On Fano Manifolds with a Birational Contraction Sending a Divisor to a Curve

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1. Introduction

Let X be a smooth, complex Fano variety of dimension n. The Picard number ρ_X of X is equal to the second Betti number of X, and is bounded in any fixed dimension, because X can vary only in a finite number of families (see [De, Chap. 5] and references therein). If $n = 3$ and $\rho_X \geq 6$, then $X \cong S \times \mathbb{P}^1$ where S is a Del Pezzo surface, so that $\rho_X \leq 10$ [MoMu, Thm. 2]. Starting from dimension 4, the maximal value of ρ_X is unknown.

Let's assume that $n \geq 4$. Bounds on the Picard number are known when X has some special extremal contraction. For instance, if X has a birational elementary contraction sending a divisor to a point, then $\rho_X \leq 3$ ([T2, Prop. 5]; see also Proposition 3.1). In fact such X are classified in the toric case [Bo], in the case of a blow-up of a point [BoCamW], and more generally when the exceptional divisor is $\mathbb{P}^{n-1}$ [T2]. Concerning the fiber type case, we know that $\rho_X \leq 11$ when X has an elementary contraction onto a surface or a 3-fold [Ca2, Thm. 1.1].

In this paper we consider the case of a birational elementary contraction of type $(n - 1, 1)$—that is, sending a divisor to a curve. Such Fano varieties have been classified in the toric case by Sato [S], and Tsukioka [T1; T3] has obtained classification results for some cases (see Remark 4.3). Our main result is the following.

THEOREM 1.1.   *Let X be a smooth Fano variety of dimension n $\geq$ 4, and suppose that X has a birational elementary contraction sending a divisor E to a curve.*

*   Then $\rho_X \leq 5$. Moreover, if $\rho_X = 5$ then we have $E \cong W \times \mathbb{P}^1$ for W a smooth Fano variety, and there exist*:

- *a smooth projective variety Y, with $\rho_Y = 4$, such that X is the blow-up of Y in a subvariety isomorphic to W with exceptional divisor E; and*
- *a smooth Fano variety Z, with $\rho_Z = 3$, having a birational elementary contraction sending a divisor E_Z to a curve and such that X is the blow-up of Z in two fibers of this contraction and E is the proper transform of E_Z.*

This theorem follows from Theorem 4.2 and Proposition 4.8. There are examples with $\rho_X = 5$ in every dimension $n \geq 4$; see Example 4.10. In dimension 4, we get the following.

COROLLARY 1.2. *Let X be a smooth Fano 4-fold, and suppose that X has a bi-rational elementary contraction sending a divisor E to a curve.*
Then $\rho_X \le 5$, and if $\rho_X = 5$ then we have one of the following possibilities:

(i) $E \cong \mathbb{P}^2 \times \mathbb{P}^1$, $\mathcal{N}_{E/X} \cong \mathcal{O}(-1,-1)$;
(ii) $E \cong \mathbb{P}^2 \times \mathbb{P}^1$, $\mathcal{N}_{E/X} \cong \mathcal{O}(-2,-1)$;
(iii) $E \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{N}_{E/X} \cong \mathcal{O}(-1,-1,-1)$, *and two of the rulings are numerically equivalent in X.*

Finally, we sum up what we know about the Picard number of Fano 4-folds with respect to their elementary contractions. Notice that the known examples with largest Picard number are products of Del Pezzo surfaces and have $\rho_X = 18$.

We recall that a contraction φ is *of type (a, b)* if

$$\dim \operatorname{Exc}(\varphi) = a \quad \text{and} \quad \dim \varphi(\operatorname{Exc}(\varphi)) = b,$$

where $\operatorname{Exc}(\varphi)$ is the locus where φ is not an isomorphism.

COROLLARY 1.3. *Let X be a smooth Fano 4-fold with $\rho_X \ge 4$. The possible elementary contractions of X are described in the following table, where S is a Del Pezzo surface and the question mark indicates the absence of any examples.*

Type	$(4, 2)$	$(4, 3)$	$(3, 1)$	$(3, 2)$	$(2, 0)$
$\rho_X = 4, 5$	*only if $X \cong \mathbb{P}^2 \times S$*	*yes*	*yes*	*yes*	*yes*
$\rho_X = 6$	*only if $X \cong \mathbb{P}^2 \times S$*	*yes*	*no*	*yes*	*yes*
$7 \le \rho_X \le 10$	*only if $X \cong \mathbb{P}^2 \times S$*	*only if $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times S$ or $\mathbb{F}_1 \times S$*	*no*	*yes*	*?*
$\rho_X = 11$	*no*	*only if $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times S$ or $\mathbb{F}_1 \times S$*	*no*	*yes*	*?*
$\rho_X \ge 12$	*no*	*no*	*no*	*yes*	*?*

We outline the technique used to prove Theorem 1.1. Given the divisor E, the classical approach is to choose an extremal ray R of $\operatorname{NE}(X)$ such that $E \cdot R > 0$ and then study the associated contraction. However, this is not enough to get a bound on ρ_X in all cases, in particular when R is small. One must iterate this procedure and run a "Mori program" for $-E$, that is, contract or flip birational extremal rays having positive intersection with E until one gets a fiber type contraction. This is possible thanks to [BCHM], where it is shown that Fano varieties are Mori dream spaces, and to [HuK], where properties of Mori dream spaces are studied.

In Section 3, we use this method to study a Fano variety X containing a prime divisor D such that the numerical classes of curves contained in D span a 2-dimensional linear subspace in $\mathcal{N}_1(X)$. This is enough to get $\rho_X \le 3$ in some cases (see Theorem 3.2).

In Section 4 we consider the exceptional divisor E of an elementary contraction φ of type $(n-1, 1)$. We apply to E the results of the preceding section, and we need a detailed analysis of the geometry of E and X to conclude. We first show that if there is a unique extremal ray having negative intersection with E (corresponding

to φ), then $\rho_X \leq 4$ (Theorem 4.2). Then we consider the case where there is a second extremal ray R such that $E \cdot R < 0$ and show that $\rho_X \leq 5$ (Proposition 4.8). Finally, we give some examples with $\rho_X = 5$.

2. Preliminaries

In this section we recall some notions and results that we need in the sequel.

CONTRACTIONS.   Let X be a normal irreducible variety of dimension n. A *contraction* of X is a projective morphism $\varphi \colon X \to Y$, with connected fibers, onto a normal variety Y (without hypotheses on the anticanonical degree of curves in fibers).

Suppose that X has terminal singularities, so that K_X is $\mathbb{Q}$-Cartier. A contraction φ is a *Mori contraction* if $-K_X$ is φ-ample.

NUMERICAL EQUIVALENCE CLASSES AND THE CONE OF CURVES.   Let X be an irreducible projective variety. We denote by $\mathcal{N}_1(X)$ the vector space of 1-cycles in X, with real coefficients, modulo numerical equivalence; its dimension is the Picard number ρ_X of X. The cone of curves $\mathrm{NE}(X)$ is the convex cone in $\mathcal{N}_1(X)$ generated by numerical classes of effective curves, and $\overline{\mathrm{NE}}(X)$ is its closure in $\mathcal{N}_1(X)$. We denote by $[C] \in \mathrm{NE}(X)$ the numerical class of a curve $C \subset X$.

If R is a half-line in $\mathcal{N}_1(X)$ and D a $\mathbb{Q}$-Cartier divisor in X, we will say that $D \cdot R > 0$, $D \cdot R = 0$, or $D \cdot R < 0$ if, for any nonzero element $\gamma \in R$, we have (respectively) $D \cdot \gamma > 0$, $D \cdot \gamma = 0$, or $D \cdot \gamma < 0$.

If $\varphi \colon X \to Y$ is a contraction, then the push-forward of 1-cycles gives a surjective linear map

$$\varphi_* \colon \mathcal{N}_1(X) \longrightarrow \mathcal{N}_1(Y),$$

and we set $\mathrm{NE}(\varphi) := \mathrm{NE}(X) \cap \ker \varphi_*$. We say that φ is *elementary* if $\rho_X - \rho_Y = 1$.

Suppose that X is $\mathbb{Q}$-factorial and that φ is elementary with $\dim \mathrm{Exc}(\varphi) = n - 1$. Then $\mathrm{Exc}(\varphi)$ is an irreducible divisor and $\mathrm{Exc}(\varphi) \cdot \mathrm{NE}(\varphi) < 0$.

For any irreducible closed subset Z of X, let $i \colon Z \hookrightarrow X$ be the inclusion, and consider the push-forward of 1-cycles $i_* \colon \mathcal{N}_1(Z) \to \mathcal{N}_1(X)$. We define

$$\mathcal{N}_1(Z, X) := i_*(\mathcal{N}_1(Z)) \subseteq \mathcal{N}_1(X).$$

Equivalently, $\mathcal{N}_1(Z, X)$ is the linear subspace of $\mathcal{N}_1(X)$ spanned by classes of curves contained in Z. Working with $\mathcal{N}_1(Z, X)$ instead of $\mathcal{N}_1(Z)$ means that we consider curves in Z modulo numerical equivalence in X, instead of numerical equivalence in Z. Notice that $\dim \mathcal{N}_1(Z, X) \leq \rho_Z$.

ONE-DIMENSIONAL FIBERS IN MORI CONTRACTIONS.   The following theorem collects results due to several people; see [AW1, Lemma 2.12, Thm. 4.1] and the references therein. Notice that X_0 does not need to be complete.

