-1} = \mathrm{Def}(Y, X)(\varepsilon_{n-1})(\mathcal{X}_n).$$

Then the obstruction to lifting $\mathcal{X}_n$ to a deformation $\mathcal{X}_{n+1}$ over A_{n+1} is the element $\mathrm{ob}(\mathcal{X}_n) \in \mathrm{Ext}^2_{\mathcal{X}_n}(\hat{\Omega}_{\mathcal{X}_n/A_n}, \mathcal{O}_{\hat{X}}) = \mathrm{Ext}^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$ *represented by the extension*

$$0 \to \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\mathcal{X}_n} \to E \to \hat{\Omega}_{\mathcal{X}_n/A_n} \to 0,$$

where

$$E = (\hat{\Omega}_{\mathcal{Y}_{n-1}/A_{n-1}} \otimes_{B_{n-1}} A_{n-1}) \times_{\hat{\Omega}_{\mathcal{X}_{n-1}/A_{n-1}}} \hat{\Omega}_{\mathcal{X}_n/A_n}.$$

Therefore, if $\mathrm{Ext}^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}) = 0$ *and if Y and X satisfy the conditions of Proposition 5.3, then the hull of* $\mathrm{Def}(Y, X)$ *is smooth.*

In practice it is easier to verify vanishing for cohomology than for the Ext groups. Next we shall give some cohomological conditions for the vanishing of $\mathrm{Ext}^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})$, but first we give a definition.

DEFINITION 8.2. Let X be a pure scheme, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. Then we denote by $\mathrm{Ob}^3(X)$ the cokernel of the local-to-global obstruction map $H^0(T^1(X)) \to H^2(\hat{T}_X)$ of Proposition 4.2.

COROLLARY 8.3. *There are three successive obstructions in $H^0(\mathcal{E}xt^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}))$, $H^1(T^1(X))$, and $\mathrm{Ob}^3(X)$ to the lifting of X_n to A_{n+1}. Therefore, if*

$$H^0(\mathcal{E}xt^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})) = H^1(T^1(X)) = \mathrm{Ob}^3(X) = 0$$

and if $\mathrm{Def}_Y(X)$ *has a hull, then its hull is smooth and of dimension*

$$h^1(\hat{T}_X) + h^0(T^1(X)) - h^2(\hat{T}_X).$$

Proof. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(\mathcal{E}xt^q_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})) \implies E^{p+q} = \mathrm{Ext}^{p+q}_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})).$$

Then there are exact sequences

$$0 \to E_1^2 \to E^2 \to E_2^{0,2},$$

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E_1^2 \to E_2^{1,1} \to E_2^{3,0}.$$

The claim now follows once we consider that $E^2 = \mathrm{Ext}^{p+q}_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}))$, $E_2^{0,2} = H^0(\mathcal{E}xt^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}))$, and $E_2^{2,0} = H^2(\hat{T}_X)$. □

COROLLARY 8.4. *Suppose that* $\mathrm{Def}(Y, X)$ *has a hull and that*

$$H^0(\mathcal{E}xt^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})) = H^1(T^1(X)) = H^2(\hat{T}_X) = H^0(T^1(X)) = 0.$$

Then every deformation of X is formally locally trivial.

The conclusion follows because, by Corollary 8.3, the hull of $\mathrm{Def}(Y, X)$ is smooth and is the same as the hull of the locally trivial deformations $\mathrm{Def}'(Y, X)$.

REMARK 8.5. The simplest case if $\mathrm{Ob}^3(X) = 0$ is when $H^2(\hat{T}_X) = 0$. This happens in particular when there is a morphism $f\colon X \to S$, where S is affine, f is proper with fibers of dimension ≤ 1, and $Y = f^{-1}(s)$ for some $s \in S$. Then, by the formal functions theorem, $H^2(\hat{T}_X) = 0$. This is the case of 3-fold flips and divisorial contractions with at most 1-dimensional fibers.

9. Local to Global

Let X be a scheme, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. In Section 8 we obtained obstructions to the lifting of a deformation $X_n \in \mathrm{Def}(Y, X)(A_n)$ to a deformation $X_{n+1} \in \mathrm{Def}(Y, X)(A_{n+1})$ for the case where X is pure and reduced. However, our methods were global and did not yield any information about the local structure of X_{n+1}. In this section we will study the problem of when local liftings of X_n globalize to give a deformation X_{n+1} of X over A_{n+1} or, more generally, when local deformations of X exist globally.

Ideally one should study the local-to-global map $\pi : \mathrm{Def}(Y, X) \to \mathrm{Def}_{\mathrm{loc}}(Y, X)$. If $X = Y$, X has isolated singularities, and $H^2(T_X) = 0$, then π is known to be smooth. This is no longer necessarily true if X has positive-dimensional singular locus. The reason is the same as that given for the failure of $\mathrm{Def}_{\mathrm{loc}}(Y, X)$ to have a hull: the presence of local automorphisms that do not lift to higher order. However, under strong restrictions on the singularities of X, π is smooth.

PROPOSITION 9.1. *Suppose the assumptions in Theorem 5.5 hold, and suppose also that* $H^2(\hat{T}_X) = 0$. *Then* π *is smooth.*

Proof. As before, we demonstrate the case $X = Y$ (the general case is proved similarly). For convenience, set $D = \mathrm{Def}_{\mathrm{loc}}(Y, X)$ and $D_{\mathrm{loc}} = \mathrm{Def}_{\mathrm{loc}}(Y, X)$. Then it suffices to show that, for any small extension

$$0 \to J \to B \xrightarrow{g} A \to 0,$$

the natural map

$$D(B) \to D(A) \times_{D_{\mathrm{loc}}(A)} D_{\mathrm{loc}}(B)$$

is surjective.

Let $X_A \in D(A)$, $s_A = \pi(X_A) \in D_{\mathrm{loc}}(A)$, and $s_B \in D_{\mathrm{loc}}(B)$ be such that $D_{\mathrm{loc}}(g)(s_B) = s_A$. By the definition of D_{loc}, s_B and s_A are equivalent to an open cover $\{U_i\}$ of X, a collection of deformations U_i^B and U_i^A of U_i over B and A (respectively) for which $U_i^B \otimes_B A \cong U_i^A$, B-isomorphisms $\phi_{ij}^B \colon U_i^B|_{U_i} \cap U_j \to U_j^B|_{U_j \cap U_i}$, and A-isomorphisms $\phi_{ij}^A \colon U_i^A|_{U_i \cap U_j} \to U_j^A|_{U_j \cap U_i}$ such that, for any i, j, k, $\phi_{ij}^A \phi_{jk}^A \phi_{ki}^A$ is the identity automorphism of $U_{ijk}^A = U_i^A \cap U_j^A \cap U_k^A$.

By assumption, we may take U_i in such a way that $U_i \cap U_j$ satisfies (H_4). Hence we may take the ϕ_{ij}^B such that, on $U_{ijk} = U_i \cap U_j \cap U_k$, the restriction of $\phi_{ijk}^B = \phi_{ij}^B \phi_{jk}^B \phi_{ki}^B$ on U_{ijk}^A is the identity automorphism of U_{ijk}^A. Hence ϕ_{ijk}^B corresponds to a B-derivation $d_{ijk} \in \mathrm{Hom}_{U_i^B}(\Omega_{U_i^B/B}, \mathcal{O}_{U_i}) = \mathrm{Hom}_{U_i}(\Omega_{U_i}, \mathcal{O}_{U_i})$. On the 4-fold intersections $U_{ijks} = U_i \cap U_j \cap U_k \cap U_s$, the ϕ_{ijk}^B satisfy a cocycle condition and hence we get an element of $H^2(\mathcal{H}om_X(\Omega_X, \mathcal{O}_X)) = H^2(T_X)$. If this element vanishes, then the ϕ_{ij}^B can be modified in such a way that $\phi_{ij}^B \phi_{jk}^B \phi_{ki}^B$ is the identity automorphism of $U_i^B \cap U_j^B \cap U_k^B$ and hence the U_i^B glue to a global deformation X_B. $\square$

In order to circumvent the failure of the local-to-global map $\pi \colon \mathrm{Def}(Y, X) \to \mathrm{Def}_{\mathrm{loc}}(Y, X)$ to be smooth, we must gain some control of the automorphisms of deformations. Bearing this in mind, and following the ideas of Lichtenbaum and Schlessinger [LiS], we establish the following definitions.

DEFINITION 9.2. Let

$$0 \to J \to B \to A \to 0 \tag{9.1}$$

be a small extension of Artin rings, and let $X_A \in \mathrm{Def}(Y, X)(A)$. Let (X_B^i, ϕ_i), $i = 1, 2$, be pairs, where $X_B^i \in \mathrm{Def}(Y, X)(B)$ and the $\phi_i \colon X_A \to X_B^i \otimes_B A$ are isomorphisms. We say that the pair (X_B^1, ϕ_1) is *isomorphic* to the pair (X_B^2, ϕ_2) if and only if there is a B-isomorphism $\psi \colon X_B^1 \to X_B^2$ such that $\psi\phi_1 = \phi_2$.

(1) We define $\mathrm{Def}(X_A/A, B)$ to be the set of isomorphism classes of pairs $[X_B, \phi]$ of deformations $X_B \in \mathrm{Def}(Y, X)(B)$ and marking isomorphisms $\phi \colon X_A \to X_B \otimes_B A$.
(2) Let $\underline{\mathrm{Def}}(X_A/A, B)$ be the sheaf of sets associated to the presheaf F on X such that $F(U) = \mathrm{Def}(U_A/A, B)$, where $U_A = X_A|_U$. Then we define

$$\mathrm{Def}_{\mathrm{loc}}(X_A/A, B) = H^0(\underline{\mathrm{Def}}(X_A/A, B)).$$

Note that there is a natural map

$$\pi \colon \mathrm{Def}(X_A/A, B) \to \mathrm{Def}_{\mathrm{loc}}(X_A/A, B).$$

Note also that, since any square zero extension of local Artin k-algebras can be obtained by a sequence of successive small extensions, we do not lose anything by working only with small extensions.

REMARK 9.3. Let $X_n \in \mathrm{Def}(Y, X)(A_n)$. Then, in the notation of Section 6, $\mathbb{T}^1(X_n/A_n) = \mathrm{Def}(X_n/A_n, B_{n+1})$ and $T^1(X_n/A_n) = \mathrm{Def}_{\mathrm{loc}}(X_n/A_n, B_{n+1})$.

THEOREM 9.4. *Let X be a scheme defined over a field k, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. Let*

$$0 \to J \to B \to A \to 0$$

be a small extension of local Artin k-algebras, and let $X_A \in \mathrm{Def}(Y, X)(A)$. Then the following statements hold.

(1) $\mathrm{Def}(X_A/A, B)$ *and* $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$ *are* $\mathbb{T}^1(Y, X) \otimes J$ *and* $H^0(T^1(X) \otimes J)$ *homogeneous spaces, respectively.*
(2) *Let $s_B \in \mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$. Then the set $\pi^{-1}(s_B)$ is a homogeneous space over $H^1(\hat{T}_X \otimes J)$.*
(3) *There is a sequence*

$$0 \longrightarrow H^1(\hat{T}_X \otimes J) \xrightarrow{\alpha} \mathrm{Def}(X_A/A, B) \xrightarrow{\pi} \mathrm{Def}_{\mathrm{loc}}(X_A/A, B) \xrightarrow{\partial} H^2(\hat{T}_X \otimes J)$$

that is exact in the following sense. Let $s_B \in \mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$. Then s_B is in the image of π if and only if $\partial(s_B) = 0$. Moreover, let $X_B, X'_B \in \mathrm{Def}(X_A/A, B)$ such that $\pi(X_A) = \pi(X'_A)$. Then there is a $\gamma \in H^1(\hat{T}_X \otimes J)$ such that $X'_A = \gamma \cdot X_A$, where by "$\cdot$" we denote the action of $H^1(\hat{T}_X \otimes J)$ on $\pi^{-1}(s_B)$.

Proof. We will prove the theorem only for the case $X = Y$. The local algebraizability conditions embedded in the definition of $\mathrm{Def}(Y, X)$ ensure that, with some effort, all steps of the proof can be carried out in the case when $Y \neq X$ and $X - Y$ is smooth. The proof of the theorem proceeds in two steps.

Step 1. In this step we obtain descriptions of $\mathrm{Def}(X_A/A, B)$ and $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$ using cotangent sheaf cohomology and spaces of infinitesimal extensions, which we describe next. Let X be an S-scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. We denote by $\mathrm{Ex}(X/S, \mathcal{F})$ the space of square zero extensions

$$0 \to \mathcal{F} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0$$

of S-schemes [Gr1]. Note that there is always a natural map

$$\mathrm{Ex}(X/S, \mathcal{F}) \to H^0(T^1(X/S, \mathcal{F})),$$

where $T^1(X/S, \mathcal{F})$ is the first cotangent cohomology sheaf of Schlessinger [LiS]. This map is an isomorphism if X and S are affine.

The sequence $B \to A \to \mathcal{O}_{X_A}$ gives the exact sequences

$$0 \longrightarrow T^1(X_A/A, J \otimes_A \mathcal{O}_{X_A}) \longrightarrow T^1(X_A/B, J \otimes_A \mathcal{O}_{X_A}) \xrightarrow{\nu} T^1(A/B, J \otimes_A \mathcal{O}_{X_A}) \longrightarrow T^2(X_A/A, J \otimes_A \mathcal{O}_{X_A}) \quad (9.2)$$

and

$$0 \to \mathrm{Ex}(X_A/A, J \otimes_A \mathcal{O}_{X_A}) \to \mathrm{Ex}(X_A/B, J \otimes_A \mathcal{O}_{X_A}) \xrightarrow{\mu} \mathrm{Ex}(A/B, J \otimes_A \mathcal{O}_{X_A}) \quad (9.3)$$

(see [Gr1; LiS]). After taking global sections on sequence (9.2) we have

$$0 \to H^0(T^1(X_A/A, J \otimes_A \mathcal{O}_{X_A})) \to H^0(T^1(X_A/B, J \otimes_A \mathcal{O}_{X_A})) \xrightarrow{\lambda} H^0(T^1(A/B, J \otimes_A \mathcal{O}_{X_A})). \quad (9.4)$$

By a slight abuse of notation, we shall use $[\mathcal{I}]$ to denote both the elements of $\mathrm{Ex}(A/B, J \otimes_A \mathcal{O}_{X_A})$ and of $H^0(T^1(A/B, J \otimes_A \mathcal{O}_{X_A}))$ corresponding to the square zero extension

$$0 \to J \otimes_A \mathcal{O}_{X_A} \to B \to A \to 0.$$

We now claim that

(1) $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B) = \lambda^{-1}([\mathcal{I}])$ and
(2) $\mathrm{Def}(X_A/A, B) = \mu^{-1}([I])$.

Indeed, an element of $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$ is equivalent to an open cover $\{U_i\}$ of X and pairs $[U_B^i, \phi_A^i] \in \mathrm{Def}(U_A^i/A, B)$, where $U_A^i = X_A|_{U_i}$, such that $[U_B^i|_{U_i \cap U_j}, \phi_A^i|_{U_i \cap U_j}] = [U_B^J|_{U_i \cap U_j}, \phi_A^J|_{U_i \cap U_j}]$ for any i, j. These give square zero extensions $[e_i] \in T^1(U_A^i/B, J \otimes_A \mathcal{O}_{U_A^i})$ and

$$0 \to J \otimes_A \mathcal{O}_{U_A^i} \to \mathcal{O}_{U_B^i} \to \mathcal{O}_{U_A^i} \to 0,$$

which are isomorphic on the overlaps $U_i \cap U_j$ and hence glue to an element $[e] \in H^0(T^1(X_A/B, J \otimes_A \mathcal{O}_{X_A}))$. Moreover, the facts that U_B^i is flat over B and $U_B^i \otimes_B A = U_A^i$ imply that $\lambda([e]) = [\mathcal{I}]$ [LiS]. Therefore, $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B) = \lambda^{-1}([\mathcal{I}])$. A similar argument shows also that $\mathrm{Def}(X_A/A, B) = \mu^{-1}([I])$.

Step 2. This is the main part of the proof of the theorem. Combining the results of the claim and the exact sequences (9.3) and (9.4), it follows that $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$ and $\mathrm{Def}(X_A/A, B)$ are $H^0(T^1(X_A/A, J \otimes_A \mathcal{O}_{X_A})) = H^0(T^1(X) \otimes J)$ [LiS] and $\mathrm{Ex}(X_A/A, J \otimes_A \mathcal{O}_{X_A}) = \mathbb{T}^1(X) \otimes J$ [Gr1] homogeneous spaces. This shows Theorem 9.4(1).

We proceed to show part (2) of the theorem. In what follows we use the following notation. Let $\{U_i\}_{i \in I}$ be an open cover of X. Then, for any choice of indices $i_1, \ldots, i_k$, we set $U_{i_1 i_2 \cdots i_k} = U_{i_1} \cap \cdots \cap U_{i_k}$. Also if X_R is a deformation of X over an Artin ring R, we set $X_R^{i_1 \cdots i_k} = X_R|_{U_{i_1} \cap \cdots \cap U_{i_k}}$.