THEOREM 2.1.   *Let X_0 be a smooth variety, $\varphi_0 \colon X_0 \to Y_0$ a Mori contraction, and F a fiber of φ_0 having a 1-dimensional irreducible component F_0. Then Y_0 is*

smooth in $\varphi_0(F)$ *and either* (i) $F = F_0 \cong \mathbb{P}^1$ *or* (ii) φ_0 *is of fiber type and F has two irreducible components, both of which are isomorphic to $\mathbb{P}^1$.*

Suppose in particular that every fiber of φ_0 has dimension at most 1, so that Y_0 is smooth. If φ_0 is of fiber type, we will say that φ_0 is a *conic bundle*. If instead φ_0 is birational, then it is the blow-up of a smooth, codimension-2 subvariety of Y_0; we will say that φ_0 is of type $(n-1, n-2)^{\mathrm{sm}}$.

Concerning the singular case, we have the following.

THEOREM 2.2 [I, Lemma 1.1]. *Let X be a projective variety with terminal singularities, and let $\varphi\colon X \to Y$ be an elementary birational Mori contraction with fibers of dimension at most 1. If F_0 is an irreducible component of a nontrivial fiber of φ and if F_0 contains a Gorenstein point of X, then $F_0 \cong \mathbb{P}^1$ and $-K_X \cdot F_0 \leq 1$.*

FANO VARIETIES AND MORI DREAM SPACES.   The notion of Mori dream space was introduced and studied in [HuK]. In that paper the authors show that Fano 3-folds are Mori dream spaces [HuK, Cor. 2.16] and conjecture the same to hold in arbitrary dimension. This was confirmed in [BCHM] as an application of fundamental results on the minimal model program.

THEOREM 2.3 [BCHM, Cor. 1.3.1]. *Any smooth Fano variety is a Mori dream space.*

(In fact one can also allow singularities, but here we consider only the smooth case.)

Being a Mori dream space implies many important features with respect to Mori theory. In the following remarks we recall some consequences of Theorem 2.3 that will be used in the sequel.

REMARK 2.4.   Let X be a smooth Fano variety, and let $X \dashrightarrow Y$ be a "rational contraction" in the sense of [HuK]. This means that there exist a normal and $\mathbb{Q}$-factorial projective variety X' and a factorization

$$X \dashrightarrow X' \longrightarrow Y$$

such that $X \dashrightarrow X'$ is an isomorphism in codimension 1 and $X' \longrightarrow Y$ is a contraction.

Many well-known properties of X hold for Y, too. The Mori cone $\mathrm{NE}(Y)$ is closed and polyhedral. For any contraction $\psi\colon Y \to Z$, $\mathrm{NE}(\psi)$ is a face of $\mathrm{NE}(Y)$ that determines ψ uniquely. Conversely, for every face F of $\mathrm{NE}(Y)$ there exists a contraction ψ of Y such that $F = \mathrm{NE}(\psi)$. Finally, ψ is elementary if and only if $\mathrm{NE}(\psi)$ has dimension 1; we will call a 1-dimensional face of $\mathrm{NE}(Y)$ an *extremal ray*.

This follows from the very definition of Mori dream space. Indeed, X' is a "small $\mathbb{Q}$-factorial modification of X"; thus, by [HuK, Def. 1.10, Prop. 1.11(2)], the foregoing properties hold for X'. Then it is not difficult to deduce the same for Y.

If $R = \mathrm{NE}(\psi)$ is an extremal ray of $\mathrm{NE}(Y)$, then we say that R is birational, divisorial, small, of fiber type, or of type (a, b) if the contraction ψ is. Moreover, we set $\mathrm{Locus}(R) := \mathrm{Exc}(\psi)$.

Consider the special case where $\varphi \colon X \to Y$ is an elementary contraction. Then the extremal rays of $\mathrm{NE}(Y)$ are in bijection (via φ_*) with the 2-dimensional faces of $\mathrm{NE}(X)$ containing the ray $\mathrm{NE}(\varphi)$; see [Ca2, 2.5].

REMARK 2.5. Let Y be as in Remark 2.4. Suppose that Y is $\mathbb{Q}$-factorial, and consider a prime divisor $D \subset Y$. There exists at least one extremal ray of $\mathrm{NE}(Y)$ having positive intersection with D. Looking at the associated contraction, one finds an elementary contraction

$$\psi \colon Y \longrightarrow Z$$

such that $D \cdot \mathrm{NE}(\psi) > 0$; in particular, D intersects every nontrivial fiber of ψ.

If ψ is of fiber type, then $\psi(D) = Z$; hence

$$\psi_*(\mathcal{N}_1(D, Y)) = \mathcal{N}_1(Z)$$

and we have $\rho_Z \leq \dim \mathcal{N}_1(D, Y)$ and $\rho_Y \leq \dim \mathcal{N}_1(D, Y) + 1$.

If ψ is birational, then $\mathrm{Exc}(\psi) \cap D \neq \emptyset$; however, $\mathrm{Exc}(\psi) \neq D$ (for otherwise $D \cdot \mathrm{NE}(\psi) < 0$) and so $\psi(D) \subset Z$ is a divisor. We have two possibilities: either $\mathrm{NE}(\psi) \subset \mathcal{N}_1(D, Y)$ and $\dim \mathcal{N}_1(\psi(D), Z) = \dim \mathcal{N}_1(D, Y) - 1$, or $\mathrm{NE}(\psi) \not\subset \mathcal{N}_1(D, Y)$ and $\dim \mathcal{N}_1(\psi(D), Z) = \dim \mathcal{N}_1(D, Y)$. In this last case ψ must be finite on D, and it follows that every nontrivial fiber of ψ is a curve.

REMARK 2.6. Let X be a smooth Fano variety and D a prime divisor in X. By [HuK, Prop. 1.11(1)] there exists a finite sequence

$$X = X_0 \xrightarrow{\sigma_0} X_1 \dashrightarrow \cdots \dashrightarrow X_{k-1} \xrightarrow{\sigma_{k-1}} X_k, \tag{2.7}$$

where:

- every X_i is projective, normal, and $\mathbb{Q}$-factorial;
- for $i = 0, \ldots, k - 1$, if $D_i \subset X_i$ is the proper transform of D then there exists a birational extremal ray R_i of $\mathrm{NE}(X_i)$ such that (i) $D_i \cdot R_i > 0$ and (ii) σ_i is either the contraction of R_i (if divisorial) or its flip (if small);
- there exists an extremal ray of fiber type R_k of $\mathrm{NE}(X_k)$ with $D_k \cdot R_k > 0$.

See [KoMo, Def. 3.33, Def. 6.5] for the definition of flip. In the terminology of [KoMo; HuK] we are considering $(-D)$-flips, and (2.7) is a Mori program for $-D$: since D is effective, $-D$ can never become nef and so the program necessarily ends with a fiber type contraction. Notice that the choice of the extremal rays R_i is arbitrary among those that have positive intersection with D_i.

3. Divisors with Picard Number 2

Let X be a smooth Fano variety and let $D \subset X$ be a prime divisor. We recall that $\mathcal{N}_1(D, X)$ is the linear subspace of $\mathcal{N}_1(X)$ spanned by classes of curves contained in D, so that $\dim \mathcal{N}_1(D, X) \leq \rho_D$. The following result is proven in [T2] under the assumption that $\rho_D = 1$; however the proof can be easily adapted to the case $\dim \mathcal{N}_1(D, X) = 1$ (see [Ca2, Prop. 3.16]).

PROPOSITION 3.1 [T2, Prop. 5].   *Let X be a smooth Fano variety of dimension $n \geq 3$, and let $D \subset X$ be a prime divisor with $\dim \mathcal{N}_1(D, X) = 1$. Then $\rho_X \leq 3$.*

In particular, we have that $\rho_X \leq 3$ when X has an elementary contraction of type $(n-1, 0)$.

   In this section we consider the case where $\dim \mathcal{N}_1(D, X) = 2$. Our goal is to prove the following two results, which give a bound on ρ_X in some cases.

THEOREM 3.2.   *Let X be a smooth Fano variety of dimension $n \geq 3$, and let $D \subset X$ be a prime divisor with $\dim \mathcal{N}_1(D, X) = 2$. Let $\varphi \colon X \to Y$ be an elementary contraction of X with $D \cdot \mathrm{NE}(\varphi) > 0$. Then one of the following holds:*

   (i) $\rho_X = 2$;
  (ii) $\rho_X = 3$ *and φ is either a conic bundle or of type $(n-1, 0)$ or of type $(n-1, n-2)^{\mathrm{sm}}$ or small*;
 (iii) *φ is of type $(n-1, n-2)^{\mathrm{sm}}$ and $\mathrm{NE}(\varphi) \not\subset \mathcal{N}_1(D, X)$*;
 (iv) *φ is small and there exists a smooth prime divisor $D' \subset X$, disjoint from $\mathrm{Exc}(\varphi)$ and with a $\mathbb{P}^1$-bundle structure, such that for any fiber f we have $D' \cdot f = -1$, $D \cdot f > 0$, and $f \not\subset D$.*

In the last case, we do not know whether the numerical class $[f]$ lies on an extremal ray of $\mathrm{NE}(X)$. However X is the blow-up of a (possibly nonprojective) complex manifold in a smooth codimension-2 subvariety with exceptional divisor D'.