Let $s_B \in \mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$. First we exhibit the action of $H^1(T_X \otimes J) = H^1(\mathcal{H}om_{X_A}(\Omega_{X_A/A}, J \otimes_A \mathcal{O}_{X_A}))$ on $\pi^{-1}(s_B)$. Let $[X_B, \phi] \in \pi^{-1}(s_B)$ and $\gamma \in H^1(\mathcal{H}om_{X_A}(\Omega_{X_A/A}, J \otimes_A \mathcal{O}_{X_A}))$. The element s_B is equivalent to giving an open cover $\{U_i\}_{i \in I}$ of X; elements $[U_B^i, \phi^i] \in \mathrm{Def}(U_A^i/A, B)$; and, for all i, j, isomorphisms $\phi^{ij} \colon U_B^i|_{U_i \cap U_j} \to U_B^j|_{U_i \cap U_j}$ such that $\phi^{ij}\phi^i = \phi^j$ on $U_i \cap U_j$. The element $[X_B, \phi] \in \mathrm{Def}(X_A/A, B)$ is equivalent to giving elements $[U_B^i, \psi^i] \in \mathrm{Def}(U_A^i/A, B)$ for all i and, for all i, j, isomorphisms $\psi^{ij} \colon U_B^i|_{U_i \cap U_j} \to U_B^j|_{U_i \cap U_j}$

such that $\psi^{ij}\psi^{i} = \psi^{j}$ on $U_i \cap U_j$ and $\psi^{jk}\psi^{ij} = \psi^{ik}$ on the triple intersections $U_i \cap U_j \cap U_k$. The cohomology class γ is equivalent to a collection $\gamma_{ij} \in \mathrm{Hom}_{X_A^{ij}}(\Omega_{X_A^{ij}/A}, J \otimes \mathcal{O}_{X_A^{ij}}) = \mathrm{Hom}_{U_B^{ij}}(\Omega_{U_B^{ij}/B}, J \otimes \mathcal{O}_{X_A^{ij}})$, where $U_B^{ij} = U_B^{i}|_{U_i \cap U_j}$, that satisfies the cocycle condition on the triple intersections. Therefore, γ is equivalent to a collection of B-derivations $d_{ij}\colon \mathcal{O}_{U_B^{ij}} \to J \otimes \mathcal{O}_{X_A^{ij}}$ satisfying the cocycle condition on the triple intersections. Then we define $\gamma \cdot [X_B, \phi]$ to be the element of $\pi^{-1}(s_B)$ that is defined by the data $[U_B^i, \psi^i]$ and glueing isomorphisms $\psi^{ij} + d_{ij}\colon U_B^i|_{U_i \cap U_j} \to U_B^j|_{U_i \cap U_j}$.

It remains to show that $\pi^{-1}(s_B)$ is an $H^1(T_X \otimes J)$-homogeneous space—in other words, that $H^1(T_X \otimes J)$ acts transitively on $\pi^{-1}(s_B)$. Let $[X_B, \psi], [X'_B, \psi'] \in \pi^{-1}(s_B)$. Then there exist an open cover $\{U_i\}_{i \in I}$ of X and isomorphisms $\lambda_i\colon X_B|_{U_i} \to X'_B|_{U_i}$, for all $i \in I$, such that $\lambda_i \psi = \psi'$ on U_i. Then, on U_{ij}, $\lambda_{ij} = \lambda_j^{-1}\lambda_i$ is an automorphism of X_B^{ij} over X_A^{ij}. Therefore, λ_{ij} corresponds to a B-derivation $d_{ij} \in \mathrm{Der}_B(\mathcal{O}_{X_B^{ij}}, J \otimes \mathcal{O}_{X_A^{ij}}) = \mathrm{Hom}_{X_B^{ij}}(\Omega_{X_B^{ij}/B}, J \otimes \mathcal{O}_{X_A^{ij}}) = \mathrm{Hom}_{X_A^{ij}}(\Omega_{X_A^{ij}/A}, J \otimes \mathcal{O}_{X_A^{ij}})$. These satisfy the cocycle condition on triple intersections and hence give an element $\gamma \in H^1(\mathcal{H}om_{X_A}(\Omega_{X_A/A}, J \otimes_A \mathcal{O}_{X_A})) = H^1(T_X \otimes J)$. Now, from the definition of the action of $H^1(T_X \otimes J)$ on $\pi^{-1}(s_B)$, it is clear that $\gamma \cdot [X_B, \psi] = [X'_B, \psi']$; therefore, the action is transitive.

Now we show part (3). Taking into consideration the previous two parts, it suffices to construct the map ∂ and to show that $\mathrm{Ker}(\partial) \subset \mathrm{Im}(\pi)$. Let $s_B \in \mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$ as before. Then, for any $i, j, k \in I$, $\phi_{ijk} = \phi_{ki}\phi_{jk}\phi_{ij}$ is a B-automorphism of $U_B^i|_{U_{ijk}}$ over X_A^{ijk}. Therefore, ϕ_{ijk} corresponds to a B-derivation $d_{ijk} \in \mathrm{Der}_B(\mathcal{O}_{U_B^i|_{U_{ijk}}}, J \otimes \mathcal{O}_{X_A^{ijk}}) = \mathrm{Hom}_{X_A^{ijk}}(\Omega_{X_A^{ijk}/A}, J \otimes \mathcal{O}_{X_A^{ijk}})$. These satisfy the cocycle condition on the 4-fold intersections and thus give an element of $H^2(\mathcal{H}om_{X_A}(\Omega_{X_A/A}, J \otimes_A \mathcal{O}_{X_A})) = H^2(T_X \otimes J)$. This defines the map ∂. If $\partial(s_B) = 0$, then the isomorphisms ϕ_{ij} can be modified so that ϕ_{ijk} is the identity automorphism of $U_B^i|_{U_{ijk}}$ and therefore the U_B^i and ϕ^i glue to a global deformation X_B and the isomorphism $\phi\colon X_A \to X_B \otimes_B A$. Hence $s_B = \pi([X_B, \phi])$, as claimed. □

COROLLARY 9.5. *With assumptions as in Theorem 9.4, there are two successive obstructions in* $H^0(T^2(X) \otimes J)$ *and* $H^1(T^1(X) \otimes J)$ *in order for*

$$\mathrm{Def}_{\mathrm{loc}}(X_A/A, B) \neq \emptyset$$

(*i.e., for* X_A *to lift locally to* B). *If these obstructions vanish then there is another obstruction in* $H^2(\hat{T}_X \otimes J)$ *in order for* $\mathrm{Def}(X_A/A, B) \neq \emptyset$ (*i.e., for the local deformations to globalize*).

Proof. We show only the case $X = Y$; the general case is similar. Let $Q = \mathrm{Im}(\nu)$, where ν is the map in the long exact sequence (9.2). Then there are two exact sequences,

$$0 \to H^0(T^1(X_A/A, J \otimes \mathcal{O}_{X_A})) \to H^0(T^1(X_A/B, J \otimes \mathcal{O}_{X_A})) \xrightarrow{\alpha} H^0(Q) \xrightarrow{\beta} H^1(T^1(X_A/A, J \otimes \mathcal{O}_{X_A})) \quad (9.5)$$

and

$$0 \to H^0(Q) \to H^0(T^1(A/B, J \otimes \mathcal{O}_{X_A})) \to H^0(T^2(X_A/A, J \otimes \mathcal{O}_{X_A})). \quad (9.6)$$

By Step 1 of the proof of Theorem 9.4, $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B) = \lambda^{-1}([I])$, where $\lambda = \beta\alpha$. It is now clear from the preceding exact sequences that there are two successive obstructions in $H^0(T^2(X_A/A, J \otimes \mathcal{O}_{X_A})) = H^0(T^2(X) \otimes J)$ and $H^1(T^1(X_A/A, J \otimes \mathcal{O}_{X_A})) = H^1(T^1(X) \otimes J)$ so that $\lambda^{-1}([I]) \neq \emptyset$. If these obstructions vanish, then by Theorem 9.4(3) it follows that there is another obstruction in $H^2(T_X \otimes J)$ so that $\mathrm{Def}(X_A/A, B) \neq \emptyset$. □

The spaces $\mathrm{Def}(X_A/A, B)$ and $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$ do not, in general, have any vector space structures over the ground field k. This complicates any calculation involving them. However, if B is the trivial extension of A by J, then these spaces do have natural k-vector space structures.

REMARK 9.6. A variant of Theorem 9.4 is already known in the case $X = Y$, and the obstructions in Corollary 9.5 are also well known [H3; LiS]. However, to our knowledge, the $\mathrm{Def}_{\mathrm{loc}}$ space and the global-to-local sequence of Theorem 9.4(3) have not been considered earlier, and this distinguishes our statement from those already found in the literature.

REMARK 9.7. Theorem 9.4 establishes a relation between the local and global deformation spaces $\mathrm{Def}(X_A/A, B)$ and $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B)$. However, the obstructions obtained in Corollary 9.5 are not satisfactory in many ways. We explain why. Recall quickly how the obstructions work. In the notation of the corollary, given a deformation X_A of X over A, if the obstruction in $H^0(T^2(X))$ vanishes then we can lift X_A locally to B—in other words, there exist an open cover $\{U^i\}$ of X and liftings U^i_B of $X_A|_{U^i}$ over B. Then, if the second obstruction in $H^1(T^1(X))$ vanishes, the local liftings can be modified in order to agree on overlaps. This does allow us to find obstructions in order for $\mathrm{Def}_{\mathrm{loc}}(X_A/A, B) \neq \emptyset$, but we lose all local information about the liftings. To gain some control over the singularities of a lifting of X_A, we would like to choose a particular lifting U^i_B of $X_A|_{U^i}$ and then find obstructions to globalize it. This requires more careful study, and additional obstructions will appear. For general choice of the rings A and B the method is probably quite tricky, but for the purposes of this paper (where mainly 1-parameter deformations are studied) we will consider only deformations over the rings A_n. Our main tool is again the T^1-lifting property.

9.1. Local to Global and the T^1-lifting Property

Let X_n be a deformation of X over A_n. Here we present a method of lifting X_n to a deformation X_{n+1} of X over A_{n+1} that allows us to control the singularities of X_{n+1}.

Let $X_{n-1} = X_n \otimes_{A_n} A_{n-1}$ and $Y_{n-1} = X_n \otimes_{A_n} B_{n-1} \in \mathbb{T}^1(X_{n-1}/A_{n-1})$, where B_{n-1} is an A_n-algebra via the map $\varepsilon_{n-1}\colon A_n \to B_{n-1}$ defined in Section 6. Then, according to the T^1-lifting property (Theorem 6.4), X_n lifts to A_{n+1} if and only if Y_{n-1} is in the image of the natural map $\tau_n\colon \mathbb{T}^1_D(X_n/A_n) \to \mathbb{T}^1_D(X_{n-1}/A_{n-1})$. Theorem 8.1 obtained an explicit obstruction element for this to occur; however, as mentioned earlier, it does not offer any local information about the possible liftings. Local information is carried by the sheaves $T^1(X_n/A_n)$. These are related to $\mathbb{T}^1(X_n/A_n)$ by the following natural commutative diagram:

$$\begin{array}{ccc} \mathbb{T}^1_D(X_n/A_n) & \xrightarrow{\phi_n} & H^0(T^1_D(X_n/A_n)) \\ {\scriptstyle\tau_n}\downarrow & & \downarrow{\scriptstyle\sigma_n} \\ \mathbb{T}^1_D(X_{n-1}/A_{n-1}) & \xrightarrow{\phi_{n-1}} & H^0(T^1_D(X_{n-1}/A_{n-1})) \end{array} \tag{9.7}$$

The idea is as follows. Let $s_{n-1} = \phi_{n-1}(Y_{n-1})$. Instead of lifting Y_{n-1} directly through τ_n, we will obtain obstructions in order for s_{n-1} to be in the image of σ_n. If these obstructions vanish, then we choose a particular element $s_n \in H^0(T^1(X_n/A_n))$ such that $\sigma_n(s_n) = s_{n-1}$ and so obtain obstructions for the existence of a global $Y_n \in \mathbb{T}^1(X_n/A_n)$ such that $\phi_n(Y_n) = s_n$. In this way we can control the local structure of Y_n. Then, according to Proposition 6.5, there is a lifting X_{n+1} of X_n over A_{n+1} such that $X_{n+1} \otimes_{A_{n+1}} B_n = Y_n$, where again B_n is an A_{n+1}-algebra via $\varepsilon_n\colon A_{n+1} \to B_n$. Now suppose that by this process we have obtained a formal deformation $f_n\colon X_n \to \mathrm{Spec}(A_n)$ for n. Suppose that it is induced by an algebraic deformation $f\colon \mathcal{X} \to \mathrm{Spec}\, A$. We will see next that the sections s_n carry a lot of information about the singularities of $\mathcal{X}$. In particular, smoothings can be detected by them, as shown by the next two propositions.

PROPOSITION 9.8. *Let $f\colon \mathcal{X} \to \Delta$ be a deformation of a pure and reduced scheme X over the spectrum of a discrete valuation ring (A, m_A). Let $f_n\colon X_n \to \mathrm{Spec}\, A_n$ be the associated formal deformation and let $Y_n = X_{n+1} \otimes_{A_{n+1}} B_n \in \mathbb{T}^1(X_n/A_n)$. Moreover, let $e \in \mathbb{T}^1(\mathcal{X}/\Delta)$ be the element that is represented by the extension*

$$0 \to \mathcal{O}_{\mathcal{X}} = f^*\omega_\Delta \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}/\Delta} \to 0. \tag{9.8}$$

Then $e_n = Y_n$ in $\mathbb{T}^1(X_n/A_n)$, where $e_n = e \otimes_A A_n$.

Proof. By Proposition 7.2, $\mathbb{T}^1(X_n/A_n) = \mathrm{Ext}^1_{X_n}(\Omega_{X_n/A_n}, \mathcal{O}_{X_n})$ and $\mathbb{T}^1(\mathcal{X}/\Delta) = \mathrm{Ext}^1_{\mathcal{X}}(\Omega_{\mathcal{X}/\Delta}, \mathcal{O}_{\mathcal{X}})$. It follows from their definition that Y_n and e_n are represented by the extensions

$$0 \to \mathcal{O}_{X_n} \xrightarrow{\alpha} (\Omega_{X_{n+1}\otimes_{A_{n+1}} B_n/A_n}) \otimes_{B_n} A_n \to \Omega_{X_n/A_n} \to 0$$

and

$$0 \to \mathcal{O}_{X_n} \xrightarrow{\beta} \Omega_{\mathcal{X}} \otimes_A A_n \to \Omega_{X_n/A_n} \to 0,$$

respectively, where $\alpha(1) = d(1 \otimes x) \otimes 1$ and $\beta(1) = dt \otimes 1$ for t a generator of the maximal ideal of m_R. It is now easy to see that the two extensions are isomorphic via the mapping

$$\Phi\colon \Omega_{\mathcal{X}} \otimes_A A_n \to (\Omega_{X_{n+1}\otimes_{A_{n+1}} B_n/A_n}) \otimes_{B_n} A_n$$

defined by $\Phi(dz \otimes a) = d(\bar{z} \otimes 1) \otimes a$, where $z \in \mathcal{O}_{\mathcal{X}}$, $a \in A$, and $\bar{z}$ is the class of z in $\mathcal{O}_{X_n}$. □

PROPOSITION 9.9. *With assumptions as in Proposition 9.8, assume in addition that X has complete intersection singularities. Then f is a smoothing of X if and only if there are $k, n \in \mathbb{Z}_{>0}$, $k < n$, such that*

$$t^k T^1(X_n/A_n) \subset \mathcal{O}_{X_n} \cdot s_n.$$

Proof. Dualizing the exact sequence (9.8) yields the exact sequence

$$\mathcal{O}_{\mathcal{X}} \xrightarrow{\alpha} T^1(\mathcal{X}/\Delta) \to T^1(\mathcal{X}) \to 0,$$

where $\alpha(1) = e$. Therefore, $T^1(\mathcal{X}) = \mathrm{Coker}(\alpha) = T^1(\mathcal{X}/\Delta)/\mathcal{O}_{\mathcal{X}} \cdot e$.

Suppose that f is a smoothing. Then $T^1(\mathcal{X})$ is supported over m_A and hence there is a $k \in \mathbb{Z}_{>0}$ such that $t^k(T^1(\mathcal{X}/\Delta)/\mathcal{O}_{\mathcal{X}} \cdot e) = 0$. Reducing it modulo m_A^n and using Proposition 9.8, we get the claim.

Conversely, suppose there exist $k, n \in \mathbb{Z}_{>0}$ such that $t^k T^1(X_n/A_n) \subset \mathcal{O}_{X_n} \cdot s_n$. Let $\mathcal{F} = T^1(\mathcal{X}/\Delta)/\mathcal{O}_{\mathcal{X}} \cdot e$ and $\mathcal{F}_n = T^1(X_n/A_n)/\mathcal{O}_{X_n} \cdot s_n$. Then, by Lemma 12.2 and Proposition 9.8, it follows that $\mathcal{F}/t^{n+1}\mathcal{F} = \mathcal{F}_n$, where t is a generator of the maximal ideal m_A of A. Now, by assumption, $t^k(\mathcal{F}/t^{n+1}\mathcal{F}) = 0$ and hence $t^k\mathcal{F} = t^{n+1}\mathcal{F} = \mathcal{F}_n$; therefore, by Nakayama's lemma, $t^k\mathcal{F} = 0$. Hence $T^1(\mathcal{X})$ is supported over m_A and so, by Lemma 11.9, f is a smoothing. □

Even though our previous discussion was for the case when $X = Y$, it is also valid in the general case.

9.2. The Maps σ_n and ϕ_n

Here we study the maps σ_n and ϕ_n in diagram (9.7). In particular we obtain conditions under which they are surjective.

PROPOSITION 9.10. *With assumptions as in Proposition 7.9, there are canonical exact sequences*

$$0 \to H^0(T^1(X)/\mathcal{F}_n) \to H^0(T^1(\mathcal{X}_n/A_n)) \xrightarrow{\sigma_n} H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) \to Q_n \to 0,$$

$$0 \to L_n \to Q_n \to H^0(\mathcal{E}xt^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})), \quad \textit{and}$$

$$0 \to L_n \to H^1(T^1(X)/\mathcal{F}_n) \to H^1(T^1(\mathcal{X}_n/A_n))$$

as well as a noncanonical sequence

$$0 \to H^0(T^1(X)/\mathcal{F}_n) \to H^0(T^1(\mathcal{X}_n/A_n)) \xrightarrow{\sigma_n} H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) \to H^1(T^1(X)/\mathcal{F}_n) \oplus H^0(\mathcal{E}xt_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}})),$$

where $\mathcal{F}_n \subset T^1(X)$ is the cokernel of the map

$$\hat{T}_{\mathcal{X}_n/A_n} \to \hat{T}_{\mathcal{X}_{n-1}/A_{n-1}}.$$

Proof. By Proposition 7.9 there exists an exact sequence

$$0 \to T^1(X)/\mathcal{F}_n \to T^1(\mathcal{X}_n/A_n) \xrightarrow{h_n} T^1(\mathcal{X}_{n-1}/A_{n-1}) \xrightarrow{\mu_n} T^2(Y, X).$$

Let $M_n = \operatorname{Ker}(\mu_n)$. Then this sequence breaks into two short exact sequences:

$$0 \to T^1(X)/\mathcal{F}_n \to T^1(\mathcal{X}_n/A_n) \xrightarrow{h_n} M_n \to 0;$$

$$0 \to M_n \to T^1(\mathcal{X}_{n-1}/A_{n-1}) \xrightarrow{\mu_n} T^2(Y, X).$$

Then we obtain the following exact sequences in cohomology:

$$0 \to H^0(T^1(X)/\mathcal{F}_n) \xrightarrow{f_1} H^0(T^1(\mathcal{X}_n/A_n)) \xrightarrow{f_2} H^0(M_n)$$
$$\xrightarrow{f_3} H^1(T^1(X)/\mathcal{F}_n) \xrightarrow{f_4} H^1(T^1(\mathcal{X}_n/A_n));$$

$$0 \to H^0(M_n) \xrightarrow{g_1} H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) \xrightarrow{g_2} H^0(T^2(Y, X)).$$

We wish to understand the kernel and Cokernel of the map $\sigma_n = g_1 \circ f_2$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(T^1(X)/\mathcal{F}_n) & \xrightarrow{f_1} & H^0(T^1(\mathcal{X}_n/A_n)) & \xrightarrow{f_2} & \operatorname{Im}(f_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \phi_n} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & 0 & \longrightarrow & H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) & = & H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) & \longrightarrow & 0
\end{array}$$

where β is the restriction of g_1 on $\operatorname{Im}(f_2)$. The snake lemma now gives that $\ker(\mu_n) = H^0(T^1(X)/\mathcal{F}_n)$ and $\operatorname{Coker}(\beta) = \operatorname{Coker}(h)$. Hence there is an exact sequence

$$0 \to H^0(T^1(X)/\mathcal{F}_n) \to H^0(T^1(\mathcal{X}_n/A_n))$$
$$\to H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) \to Q_n \to 0, \quad (9.9)$$

where $Q_n = \operatorname{Coker}(\beta)$. Now the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im}(f_2) & = & \operatorname{Im}(f_2) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow & & \\
0 & \longrightarrow & H^0(M_n) & \longrightarrow & H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) & \longrightarrow & H^0(T^2(Y, X)) & \longrightarrow & 0
\end{array}$$

implies that there is an exact sequence

$$0 \to L_n \to Q_n \to H^0(T^2(Y, X)) \quad (9.10)$$

with $L_n = \operatorname{Coker}[\operatorname{Im}(f_2) \to H^0(M_n)$ and thus there is another exact sequence

$$0 \to L_n \to H^1(T^1(X)/\mathcal{F}_n) \to H^1(T^1(\mathcal{X}_n/A_n)). \quad (9.11)$$

The proposition now follows from (9.9), (9.10), and (9.11). □

COROLLARY 9.11. *There are two successive obstructions in* $H^0(\mathcal{E}xt^2_{\hat{X}}(\hat{\Omega}_X, \mathcal{O}_{\hat{X}}))$ *and* $H^1(T^1(X)/\mathcal{F}_n)$ *to an element* s_{n-1} *of* $H^0(T^1(X_{n-1}/A_{n-1}))$ *being in the image of* σ_n.

The exact sequences in Proposition 9.10 are not very enlightening in general. However, if X has local complete intersection singularities, then they are greatly simplified.

COROLLARY 9.12. *Suppose that X has local complete intersection singularities or, more generally, that $H^0(\mathcal{E}xt^2_X(\Omega_X, \mathcal{O}_X)) = 0$. Then there is an exact sequence*

$$0 \to H^0(T^1(X)/\mathcal{F}_n) \to H^0(T^1(\mathcal{X}_n/A_n)) \xrightarrow{\sigma_n} H^0(T^1(\mathcal{X}_{n-1}/A_{n-1})) \xrightarrow{\partial} H^1(T^1(X)/\mathcal{F}_n).$$

Next we study the local-to-global map ϕ_n. If X is pure and reduced, then the diagram (9.7) is part of the commutative diagram with exact rows

$$\begin{array}{ccccccc} H^1(\hat{T}_{X_n/A_n}) & \xrightarrow{\psi_n} & \mathbb{T}^1(X_n/A_n) & \xrightarrow{\phi_n} & H^0(T^1(X_n/A_n)) & \xrightarrow{\partial_n} & H^2(\hat{T}_{X_n/A_n}) \\ \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \sigma_n} & & \downarrow{\scriptstyle \lambda_n} \\ H^1(\hat{T}_{X_{n-1}/A_{n-1}}) & \xrightarrow{\psi_{n-1}} & \mathbb{T}^1(X_{n-1}/A_{n-1}) & \xrightarrow{\phi_{n-1}} & H^0(T^1(X_{n-1}/A_{n-1})) & \xrightarrow{\partial_{n-1}} & H^2(\hat{T}_{X_{n-1}/A_{n-1}}) \end{array} \tag{9.12}$$

where ψ_n and ψ_{n-1} are injective. Hence the obstruction for an element $s_n \in H^0(T^1(X_n/A_n))$ to being in the image of ϕ_n is the element $\partial_n(s_n) \in H^2(\hat{T}_{X_n/A_n})$. If X has isolated singularities then it is well known that there are successive obstructions in $H^2(\hat{T}_X)$ in order for $\partial_n(s_n)$ to be zero. However, in the general case this is not so, and once more the reason is the inability to lift local automorphisms. The best that we can do in this case is to find conditions for the map ϕ_n to be surjective.

PROPOSITION 9.13. *Let X be a pure and reduced scheme over a field k, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. Let $X_n \in$ $\mathrm{Def}(Y, X)(A_n)$, and let $\mathcal{F}_k = \mathrm{Coker}[\hat{T}_{X_k/A_k} \to \hat{T}_{X_{k-1}/A_{k-1}}] \subset T^1(X)$. If $H^2(\hat{T}_X) = H^1(\mathcal{F}_k) = 0$ for all $k \le n$, then ϕ_n is surjective.*

Note that, if the singularities of X are isolated, then $H^1(\mathcal{F}_k) = 0$ and the proposition is the familiar result about isolated singularities. Admittedly it is not easy to check the conditions of the proposition, but at least the sheaves $\mathcal{F}_k$ are all subsheaves of $T^1(X)$, which depends only on X.

Proof of Proposition 9.13. The long exact sequence described in Proposition 7.9 gives the following short exact sequences:

$$0 \to \hat{T}_X \to \hat{T}_{X_n/A_n} \to Q_n \to 0;$$

$$0 \to Q_n \to \hat{T}_{X_{n-1}/A_{n-1}} \to \mathcal{F}_n \to 0.$$

These give the exact sequences

$$\cdots \to H^2(\hat{T}_X) \to H^2(\hat{T}_{X_n/A_n}) \to H^2(Q_n) \to \cdots,$$

$$\cdots \to H^1(\mathcal{F}_n) \to H^2(Q_n) \to H^2(\hat{T}_{X_{n-1}/A_{n-1}}) \to \cdots.$$

The claim now follows by induction on n. □

So far we have found conditions in order for ϕ_n and σ_n to be surjective. Returning to our original problem and starting with a deformation X_n of X over A_n, we want to lift $Y_{n-1} = X_n \otimes_{A_n} B_{n-1}$ to a Y_n in $\mathbb{T}^1(X_n/A_n)$. Let $s_{n-1} = \phi_{n-1}(Y_{n-1})$. If the obstructions in Corollary 9.12 and Proposition 9.13 vanish, then there is a $Y_n' \in \mathbb{T}^1(X_n/A_n)$ such that $\phi_{n-1}(\tau_n(Y_n') - Y_{n-1}) = 0$. Hence, in order to obtain a lifting Y_n of Y_{n-1}, we need to lift the locally trivial deformation $Z_{n-1} = \tau_n(Y_n') - Y_{n-1}$. It is well known that if X has isolated singularities then the obstruction to lifting Z_{n-1} to a locally trivial deformation Z_n over A_n is in $H^2(T_X)$ (this also follows immediately from the next proposition). In general, though, this is not true. Again the best that we can do is to find conditions for τ_n to be surjective.

PROPOSITION 9.14. *With assumptions as in Proposition 9.13, if*

$$H^1(\mathcal{F}_n) = H^2(\hat{T}_X) = 0$$

then every locally trivial lifting Z_{n-1} of X_{n-1} over B_{n-1} lifts to a locally trivial lifting Z_n of X_n over B_n.

Proof. From diagram (9.7) it follows that the isomorphism classes of locally trivial liftings of X_k over B_k are in one-to-one correspondence with $H^1(\hat{T}_{X_k/A_k})$. Hence the statement of the proposition is equivalent to saying that if $H^1(\mathcal{F}_n) = H^2(\hat{T}_X) = 0$ then the natural map

$$\mu_n \colon H^1(\hat{T}_{X_n/A_n}) \to H^1(\hat{T}_{X_{n-1}/A_{n-1}})$$

is surjective. This follows from arguments that are similar to those used in the proof of Proposition 9.13. □

The previous discussion suggests that we must study the sheaves $\mathcal{F}_n$ and the quotients $T^1(X)/\mathcal{F}_n$. There are two main cases. The first is when $T^1(X)/\mathcal{F}_n$ has finite support for all n (and hence no higher cohomology) and σ_n is surjective for all n. Here the only obstruction to the lifting of X_n to A_{n+1} is in $H^2(\hat{T}_X)$. This case is treated in Lemma 12.2.

The second case is when we know that $H^2(\mathcal{F}_n) = 0$ for all n. The simplest cases of this occurring are when the singular locus of X is 1-dimensional and when there is a proper morphism $f \colon X \to Z$ with 1-dimensional fibers and Z affine (e.g., the cases of flipping, flopping, and divisorial contractions with 1-dimensional fibers). In this case we will show that $H^1(T^1(X)/\mathcal{F}_n)$ is a quotient of $H^1(T^1(X))$ and hence we can at least find a uniform bound for its dimension, which is finite if X has proper singular locus. Indeed, there is an exact sequence

$$0 \to \mathcal{F}_n \to T^1(X) \to T^1(X)/\mathcal{F}_n \to 0$$

that induces the exact sequence

$$H^1(\mathcal{F}_n) \to H^1(T^1(X)) \to H^1(T^1(X)/\mathcal{F}_n) \to H^2(\mathcal{F}_n).$$

Since $H^2(\mathcal{F}_n) = 0$, it follows that $H^1(\mathcal{F}_n)$ is a quotient of $H^1(T^1(X))$.

Thus we have shown the following result.

COROLLARY 9.15. *Suppose that the singular locus of X is 1-dimensional or that there is a proper morphism $f \colon X \to Z$ with 1-dimensional fibers and Z affine.*

If $H^1(T^1(X)) = 0$, *then the map* $\sigma_n\colon H^0(T^1(X_n/A_n)) \to H^0(T^1(X_{n-1/A_{n-1}})$ *is surjective for all* n.

10. $\mathbb{Q}$-Gorenstein Deformations

Let X be a $\mathbb{Q}$-Gorenstein scheme, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. In this section we extend the results obtained in the previous sections regarding the usual deformation functor $\mathrm{Def}(Y, X)$ to the case of the $\mathbb{Q}$-Gorenstein deformation functor $\mathrm{Def}^{qG}(Y, X)$. Toward this end, we will locally compare the $\mathbb{Q}$-Gorenstein deformations of X to the deformations of its index-1 cover $\tilde{X}$. The key property that enables us to do so is that locally every $\mathbb{Q}$-Gorenstein deformation of X lifts to a deformation of $\tilde{X}$ [KoSh].

Let

$$0 \to J \to B \to A \to 0$$

be a small extension of Artin rings and let $X_A \in \mathrm{Def}^{qG}(Y, X)(A)$. Then, in complete analogy with the case of $\mathrm{Def}(Y, X)$ (Definition 9.2), we define $\mathrm{Def}^{qG}(X_A/A, B)$, $\underline{\mathrm{Def}}^{qG}(X_A/A, B)$, and $\mathrm{Def}^{qG}_{\mathrm{loc}}(X_A/A, B) = H^0(\underline{\mathrm{Def}}(X_A/A, B))$.

We need the following technical result.

LEMMA 10.1. *Let* B *be an* A*-algebra,* M *a* B*-module, and* G *a group acting on them compatibly with the algebra structure; in other words, for any* $g \in G$, *the map* $\phi_g\colon B \to B$ *defined by* $\phi_g(b) = g \cdot b$ *is an* A*-algebra isomorphism and* $g \cdot (bm) = (g \cdot b)(g \cdot m)$ *for any* $b \in B$ *and* $m \in M$. *Then there is an action of* G *on* $T^i(B/A, M)$, $i = 0, 1, 2$. *If* $A = k$ *is a field, then* G *also acts on* $\bigcup_{C \in \mathrm{Art}(k)} \mathrm{Def}(B)(C)$, *where* $\mathrm{Def}(B)(C)$ *is the set of all deformations of* B *over* C.

Proof. For any $g \in G$, there is an induced isomorphism $\phi_g\colon B \to B$ of B given by $\phi_g(b) = g^{-1} \cdot b$ for any $b \in B$. This yields an isomorphism

$$\phi_g^*\colon T^i(B/A, M) \to T^i(B/A, M^*),$$

where M^* is M as an abelian group but where the B-module structure is given by $b \cdot m = (g^{-1} \cdot b)m$. The map $\psi_g\colon M^* \to M$ given by $\psi_g(m) = g \cdot m$ is a B-module homomorphism inducing an isomorphism

$$\psi_g^*\colon T^i(B/A, M^*) \to T^i(B/A, M).$$

Now the map $f_g = \psi_g^* \circ \phi_g^*\colon T^i(B/A, M) \to T^i(B/A, M)$ gives the G-action on $T^i(B/A, M)$.

We can describe $T^i(B/A, M)$, $i = 1, 2$, as the spaces of infinitesimal one- and two-term extensions of B by M, respectively. It is useful to describe the action of G on $T^i(B/A, M)$ when the latter is viewed this way.