LEMMA 3.3.   *Let X be a smooth Fano variety of dimension $n \geq 3$, and let $D \subset X$ be a prime divisor with $\dim \mathcal{N}_1(D, X) = 2$. Suppose there exists an elementary divisorial contraction $\varphi \colon X \to Y$ such that $D \cdot \mathrm{NE}(\varphi) = 0$ and $\mathrm{Exc}(\varphi) \cap D \neq \emptyset$.*
   Then either $\rho_X \leq 4$ or there exists an extremal ray $R \neq \mathrm{NE}(\varphi)$, of type $(n-1, n-2)^{\mathrm{sm}}$, such that $R \cdot \mathrm{Exc}(\varphi) < 0$ and $R + \mathrm{NE}(\varphi)$ is a face of $\mathrm{NE}(X)$.

Observe that if X is a toric Fano variety and if $D \subset X$ is a prime divisor that is closed with respect to the torus action, then $\rho_X \leq 3 + \dim \mathcal{N}_1(D, X) = 3 + \rho_D$ by [Ca1, Thm. 2.4]; in particular, $\rho_X \leq 5$ when $\dim \mathcal{N}_1(D, X) = 2$. However in general one can not expect a similar bound, as the following example shows.

EXAMPLE 3.4.   Consider a Del Pezzo surface S with $\rho_S = 9$, and let $X = S \times \mathbb{P}^{n-2}$. Then $\rho_X = 10$ and X contains divisors $D = C \times \mathbb{P}^{n-2}$, where $C \subset S$ is an irreducible curve, with $\dim \mathcal{N}_1(D, X) = 2$.

Before proving Theorem 3.2 and Lemma 3.3, we need some preliminary properties. We fix a smooth Fano variety X of dimension $n \geq 3$ and a prime divisor $D \subset X$, and we carry out Mori's program for $-D$ as explained in Remark 2.6. We stop at X_m when we get either a contraction of fiber type or a birational extremal ray R_m that is not contained in $\mathcal{N}_1(D_m, X_m)$. Thus we obtain a sequence analogous to (2.7):
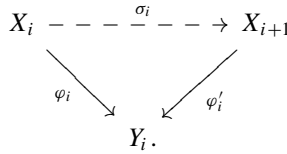
$$X = X_0 \xrightarrow{\;\sigma_0\;} X_1 \dashrightarrow \cdots \dashrightarrow X_{m-1} \xrightarrow{\;\sigma_{m-1}\;} X_m, \qquad\qquad (3.5)$$

where moreover $R_i \subset \mathcal{N}_1(D_i, X_i)$ for $i = 0, \ldots, m-1$ and there exists an extremal ray R_m of $\mathrm{NE}(X_m)$ with $D_m \cdot R_m > 0$ that is either of fiber type or birational with $R_m \not\subset \mathcal{N}_1(D_m, X_m)$.

LEMMA 3.6.   *For every $i = 0, \ldots, m-1$, we have*:

$$\dim \mathcal{N}_1(D_{i+1}, X_{i+1}) = \begin{cases} \dim \mathcal{N}_1(D_i, X_i) - 1 & \text{if } R_i \text{ is divisorial}; \\ \dim \mathcal{N}_1(D_i, X_i) & \text{if } R_i \text{ is small}. \end{cases}$$

Proof. By construction we have $R_i \subset \mathcal{N}_1(D_i, X_i)$, so the statement is clear if R_i is divisorial. Suppose that R_i is small; let $\varphi_i \colon X_i \to Y_i$ be its contraction, let $\varphi_i' \colon X_{i+1} \to Y_i$ be the flip of φ_i, and let $R_i' := \mathrm{NE}(\varphi_i')$:

$$X_i \; - - - \overset{\sigma_i}{-} - - \to X_{i+1}$$

$$\varphi_i \searrow \qquad \swarrow \varphi_i'$$

$$Y_i .$$

Then $D_{i+1} \cdot R_i' < 0$ (see [KoMo, Cor. 6.4(4)]); hence $R_i' \subset \mathcal{N}_1(D_{i+1}, X_{i+1})$. This implies the statement because $\varphi_i(D_i) = \varphi_i'(D_{i+1})$ and

$$\dim \mathcal{N}_1(D_i, X_i) = \dim \mathcal{N}_1(\varphi_i(D_i), Y_i) + 1 = \dim \mathcal{N}_1(D_{i+1}, X_{i+1}). \qquad \square$$

COROLLARY 3.7.   *Suppose that in (3.5) the ray R_m is of fiber type. Then*

$$\rho_X \leq 1 + \dim \mathcal{N}_1(D, X).$$

Proof. We have

$$\rho_{X_{i+1}} = \begin{cases} \rho_{X_i} - 1 & \text{if } R_i \text{ is divisorial}; \\ \rho_{X_i} & \text{if } R_i \text{ is small}. \end{cases}$$

Therefore, by Lemma 3.6 it follows that $\rho_{X_i} - \dim \mathcal{N}_1(D_i, X_i)$ is constant; in particular, $\rho_X - \dim \mathcal{N}_1(D, X) = \rho_{X_m} - \dim \mathcal{N}_1(D_m, X_m)$. If R_m is of fiber type then $\rho_{X_m} \leq 1 + \dim \mathcal{N}_1(D_m, X_m)$ (see Remark 2.5), which gives the statement. $\square$

Let $A_1 \subset X_1$ be the indeterminacy locus of σ_0^{-1}, and for $i \in \{2, \ldots, m\}$ let $A_i \subset X_i$ be the union of the proper transform of $A_{i-1} \subset X_{i-1}$ with the indeterminacy locus of σ_{i-1}^{-1}. Then $X_i \setminus A_i$ is isomorphic to an open subset of X, and

$$\mathrm{Sing}(X_i) \subseteq A_i \subset D_i.$$

Notice, moreover, that $\dim A_i > 0$ whenever R_{i-1} is small.

LEMMA 3.8.   *Let $i \in \{1, \ldots, m\}$, and assume that $-K_{X_j} \cdot R_j > 0$ for every $j = 0, \ldots, i-1$. Then $X_1, \ldots, X_i$ have terminal singularities. Moreover, if $C \subset X_i$ is an irreducible curve not contained in A_i and if $C_0 \subset X$ is its proper transform, then*

$$-K_{X_i} \cdot C \geq -K_X \cdot C_0,$$

with strict inequality whenever $C \cap A_i \neq \emptyset$.

Proof. We assume that the statement holds for $i - 1$ and consider $\sigma_{i-1} \colon X_{i-1} \dashrightarrow X_i$. Suppose that σ_{i-1} is a flip, and consider a common resolution of X_{i-1} and X_i:

$$\begin{array}{ccc} & \hat{X} & \\ {}^{f}\swarrow & & \searrow^{g} \\ X_{i-1} \dashrightarrow[\sigma_{i-1}] & & X_i. \end{array}$$

Let $G_1, \ldots, G_r \subset \hat{X}$ be the exceptional divisors, and write

$$K_{\hat{X}} = f^*(K_{X_{i-1}}) + \sum_{j=1}^{r} a_j G_j = g^*(K_{X_i}) + \sum_{j=1}^{r} b_j G_j \quad \text{with } a_j, b_j \in \mathbb{Q}.$$

Since X_{i-1} has terminal singularities and since $-K_{X_{i-1}} \cdot R_{i-1} > 0$, it follows from [KoMo, Lemma 3.38] that $b_j \geq a_j > 0$ for every $j = 1, \ldots, r$; thus also X_i has terminal singularities.

The curve $C \subset X_i$ is not contained in A_i; hence C intersects the open subset where X_{i-1} and X_i are nonsingular and isomorphic. If $\tilde{C} \subset X_{i-1}$ and $\hat{C} \subset \hat{X}$ are the proper transforms of C, then $G_j \cdot \hat{C} \geq 0$ for every j and we have

$$-K_{X_i} \cdot C = -K_{X_{i-1}} \cdot \tilde{C} + \sum_{j=1}^{r} (b_j - a_j) G_j \cdot \hat{C} \geq -K_{X_{i-1}} \cdot \tilde{C} \geq -K_X \cdot C_0.$$

Now suppose that $C \cap A_i \neq \emptyset$. If $\tilde{C} \cap A_{i-1} \neq \emptyset$, then $-K_{X_{i-1}} \cdot \tilde{C} > -K_X \cdot C_0$. Otherwise $\tilde{C}$ must intersect $\text{Locus}(R_{i-1})$, so there exists a j_0 such that $f(G_{j_0}) \subseteq \text{Locus}(R_{i-1})$ and $G_{j_0} \cdot \hat{C} > 0$. Again by [KoMo, Lemma 3.38] we have $b_{j_0} > a_{j_0}$; therefore, $-K_{X_i} \cdot C > -K_{X_{i-1}} \cdot \tilde{C}$ and we are done.

The case where σ_i is a divisorial contraction is similar and shorter. $\square$

LEMMA 3.9. *Let $i \in \{1, \ldots, m\}$, and assume that $-K_{X_j} \cdot R_j > 0$ for every $j = 0, \ldots, i - 1$. Consider a birational elementary contraction $\psi \colon X_i \to Y$ such that $D_i \cdot \text{NE}(\psi) > 0$ and $\text{NE}(\psi) \not\subset \mathcal{N}_1(D_i, X_i)$.*

Then $\text{Exc}(\psi)$ is disjoint from A_i, and $\psi|_{X_i \setminus A_i}$ is a Mori contraction of type $(n - 1, n - 2)^{\text{sm}}$.

Proof. Let F be a nontrivial fiber of ψ. Then F must meet D_i, but ψ is finite on D_i. Thus F is a curve that intersects D_i in finitely many points; in particular, F cannot be contained in A_i.