Let (E) be a one-term infinitesimal extension

$$0 \to M \to C \to B \to 0$$

of B by M. Then, for any $g \in G$, let (E') be the pullback extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow \phi_g & & \\ 0 & \longrightarrow & M & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

and let (E'') be the pushout extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & C' & \longrightarrow & B & \longrightarrow & 0 \\ & & \psi_g \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & C'' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Then $g \cdot [E] = [E'']$ in $T^1(B/A, M)$. The action on two-term extensions is defined exactly analogously. Next we will show that G acts on $\bigcup_{C \in \text{Art}(k)} \text{Def}(B)(C)$. So let $C \in \text{Art}(k)$ be a finite local Artin k-algebra and R_C a deformation of B over C. We proceed by induction on the length of C. If $\text{length}(C) = 1$, then $R_C \in T^1(B/k, B)$ and the action is already defined. Now any C appears as an extension

$$0 \to k \to C \to C' \to 0.$$

Let $R_{C'} = R_C \otimes_C C'$. Then, by induction, $g \cdot R_{C'}$ is defined and there is an isomorphism $g \cdot R_{C'} \to R_{C'}$ (not over C' in general). Define $g \cdot R_C$ to be the extension obtained by pulling back

$$0 \to B \to R_C \to R_{C'} \to 0$$

via the map $g \cdot R_{C'} \to R_{C'}$. □

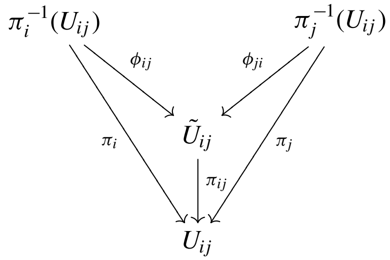
Construction of the Sheaves $T^i_{qG}(X_A/B, \mathcal{F})$

Let $X_A \to \text{Spec}\, A \to \text{Spec}\, B$ be morphisms of schemes such that X_A is a Cohen–Macauley and relatively Gorenstein in codimension 1 and such that there is an $n \in \mathbb{Z}$ with $\omega^{[n]}_{X_A/A}$ invertible. Let $\mathcal{F}$ be a coherent sheaf on X_A. Next we will define coherent sheaves $T^i_{qG}(X_A/B, \mathcal{F})$.

Let $X_A = \bigcup_i U_i$ be an affine cover of X_A, and let $\pi_i \colon \tilde{U}_i \to U_i$ be the index-1 cover of U_i. Let r_i be the index of U_i and let $\mathcal{F}_i = \mathcal{F}|_{U_i}$. Then π_i is Galois with Galois group the group of r_i roots of unity μ_{r_i}. Hence, by Lemma 10.1, μ_i acts on $T^k(\tilde{U}_i, \pi_i^* \mathcal{F}_i)$, $k \geq 0$. Let $T^k_{qG}(U_i/B, \mathcal{F}_i) = (T^k(\tilde{U}_i/B, \pi_i^* \mathcal{F}_i))^{\mu_i}$. This is a coherent sheaf on U_i. We will show that these sheaves glue to a coherent sheaf $T^k_{qG}(X_A/B, \mathcal{F})$. It suffices to show that there are isomorphisms

$$\phi_{ij} \colon T^k_{qG}(U_i/B, \mathcal{F}_i)|_{U_{ij}} \to T^k_{qG}(U_j/B, \mathcal{F}_i)|_{U_{ij}},$$

where $U_{ij} = U_i \cap U_j$. Let r_{ij} be the index of U_{ij}. Then $r_{ij} | r_j$ and $r_{ij} | r_i$. Let $\pi_{ij} \colon \tilde{U}_{ij} \to U_{ij}$ be the index-1 cover of U_{ij}. Then, from the uniqueness and the construction of the index-1 cover it follows that there are factorizations



where ϕ_{ij} and ϕ_{ji} are étale of degrees $s_{ij} = r_i/r_{ij}$ and $s_{ji} = r_j/r_{ij}$, respectively. Then

$$T^k(\pi_i^{-1}(U_{ij})/Y, \pi_i^*\mathcal{F}_i) = \phi_{ij}^* T^k(\tilde{U}_{ij}/Y, \pi_{ij}^*\mathcal{F}_i)$$

and therefore

$$(T^k(\pi_i^{-1}(U_{ij})/Y, \pi_i^*\mathcal{F}_i))^{\mu_{s_{ij}}} = T^k(\tilde{U}_{ij}/Y, \pi_{ij}^*\mathcal{F}_i).$$

Hence

$$\begin{aligned}(T^k(\pi_i^{-1}(U_{ij})/Y, \pi_i^*\mathcal{F}_i))^{\mu_{r_i}} &= ((T^k(\pi_i^{-1}(U_{ij}), \pi_i^*\mathcal{F}_i))^{\mu_{s_{ij}}})^{\mu_{r_{ij}}} \\ &= (T^k(\tilde{U}_{ij})/Y, \pi_{ij}^*\mathcal{F}_{ij})^{\mu_{r_{ij}}}.\end{aligned}$$

Similarly, it follows that

$$(T^k(\pi_j^{-1}(U_{ij})/Y, \pi_j^*\mathcal{F}_j))^{\mu_{r_j}} = (T^k(\tilde{U}_{ij})/Y, \pi_{ij}^*\mathcal{F}_{ij})^{\mu_{r_{ij}}}$$

and hence

$$T^k_{qG}(U_i/B, \mathcal{F}_i)|_{U_{ij}} = T^k_{qG}(U_j/B, \mathcal{F}_j)|_{U_{ij}}.$$

Therefore, the sheaves $T^k_{qG}(U_i/B, \mathcal{F}_i)$ glue to a global sheaf $T^k_{qG}(X_A/B, \mathcal{F})$.

The next proposition shows that T^0_{qG} and T^0 agree under certain conditions.

PROPOSITION 10.2. *Suppose that $\mathcal{F}$ is a locally free coherent sheaf on X_A. Then*

$$T^0_{qG}(X_A/B, \mathcal{F}) \cong T^0(X_A/B, \mathcal{F}).$$

Proof. Let $\{U_i\}$ be an affine cover of X_A and let $\mathcal{F}_i = \mathcal{F}|_{U_i}$. Let $\pi_i : \tilde{U}_i \to U_i$ be the index-1 cover and G_i the corresponding Galois group. Then $T^0_{qG}(U_i/B, \mathcal{F}_i) = T^0(\tilde{U}_i/B, \mathcal{F}_i)^{G_i}$ and, moreover, $T^0(\tilde{U}_i/B, \mathcal{F}_i) = \mathrm{Hom}_{\tilde{U}_i}(\Omega_{\tilde{U}_i/B}, \pi_i^*\mathcal{F}_i)\tilde{}$. The G_i-action is given as follows. Let $g \in G_i$ and $f \in \mathrm{Hom}_{\tilde{U}_i}(\Omega_{\tilde{U}_i/B}, \pi_i^*\mathcal{F}_i)$. Then $g \cdot f$ is the $\mathcal{O}_{\tilde{U}_i}$-sheaf homomorphism defined by $(g \cdot f)(x) = g^{-1} \cdot f(g \cdot x)$. The natural map $\pi_i^*\Omega_{U_i/B} \to \Omega_{\tilde{U}_i/B}$ induces a homomorphism

$$\begin{aligned}\phi \colon &\mathrm{Hom}_{\tilde{U}_i}(\Omega_{\tilde{U}_i/B}, \pi_i^*\mathcal{F}_i) \\ &\to \mathrm{Hom}_{\tilde{U}_i}(\pi^*\Omega_{U_i/B}, \pi_i^*\mathcal{F}_i) = \mathrm{Hom}_{U_i}(\Omega_{U_i/B}, \pi_i^*\mathcal{F}_i). \quad (10.1)\end{aligned}$$

Now, since $\mathcal{F}$ is assumed to be locally free, it follows that both modules in the sequence (10.1) are reflexive. Furthermore, since X_A is Gorenstein in codimension 1, it follows that π_i is étale in codimension 1 and therefore ϕ is an isomorphism. Hence, taking G_i-invariants, we get an isomorphism

$$T^0_{qG}(U_i/B, \mathcal{F}_i) \to (\mathrm{Hom}_{U_i}(\Omega_{U_i/B}, \pi_i^*\mathcal{F}_i)^{G_i})\tilde{}.$$

We now claim that $\mathrm{Hom}_{U_i}(\Omega_{U_i/B}, \pi_i^*\mathcal{F}_i)^{G_i} = \mathrm{Hom}_{U_i}(\Omega_{U_i/B}, \mathcal{F}_i)$. The natural injection $\mathcal{F}_i \to \pi_i^*\mathcal{F}_i$ gives a natural injection

$$\psi\colon \mathrm{Hom}_{U_i}(\Omega_{U_i/B}, \mathcal{F}_i) \to \mathrm{Hom}_{U_i}(\Omega_{U_i/B}, \pi_i^*\mathcal{F}_i)^{G_i}.$$

Now let $f \in \mathrm{Hom}_{U_i}(\Omega_{U_i/B}, \pi_i^*\mathcal{F}_i)^{G_i}$. The definition of the G_i-action shows that $\mathrm{Im}(f) \subset (\pi_i^*\mathcal{F}_i)^{G_i} = \mathcal{F}_i$. Hence ψ is surjective and thus is an isomorphism. As a result, for any U_i we have an isomorphism

$$g_i\colon T^0_{qG}(U_i/B, \mathcal{F}_i) \to T^0(U_i/B/\mathcal{F}_i).$$

Following the construction of the sheaves T^i_{qG}, we see that these isomorphisms glue to a global isomorphism

$$g\colon T^0_{qG}(X_A/B, \mathcal{F}) \to T^0(X_A/B, \mathcal{F})$$

as claimed. □

PROPOSITION 10.3. *Let X be a $\mathbb{Q}$-Gorenstein scheme defined over a field k. Let $X_A \to \operatorname{Spec} A$ be a $\mathbb{Q}$-Gorenstein deformation of X over a finite local Artin k-algebra A. Let*

$$0 \to J \to B \to A \to 0$$

be an extension of finite local Artin k-algebras with $J^2 = 0$. Then there is a k-isomorphism

$$T^1_{qG}(X_A/B, J \otimes \mathcal{O}_{X_A}) \to \mathrm{Def}^{qG}(X_A/A, B),$$

where $\mathrm{Def}^{qG}(X_A/A, B)$ is the space of isomorphism classes of $\mathbb{Q}$-Gorenstein liftings X_B of X_A over B.

Proof. Let r be the index of X, $\pi_A\colon \tilde{X}_A \to X_A$ the index-1 cover of X_A, and $\pi\colon \tilde{X} \to X$ the index-1 cover of X. Then $\tilde{X}_A$ is a deformation of $\tilde{X}$ over A. An element of $T^1_{qG}(X_A/B, J \otimes \mathcal{O}_{X_A})$ corresponds to a μ_r-invariant square zero extension

$$0 \to J \otimes \mathcal{O}_{\tilde{X}_A} \to \mathcal{O}_{\tilde{X}_B} \to \mathcal{O}_{\tilde{X}_A} \to 0. \tag{10.2}$$

Taking invariants yields an extension

$$0 \to J \otimes \mathcal{O}_{X_A} \to \mathcal{O}_{X_B} \to \mathcal{O}_{X_A} \to 0 \tag{10.3}$$

and hence a $\mathbb{Q}$-Gorenstein lifting X_B of X_A over B. This defines a map

$$\phi\colon T^1_{qG}(X_A/B, J \otimes \mathcal{O}_{X_A}) \to \mathrm{Def}^{qG}(X_A/A, B).$$

Next we show that ϕ is surjective. Indeed, let X_B be a $\mathbb{Q}$-Gorenstein lifting of X_A over B. Then there is a square zero extension as in (10.3). Let $\pi_B\colon \tilde{X}_B \to X_B$ be the index-1 cover of X_B. As before, this is a lifting of $\tilde{X}_A$ over B. Hence there is a μ_r-invariant extension as in (10.2), and therefore ϕ is surjective.

It remains to show that ϕ is injective. Since X is Gorenstein in codimension 1, it follows that $\pi\colon \tilde{X} \to X$ is étale in codimension 1. Let $U \subset X$ be the Gorenstein locus. Then $\pi^{-1}(U) \to U$ is étale and $\operatorname{codim}(\tilde{X} - \pi^{-1}(U), \tilde{X}) \geq 2$. Therefore, the natural map

$$\operatorname{Def}(\tilde{X}) \to \operatorname{Def}(\pi^{-1}(U))$$

is injective [Ar2, Lemma 9.1] and hence ϕ is injective, too. □

The next corollary is an immediate consequence of Proposition 10.3.

COROLLARY 10.4. *Let X be a $\mathbb{Q}$-Gorenstein scheme defined over a field k, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. Let $X_n \in \operatorname{Def}_Y^{qG}(X)(A_n)$. Then*

(1) $T^1_{qG}(Y, X) = T^1_{qG}(X/k, \mathcal{O}_X)$ *and*
(2) $T^1_{qG}(X_n/A_n) = T^1_{qG}(X_n/A_n, \mathcal{O}_{X_n})$.

Most of the functorial properties of the usual T^i sheaves hold for the T^i_{qG} as well. Next we present a few that are of interest to us.

PROPOSITION 10.5. *Let X be a $\mathbb{Q}$-Gorenstein scheme defined over a field k. Let $A \in \operatorname{Art}(k)$ and let $X_A \to \operatorname{Spec}(A)$ be a $\mathbb{Q}$-Gorenstein deformation of X over A. Then the following statements hold.*

(1) *Let $A \to B$ be a morphism of Artin local k-algebras, $X_B = X_A \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B)$ the fiber product, and $\mathcal{F}_B$ an $\mathcal{O}_{X_B}$-module. Then there are natural isomorphisms*
$$T^i_{qG}(X_B/B, \mathcal{F}_B) \cong T^i_{qG}(X_A/A, j_*\mathcal{F}_B),$$
where $j \colon X_B \to X_A$ is the projection map.

(2) *Let $C \to B \to A$ be a sequence of ring homomorphisms, and let $\mathcal{F}$ be an $\mathcal{O}_{X_A}$-module. Then there is an exact sequence*
$$\cdots \to T^i_{qG}(X_A/B, \mathcal{F}) \to T^i_{qG}(X_A/C, \mathcal{F}) \to T^i(B/C, \mathcal{F}) \to T^{i+1}_{qG}(X_A/B, \mathcal{F}) \to \cdots .$$

(3) *Let*
$$0 \to J \to B \to A \to 0$$
be a square zero extension of Artin local k-algebras, and let X_B be a $\mathbb{Q}$-Gorenstein lifting of X_A over B. Then there is an exact sequence
$$\cdots \to T^i_{qG}(X_A/A, J \otimes_A \mathcal{O}_{X_A}) \to T^i_{qG}(X_B/B, \mathcal{O}_{X_B}) \to T^i_{qG}(X_A/A, \mathcal{O}_{X_A}) \to T^{i+1}_{qG}(X_A/A, J \otimes_A \mathcal{O}_{X_A}) \to \cdots .$$

The proof of the proposition follows immediately from the corresponding statements for the usual T^i after passing, as before, to the index-1 covers.

Next we show that $T^2_{qG}(X) = T^2_{qG}(X/k, \mathcal{O}_X)$ is an obstruction sheaf for $\operatorname{Def}_Y^{qG}(X)$ if $X - Y$ is smooth. For the sake of simplicity we only present the case $X = Y$.

PROPOSITION 10.6. *Let X be a $\mathbb{Q}$-Gorenstein scheme defined over a field k. Let*
$$0 \to J \to B \to A \to 0$$

be a square zero extension of finite Artin local k-algebras such that $m_B J = 0$ *and* $X_A \in \mathrm{Def}^{qG}(X)(A)$. *Then there is a section* $\mathrm{ob}(X_A) \in H^0(T^2_{qG}(X)) \otimes_k J$ *such that, for any affine open* $U_A \subset X_A$, $\mathrm{ob}(X_A)|_U \in T^2_{qG}(U) \otimes_k J$ *is the obstruction to lifting* U_A *to a* $\mathbb{Q}$*-Gorenstein deformation* U_B *of* U *over* B, *where* $U = U_A \otimes_A k$.

Proof. This a local result, so we may assume that X is affine of index r. Let $\pi : \tilde{X} \to X$ be the index-1 cover of X. Then $T^2_{qG}(X) = (T^2(\tilde{X}))^{\mu_r}$. Let

$$0 \to J \to B \to A \to 0$$

be an extension of finite local Artin algebras, and let X_A be a $\mathbb{Q}$-Gorenstein deformation of X over A. Let $\pi_A \colon \tilde{X}_A \to X_A$ be the index-1 cover. Then $\tilde{X}_A$ is a deformation of $\tilde{X}$ over A [KoSh] and, by Lemma 10.1, the obstruction $\mathrm{ob}(\tilde{X}_A) \in T^2(\tilde{X}) \otimes_k J$ is μ_r-invariant and hence it is, in fact, in $T^2_{qG}(X) \otimes_k J$. Thus, if $\mathrm{ob}(\tilde{X}_A) = 0$, then there is a deformation $\tilde{X}'_B$ of $\tilde{X}$ over B that lifts $\tilde{X}_A$. This deformation may not be μ_r-invariant, but $\tilde{X}_B = \frac{1}{r}\sum_{i=0}^{r-1} \zeta^i \cdot \tilde{X}'_B$ is, where ζ is a primitive r-root of unity. Then $X_B = \tilde{X}_B/\mu_r$ is a lifting of X_A over B. □

Having developed the theory of $\mathbb{Q}$-Gorenstein cotangent sheaves $T^i_{qG}(X)$, we can now repeat most of the arguments verbatim for the usual deformation functor $\mathrm{Def}(Y, X)$ in Section 9. In particular we have the following.