Using Lemma 3.8, we see that $-K_{X_i} \cdot F > 0$ (namely, ψ is a Mori contraction); moreover, $\dim(F \cap \text{Sing}(X_i)) \leq 0$. We can now apply Theorem 2.2 to deduce that $-K_{X_i} \cdot F_0 \leq 1$ for any irreducible component F_0 of F.

Again by Lemma 3.8, this shows that F cannot intersect A_i. In particular, $\text{Exc}(\psi)$ is contained in the smooth locus of X_i, and the statement follows. $\square$

LEMMA 3.10. *Let $i \in \{1, \ldots, m\}$, and assume that $-K_{X_j} \cdot R_j > 0$ for every $j = 0, \ldots, i - 1$.*

If $\dim \mathcal{N}_1(D_i, X_i) = 1$ *and* $\dim A_i > 0$, *then* $i = m$, $\rho_{X_m} \leq 2$, *and every elementary contraction* $\psi \colon X_m \to Y$ *such that* $D_m \cdot \mathrm{NE}(\psi) > 0$ *is of fiber type. In particular,* R_m *is of fiber type.*

Proof. Let $\psi \colon X_i \to Y$ be an elementary contraction such that $D_i \cdot \mathrm{NE}(\psi) > 0$.

If ψ is birational then we cannot have $\mathrm{NE}(\psi) \subset \mathcal{N}_1(D_i, X_i)$, for otherwise $\psi(D_i)$ is a point and $D_i = \mathrm{Exc}(\psi)$, which contradicts $D_i \cdot \mathrm{NE}(\psi) > 0$. On the other hand, if $\mathrm{NE}(\psi) \not\subset \mathcal{N}_1(D_i, X_i)$, then Lemma 3.9 implies that $\mathrm{Exc}(\psi)$ is a divisor disjoint from A_i. This also is impossible, because $\mathrm{Exc}(\psi) \cap D_i \neq \emptyset$, hence there exists a curve $C \subset D_i$ with $\mathrm{Exc}(\psi) \cdot C > 0$. Since all curves in D_i are numerically proportional, the same must hold for every curve $C \subset D_i$. Now choosing $C \subseteq A_i$, we get $\mathrm{Exc}(\psi) \cap A_i \neq \emptyset$.

Therefore ψ is of fiber type, $\rho_{X_i} \leq 2$, and it must be that $i = m$. $\qquad\square$

Proof of Theorem 3.2. We assume that $\rho_X \geq 3$ and show that one of (ii), (iii), or (iv) holds. Let's consider the possibilities for φ.

If φ is of fiber type, then $\rho_X = 3$ (see Remark 2.5). Moreover, $\ker \varphi_* \not\subset \mathcal{N}_1(D, X)$; hence φ is finite on D and is a conic bundle, and we are in (ii).

Suppose now that φ is birational. If φ is of type $(n-1, 0)$, then $\rho_X = 3$ by Proposition 3.1, so again we are in (ii). If instead $\mathrm{NE}(\varphi) \not\subset \mathcal{N}_1(D, X)$, then φ is of type $(n-1, n-2)^{\mathrm{sm}}$ and we are in (iii). Thus we assume that φ is not of type $(n-1, 0)$ and that $\mathrm{NE}(\varphi) \subset \mathcal{N}_1(D, X)$.

Consider the sequence (3.5). We can assume that $R_0 = \mathrm{NE}(\varphi)$ so that $m \geq 1$. Then $\dim A_1 > 0$, because if φ is divisorial then $A_1 = \varphi(\mathrm{Exc}(\varphi))$.

Suppose that R_m is of fiber type. Then Corollary 3.7 gives $\rho_X = 3$ and, in order to get (ii), we need only show that φ is either small or of type $(n-1, n-2)^{\mathrm{sm}}$.

Let's assume that φ is divisorial. Then $\dim \mathcal{N}_1(D_1, X_1) = 1$, and Lemma 3.10 yields that $m = 1$. Thus we have

$$X \xrightarrow{\ \varphi\ } X_1 \xrightarrow{\ \psi\ } Y,$$

where ψ is the contraction of R_1 and is of fiber type. We have $\rho_Y = 1$, so $Y = \psi(D_1)$ is not a point. Since all curves contained in D_1 are numerically proportional, it follows that ψ must be finite on D_1. Then every fiber of ψ has dimension 1 and $\dim Y = n - 1$.

Notice that ψ is finite on $A_1 = \varphi(\mathrm{Exc}(\varphi))$ because $A_1 \subset D_1$. Choose a point $x_1 \in A_1$. The fiber $\psi^{-1}(\psi(x_1))$ has dimension 1 and is not contained in A_1, so $\varphi^{-1}(\psi^{-1}(\psi(x_1)))$ has some 1-dimensional irreducible component. Then Theorem 2.1 applied to $\psi \circ \varphi$ yields that $\varphi^{-1}(\psi^{-1}(\psi(x_1)))$ is 1-dimensional and has exactly two irreducible components. This means that $\psi^{-1}(\psi(x_1)) \cap A_1 = \{x_1\}$ (i.e., ψ is injective on A_1), and the two components are $\varphi^{-1}(x_1)$ and the proper transform of $\psi^{-1}(\psi(x_1))$.

Therefore, every nontrivial fiber of φ is 1-dimensional; hence X_1 is smooth and φ is of type $(n-1, n-2)^{\mathrm{sm}}$. In fact, it is not difficult to show that X_1 is Fano and that ψ is a smooth morphism.

Let's now consider the case where R_m is birational and show that this gives (iv). We claim that $-K_{X_i} \cdot R_i > 0$ for every $i = 0, \ldots, m-1$. Indeed this is true for $i = 0$. Fix $i \in \{1, \ldots, m-1\}$ and assume that $-K_{X_j} \cdot R_j > 0$ for $j = 0, \ldots, i-1$.

Observe that $\dim A_i > 0$. This is clear if $i = 1$ or if R_{i-1} is small. Suppose that $i > 1$ and that R_{i-1} is divisorial, so that σ_{i-1} is its contraction. Since $\dim \mathcal{N}_1(D, X) = 2$, it follows from Lemma 3.6 that there is at most one divisorial ray among $R_0, \ldots, R_{m-1}$. Thus R_{i-2} is small, and A_{i-1} contains the indeterminacy locus L of σ_{i-2}^{-1}, which is the locus of a small extremal ray of $\mathrm{NE}(X_{i-1})$. Then σ_{i-1} is finite on L and $\sigma_{i-1}(L) \subset A_i$ has positive dimension. Therefore, Lemma 3.10 implies that $\dim \mathcal{N}_1(D_i, X_i) > 1$; thus $\dim \mathcal{N}_1(D_i, X_i) = 2$ and R_{i-1} is small.

Let R_{i-1}' be the small extremal ray of $\mathrm{NE}(X_i)$ whose contraction is the flip of R_{i-1} in X_{i-1}. Then $-K_{X_i} \cdot R_{i-1}' < 0$ and $D_i \cdot R_{i-1}' < 0$ (see [KoMo, Cor. 6.4(4)]), so $R_{i-1}' \subset \mathcal{N}_1(D_i, X_i)$ and

$$\mathcal{N}_1(D_i, X_i) \cap \mathrm{NE}(X_i) = R_i + R_{i-1}'.$$

Since by Lemma 3.8 the divisor D_i contains curves of positive anticanonical degree and since $-K_{X_i} \cdot R_{i-1}' < 0$, we must have $-K_{X_i} \cdot R_i > 0$. We have also shown that $R_0, \ldots, R_{m-1}$ are small; in particular, φ is small.

Now it follows from Lemma 3.9 that $\mathrm{Locus}(R_m) \cap A_m = \emptyset$ and that R_m is of type $(n-1, n-2)^{\mathrm{sm}}$. Therefore, the proper transform of $\mathrm{Locus}(R_m)$ in X yields a divisor D' as in (iv), and we are done.   □

We need one more lemma before proving Lemma 3.3.

LEMMA 3.11.   *Let X be a smooth Fano variety of dimension n, and let $\varphi_1 \colon X \to Y_1$ be a divisorial elementary contraction. Let $\psi \colon Y_1 \to Z$ be an elementary birational contraction with fibers of dimension at most 1. Consider the elementary contraction $\varphi_2 \colon X \to Y_2$ such that $\mathrm{NE}(\psi \circ \varphi_1) = \mathrm{NE}(\varphi_1) + \mathrm{NE}(\varphi_2)$, and set $E_i := \mathrm{Exc}(\varphi_i) \subset X$ for $i = 1, 2$:*

$$\begin{array}{ccc}
X & \xrightarrow{\ \varphi_2\ } & Y_2 \\
{\scriptstyle \varphi_1}\Big\downarrow & & \Big\downarrow \\
Y_1 & \xrightarrow[\ \psi\ ]{} & Z.
\end{array}$$

Then Y_2 is smooth, φ_2 is of type $(n-1, n-2)^{\mathrm{sm}}$, and $\mathrm{Exc}(\psi) = \varphi_1(E_2)$. Moreover, one of the following holds:

 (i) *ψ is a divisorial Mori contraction, $\mathrm{Exc}(\psi) \cap \varphi_1(E_1)$ is a union of fibers of ψ, $E_1 \cdot \mathrm{NE}(\varphi_2) = 0$, and $E_1 \neq E_2$;*
 (ii) *ψ is small, $\mathrm{Exc}(\psi) = \varphi_1(E_1)$, $E_1 \cdot \mathrm{NE}(\varphi_2) < 0$, and $E_1 = E_2$.*

Proof. Let F be a nontrivial fiber of ψ; then $(\varphi_1)^{-1}(F)$ is a fiber of $\psi \circ \varphi_1 \colon X \to Z$. By Theorem 2.1, if $(\varphi_1)^{-1}(F)$ has an irreducible component of dimension 1 then $(\varphi_1)^{-1}(F) \cong \mathbb{P}^1$. This means that either $F \subseteq \varphi_1(E_1)$ or $F \cap \varphi_1(E_1) = \emptyset$. Therefore, $\mathrm{Exc}(\psi) \cap \varphi_1(E_1)$ is a union of fibers of ψ.

Now let F' be a nontrivial fiber of φ_2. Then $\varphi_1(F')$ is contained in a nontrivial fiber of ψ, thus $\varphi_1(F') \subseteq \mathrm{Exc}(\psi)$ and $\dim \varphi_1(F') = 1$. But φ_1 is finite on F', so $\dim F' = 1$ and φ_2 is birational with fibers of dimension at most 1. Thus Y_2 is

smooth and φ_2 is of type $(n-1, n-2)^{\text{sm}}$. We also have $\varphi_1(E_2) \subseteq \text{Exc}(\psi)$. Notice that if F' intersects E_1, then $\varphi_1(F')$ intersects $\varphi_1(E_1)$ and hence $\varphi_1(F') \subseteq \varphi_1(E_1)$.

Suppose that $E_1 \neq E_2$. Then $\varphi_1(E_2)$ is a divisor contained in $\text{Exc}(\psi)$; hence ψ is divisorial with exceptional locus $\varphi_1(E_2)$ (notice that Y_1 is $\mathbb{Q}$-factorial because φ_1 is divisorial). Since $\varphi_1(E_2)$ cannot be contained in $\varphi_1(E_1)$, it follows that ψ has nontrivial fibers that are disjoint from $\varphi_1(E_1)$ and so is a Mori contraction. Moreover, there must be fibers of φ_2 that are disjoint from E_1, hence $E_1 \cdot \text{NE}(\varphi_2) = 0$.

Assume $E_1 = E_2$, so that $E_1 \cdot \text{NE}(\varphi_2) < 0$. Clearly the exceptional locus of $\psi \circ \varphi_1$ contains E_1. However, every curve in $\text{NE}(\psi \circ \varphi_1) = \text{NE}(\varphi_1) + \text{NE}(\varphi_2)$ has negative intersection with E_1 and hence is contained in E_1; namely, $\text{Exc}(\psi \circ \varphi_1) = E_1$. This yields $\text{Exc}(\psi) = \varphi_1(E_1)$, and ψ is small. $\qquad\square$

Proof of Lemma 3.3. We set $E := \text{Exc}(\varphi) \subset X$, $A := \varphi(E) \subset Y$, and $D_Y := \varphi(D) \subset Y$. Observe that there are nontrivial fibers of φ disjoint from D and others contained in D. We thus have $\dim \mathcal{N}_1(D_Y, Y) = 1$, but A is not contained in D_Y; this is the main difference with respect to the situation of Lemma 3.10.

Let $\psi \colon Y \to Z$ be an elementary contraction of Y with $D_Y \cdot \text{NE}(\psi) > 0$, as in Remark 2.5. If ψ is of fiber type, then $\rho_Z \leq 1$ and $\rho_X \leq 3$. Suppose that ψ is birational. As in the proof of Lemma 3.10, we see that ψ is finite on D_Y and that its fibers have dimension at most 1. Hence Lemma 3.11 applies; in particular, $\text{Exc}(\psi) \cap A$ is a union of fibers of ψ. If ψ is not divisorial, then Lemma 3.11(ii) gives an extremal ray R as in the statement. If ψ is divisorial, then we are in Lemma 3.11(i) and so ψ is a Mori contraction and Z is $\mathbb{Q}$-factorial.

Set $D_Z := \psi(D_Y) \subset Z$. Then D_Z is a prime divisor in Z with $\dim \mathcal{N}_1(D_Z, Z) = 1$ and $D_Z \supset \psi(\text{Exc}(\psi))$. Let $\xi \colon Z \to W$ be an elementary contraction of Z with $D_Z \cdot \text{NE}(\xi) > 0$, as in Remark 2.5. If ξ is of fiber type, then $\rho_W \leq 1$ and $\rho_X \leq 4$. Suppose that ξ is birational; as before, it is finite over D_Z and has fibers of dimension at most 1. Set $\eta := \xi \circ \psi$, and let $\psi_1 \colon Y \to Z_1$ be the elementary contraction of Y such that $\text{NE}(\eta) = \text{NE}(\psi) + \text{NE}(\psi_1)$:

$$
\begin{array}{ccccc}
X & \xrightarrow{\ \varphi\ } & Y & \xrightarrow{\ \psi_1\ } & Z_1 \\
& & {\scriptstyle\psi}\downarrow & {\scriptstyle\eta}\searrow & \downarrow \\
& & Z & \xrightarrow[\ \xi\ ]{} & W.
\end{array}
$$

Again, ψ_1 is birational with fibers of dimension at most 1, so Lemma 3.11 applies. Either ψ_1 is not divisorial and we again have an extremal ray R as in the statement, or ψ_1 is a divisorial Mori contraction and $\text{Exc}(\psi_1) \cap A$ is a union of fibers of ψ_1. We show that this last case leads to a contradiction.

Every curve in $\text{NE}(\eta)$ has positive anticanonical degree, so η is a Mori contraction. If $\text{Exc}(\psi) = \text{Exc}(\psi_1)$, then every curve in $\text{NE}(\eta)$ has negative intersection with $\text{Exc}(\psi)$; hence $\text{Exc}(\eta) = \text{Exc}(\psi)$ and $\text{Exc}(\xi) = \psi(\text{Exc}(\psi))$. This is impossible, however, because ξ is finite on D_Z, which contains $\psi(\text{Exc}(\psi))$.

Therefore, $\text{Exc}(\psi) \neq \text{Exc}(\psi_1)$. Then $\psi(\text{Exc}(\psi_1))$ is a divisor contained in $\text{Exc}(\xi)$, which means that $\text{Exc}(\xi) = \psi(\text{Exc}(\psi_1))$ and ξ is divisorial. As in the

proof of Lemma 3.10, we see that $\mathrm{Exc}(\xi)$ must intersect every curve contained in D_Z and that $\dim \psi(\mathrm{Exc}(\psi)) = n - 2 \geq 1$; hence $\mathrm{Exc}(\xi) \cap \psi(\mathrm{Exc}(\psi)) \neq \emptyset$. Then $\dim(\psi(\mathrm{Exc}(\psi)) \cap \mathrm{Exc}(\xi)) \geq n - 3$ and, since ξ is finite on $\psi(\mathrm{Exc}(\psi))$, we get

$$\dim \xi(\psi(\mathrm{Exc}(\psi)) \cap \mathrm{Exc}(\xi)) \geq n - 3.$$

We claim that

$$\dim \xi(\psi(\mathrm{Sing}(Y)) \cap \mathrm{Exc}(\xi)) \leq n - 4. \tag{3.12}$$

First we show that (3.12) allows us to conclude the proof. Since both ψ and ξ are Mori contractions with fibers of dimension at most 1, it follows that $\mathrm{Sing}(W) \subseteq \eta(\mathrm{Sing}(Y))$. Thus (3.12) implies that there exists a point $w_0 \in W \setminus \mathrm{Sing}(W)$ such that the fiber $\xi^{-1}(w_0)$ has dimension 1 and intersects $\psi(\mathrm{Exc}(\psi))$. Restricting η to a contraction $Y \setminus \eta^{-1}(\mathrm{Sing}(W)) \to W \setminus \mathrm{Sing}(W)$, we can now apply Theorem 2.1 to $\eta^{-1}(w_0)$, as in the proof of Lemma 3.11, and derive a contradiction.

Let's show (3.12). If $\dim A = n - 2$, then $\dim \mathrm{Sing}(Y) \leq n - 4$ and so (3.12) holds. If $\dim A \leq n - 3$ then we still have $A \supseteq \mathrm{Sing}(Y)$, so it is enough to show that $\dim \xi(\psi(A) \cap \mathrm{Exc}(\xi)) \leq n - 4$. This is clear if $\psi(A)$ is not contained in $\mathrm{Exc}(\xi)$. If instead $\psi(A) \subseteq \mathrm{Exc}(\xi) = \psi(\mathrm{Exc}(\psi_1))$, then

$$A = \psi^{-1}(\psi(A)) \subseteq \psi^{-1}(\mathrm{Exc}(\xi)) \subseteq \mathrm{Exc}(\psi) \cup \mathrm{Exc}(\psi_1).$$

Because A is irreducible, it is contained either in $\mathrm{Exc}(\psi)$ or in $\mathrm{Exc}(\psi_1)$, and it is a union of fibers of both ψ and ψ_1. In all cases we have $\dim \eta(A) \leq n - 4$, and we are done. $\qquad\square$

4. Elementary Contractions of Type $(n - 1, 1)$

Throughout this section, we fix the following notation.

4.1. We use X to denote a smooth Fano variety of dimension $n \geq 4$, and R_1 is an extremal ray of type $(n - 1, 1)$. For any integer $i \in \mathbb{Z}_{\geq 0}$, if R_i is an extremal ray of $\mathrm{NE}(X)$ then we denote by $\varphi_i \colon X \to Y_i$ the associated contraction and set $E_i := \mathrm{Exc}(\varphi_i)$.