THEOREM 10.7. *Let* X *be a* $\mathbb{Q}$*-Gorenstein scheme defined over a field* k, *and let* $Y \subset X$ *be a closed subscheme of* X *such that* $X - Y$ *is smooth. Let*

$$0 \to J \to B \to A \to 0$$

be a small extension of local Artin k-algebras and let $X_A \in \mathrm{Def}^{qG}(Y, X)(A)$. *Then the following statements hold.*

(1) *The spaces* $\mathrm{Def}^{qG}(X_A/A, B)$ *and* $\mathrm{Def}^{qG}_{\mathrm{loc}}(X_A/A, B)$ *are* $\mathbb{T}^1_{qG}(Y, X) \otimes J$ *and* $H^0(T^1_{qG}(X) \otimes J)$ *homogeneous spaces, respectively.*
(2) *Let* $s_B \in \mathrm{Def}^{qG}_{\mathrm{loc}}(X_A/A, B)$. *Then the set* $\pi^{-1}(s_B)$ *is a homogeneous space over* $H^1(\hat{T}_X \otimes J)$.
(3) *There is a sequence*

$$0 \to H^1(\hat{T}_X \otimes J) \xrightarrow{\alpha} \mathrm{Def}^{qG}(X_A/A, B) \xrightarrow{\pi} \mathrm{Def}^{qG}_{\mathrm{loc}}(X_A/A, B) \xrightarrow{\partial} H^2(\hat{T}_X \otimes J)$$

that is exact in the following sense. Let $s_B \in \mathrm{Def}^{qG}_{\mathrm{loc}}(X_A/A, B)$. *Then* s_B *is in the image of* π *if and only if* $\partial(s_B) = 0$. *Moreover, let* $X_B, X'_B \in \mathrm{Def}^{qG}(X_A/A, B)$ *be such that* $\pi(X_A) = \pi(X'_A)$. *Then there is a* $\gamma \in H^1(\hat{T}_X \otimes J)$ *such that* $X'_A = \gamma \cdot X_A$, *where by "·" we denote the action of* $H^1(\hat{T}_X \otimes J)$ *on* $\pi^{-1}(s_B)$.

COROLLARY 10.8. *With assumptions as in Theorem 10.7, there are two successive obstructions in* $H^0(T^2_{qG}(X) \otimes J)$ *and* $H^1(T^1_{qG}(X) \otimes J)$ *in order for* $\mathrm{Def}^{qG}_{\mathrm{loc}}(X_A/A, B) \neq \emptyset$ *(i.e., for* X_A *to lift locally to* B*). If these obstructions vanish, then there is another obstruction in* $H^2(\hat{T}_X \otimes J)$ *in order for* $\mathrm{Def}(X_A/A, B) \neq \emptyset$ *(i.e., for the local obstructions to globalize).*

The local lifting method and the results that were described in Section 9.1 apply immediately to the $\mathbb{Q}$-Gorenstein case as well. For the convenience of the reader, we state the main technical tools needed to apply it.

PROPOSITION 10.9. *Let X be a $\mathbb{Q}$-Gorenstein scheme defined over a field k, and let $Y \subset X$ be a closed subscheme of X such that $X - Y$ is smooth. Let $\mathcal{X}_n \in \mathrm{Def}_Y^{qG}(X)(A_n)$ and $\mathcal{X}_{n-1} = \mathcal{X}_n \otimes_{A_n} A_{n-1}$. Then there is an exact sequence*

$$0 \to \hat{T}_X \to \hat{T}_{X_n/A_n} \to \hat{T}_{X_{n-1}/A_{n-1}} \\ \to T^1_{qG}(X) \to T^1_{qG}(X_n/A_n) \to T^1_{qG}(X_{n-1}/A_{n-1}) \to T^2_{qG}(X).$$

Proof. Use Proposition 10.5 and Proposition 10.2 on the extension

$$0 \to k \to A_n \to A_{n-1} \to 0.$$ □

PROPOSITION 10.10. *With assumptions as in Proposition 10.9, there are canonical exact sequences*

$$0 \to H^0(T^1_{qG}(X)/\mathcal{F}_n) \to H^0(T^1_{qG}(\mathcal{X}_n/A_n)) \\ \to H^0(T^1_{qG}(\mathcal{X}_{n-1}/A_{n-1})) \to Q_n \to 0,$$

$$0 \to L_n \to Q_n \to H^0(T^2_{qG}(X)), \quad \text{and}$$

$$0 \to L_n \to H^1(T^1_{qG}(X)/\mathcal{F}_n) \to H^1(T^1_{qG}(\mathcal{X}_n/A_n))$$

in addition to a noncanonical sequence

$$0 \to H^0(T^1_{qG}(X)/\mathcal{F}_n) \to H^0(T^1_{qG}(\mathcal{X}_n/A_n)) \xrightarrow{\phi_n} H^0(T^1_{qG}(\mathcal{X}_{n-1}/A_{n-1})) \\ \to H^1(T^1_{qG}(X)/\mathcal{F}_n) \oplus H^0(T^2_{qG}(X)),$$

where $\mathcal{F}_n \subset T^1_{qG}(X)$ is the cokernel of the map $\hat{T}_{\mathcal{X}_n/A_n} \to \hat{T}_{\mathcal{X}_{n-1}/A_{n-1}}$.

COROLLARY 10.11. *Suppose that the index-1 cover of every singular point of X has local complete intersection singularities. Then there is an exact sequence*

$$0 \to H^0(T^1_{qG}(X)/\mathcal{F}_n) \to H^0(T^1_{qG}(\mathcal{X}_n/A_n)) \\ \xrightarrow{\sigma_n} H^0(T^1_{qG}(\mathcal{X}_{n-1}/A_{n-1})) \xrightarrow{\partial} H^1(T^1_{qG}(X)/\mathcal{F}_n).$$

11. From Formal to Algebraic

For geometric applications we are interested in algebraic deformations $f : \mathcal{X} \to S$ of a scheme X of finite type over a field k. However, the methods of this paper are formal and so produce only formal deformations of X. It is therefore of interest to know under what conditions a formal deformation is algebraic as well as which properties of an algebraic deformation can be read from the associated formal deformation.

The problem of whether a formal deformation is algebraic is a difficult one. An affirmative answer is known for the cases where X is affine with isolated singularities [Ar2, Thm. 5.1] and where X is projective with $H^2(X, \mathcal{O}_X) = 0$ ([Se, Thm. 2.5.13]; see also [Gr2]). This problem is extensively studied in [Ar1].

In general it is difficult to compare the properties of an algebraic deformation and its associated formal deformation. For example, it is possible that the formal deformation is trivial but the global one is not [Se, Ex. 1.2.5]. In this section we state criteria for recognizing the properties of being locally trivial and smoothing from certain properties of the corresponding formal deformation. Then we define the notion of formal smoothing, which we will use in Section 12.

The next theorem by Artin is the key to the relation between locally formally trivial and locally trivial deformations.

THEOREM 11.1 [Ar1, Cor. 2.6]. *Let X_1 and X_2 be S-schemes of finite type, and let $x_i \in X_i$ be points, $i = 1, 2$. If the complete local rings $\hat{\mathcal{O}}_{X_i, x_i}$ are $\mathcal{O}_S$-isomorphic, then X_1 and X_2 are locally isomorphic for the étale topology.*

COROLLARY 11.2. *Let $f: \mathcal{X} \to S$ be a flat morphism of schemes of finite type. Moreover, assume that f is either proper or a morphism of local schemes. Let $s \in S$ and suppose that the corresponding formal deformation $X_n \to S_n$, where $X_n = \mathcal{X} \times_S S_n$ and $S_n = \operatorname{Spec}(\mathcal{O}_{S,s}/m_s^{n+1})$, is locally trivial. Then there exist a neighborhood $s \in U \subset S$ and an étale cover $\{V_i\}$ of $f^{-1}U$ such that $V_i \to U$ is trivial.*

In particular, with assumptions as in the previous theorem, if the fiber over s (i.e., $\mathcal{X}_s$) is singular then the general fiber is singular, too, and hence f is not a smoothing.

Proof. If f is a flat family of local schemes, then the corollary follows immediately from Theorem 11.1. Now suppose that f is proper. Let $X_s = \mathcal{X} \times_S \operatorname{Spec} k(s)$ and let $\hat{\mathcal{X}}$ be the formal completion of $\mathcal{X}$ along X_s. Then the assumptions imply that $\hat{\mathcal{X}}$ is locally trivial. In particular, it follows that $\hat{\mathcal{O}}_{\mathcal{X},P} \cong \hat{\mathcal{O}}_{\mathcal{Y},P}$, where $\mathcal{Y} = X_s \times S$, $P \in X_s$, and $\hat{\mathcal{O}}_{\mathcal{X},P}, \hat{\mathcal{O}}_{\mathcal{Y},P}$ are the completions of $\mathcal{O}_{\mathcal{X},P}, \mathcal{O}_{\mathcal{Y},P}$ at the maximal ideals $m_{\mathcal{X},P}, m_{\mathcal{Y},P}$ of $\mathcal{O}_{\mathcal{X},P}, \mathcal{O}_{\mathcal{Y},P}$. Hence, by Theorem 11.1 there is an étale cover $\{V_i\}$ of X_s in $\mathcal{X}$ such that $V_i \to S$ is trivial. Let $Z = \mathcal{X} - \bigcup_i V_i$. Then, since f is proper, $Y = f(Z)$ is closed in S and $U = S - Y$ has the required properties. □

Let $f: \mathcal{X} \to \mathcal{S}$ be a deformation of a scheme X over the spectrum of a discrete valuation ring. Next we will obtain criteria on the corresponding formal deformation $f_n: X_n \to \operatorname{Spec} A_n$ in order for f to be a smoothing. First we define the relative differentials of a morphism of formal schemes.

DEFINITION 11.3 [LNSa]. Let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{S}$ be a morphism of formal schemes. Let $\mathfrak{I}, \mathfrak{J}$ be ideals of definition of $\mathfrak{X}, \mathfrak{S}$ (respectively) such that $\mathfrak{f}^*\mathfrak{J} \cdot \mathcal{O}_{\mathfrak{X}} \subset \mathfrak{I}$. Let $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^{n+1})$ and $S_n = (\mathfrak{S}, \mathcal{O}_{\mathfrak{S}}/\mathfrak{J}^{n+1})$ be the corresponding schemes, and let $f_n\colon X_n \to S_n$ be the corresponding morphism. Then $\varprojlim \Omega_{X_n/S_n}$ and $\varprojlim \omega_{X_n/S_n}$ are sheaves of $\mathcal{O}_{\mathfrak{X}} = \varprojlim \mathcal{O}_{X_n}$-modules, and we define the sheaf of formal relative differentials

$$\Omega_{\mathfrak{X}/\mathfrak{S}} = \varprojlim \Omega_{X_n/S_n}$$

and the formal dualizing sheaf

$$\omega_{\mathfrak{X}/\mathfrak{S}} = \varprojlim \omega_{X_n/S_n}.$$

If $\mathfrak{f}$ is of pseudo-finite type, then both are coherent. In this case we also define

$$T^1(\mathfrak{X}/\mathfrak{S}) = \mathcal{E}xt^1_{\mathfrak{X}}(\Omega_{\mathfrak{X}/\mathfrak{S}}, \mathcal{O}_{\mathfrak{X}}),$$

the first-order formal relative cotangent sheaf. For the basic properties of $\Omega_{\mathfrak{X}/\mathfrak{S}}$, see [TLóR].

Next we define the notion of a formal $\mathbb{Q}$-Gorenstein deformation $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ and the corresponding sheaf $T^1_{qG}(\mathfrak{X}/\mathfrak{S})$.

DEFINITION 11.4. Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be a flat morphism of formal schemes.

(1) We say that $\mathfrak{f}$ is a *formal $\mathbb{Q}$-Gorenstein deformation* if there are ideals of definition $\mathfrak{I}, \mathfrak{J}$ of $\mathfrak{X}, \mathfrak{S}$ (respectively) such that $\mathfrak{f}^*\mathfrak{J} \cdot \mathcal{O}_{\mathfrak{X}} \subset \mathfrak{I}$ and the corresponding deformations of schemes $f_n\colon X_n \to S_n$, where $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^{n+1})$ and $S_n = (\mathfrak{S}, \mathcal{O}_{\mathfrak{S}}/\mathfrak{J}^{n+1})$ are $\mathbb{Q}$-Gorenstein.
(2) Suppose that $\mathfrak{f}$ is a formal $\mathbb{Q}$-Gorenstein deformation. Then, with notation as in (1), let $\{U_i\}$ be an affine open cover of X and let $X_{i,n} = X_n|_{U_i}$. Then the deformation $X_{i,n} \to S_n$ is induced by a deformation $\tilde{X}_{i,n} \to S_n$, where $\pi_{i,n}\colon \tilde{X}_{i,n} \to X_{i,n}$ is the index-1 cover [KoSh]. These form an inverse system, and setting $\tilde{\mathfrak{X}}_i = \varprojlim \tilde{X}_{i,n}$ yields a map of formal schemes $\pi_i\colon \tilde{\mathfrak{X}}_i \to \mathfrak{X}|_{U_i}$, which we call the *formal index-1 cover*. Then, as in the usual scheme case, the covering groups G_i act on $T^1(\tilde{\mathfrak{X}}_i/\mathfrak{S})$ and we define $T^1_{qG}(\mathfrak{X}_i, \mathfrak{S}) = T^1(\tilde{\mathfrak{X}}_i/\mathfrak{S})^{G_i}$. These glue together to form a coherent sheaf $T^1_{qG}(\mathfrak{X}/\mathfrak{S})$ on $\mathfrak{X}$.

NOTATION 11.5. Let $\mathfrak{F}$ be a coherent sheaf on a formal scheme $\mathfrak{X}$. We denote by $\mathrm{Fitt}_k(\mathfrak{F}) \subset \mathcal{O}_{\mathfrak{X}}$ the k-fitting ideal of $\mathfrak{F}$. These ideals measure the obstruction for $\mathfrak{F}$ to be locally generated by k elements. In fact, $\mathfrak{F}$ is locally generated by k elements if and only if $\mathrm{Fitt}_k(\mathfrak{F}) = \mathcal{O}_{\mathfrak{X}}$. Moreover, fitting ideals commute with base change and completion [E, Prop. 20.6].

Next we define the notion of a formal smoothing.

DEFINITION 11.6. Let X be a proper equidimensional scheme of finite type over a separable field k. Then a formal deformation $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ for $\mathfrak{S} = \mathrm{Specf}\, k[[t]]$ is called a *formal smoothing* of X if and only if there is a $k \in \mathbb{Z}_{>0}$ such that $\mathfrak{I}^k \subset \mathrm{Fitt}_n(\Omega_{\mathfrak{X}/\mathfrak{S}})$, where $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of $\mathfrak{X}$ and $n = \dim X$.

REMARK 11.7. In the previous definition we required that X be equidimensional in order to control the dimension of the components of the general fiber. However, it is not a very restrictive condition because almost all singularities of interest in applications (e.g., moduli of canonically polarized varieties and the minimal model program) are Cohen–Macauley and hence equidimensional.

The next proposition shows that formal smoothness implies smoothness in the case of algebraic deformations.

PROPOSITION 11.8. *Let X be a proper equidimensional scheme of dimension n that is of finite type over a separable field k. Let $f: \mathcal{X} \to S$ be a deformation of X over the spectrum of a discrete valuation ring (A, m), and let $\mathfrak{f}: \mathfrak{X} \to \mathfrak{S}$ be the associated formal deformation. Then f is a smoothing of X if and only if $\mathfrak{f}$ is a formal smoothing of X.*

Proof. Since f is proper, it follows that the general fiber $\mathcal{X}_g = \mathcal{X} \times_S \operatorname{Spec} K(A)$ is equidimensional of dimension n. Assume that $\mathfrak{f}$ is formally smooth. Then—since $\operatorname{Fitt}_n(\Omega_{\mathfrak{X}/\mathfrak{S}}) = \operatorname{Fitt}_n(\Omega_{\mathcal{X}/S})^\wedge$, the formal completion of $\operatorname{Fitt}_n(\Omega_{\mathcal{X}/S})$ along X—the assumption implies that $\mathcal{O}_{\mathcal{X}} / \operatorname{Fitt}_n(\Omega_{\mathcal{X}/S})$ is supported on the central fiber. Therefore, $\operatorname{Fitt}_n(\Omega_{\mathcal{X}_g}) = \mathcal{O}_{\mathcal{X}_g}$ and hence $\Omega_{\mathcal{X}_g/K(A)}$ is locally generated by n elements.

Let $\mathcal{X}_g^n$ be an irreducible component of $\mathcal{X}_g$ and let $P \in \mathcal{X}_g$ be a closed point. Then, since $\mathcal{X}_g$ is Noetherian, we have $\dim \mathcal{O}_{\mathcal{X}_g, P} = n$. Let $m_P \subset \mathcal{O}_{\mathcal{X}_g, P} = n$ be the maximal ideal. Then there is an exact sequence

$$m_P/m_P^2 \to \Omega_{\mathcal{X}_g/K(A)} \otimes (\mathcal{O}_{\mathcal{X}_g, P}/m_P) \to \Omega_{K(\mathcal{O}_{\mathcal{X}_g, P})/K(A)} \to 0,$$

which is exact on the left as well because k is separable. Therefore, $\dim(m_P/m_P^2) = \dim \mathcal{O}_{\mathcal{X}_g, P}$ and hence $\mathcal{O}_{\mathcal{X}_g, P}$ is regular. In fact, the proof shows that it is geometrically regular and therefore $\mathcal{O}_{\mathcal{X}_g, P}$ is smooth. Hence $\mathcal{X}_g$ is smooth and irreducible. The converse is proved similarly. □

If X has complete intersection singularities or if X is $\mathbb{Q}$-Gorenstein and the index-1 cover of any of its singular points has complete intersection singularities, then it is possible to give simpler criteria, which we will use in Section 12.