Our goal is to bound ρ_X; notice that $\rho_X \geq 2$ by our assumptions.
   We observe first of all that, since $\varphi_1(E_1)$ is a curve, we have

$$\dim \mathcal{N}_1(\varphi_1(E_1), Y_1) = 1 \quad \text{and} \quad \dim \mathcal{N}_1(E_1, X) = 2;$$

thus we can apply to E_1 the results of the preceding section. Indeed, there exists an extremal ray R_2 with $E_1 \cdot R_2 > 0$, and by Theorem 3.2 we can conclude at once that $\rho_X \leq 3$ unless R_2 is small or of type $(n - 1, n - 2)^{\mathrm{sm}}$. More precisely, we show the following.

THEOREM 4.2. *Let X and R_1 be as in 4.1, and let R_2 be an extremal ray with $E_1 \cdot R_2 > 0$. Then one of the following holds.*
(i) $\rho_X \leq 4$. *More precisely, we have the possibilities*:
   • φ_2 *is of type* $(n, n - 1)$, $(n, n - 2)$, *or* $(n - 1, n - 3)$ *and* $\rho_X = 2$;
   • φ_2 *is a conic bundle and* $\rho_X = 3$;

- φ_2 is of type $(n-2, n-4)$ *and* $\rho_X \le 3$;
- $n = 4$, φ_2 *is of type* $(2,0)$, *and* $\rho_X = 4$;
- φ_2 *is of type* $(n-1, n-2)$ *and* $\rho_X \le 4$.

(ii) φ_2 *is of type* $(n-1, n-2)^{\mathrm{sm}}$; $E_2 \cdot R_1 = 0$; *and there exists an extremal ray* $R_0 \ne R_1$ *such that* $E_1 \cdot R_0 < 0$.

Case (ii) will be treated in Proposition 4.8, where we will show that $\rho_X \le 5$.

REMARK 4.3 (Classification results by Tsukioka). Suppose that Y_1 is smooth and φ_1 is the blow-up of a smooth curve. When φ_2 is of type $(n, n-2)$, the possible X and Y_1 are classified in [T1]. Moreover if $n = 4$, φ_2 is of type $(3,1)$, and E_2 is smooth, then it is shown in [T3] that $Y_1 \cong \mathbb{P}^4$ and $\varphi_1(E_1)$ is an elliptic curve of degree 4 in $\mathbb{P}^4$.

Proof of Theorem 4.2. First notice that every nontrivial fiber F of φ_2 has dimension at most 2. In fact, $F \cap E_1$ is nonempty and φ_1 is finite on it, so that

$$\dim F - 1 \le \dim F \cap E_1 = \dim \varphi_1(F \cap E_1) \le \dim \varphi_1(E_1) = 1.$$

This, together with Theorem 3.2, implies the statement—unless we are in cases (iii) or (iv) of Theorem 3.2.

First we consider case (iii). Assume that φ_2 is of type $(n-1, n-2)^{\mathrm{sm}}$ and $R_2 \not\subset \mathcal{N}_1(E_1, X)$. We treat separately the two possibilities $E_2 \cdot R_1 = 0$ and $E_2 \cdot R_1 > 0$.
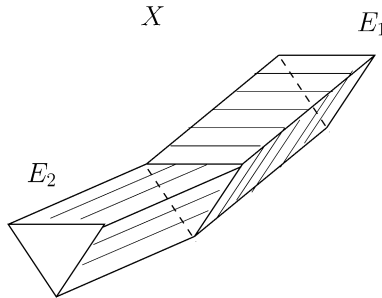

Figure 1 The case $R_2 \not\subset \mathcal{N}_1(E_1, X)$ and $E_2 \cdot R_1 = 0$

Suppose that $E_2 \cdot R_1 = 0$ (Figure 1). Then E_2 must contain some fiber F' of φ_1 of dimension $n-2$. Since φ_2 is finite on F', we have $\varphi_2(F') = \varphi_2(E_2)$ and so

$$\mathcal{N}_1(E_2, X) = \mathbb{R}R_2 + \mathcal{N}_1(F', X) = \mathbb{R}(R_1 + R_2).$$

Then Lemma 3.3 applies to E_2 and φ_1, and thus either $\rho_X \le 4$ or we have (ii). See Remark 4.7 for a more precise description of this case.

Assume now that $E_2 \cdot R_1 > 0$ (Figure 2) and consider $D := \varphi_2(E_1) \subset Y_2$ and $A := \varphi_2(E_2) \subset D$. If $C \subset Y_2$ is an irreducible curve not contained in A then $-K_{Y_2} \cdot C \ge 1$, with strict inequality (by Lemma 3.8) whenever $C \cap A \ne \emptyset$.
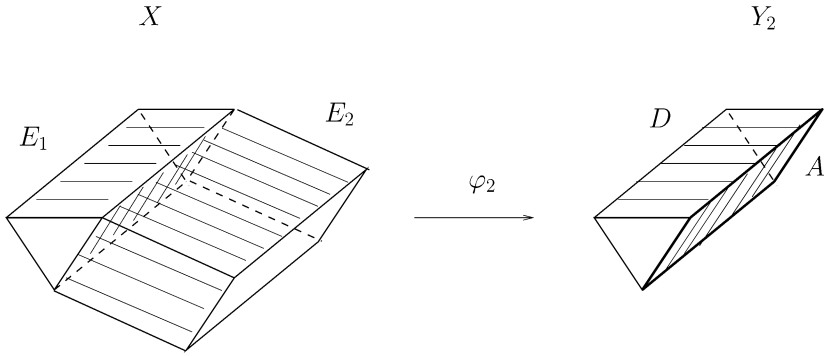
Figure 2 The case $R_2 \not\subset \mathcal{N}_1(E_1, X)$ and $E_2 \cdot R_1 > 0$

We first suppose that Y_2 is Fano and apply Theorem 3.2 to $D \subset Y_2$ (choosing any extremal ray of NE(Y) that has positive intersection with D). If $\rho_{Y_2} \leq 3$, then $\rho_X \leq 4$. Otherwise we are in case (iii) or (iv) of Theorem 3.2, and there exists a smooth prime divisor $D' \subset Y_2$ with a $\mathbb{P}^1$-bundle structure such that, for any fiber f, we have $-K_{Y_2} \cdot f = 1$, $f \cdot D > 0$, and $f \not\subset D$. Then $f \cap A = \emptyset$; that is, D' cannot intersect A, so its inverse image D'' in X is a prime divisor that intersects E_1 but is disjoint from E_2. This is impossible, because either $D'' \cdot R_1 = 0$ and D'' contains some nontrivial fiber of φ_1, or $D'' \cdot R_1 > 0$ and $D'' \cap C \neq \emptyset$ for some irreducible curve $C \subset E_2$ with $[C] \in R_1$.
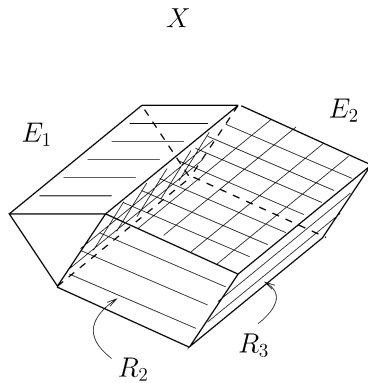

Figure 3 The case Y_2 not Fano

Assume that Y_2 is not Fano (Figure 3). This means that there exists some extremal ray of NE(Y_2) with nonpositive anticanonical degree. Let's consider the associated contraction

$$\tilde{\psi} \colon Y_2 \longrightarrow \tilde{Z},$$

and notice that Exc$(\tilde{\psi}) \subseteq A \subset D$. Then any nontrivial fiber of $\tilde{\psi}$ must be 1-dimensional. In fact, if $\tilde{\psi}$ had a fiber F with dim $F \geq 2$, then we would have

$$\dim(\varphi_2^{-1}(F) \cap E_1) \geq 2;$$

thus $R_1 \subset \mathcal{N}_1(\varphi_2^{-1}(F), X)$ and $(\varphi_2)_*(R_1) \subset \mathcal{N}_1(F, Y_2)$. This implies that $\mathrm{NE}(\tilde{\psi}) = (\varphi_2)_*(R_1)$, which is impossible because $\mathrm{Exc}(\tilde{\psi})$ should contain all D.

Therefore $\tilde{\psi}$ is small with fibers of dimension at most 1. By Lemma 3.11 we see that there exists an extremal ray R_3 of $\mathrm{NE}(X)$ of type $(n-1, n-2)^{\mathrm{sm}}$ such that $E_2 \cdot R_3 < 0$ and $R_2 + R_3$ is a face of $\mathrm{NE}(X)$; in particular, $\rho_X \geq 3$.

We need to establish some features of E_2 in order to show the following:

$$E_1 \cdot R_3 = 0 \quad \text{and} \quad [C] \in R_1 + R_2 + R_3 \text{ for every curve } C \subset E_2. \qquad (4.4)$$

Observe first of all that E_2 is smooth and that $\varphi_2|_{E_2}$ and $\varphi_3|_{E_2}$ are $\mathbb{P}^1$-bundles. Moreover, we have $\mathcal{N}_1(E_2, X) = \mathbb{R}(R_1 + R_2 + R_3)$. Indeed, R_1, R_2, and R_3 are contained in $\mathcal{N}_1(E_2, X)$. On the other hand, since E_1 meets every fiber of $\varphi_2|_{E_2}$, we have $\varphi_2(E_1 \cap E_2) = \varphi_2(E_2)$ and so

$$\mathbb{R}(R_1 + R_2 + R_3) \subseteq \mathcal{N}_1(E_2, X) \subseteq \mathbb{R}R_2 + \mathcal{N}_1(E_1, X);$$

however, since $\dim(\mathbb{R}R_2 + \mathcal{N}_1(E_1, X)) = 3$, these inclusions are equalities.

Let T be the normalization of $\varphi_1(E_2)$, $\xi \colon E_2 \to T$ the contraction induced by $(\varphi_1)|_{E_2}$ and let $i \colon E_2 \hookrightarrow X$ be the inclusion:

$$
\begin{CD}
E_2 @>i>> X \\
@V\xi VV @VV\varphi_1 V \\
T @>>> Y_1.
\end{CD}
$$

It is easy to see that $i_*(\ker \xi_*) = \ker(\varphi_1)_* = \mathbb{R}R_1$. Since in general i_* is not injective, ξ does not need to be an elementary contraction; however, ξ is birational with $\mathrm{Exc}(\xi) = E_1 \cap E_2$, and $\xi(\mathrm{Exc}(\xi)) \subset T$ is a curve.

Notice also that ρ_T is the codimension of $\ker \xi_*$ in $\mathcal{N}_1(E_2)$ and, since $i_*(\ker \xi_*)$ has codimension 2 in $\mathcal{N}_1(E_2, X)$, we see that $\rho_T \geq 2$; and $\rho_T = 2$ if and only if $\ker \xi_* \supseteq \ker i_*$.

The diagram

$$
\begin{CD}
E_2 @>\xi>> T \\
@V\varphi_2|_{E_2} VV \\
A
\end{CD}
$$

gives a proper covering family of irreducible rational curves in T (see [De, Sec. 5.4] and references therein). This family of curves induces an equivalence relation (E_2-equivalence, in the terminology of [De]) on T as a set, where two points $t_1, t_2 \in T$ are *equivalent* if there exist $F_1, \ldots, F_m$ fibers of $\varphi_2|_{E_2}$ such that $\xi(F_1 \cup \cdots \cup F_m)$ is connected and contains both t_1 and t_2. Then there exists a dense open subset $T_0 \subseteq T$, closed for the equivalence relation, and a proper morphism $\alpha_0 \colon T_0 \to C_0$, where C_0 is a normal variety, such that every fiber of α_0 is an equivalence class (see e.g. [De, Thm. 5.9]).

Let $S \subset T_0$ be a fiber of α_0. We know that $\dim \mathcal{N}_1(S, T) = 1$ by [Ko, Prop. IV.3.13.3], and since $\rho_T > 1$ we know that $S \subsetneq T$ and $\dim C_0 > 0$. Moreover, $\xi^{-1}(S)$ is a union of fibers of φ_2 and thus intersects $E_1 \cap E_2 = \text{Exc}(\xi)$, so that $S \cap \xi(\text{Exc}(\xi)) \neq \emptyset$. Hence $\xi(\text{Exc}(\xi))$ intersects every fiber of α_0, which means that $\dim C_0 = 1$ and every fiber of α_0 has codimension 1.

Now if C is the smooth projective curve containing C_0 as an open subset, then it is not difficult to see that the rational map $\alpha_0 \colon T \dashrightarrow C$ extends to a contraction $\alpha \colon T \to C$ whose fibers are equivalence classes. We thus have the following diagram:

$$
\begin{array}{ccc}
E_2 & \xrightarrow{\ \xi\ } & T \\
\varphi_2|_{E_2} \downarrow & & \downarrow \alpha \\
A & \longrightarrow & C.
\end{array}
$$

We deduce that $\rho_T = 2$ and $\ker \xi_* \supseteq \ker i_*$. We refer the interested reader to [BoCaD] and [Ca2, Sec. 4] for related results.

We have

$$\mathbb{R}(R_1 + R_2) \subseteq i_*(\ker(\alpha \circ \xi)_*),$$

and since $\ker(\alpha \circ \xi)_*$ is a hyperplane in $\mathcal{N}_1(E_2)$ and contains $\ker i_*$, its image under i_* must be $\mathbb{R}(R_1 + R_2)$. In particular we see that $\text{NE}((\varphi_3)|_{E_2})$ cannot be contained in $\ker(\alpha \circ \xi)_*$.

Let's show that $E_1 \cdot R_3 = 0$. In fact, if $E_1 \cdot R_3 > 0$ then, reasoning as for R_2, we get a second contraction $\alpha' \colon T \to C'$ where C' is another smooth curve. Moreover, $\text{NE}((\varphi_3)|_{E_2})$ is contained in $\ker(\alpha' \circ \xi)_*$, so $\alpha \circ \xi \neq \alpha' \circ \xi$ and $\alpha \neq \alpha'$. However, $\dim T = n - 1 \geq 3$, and the fibers of α and α' are Cartier divisors that should intersect only in finitely many points, which is impossible.

Thus $E_1 \cdot R_3 = 0$, $\mathcal{N}_1(E_1, X) = \mathbb{R}(R_1 + R_3)$, and

$$\mathcal{N}_1(E_1, X) \cap \text{NE}(X) = R_1 + R_3. \tag{4.5}$$

Then $\mathcal{N}_1(E_1, X)$ cannot contain other extremal rays, and $R_1 + R_3$ is a face of $\text{NE}(X)$ by the following remark.

REMARK 4.6.   Let X be as in 4.1, and consider a divisorial extremal ray S_1 of $\text{NE}(X)$ with exceptional divisor G_1 such that $G_1 \cdot S \geq 0$ for every extremal ray $S \neq S_1$. Let S_2 be a birational extremal ray of $\text{NE}(X)$ with $G_1 \cdot S_2 = 0$. Then $S_1 + S_2$ is a face of $\text{NE}(X)$ whose contraction is birational. This is probably well known; similar properties can be found in [N].

Indeed, let $C_i \subset X$ be a curve with $[C_i] \in S_i$ for $i = 1, 2$. If $S_1 + S_2$ were not a face of $\text{NE}(X)$, then we should have

$$\lambda_1 C_1 + \lambda_2 C_2 \equiv \sum_{j=3}^{m} \lambda_j C_j,$$

where $\lambda_j \in \mathbb{Q}_{>0}$ for every $j = 1, \ldots, m$ and where, for $j \geq 3$, $[C_j]$ belongs to an extremal ray S_j with $G_1 \cdot S_j \geq 0$. Then, intersecting with G_1 yields a contradiction.

Moreover, if $C \subset X$ is an irreducible curve with $[C] \in S_1 + S_2$, then either $C \cdot G_1 < 0$ or $[C] \in S_2$, so that $C \subset G_1 \cup \mathrm{Locus}(S_2)$.

We continue now with the proof of (4.4) and consider the 3-dimensional cone

$$i_*(\overline{\mathrm{NE}}(E_2)) \subseteq \mathcal{N}_1(E_2, X) \cap \mathrm{NE}(X),$$

which contains R_1, R_2, and R_3. Since $R_1 + R_3$ and $R_2 + R_3$ are faces of $\mathrm{NE}(X)$, they are faces of $i_*(\overline{\mathrm{NE}}(E_2))$, too. On the other hand, $\mathrm{NE}(\alpha \circ \xi)$ is a face of $\overline{\mathrm{NE}}(E_2)$, and since $\ker(\alpha \circ \xi)_* \supseteq \ker i_*$, it follows that $i_*(\mathrm{NE}(\alpha \circ \xi))$ is a face S of $i_*(\overline{\mathrm{NE}}(E_2))$ that is contained in $i_*(\ker(\alpha \circ \xi)_*) = \mathbb{R}(R_1 + R_2)$ and contains both R_1 and R_2. Therefore, $S = R_1 + R_2$ and so

$$i_*(\overline{\mathrm{NE}}(E_2)) = R_1 + R_2 + R_3,$$

which implies (4.4).

Now let's consider $\varphi_1 \colon X \to Y_1$ and the divisor $\varphi_1(E_2) \subset Y_1$. Let $\eta \colon Y_1 \to W$ be an elementary contraction with $\varphi_1(E_2) \cdot \mathrm{NE}(\eta) > 0$, as in Remark 2.5. Moreover, let R_4 be the extremal ray of $\mathrm{NE}(X)$ such that $R_1 + R_4$ is a face and $(\varphi_1)_*(R_4) = \mathrm{NE}(\eta)$. Then, since $\dim \mathcal{N}_1(\varphi_1(E_2), Y_1) = 2$, if η is of fiber type we get $\rho_W \leq 2$ and $\rho_X \leq 4$.