We will need the next easy lemma.

LEMMA 11.9. *Let X be a local complete intersection scheme of finite type over a field k. Then, if $\mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X) = 0$, X is smooth.*

Proof. We may assume that X is affine. Then, since it is complete intersection, there exists an exact sequence

$$0 \to \mathcal{O}_X^k \to \mathcal{O}_X^m \to \Omega_X \to 0$$

such that $m - k = \dim X$. Since $\mathcal{E}xt^1_X(\Omega_X, \mathcal{O}_X) = 0$, it follows that the previous sequence is split exact. Hence

$$\mathcal{O}_X^m = \mathcal{O}_X^k \oplus \Omega_X$$

and therefore Ω_X is free and of rank equal to the dimension of X. Hence X is smooth. □

PROPOSITION 11.10. *Let X be a local complete intersection scheme, and let $f\colon \mathcal{X} \to S$ be a deformation of X over the spectrum of a discrete valuation ring (A, m_A). Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be the corresponding formal deformation. Assume that f is proper and of finite type. Let $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ be an ideal of definition of $\mathfrak{X}$. Then the following statements are equivalent*:

(1) *the family $f\colon \mathcal{X} \to S$ is a smoothing of X*;
(2) *there is an $m \in \mathbb{N}$ such that $\mathfrak{I}^m T^1(\mathfrak{X}/\mathfrak{S}) = 0$*;
(3) *there is a $k \in \mathbb{N}$ such that, for all $n \geq k$,*

$$T^1(X_{n+1}/A_{n+1}) = T^1(X_n/A_n),$$

where $X_n = \mathcal{X} \times_S S_n$, $S_n = \operatorname{Spec} A_n$, and $A_n = A/m_A^{n+1}$.

Proof. First we show that (1) implies (2). In this case, $\mathfrak{X} = \hat{\mathcal{X}}$ is the completion of $\mathcal{X}$ along X. Then $\Omega_{\mathfrak{X}/\mathfrak{S}} = \hat{\Omega}_{\mathcal{X}/S}$ [TLóR] and hence

$$T^1(\mathfrak{X}/\mathfrak{S}) = \mathcal{E}xt^1_{\mathfrak{X}}(\Omega_{\mathfrak{X}/\mathfrak{S}}, \mathcal{O}_{\mathfrak{X}}) = \mathcal{E}xt^1_{\mathcal{X}}(\Omega_{\mathcal{X}/S}, \mathcal{O}_{\mathcal{X}})^\wedge = T^1(\mathcal{X}/S)^\wedge.$$

Now, by Lemma 11.9, $\mathcal{X} \to S$ is a smoothing if and only if $T^1(\mathcal{X}/S)$ is supported on X. Since $T^1(\mathcal{X}/S)$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module, this is equivalent to saying that there is an $m \in \mathbb{N}$ such that $I^m T^1(\mathcal{X}/S) = 0$, where I is the ideal sheaf of X in $\mathcal{X}$. Hence $\mathfrak{I}^m T^1(\mathfrak{X}/\mathfrak{S}) = 0$, where $\mathfrak{I} = \hat{I}$. Conversely, if $\mathfrak{I}^m T^1(\mathfrak{X}/\mathfrak{S}) = 0$ for some m and some ideal of definition $\mathfrak{I}$, then it also holds for all ideals of definition and in particular for $\mathfrak{I} = \hat{I}$. Hence $(I^m T^1(\mathcal{X}/S))^\wedge = 0$ and thus there is an $X \subset \mathcal{U} \subset \mathcal{X}$ (an open neighborhood of X in $\mathcal{X}$) such that $I^m T^1(\mathcal{X}/S)|_{\mathcal{U}} = 0$; therefore, since f is proper and S is local, $I^m T^1(\mathcal{X}/S) = 0$. Hence $T^1(\mathcal{X}/S)$ is supported on $\mathcal{X}$ and so f is a smoothing.

Next we show that (1) is equivalent to (3). Let t be a generator of the maximal ideal of R. Then the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \xrightarrow{t^{n+1}} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_n} \to 0$$

gives the exact sequence

$$0 \to T_{\mathcal{X}/\Delta} \xrightarrow{t^{n+1}} T_{\mathcal{X}/\Delta} \to T_{X_n/A_n} \to T^1(\mathcal{X}/\Delta) \xrightarrow{t^{n+1}} T^1(\mathcal{X}/\Delta) \to T^1(X_n/A_n) \to 0.$$

Thus f is a smoothing if and only if $T^1(\mathcal{X}/S)$ is supported on X and hence if and only if there is a $k \in \mathbb{N}$ such that $t^k T^1(\mathcal{X}/S) = 0$. Now it follows from the previous exact sequence that this is equivalent to saying that $T^1(X_{n+1}/A_{n+1}) = T^1(X_n/A_n)$ for all $n \geq k$. □

PROPOSITION 11.11. *Let X be a $\mathbb{Q}$-Gorenstein scheme such that the index-1 cover of its singular points has complete intersection singularities only. Let $f\colon \mathcal{X} \to S$ be a $\mathbb{Q}$-Gorenstein deformation of X over the spectrum of a discrete valuation ring (A, m_A), and let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{S}$ be the corresponding formal deformation. Assume that f is proper and of finite type. Let $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$ be an ideal of definition of $\mathfrak{X}$. Then the following statements are equivalent*:

(1) *the family* $f\colon \mathcal{X} \to S$ *is a smoothing of* X;
(2) *there is an* $m \in \mathbb{N}$ *such that* $\mathfrak{I}^m T^1_{qG}(\mathfrak{X}/\mathfrak{S}) = 0$ *and* $\mathfrak{I}^m \subset \operatorname{Fitt}_1(\omega_{\mathfrak{X}/\mathfrak{S}})$;
(3) *there is a* $k \in \mathbb{N}$ *such that, for all* $n \geq k$, $\mathfrak{I}^m \subset \operatorname{Fitt}_1(\omega_{\mathfrak{X}/\mathfrak{S}})$ *and*

$$T^1_{qG}(X_{n+1}/A_{n+1}) = T^1_{qG}(X_n/A_n),$$

where $X_n = \mathcal{X} \times_S S_n$, $S_n = \operatorname{Spec} A_n$, *and* $A_n = A/m_A^{n+1}$.

Proof. The proof follows the lines of that for Proposition 11.10 with a few differences that we explain next. The condition $\mathfrak{I}^m \subset \operatorname{Fitt}_1(\omega_{\mathfrak{X}/\mathfrak{S}})$ means that, generically over S, $\omega_{\mathcal{X}/S}$ is generated by one element and hence is a line bundle. Therefore, the general fiber of f is Gorenstein. Hence the index-1 cover of any singularity of $\mathcal{X}$ is étale away from the central fiber. Now, since the index-1 cover of any singular point of X is assumed to be complete intersection, it follows that the general fiber of f is also complete intersection. Now applying the arguments of the proof of Proposition 11.10 yields the claimed result. □

12. Smoothing Criteria

Let X be a proper pure and reduced scheme of finite type over a field k. Moreover, assume that the singular points of X are either complete intersection or $\mathbb{Q}$-Gorenstein with complete intersection index-1 covers. In this section we give some smoothing and nonsmoothing criteria for such schemes X. Following the methodology of this section and the methods developed in previous sections, one could also give similar criteria for algebraic germs $Y \subset X$. However, for the sake of simplicity we will only consider the case $X = Y$.

In what follows we denote by D either $\operatorname{Def}(X)$ or $\operatorname{Def}^{qG}(X)$ and by $T^i_D(X)$ either $T^i(X)$ or $T^i_{qG}(X)$.

The sheaves $T^i_D(X)$ are fundamental in the study of the deformation theory of X. However, they can be extremely complicated. The reduced part of their support is contained in the singular locus of X, but it may have embedded components. This happens even in the simplest cases. For example, if X is the pinch point given by $x^2 - y^2 z = 0$, then $T^1(X) = k[x, y, z]/(x, y^2, yz)$ and has an embedded point over the pinch point. This makes any calculation involving $T^i_D(X)$ most difficult. So it is better to consider instead the pure part of $T^i_D(X)$, which we define next. It is simply a generalization of the notion of torsion free.

Definition 12.1. Let X be a pure and reduced scheme, and let $\mathcal{F}$ be a coherent sheaf on X of dimension d. Let $\mathcal{F}_{d-1} \subset \mathcal{F}$ be the maximal subsheaf of $\mathcal{F}$ of dimension at most $d-1$. Then we define:

(1) the support of the torsion part of $\mathcal{F}$ to be the support of $\mathcal{F}_{d-1}$;
(2) the rank of $\mathcal{F}$, $\operatorname{rk}(\mathcal{F})$, by

$$\operatorname{rk}(\mathcal{F}) = \max_\xi \{\operatorname{length}(\mathcal{F}_\xi), \text{ where } \xi \text{ is a generic point of the support of } \mathcal{F}\};$$

(3) the pure part of $\mathcal{F}$, $p(\mathcal{F})$, to be the quotient $\mathcal{F}/\mathcal{F}_{d-1}$ (this is pure of dimension d).

Let $X_n \to \operatorname{Spec} A_n$ be a deformation of X over A_n, and let $X_{n-1} = X_n \otimes_{A_{n-1}} A_n$. Then, from our discussion in Sections 9 and 10, it follows that—in order to understand the obstructions to lifting X_n to a deformation X_{n+1} over A_{n+1}—it is important to study the sheaves $\mathcal{F}_n$ and $T^1_D(X)/\mathcal{F}_n$, where $\mathcal{F}_n \subset T^1_D(X)$ is the cokernel of the natural map $T_{X_n/A_n} \to T_{X_{n-1}/A_{n-1}}$. The next lemma does this in some cases.

LEMMA 12.2. *Let $\mathcal{X} \to \Delta = \operatorname{Spec}(R)$ be a deformation of X, where (R, m) is a discrete valuation ring. Let $X_n = \mathcal{X} \otimes_R R/m^{n+1}$, and (as in Proposition 9.10) let $\mathcal{F}_n \subset T^1_D(X)$ be the cokernel of the natural map $T_{X_n/A_n} \to T_{X_{n-1}/A_{n-1}}$, where $A_n = R/m^{n+1}$. Then the following statements hold.*

(1) *There is an injective map*

$$\phi\colon \hat{T}^1_D(\mathcal{X}/\Delta) \to \varprojlim_n T^1_D(X_n/A_n),$$

where $\hat{T}^1_D(\mathcal{X}/\Delta)$ is the m-adic completion of $T^1_D(\mathcal{X}/\Delta)$. Moreover, ϕ is an isomorphism at any local complete intersection point of X.

(2) *Suppose that X is unobstructed at any generic point of its singular locus and that $\mathcal{X}$ is a smoothing. Then there is an $n_0 \in \mathbb{Z}$ such that*

(a) $\operatorname{rk}(T^1_D(X)/\mathcal{F}_n) = 0$ *if $n \geq n_0$ and*

(b) $0 < \operatorname{rk}(T^1_D(X)/\mathcal{F}_n) \leq \operatorname{rk}(T^1_D(X))$ *for all $n < n_0$.*

(c) *Suppose that, at any generic point ξ of the singular locus of X, X is a hypersurface singularity $(f = 0) \subset \mathbb{C}^n$ with $\mu(f) = \tau(f)$, where $\mu(f)$ and $\tau(f)$ are (respectively) the Milnor and Tjurina numbers of f. If $\mathcal{X}$ is smooth at ξ, then* $\operatorname{rk}(T^1_D(X)/\mathcal{F}_n) = 0$ *for all n.*

Proof. Let t be a generator of the maximal ideal of R. Then the exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}} \xrightarrow{t^{n+1}} \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{X_n} \to 0$$

gives the exact sequence

$$\begin{aligned} 0 \to T_{\mathcal{X}/\Delta} \xrightarrow{t^{n+1}} T_{\mathcal{X}/\Delta} \to T_{X_n/A_n} \to T^1_D(\mathcal{X}/\Delta) \\ \xrightarrow{t^{n+1}} T^1_D(\mathcal{X}/\Delta) \to T^1_D(X_n/A_n) \to T^2_D(\mathcal{X}/\Delta), \end{aligned} \tag{12.1}$$

where $T^2_D(\mathcal{X}/\Delta)$ is a sheaf supported on the noncomplete intersection singular points of X. Then it follows that there are injections

$$\phi_n\colon T^1_D(\mathcal{X}/\Delta)/t^{n+1}T^1_D(\mathcal{X}/\Delta) \to T^1_D(X_n/A_n).$$

Passing to the inverse limits yields the claimed map ϕ. Furthermore, since the ϕ_n are isomorphisms at any complete intersection point of X, we know that ϕ is an isomorphism, too.

Suppose that $\mathcal{X}$ is a smoothing and that, at any generic point of its singular locus, X is unobstructed. Then, at any generic point ξ of the singular locus of X, $T^2_D(\mathcal{X}/\Delta)_\xi = 0$ and the argument of the proof of Proposition 11.10 shows that there is an $n_0 \in \mathbb{Z}$ such that $T^1_D(X_n/A_n)_\xi = T^1_D(X_{n-1}/A_{n-1})_\xi$ for all $n \geq n_0$. In fact, something stronger holds. Suppose there is a $k \in \mathbb{Z}$ such that $T^1_D(X_k/A_k)_\xi = T^1_D(X_{k-1}/A_{k-1})_\xi$. Then we will show that $T^1_D(X_n/A_n)_\xi = T^1_D(X_{n-1}/A_{n-1})_\xi$ for all $n \geq k$. From (12.1) it follows that

$$T^1_D(X_n/A_n)_\xi = T^1_D(\mathcal{X}/\Delta)_\xi/t^{n+1}T^1_D(\mathcal{X}/\Delta)_\xi$$

for all n and hence, since $T^1_D(X_k/A_k)_\xi = T^1_D(X_{k-1}/A_{k-1})_\xi$,

$$t^{k+1}T^1_D(\mathcal{X}/\Delta)_\xi = t^k T^1_D(\mathcal{X}/\Delta)_\xi;$$

consequently,

$$t^{n+1}T^1_D(\mathcal{X}/\Delta)_\xi = t^n T^1_D(\mathcal{X}/\Delta)_\xi$$

for all $n \geq k$. Hence $T^1_D(X_n/A_n)_\xi = T^1_D(X_{n-1}/A_{n-1})_\xi$ for all $n \geq k$.

Moreover, by Proposition 9.10 and Proposition 10.10, there is an exact sequence

$$0 \longrightarrow T^1_D(X)/\mathcal{F}_n \longrightarrow T^1_D(X_n/A_n) \xrightarrow{\phi_n} T^1_D(X_{n-1}/A_{n-1}); \qquad (12.2)$$

hence it follows that there is an $n_0 \in \mathbb{Z}$ such that, generically along the singularities of X, ϕ_n is an isomorphism for all $n \geq n_0$ but not if $n < n_0$. Therefore, $\operatorname{rk}(T^1_D(X)/\mathcal{F}_n) = 0$ if $n \geq n_0$ and $0 < \operatorname{rk}(T^1_D(X)/\mathcal{F}_n) \leq \operatorname{rk}(T^1_D(X))$ if $n < n_0$, as claimed.

Let $\xi \in X$ be a generic point of the singular locus of X and let $K = k(\mathcal{O}_{\mathcal{X},\xi})$. Suppose that, at ξ, X is a hypersurface singularity given by $(f = 0) \subset \mathbb{C}^n$ and $\mu(f) = \tau(f)$. If $\mathcal{X}$ is smooth at ξ, then $\dim_K T^1_D(\mathcal{X}/\Delta) = \mu(f)$. But since $\mu(f) = \tau(f) = \dim_K T^1_D(X)$ by assumption, it follows from (12.1) that $T^1_D(\mathcal{X}/\Delta) = T^1_D(X)$ and hence $tT^1_D(\mathcal{X}/\Delta) = 0$. Therefore, $t^n T^1_D(\mathcal{X}/\Delta) = 0$ for all n, so

$$T^1_D(X_n/A_n) = T^1_D(X_{n-1}/A_{n-1}) = T^1_D(\mathcal{X}/\Delta)$$

for all n. Hence from (12.2) it follows that $\operatorname{rk}(T^1_D(X)/\mathcal{F}_n) = 0$ for all n, as claimed. □

The next theorem gives some conditions under which X is not smoothable.

THEOREM 12.3. *Suppose that $H^0(p(T^1_D(X))) = 0$ and that, at any generic point of the singular locus of X, X is complete intersection. Let Z be the support of the torsion part of $T^1_D(X)$, and let $f : \mathcal{X} \to \Delta$ be a 1-parameter deformation of X. Then*

(1) *$X^{\text{sing}} \subset \mathcal{X}^{\text{sing}}$, where X^{sing} and $\mathcal{X}^{\text{sing}}$ are the singular parts of X and $\mathcal{X}$; in particular, $\mathcal{X}$ is not smooth.*
(2) *Suppose also that $H^1_Z(p(T^1_D(X))) = 0$ and that, at any generic point ξ of the singular locus of X, X is analytically isomorphic to $(x_1^2 + \cdots + x_k^2 = 0) \subset \mathbb{C}^n$. Then there is a proper closed subset W of the singular locus of X such that $\mathcal{X} - W$ is locally trivial. In particular, the general fiber $\mathcal{X}_g$ of f is singular and hence X is not smoothable.*

COROLLARY 12.4. *Suppose that $T^1_D(X)$ is pure and that $H^0(T^1_D(X)) = 0$. Suppose also that the general singularity of X is analytically isomorphic to*

$$(x_1^2 + \cdots + x_k^2 = 0) \subset \mathbb{C}^n.$$

Then X is not smoothable.

Corollary 12.4 applies in particular to schemes with only normal crossing singularities.

Proof of Theorem 12.3. Let $\mathcal{X} \to \Delta$ be a deformation of X over $\Delta = \mathrm{Spec}(R)$, where (R, m) is a discrete valuation ring. Suppose that $\mathcal{X}$ is not trivial at any generic point of the singular locus of X. Let $X_n = \mathcal{X} \times_\Delta \mathrm{Spec}(R/m^n)$. By our assumptions, every section of $T^1_D(X)$ vanishes generically along the singularities of X. The theorem will follow if we show that:

(1) $T^1_D(\mathcal{X}/\Delta)$ has a section s that does not vanish generically along the singular locus of X; and
(2) any section of $T^1_D(X_n/A_n)$ vanishes generically along the singular locus of X for any n.

Indeed, if there is a smoothing $\mathcal{X}$, then by (1) there is a section s of $T^1_D(\mathcal{X}/\Delta)$ that does not vanish at any generic point of the singular locus of X. But then, by Lemma 12.2(1), there is an $n \in \mathbb{Z}$ such that the image s_n of s in $T^1_D(X_n/A_n)$ does not vanish at any generic point of the singular locus of X. But this is impossible by (2).

Next we show (1). Since X is complete intersection at any generic point of the singular locus X, it follows that there is an exact sequence

$$0 \to f^*\omega_\Delta = \mathcal{O}_\mathcal{X} \to \Omega_\mathcal{X} \to \Omega_{\mathcal{X}/\Delta} \to 0. \tag{12.3}$$

This gives a section s of $\mathcal{E}xt^1_\mathcal{X}(\Omega_{\mathcal{X}/\Delta}, \mathcal{O}_\mathcal{X}) = T^1(\mathcal{X}/\Delta)$. If $\mathcal{X}$ is also $\mathbb{Q}$-Gorenstein, then this gives an element of $T^1_{qG}(\mathcal{X}/\Delta)$. Since X is pure, X_n is also pure and hence there is an exact sequence

$$0 \to \mathcal{O}_{X_n} \to \Omega_\mathcal{X} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_{X_n} \to \Omega_{X_n/A_n} \to 0 \tag{12.4}$$

that gives an element of $T^1(X_n/A_n) = \mathcal{E}xt^1_{X_n}(\Omega_{X_n/A_n}, \mathcal{O}_{X_n})$. If $\mathcal{X}$ is also $\mathbb{Q}$-Gorenstein, then this gives an element of $T^1_{qG}(X_n/A_n)$. Next we claim that the extension (12.3) is not split—nor even generically split along the singular locus of X.

Case 1. Suppose that $\mathcal{X}$ is smooth and that (12.3) is generically split along the singular locus of X. Then $\Omega_\mathcal{X} \cong \Omega_{\mathcal{X}/\Delta} \oplus \mathcal{O}_\mathcal{X}$ and hence $\Omega_{\mathcal{X}/\Delta}$ is free and so Ω_X is free, which of course is not true. Hence, in this case (12.3) is not even generically split.

Case 2. Suppose that the general singularity of X is analytically isomorphic to

$$(x_1^2 + \cdots + x_k^2 = 0) \subset \mathbb{C}^n. \tag{12.5}$$

Hence if (12.3) were generically split then, generically over the singular locus of X,

$$\mathrm{Ext}^1_\mathcal{X}(\Omega_\mathcal{X}, \mathcal{O}_\mathcal{X}) \cong \mathrm{Ext}^1_\mathcal{X}(\Omega_{\mathcal{X}/\Delta}, \mathcal{O}_\mathcal{X}). \tag{12.6}$$

Around the generic point ξ of the singular locus of X we may assume that X is the singularity given by (12.5). Thus all Ext spaces involved are now finite dimensional over $K = k(\mathcal{O}_{\mathcal{X},\xi})$. We will show by direct computation that (12.6) is impossible. In suitable local analytic coordinates, X is given by (12.5); if we use the Weierstrass preparation theorem, then $\mathcal{X}$ is given by

$$x_1^2 + \cdots + x_k^2 + t^s g(x_{k+1}, \ldots, x_n, t) = 0,$$

where $g \neq 0$ and t does not divide g. Straightforward calculations show that

$$\mathrm{Ext}^1_{\mathcal{X}}(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) = \frac{k[x_1, \ldots, x_n, t]}{(x_1, \ldots, x_k, t^s \partial g/\partial x_{k+1}, \ldots, t^s \partial g/\partial x_n, st^{s-1}g + t^s \partial g/\partial t, t^s g)}$$

and similarly

$$\mathrm{Ext}^1_{\mathcal{X}}(\Omega_{\mathcal{X}/\Delta}, \mathcal{O}_{\mathcal{X}}) = \frac{k[x_1, \ldots, x_n, t]}{(x_1, \ldots, x_k, t^s \partial g/\partial x_{k+1}, \ldots, t^s \partial g/\partial x_n, t^s g)}.$$

If $\mathrm{Ext}^1_{\mathcal{X}}(\Omega_{\mathcal{X}}, \mathcal{O}_{\mathcal{X}}) \cong \mathrm{Ext}^1_{\mathcal{X}}(\Omega_{\mathcal{X}/\Delta}, \mathcal{O}_{\mathcal{X}})$, then

$$st^{s-1}g + t^s \partial g/\partial t \in (x_1, \ldots, x_k, t^s \partial g/\partial x_{k+1}, \ldots, t^s \partial g/\partial x_n, t^s g);$$

hence there are polynomials $h_i, h \in k[x_1, \ldots, x_n, t]$ such that

$$st^{s-1}g + t^s \frac{\partial g}{\partial t} = \sum_{i=s+1}^{n} h_i t^s \frac{\partial g}{\partial x_i} + ht^s g$$

and therefore t divides g, which is impossible. This shows part (1) of the claim.

Now we show (2), proceeding by induction on n. By assumption, $n = 1$ is true. By Lemma 12.2, there is an $n_0 \in \mathbb{Z}$ such that $\mathrm{rk}((T^1_D(X)/\mathcal{F}_n) = 0$ for all $n \geq n_0$ and, since $\mathrm{rk}(T^1_D(X)) = 1$, $\mathrm{rk}((T^1_D(X)/\mathcal{F}_n) = 1$ for all $n < n_0$. Hence $p(T^1_D(X)/\mathcal{F}_n) = p(T^1_D(X))$ for all $n < n_0$.

Suppose that $n < n_0$ and construct the pushout diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & T^1_D(X)/\mathcal{F}_n & \xrightarrow{\alpha_n} & T^1(X_n/A_n) & \longrightarrow & T^1(X_{n-1}/A_{n-1}) \\
 & \downarrow \beta_n & & \downarrow \gamma_n & & \| \\
0 \longrightarrow & p(T^1_D(X)/\mathcal{F}_n) & \longrightarrow & Q_n & \longrightarrow & T^1(X_{n-1}/A_{n-1})
\end{array}$$

with $\mathrm{Ker}(\beta_n) = \mathrm{Ker}(\gamma_n)$ supported on Z. Let $M_n = \mathrm{Coker}(\alpha_n)$. Then there is a commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow 0 = H^0_Z(p(T^1_D(X))) & \longrightarrow & H^0_Z(Q_n) & \longrightarrow & H^0_Z(M_n) & \longrightarrow & H^1_Z(p(T^1_D(X))) \\
\downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 \\
0 \longrightarrow 0 = H^0(p(T^1_D(X))) & \longrightarrow & H^0(Q_n) & \longrightarrow & H^0(M_n) & \longrightarrow & H^1(p(T^1_D(X)))
\end{array}$$

Now f_3 is an isomorphism by induction and $H^1_Z(p(T^1_D(X))) = 0$ by assumption. Hence, by the five lemma, f_2 is also an isomorphism and therefore all sections of Q_n are supported on Z.

Now there is an exact sequence

$$0 \to \mathrm{Ker}(\gamma_n) \to T^1(X_n/A_n) \to Q_n \to 0,$$

and, since $\mathrm{Ker}(\beta_n) = \mathrm{Ker}(\gamma_n)$, we know that $\mathrm{Ker}(\gamma_n)$ is supported on Z. Let $U = X - Z$. Then $T^1(X_n/A_n)|_U = Q_n|_U$ and hence, since the sections of Q_n are supported on Z, the sections of $T^1(X_n/A_n)$ are also supported on Z. Thus, for all $n < n_0$, the sections of $T^1(X_n/A_n)$ are supported on Z. If $n \geq n_0$, then $\mathrm{rk}(T^1_D(X)/\mathcal{F}_n) = 0$ and hence $Z'_n = \mathrm{Supp}(T^1_D(X)/\mathcal{F}_n)$ is a proper subset of X^{sing}. By induction, all sections of $T^1(X_{n-1}/A_{n-1})$ are supported on a proper subset Z_{n-1} of X^{sing}. Let $Z_n = Z'_n \cup Z_{n-1}$ and $U_n = X - Z_n$. Then $T^1(X_n/A_n)|_{U_n} = T^1(X_{n-1}/A_{n-1})|_{U_n}$ and hence all sections of $T^1(X_n/A_n)$ are supported on Z_n. This shows (2).

It remains to show part (1) of the theorem. This is a local result, so we may assume that X is affine and $\mathcal{X}$ is smooth. Then, by Lemma 12.2, $\mathrm{rk}((T^1_D(X)/\mathcal{F}_n) = 0$ for all n. The previous proof now shows that the sections of $T^1_D(X_n/A_n)$ vanish at any generic point of the singular locus of X for all n, and part (1) follows as before. □

Next we present some smoothing criteria.

THEOREM 12.5. *Let X be a proper pure and reduced scheme of finite type over a field k of characteristic 0. Let D be either* $\mathrm{Def}(X)$ *or* $\mathrm{Def}^{qG}(X)$. *Assume that*:

(1) *X has complete intersection singularities if $D = \mathrm{Def}(X)$; or*
(2) *X is locally smoothable, and the index-1 cover of any singularity of X has complete intersection singularities, if $D = \mathrm{Def}^{qG}(X)$.*

Then, if $T^1_D(X)$ is finitely generated by its global sections and if $H^1(T^1_D(X)) = H^2(T_X) = 0$, X is D-formally smoothable.

COROLLARY 12.6. *If every deformation of X is effective, then X is D-smoothable.*

REMARK 12.7.

(1) The requirement that X be proper can be replaced by the more general requirement that $\mathrm{Def}(X)$ have a hull.
(2) The conditions of the theorem on the vanishing of the obstructions are rather restrictive, but there are some cases when they are satisfied. We mention two of them. The first is when there is a proper morphism $f\colon X \to \mathrm{Spec}\, A$ such that $\dim f^{-1}(s) \leq 1$ for all $s \in \mathrm{Spec}\, A$. Then, by the formal functions theorem, $H^2(T_X) = 0$. This is, for example, the case of birational maps with at most 1-dimensional fibers. The second case is when X is a Fano variety with only double-point normal crossing singularities such that $T^1(X)$ is finitely generated by its global sections. Then $H^1(T^1(X)) = H^2(T_X) = 0$ [Tz2].

Proof of Theorem 12.5. For simplicity we show only the case $D = \mathrm{Def}(X)$. The $\mathbb{Q}$-Gorenstein case is analogous; one need only lift the following argument to the index-1 covers.

The conditions of the theorem imply that $\mathrm{Def}(X)$ exists and is smooth. Let $s_1, \ldots, s_k \in H^0(T^1(X))$ be sections that generate $T^1(X)$. Because $\mathrm{Def}(X)$ is smooth, the sections $s_1, \ldots, s_k$ lift to a formal deformation $f_n\colon Y_n \to S_n$ of X over

$S_n = \mathrm{Spec}(S/m_S^{n+1})$, where $S = k[[t_1, \ldots, t_k]]$ and m_S is its maximal ideal. Let $f : \mathcal{Y} \to \mathcal{S}$ be the corresponding morphism of formal schemes. We will show that $\mathcal{Y}$ is smooth over Specf $K(S)$. Let $U \subset X$ be the smooth locus of X. Then $f|_U$ is smooth and hence, since X is pure, it follows that there is an exact sequence

$$0 \to f^*\hat{\Omega}_{\mathcal{S}} \to \hat{\Omega}_{\mathcal{Y}} \to \hat{\Omega}_{\mathcal{Y}/\mathcal{S}} \to 0 \tag{12.7}$$

[TLóR]. Moreover, $\hat{\Omega}_{\mathcal{S}} = \hat{\Omega}_R^{\triangle} \cong \mathcal{O}_{\mathcal{S}}^k$, where $R = k[t_1, \ldots, t_k]$. Hence $f^*\hat{\Omega}_{\mathcal{S}} = \mathcal{O}_{\mathcal{Y}}^k$, and dualizing the previous sequence yields

$$\mathrm{Hom}_{\mathcal{Y}}(\hat{\Omega}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}}) \to \mathcal{O}_{\mathcal{Y}}^k \xrightarrow{\phi} T^1(\mathcal{Y}/\mathcal{S}) \to T^1(\mathcal{Y}) \to 0. \tag{12.8}$$

By construction, however, ϕ is surjective and therefore

$$T^1(\mathcal{Y}) = \underline{\mathrm{Ext}}^1_{\mathcal{Y}}(\hat{\Omega}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}}) = 0.$$

Claim: $\mathcal{O}_{\mathcal{Y}}$ and $\mathcal{O}_{\mathcal{Y}} \otimes_S K(S)$ have smooth local rings.

The result is local and hence we may assume that X is affine and given by $\mathcal{O}_X = k[x_1, \ldots, x_m]/(f)$, where $(f) = (f_1, \ldots, f_s)$ is a complete intersection. Then $\mathcal{O}_{X_n} = S_n[x_1, \ldots, x_m]/(f_n)$, where (f_n) is a lifting of (f) on $S_n[x_1, \ldots, x_m]$ and so

$$\mathcal{O}_{\mathcal{Y}} = \varprojlim \mathcal{O}_{X_n} = \frac{S[x_1, \ldots, x_m]^\wedge}{(\bar{f}_1, \ldots, \bar{f}_s)}, \tag{12.9}$$

where $S[x_1, \ldots, x_m]^\wedge$ is the m_S-adic completion of $S[x_1, \ldots, x_m]$ and $\bar{f}_i = \varprojlim f_i^{(n)}$. Let

$$0 \to \mathcal{O}_X^r \to \mathcal{O}_X^m \to \Omega_X \to 0$$

be a presentation of Ω_X, where $m - r = \dim \mathcal{O}_X$. Then this exact sequence lifts to compatible exact sequences

$$0 \to \mathcal{O}_{X_n}^r \to \mathcal{O}_{X_n}^m \to \Omega_{X_n} \to 0.$$

Furthermore, $\mathcal{O}_{\mathcal{Y}} = \varprojlim \mathcal{O}_{X_n}$ and hence, taking inverse limits and taking into consideration that $\hat{\Omega}_{\mathcal{Y}} = \varprojlim \Omega_{\mathcal{O}_{X_n}}$ [TLóR], we obtain an exact sequence

$$0 \to \mathcal{O}_{\mathcal{Y}}^r \to \mathcal{O}_{\mathcal{Y}}^m \to \hat{\Omega}_{\mathcal{Y}} \to 0.$$

This extension is trivial because $\underline{\mathrm{Ext}}^1_{\mathcal{Y}}(\hat{\Omega}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{Y}}) = 0$. Hence

$$\mathcal{O}_{\mathcal{Y}}^m \cong \mathcal{O}_{\mathcal{Y}}^r \oplus \hat{\Omega}_{\mathcal{Y}}$$

and so $\hat{\Omega}_{\mathcal{Y}}$ is locally free and of the rank claimed. This implies that $\mathcal{O}_{\mathcal{Y}}$ has geometrically regular local rings. Indeed, let $P \in X$ be a point and m_P the maximal ideal of $\mathcal{O}_{\mathcal{Y},P}$. Then, since k is perfect, it follows that

$$m_p/m_P^2 \cong \hat{\Omega}_{\mathcal{Y}} \otimes k(P)$$

[E] and therefore $\dim_{k(P)} m_P/m_P^2 = \dim \mathcal{O}_{\mathcal{Y},P}$; hence $\mathcal{O}_{\mathcal{Y},P}$ is geometrically regular and therefore smooth. Because any localization of $\mathcal{O}_{\mathcal{Y}}$ is a localization of $\mathcal{O}_{\mathcal{Y},P}$ for some $P \in X$, it follows that $\mathcal{O}_{\mathcal{Y}}$ has smooth local rings. In particular, since any localization of $\mathcal{O}_{\mathcal{Y}} \otimes_S K(S)$ is a localization of $\mathcal{O}_{\mathcal{Y}}$, $\mathcal{O}_{\mathcal{Y}} \otimes_S K(S)$ is smooth.

Since $\hat{\Omega}_{\mathcal{Y}} \cong \mathcal{O}_{\mathcal{Y}}^d$, where $d = \dim X$, the sequence (12.8) becomes

$$\mathcal{O}_{\mathcal{Y}}^d \xrightarrow{\psi} \mathcal{O}_{\mathcal{Y}}^k \xrightarrow{\phi} T^1(\mathcal{Y}/\mathcal{S}) \to T^1(\mathcal{Y}) \to 0;$$

as in the usual scheme case, ψ is given by the Jacobian matrix $J = (\partial \bar{f}_i/x_j)$. Since $\mathcal{O}_{\mathcal{Y}} \otimes_S K(S)$ is smooth, it follows that J has maximum rank at all localizations of $\mathcal{O}_{\mathcal{Y}} \otimes_S K(S)$. Therefore, $\psi \otimes_S K(S)$ is surjective and hence $T^1(\mathcal{Y}/\mathcal{S})$ is torsion over $\mathcal{S}$. Thus there is a formal arc $\Delta = \operatorname{Specf} k[[t]] \to \operatorname{Spec} S$ such that, in the fiber $\mathcal{X} = \mathcal{Y} \times_{\operatorname{Specf} S} \operatorname{Specf} k[[t]]$, $T^1(\mathcal{X}/\Delta)$ is torsion over $k[[t]]$; hence there is an $l \in \mathbb{N}$ such that $t^l T^1(\mathcal{X}/\Delta) = 0$ and therefore $\mathcal{X} \to \Delta$ is a formal smoothing of X. □

The preceding proof also shows the following.

COROLLARY 12.8. *With assumptions as in Corollary 12.6, suppose that* $T_D^1(X) = \mathcal{O}_Z$, *where* Z *is the singular locus of* X. *Then there is a smoothing* $f : \mathcal{X} \to \Delta$ *of* X *such that*

(1) $\mathcal{X}$ *is smooth if* $D = \operatorname{Def}(X)$ *and*
(2) *the singularities of* $\mathcal{X}$ *are smooth quotients if* $D = \operatorname{Def}^{qG}(X)$.

There is one nice and simple case when $T^1(X)$ is finitely generated by its global sections.

COROLLARY 12.9. *Let* X *be a projective local complete intersection field over a field* k *of characteristic 0. Let* $X \subset Y$ *be an embedding such that* Y *is smooth. Suppose that* $\mathcal{N}_{X/Y}$ *is finitely generated by its global sections and that* $H^1(T^1(X)) = H^2(T_X) = 0$. *Then* X *is formally smoothable.*

Proof. Dualizing the conormal sequence for $X \subset Y$ yields a surjection

$$\mathcal{N}_{X/Y} \to T^1(X) \to 0.$$

Hence $T^1(X)$ is finitely generated by its global sections, too, and so X is formally smoothable. □

Next we give a similar criterion for $\mathbb{Q}$-Gorenstein deformations.

COROLLARY 12.10. *Let* X *be a projective* $\mathbb{Q}$*-Gorenstein scheme defined over a field* k *of characteristic 0. Suppose that its Gorenstein points are complete intersections and that the high-index points are complete intersection quotients. Let* $X \subset Y$ *be an embedding such that, locally around any point* $P \in X$, $P \in Y$ *is a general deformation of* $P \in X$. *Suppose that* $\mathcal{N}_{X/Y}$ *is finitely generated by its global sections and that* $H^1(T^1_{qG}(X)) = H^2(T_X) = 0$. *Then* X *has a* $\mathbb{Q}$*-Gorenstein smoothing.*

Proof. Dualizing the conormal sequence for $X \subset Y$, we obtain a sequence

$$\mathcal{N}_{X/Y} \xrightarrow{\phi} T^1(X) \to \underline{\mathrm{Ext}}^1_X(\Omega_Y \otimes \mathcal{O}_X, \mathcal{O}_X) \to 0.$$

We claim that $\mathrm{Im}(\phi) = T^1_{qG}(X)$ and hence if $\mathcal{N}_{X/Y}$ is generated by global sections then so is $T^1_{qG}(X)$. The claim is local at the singularities of Y, so we may assume that Y is affine. By assumption, Y is smooth at any index-1 point, and in this case we are done. Asssume then that Y has index $r > 1$. Let $\pi : \tilde{Y} \to Y$ be the index-1 cover; then $\tilde{X} = \pi^{-1}(X)$ is the index-1 cover of X. Moreover, since Y is the general deformation of X by assumption, it follows that $\tilde{Y}$ is smooth and hence there is a surjection

$$\mathcal{N}_{\tilde{X}/\tilde{Y}} \to T^1(\tilde{X}) \to 0.$$

Let G be the Galois group of the π. Then taking invariants yields

$$\mathcal{N}^G_{\tilde{X}/\tilde{Y}} = \mathcal{N}_{X/Y} \to T^1_{qG}(X) \to 0$$

as claimed. □

In general, if $X \subset Y$ and if $\mathcal{N}_{X/Y}$ is finitely generated by its global sections (or, even better, is ample), then X has nice deformation properties. Considering cases with respect to the singularities of X (like normal crossings) and the shape of the singular locus of X, one can derive various kinds of criteria—similar to the previous corollary for the smoothability of X—without even referring to $T^1(X)$. Let Z be the singular locus of X. In general, $T^1(X)$ is not a sheaf of $\mathcal{O}_Z$-modules. It usually has an embedded part; in fact, sometimes even Z is an embedded component of its support (this happens, for instance, if X is given by $xy + z^n = 0$ in $\mathbb{C}^4$, $n \geq 3$). So it is rather difficult to describe $T^1(X)$ directly and to check whether it is generated by its global sections. However, if the singular locus of X is 1-dimensional, then it is possible to give criteria for the finite generation of $T^1(X)$ without any reference to its embedded part.

THEOREM 12.11. *Let X be a projective scheme with singularities as in Theorem 12.5, and let Z be its reduced singular locus. Suppose that* $\dim Z = 1$ *and that*:

(1) $p(I^k_Z T^1_D(X)/I^{k+1}_Z T^1_D(X))$ *is generated by its global sections for all* $k \geq 0$;
(2) $H^1(p(I^k_Z T^1_D(X)/I^{k+1}_Z T^1_D(X))) = 0$ *for all* $k \geq 0$;
(3) $H^1(p(T^1_D(X) \otimes \mathcal{O}_Z)) = H^2(T_X) = 0$.

Then X is D-formally smoothable.

Proof. Let $\mathcal{F}_k \subset I^k_Z T^1_D(X)/I^{k+1}_Z T^1_D(X)$ be the maximal 0-dimensional subsheaf of $I^k_Z T^1_D(X)/I^{k+1}_Z T^1_D(X)$. Then there is an exact sequence

$$0 \to H^0(\mathcal{F}_k) \to H^0(I^k_Z T^1_D(X)/I^{k+1}_Z T^1_D(X)) \to H^0(p(I^k_Z T^1_D(X)/I^{k+1}_Z T^1_D(X))) \to 0.$$

Hence, if $p(I_Z^k T_D^1(X)/I_Z^{k+1} T_D^1(X))$ is generated by its global sections, then so is $I_Z^k T_D^1(X)/I_Z^{k+1} T_D^1(X)$. There are also exact sequences

$$0 \to I_Z^k T_D^1(X)/I_Z^{k+1} T_D^1(X) \to T_D^1(X)/I_Z^{k+1} T_D^1(X) \to T_D^1(X)/I_Z^k T_D^1(X) \to 0$$

for all $k \geq 0$. By induction, then, $T_D^1(X)/I_Z^k T_D^1(X)$ is finitely generated by its global sections for all k. But since $T_D^1(X)$ is supported on Z, it follows that $I_Z^m T_D^1(X) = 0$ for m sufficiently large. Hence $T_D^1(X)$ is finitely generated by its global sections and so, by Theorem 12.5, X is D-smoothable. □

If X has normal crossing singularities at any generic point of its singular locus, then $p(T^1(X))$ is an $\mathcal{O}_Z$-module and hence one need only take $k = 0$ in the conditions of the theorem.

COROLLARY 12.12. *With assumptions as in the previous theorem, suppose in addition that X has normal crossing singularities at any generic point of its singular locus, that $p(T_D^1(X))$ is finitely generated by its global sections, and that $H^1(p(T_D^1(X))) = H^2(T_X) = 0$. Then X is smoothable.*

13. Examples

In this section we apply the theory developed in the previous parts of the paper to give some examples from the theory of moduli spaces of stable surfaces and the 3-dimensional minimal model program.

1. In this example we construct a few classes of locally but not globally smoothable stable surfaces with normal crossing singularities. This means that the irreducible components of the moduli space of stable surfaces that they belong to do not contain any smooth surfaces of general type. Hence these are extra components that appear after the moduli space of surfaces of general type is compactified by adding the stable surfaces.

1.1. Let X be a projective surface with exactly one singular point P such that:

a. $K_X = kA$, where A is very ample and $k \geq 2$ is an integer;
b. $P \in X$ is analytically isomorphic to the cone over a smooth projective plane curve of degree 4.

Note that such surfaces do exist—for example, $X \subset \mathbb{P}^3$ given by $(x_0^2 + x_3^2)x_0^4 + (x_1^2 + x_3^2)x_1^4 + (x_2^2 + x_3^2)x_2^4 = 0$.

Let $f\colon Y \to X$ be the blowup of X along P. Then Y is smooth and the f-exceptional divisor is a smooth curve $E \subset \mathbb{P}^2$ of degree 4 such that $\mathcal{N}_{E/Y} = \mathcal{O}_E(-1)$ and hence $E^2 = -4$. Moreover, a straightforward calculation shows that

$$K_Y = f^* K_X - 2E.$$

Let Z be obtained by glueing two copies of Y along E. This is a surface with normal crossing singularities, and we claim that K_Z is ample and Z is not smoothable.

By [Fr] or [Tz1], $T^1(Z) = \mathcal{N}_{E/Y} \otimes \mathcal{N}_{E/Y} = \mathcal{O}_E(-2)$ and so $H^0(T^1(X)) = 0$. Hence, by Theorem 12.3, Z is not smoothable.

Next we show that K_Z is ample. For this it suffices to show that $K_Z|_{Z_i}, i = 1, 2$, is ample, where $Z_i \cong Y$ are the irreducible components of Z. It is not difficult to see that

$$K_Z|_{Z_i} = K_Y + E = f^*K_X - E.$$

This is ample if and only if $(f^*K_X - E)^2 > 0$ and $(f^*K_X - E) \cdot D > 0$ for any irreducible curve $D \subset Y$. Now

$$(f^*K_X - E)^2 = K_X^2 - 4 = k^2A^2 - 4 > 0,$$

since $k \geq 2$ and A is very ample; therefore, $A^2 > 1$. Let $D \subset Y$ be an irreducible curve and let $C = f_*D$. Then

$$(f^*K_X - E) \cdot D = K_X \cdot C - E \cdot D = kA \cdot C - m_P(C) = k \deg(C) - m_P(C),$$

where $m_P(C)$ is the multiplicity of C at P and $\deg C$ is the degree of C with respect to the embedding defined by A. Then $\deg(C) \geq m_P(C)$ and hence, since $k \geq 2$, it follows that

$$(f^*K_X - E) \cdot D > 0.$$

Therefore, K_Z is ample as claimed.

1.2. Let X be a smooth projective surface with K_X ample. Suppose that X contains a smooth curve C with $p_a(C) \geq 2$ and $K_X \cdot C > 2(p_a(C) - 1)$.

Such surfaces do exist. For example, let $C \subset \mathbb{P}^3$ be a smooth plane curve of degree $k \geq 4$ given by $f_k(x, y, z) = t = 0$, where $f_k(x, y, z)$ is a homogeneous polynomial of degree $k \geq 4$, and let $X \subset \mathbb{P}^3$ be the hypersurface of degree $d > k + 1$ given by

$$g_{d-k}(x, y, z, t) f_k(x, y, z) + th_{d-1}(x, y, z, t) = 0,$$

where $g_{d-k}(x, y, z, t)$ and $h_{d-1}(x, y, z, t)$ are homogeneous polynomials of degrees $d - k$ and $d - 1$, respectively. For general choice of $g_{d-k}(x, y, z, t)$ and $h_{d-1}(x, y, z, t)$, X is a smooth surface containing C. Furthermore, $\mathcal{O}_X(K_X) = \mathcal{O}_X(d - 4)$ and hence

$$K_X \cdot C = \deg \mathcal{O}_C(d - 4) = (d - 4)k > k^2 - 3k = 2(p_a(C) - 1),$$

since $d > k + 1$ and $k \geq 4$. Moreover, K_X is ample.

Let Z be obtained by glueing two copies of X along C. This is a surface with normal crossing singularities, and we claim that K_Z is ample and Z is not smoothable.

Let Z_1, Z_2 be the two irreducible components of Z. By construction, $Z_i \cong Z$, $i = 1, 2$. Then K_Z is ample if and only if $K_Z|_{Z_i}$ is ample, $i = 1, 2$. As in the previous example, $K_Z|_{Z_i} = K_X + C$. By construction, $K_Z + C$ is ample and hence K_Z is ample.

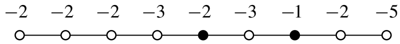
By adjunction, $\mathcal{N}_{C/X} = \omega_C \otimes \omega_X^{-1}$ and therefore

$$\deg \mathcal{N}_{C/X} = 2p_a(C) - 2 - K_X \cdot C < 0,$$

by assumption. As in the previous example, $T^1(X) = \mathcal{N}_{C/X} \otimes \mathcal{N}_{C/X}$. Hence $T^1(X)$ is a line bundle on C of negative degree. Therefore, $H^0(T^1(X)) = 0$ and, by Theorem 12.3, X is not smoothable.

2. In this example we construct a terminal 3-fold divisorial extremal neighborhood $f\colon Y \to X$ such that the general member of $|\mathcal{O}_Y|$ is not normal.

Let U be the germ of a smooth surface around the configuration of rational curves



Let $h\colon U \to \tilde{Z}$ be the contraction of all the curves except for those marked by a solid circle. Then (i) we get a map $\tilde{f} : \tilde{Z} \to T$ contracting two smooth rational curves C_1 and C_2 to a point $0 \in T$ such that $0 \in T$ is an A_5 singularity and (ii) $\tilde{Z}$ has exactly three singular points $P_1 \in C_1$, $P_2 \in C_2$, and $Q = C_1 \cap C_2$. It is easy to see that $(P_1 \in \tilde{Z}) \cong 1/9(1,5)$, $(P_2 \in \tilde{Z}) \cong 1/9(1,-5)$, and $(Q \in \bar{Z}) \cong 1/3(1,1)$. Let Z be obtained from $\tilde{Z}$ by identifying C_1 and C_2 via an involution of $C_1 + C_2$ taking P_1 to P_2 and leaving Q fixed. Let $\pi : \tilde{Z} \to Z$ be the quotient map. Then the singular locus of Z is a smooth rational curve C; $\pi^{-1}(C) = C_1 + C_2$; and Z has one singularity analytically isomorphic to $(xy = 0)/\mathbb{Z}(5,-5,1)$, has one degenerate cusp analytically isomorphic to $x^3 + y^3 + xyz = 0$, and has normal crossing singularities at all other singular points. Moreover, $\tilde{Z}$ is the normalization of Z, and there is a natural morphism $f\colon Z \to T$ contracting C to $0 \in T$.

Straightforward calculations show that $K_Z \cdot C = -1/9 < 0$. Also, since U is the minimal log-resolution of $C \subset Z$, it follows from [Tz1] that $\deg p(T^1_{qG}(Z)) = -2 - 1 + 1 + 3 = 1$ and hence we have $p(T^1_{qG}(Z)) = \mathcal{O}_{\mathbb{P}^1}(1)$. Thus, by Corollary 12.12, there exists a $\mathbb{Q}$-Gorenstein smoothing $Y \to \Delta$ of Z. Then f extends to a morphism $g\colon Y \to X$ over Δ, where X is a deformation of T [KoMo]. Now $g\colon Y \to X$ is a 3-fold extremal neighborhood and $Z \in |\mathcal{O}_Y|$ is the general member. Moreover, the neighborhood is divisorial because X is Gorenstein.

Finally, observe that the method of producing 3-fold extremal neighborhoods by deforming birational surface morphisms $f\colon Z \to T$ is fundamental in the classification of flips by Kollár and Mori [KoMo]. In principle, it could be used in higher dimensions to understand higher-dimensional flips and divisorial contractions.

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Department of Mathematics
University of Cyprus
P.O. Box 20537
Nicosia, 1678
Cyprus

tziolas@ucy.ac.cy