Suppose that η is birational (Figure 4). Let's show that η must be finite on $\varphi_1(E_2)$. If not, then there should be a curve $C \subset E_1 \cup E_2$ with $[C] \in R_4$. But $[C] \in R_1 + R_2 + R_3$ by (4.4) and (4.5), which yields either $R_4 = R_2$ or $R_4 = R_3$. In either case we would have $\mathrm{Exc}(\eta) = \varphi_1(E_2)$ and $\varphi_1(E_2) \cdot \mathrm{NE}(\eta) < 0$, a contradiction.
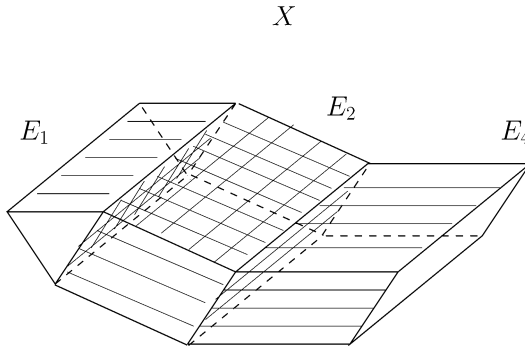
$$X$$

Figure 4   The case η birational

Thus η is finite on $\varphi_1(E_2) \supset \varphi_1(E_1)$ and must have fibers of dimension at most 1. Then, by Lemma 3.11, η is a divisorial Mori contraction with $\mathrm{Exc}(\eta) \cap \varphi_1(E_1) = \emptyset$, R_4 is of type $(n - 1, n - 2)^{\mathrm{sm}}$, and $E_4 \cap E_1 = \emptyset$. Moreover $\mathrm{Exc}(\eta)$ must intersect $\varphi_1(E_2)$, so that $E_4 \cap E_2 \neq \emptyset$. Since E_2 cannot contain curves in R_4, we have $E_2 \cdot R_4 > 0$.

If $R_2 + R_4$ is a face of $\mathrm{NE}(X)$ then $(\varphi_2)_*(R_4)$ is an extremal ray of $\mathrm{NE}(Y_2)$, with $D \cdot (\varphi_2)_*(R_4) > 0$, whose locus is either $\varphi_2(E_4)$ or the whole Y_2. However,

if $C \subset X$ is a nontrivial fiber of φ_4, then it is easy to see that $\varphi_2(C) \cdot \varphi_2(E_4) \geq 0$; thus the contraction of $(\varphi_2)_*(R_4)$ is of fiber type and, as before, we get $\rho_{Y_2} \leq 3$ and $\rho_X \leq 4$.

Finally, let's assume that $R_2 + R_4$ is not a face of $NE(X)$ and consider the divisor $\varphi_4(E_1) \subset Y_4$. There exists an extremal ray S of $NE(Y_4)$ with $\varphi_4(E_1) \cdot S > 0$. Let R_5 be the extremal ray of $NE(X)$ such that $R_4 + R_5$ is a face of $NE(X)$ and $(\varphi_4)_*(R_5) = S$. We observe that, by construction, $R_5 \neq R_2$. Since $\varphi_4^{-1}(\varphi_4(E_1)) = E_1$, we have $E_1 \cdot R_5 > 0$; hence $R_5 \neq R_1$ and $R_5 \neq R_3$.

Now we apply what we proved so far to R_5. Notice that $R_5 \not\subset \mathcal{N}_1(E_1, X)$; in particular, R_5 cannot be small. Then either (a) $\rho_X \leq 4$ or (b) R_5 is of type $(n - 1, n - 2)^{sm}$ and there exists a divisorial extremal ray $R_6 \neq R_5$ such that $E_5 \cdot R_6 < 0$ and $E_1 \cdot R_6 = 0$. We show that case (b) is impossible.

In fact we have $R_6 \subset \mathcal{N}_1(E_1, X) = \mathbb{R}(R_1 + R_3)$, and $R_6 \neq R_1$ because $Locus(R_6) = E_5 \neq E_1$, so the only possibility is that $R_6 = R_3$ and $E_5 = E_2$. If C is a curve with numerical class in R_5, then $C \subset E_2$ and hence $[C] \in R_1 + R_2 + R_3$ by (4.4). But R_5 is distinct from R_1, R_2, R_3, so we obtain a contradiction.
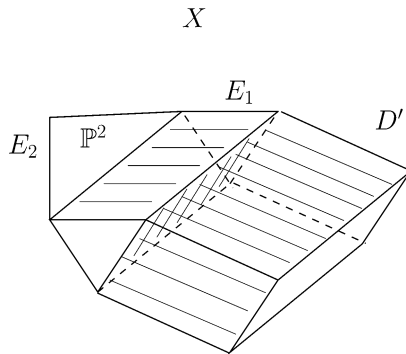
$$X$$

$$E_1$$
$$D'$$
$$E_2 \quad \mathbb{P}^2$$

Figure 5 φ_2 is $(2, 0)$, $n = 4$

We still have to consider the case (Figure 5) where φ_2 is of type $(n - 2, n - 4)$ and there exists a smooth prime divisor $D' \subset X$, disjoint from E_2 and having a $\mathbb{P}^1$-bundle structure $\xi \colon D' \to W$, such that for any fiber f of ξ we have $D' \cdot f = -1$ and $E_1 \cdot f > 0$. Notice that every nontrivial fiber of φ_1 must intersect E_2, so it cannot be contained in D'. This implies that $D' \cdot R_1 > 0$, so that D' intersects every curve contracted by φ_1. Again, since $D' \cap E_2 = \emptyset$, we see that φ_1 is finite on E_2. This gives

$$n - 3 = \dim(E_1 \cap E_2) = \dim \varphi_1(E_1 \cap E_2) \leq 1$$

and so $n = 4$.

We have $\mathcal{N}_1(E_1, X) \cap NE(X) = R_1 + R_2$; also, $D' \cdot R_1 > 0$, $D' \cdot R_2 = 0$, and $D' \cdot f < 0$. Thus $[f] \notin \mathcal{N}_1(E_1, X)$ and $\rho_X \geq 3$. Moreover, $\xi(D' \cap E_1) = W$; hence

$$\mathcal{N}_1(D', X) = \mathbb{R}[f] \oplus \mathcal{N}_1(D' \cap E_1, X) = \mathbb{R}[f] \oplus \mathbb{R}R_1 \oplus \mathbb{R}R_2$$

and $\dim \mathcal{N}_1(D', X) = 3$.

Since $D' \cdot f < 0$, there exists an extremal ray $\tilde{R}_2$ of NE(X) with $D' \cdot \tilde{R}_2 <$ 0. If $\tilde{R}_2$ were small, then by [Ka] its exceptional locus would contain $F \cong \mathbb{P}^2$. Then $W \cong \mathbb{P}^2$ and $\rho_{D'} = 2$, a contradiction. Thus $\tilde{R}_2$ is divisorial, with exceptional divisor D'. Since $\tilde{R}_2 \not\subset \mathcal{N}_1(E_1, X)$, it follows that $\tilde{R}_2$ is of type $(3, 2)^{\mathrm{sm}}$ and $E_1 \cdot \tilde{R}_2 > 0$. Applying to $\tilde{R}_2$ what we have already proved yields $\rho_X \leq 4$. In fact, it is not difficult to see that $\tilde{R}_2$ contains $[f]$.   □

REMARK 4.7. Let X and R_1 be as in 4.1, and suppose that R_2 is a birational extremal ray with $E_1 \cdot R_2 > 0$, $E_2 \cdot R_1 = 0$, and $R_2 \not\subset \mathcal{N}_1(E_1, X)$ (see Figure 1). We have seen in the proof of Theorem 4.2 that R_2 is of type $(n - 1, n - 2)^{\mathrm{sm}}$ and that $\mathcal{N}_1(E_2, X) = \mathbb{R}(R_1 + R_2)$.

Therefore $E_2 \cdot S \geq 0$ for every extremal ray $S \neq R_2$, and Y_2 is Fano by [W, Prop. 3.4]. Moreover, by Remark 4.6 we have that $R_1 + R_2$ is a face of NE(X), whose contraction is birational. Notice that the contraction of $R_1 + R_2$ cannot send E_1 to a point, for otherwise we would have $\mathcal{N}_1(E_1, X) = \mathbb{R}(R_1 + R_2)$, which is excluded by our assumptions. Thus $(\varphi_2)_*(R_1)$ is an extremal ray of NE(Y_2), whose contraction is birational, and cannot send $\varphi_2(E_1)$ to a point. This means that Y_2 has an elementary contraction of type $(n - 1, 1)$ given by $(\varphi_2)_*(R_1)$, with exceptional divisor $\varphi_2(E_1)$, and φ_2 is the blow-up of a smooth fiber of this contraction.

PROPOSITION 4.8.   *Let X and R_1 be as in 4.1, and suppose that there exists an extremal ray $R_0 \neq R_1$ with $E_1 \cdot R_0 < 0$. Then $\rho_X \leq 5$; $R_0 + R_1$ is a face of NE(X); $E_1 \cong W \times \mathbb{P}^1$, where W is smooth and Fano; Y_0 is smooth; and φ_0 is the blow-up of a smooth subvariety isomorphic to W.*

If $\rho_X = 5$ then there exists a smooth Fano variety Z, with $\rho_Z = 3$, dim $Z = n$, and having an elementary contraction of type $(n - 1, 1)$, such that X is the blow-up of Z in two fibers of this contraction.

Proof. Every nontrivial fiber of φ_0 is contained in E_1 and thus has dimension 1. Hence R_0 is of type $(n - 1, n - 2)^{\mathrm{sm}}$, Y_0 and E_1 are smooth, φ_0 is the blow-up of a smooth codimension-2 subvariety $W \subset Y_0$, and E_1 is a $\mathbb{P}^1$-bundle over W. Moreover, $\mathcal{N}_1(E_1, X) = \mathbb{R}(R_0 + R_1)$, $\mathcal{N}_1(E_1, X) \cap \mathrm{NE}(X) = R_0 + R_1$, and there are no other extremal rays with negative intersection with E_1.

For $i = 1, 2$ let C_i be a curve in R_i and H_i a nef divisor such that, for every extremal ray S of NE(X), $H_i \cdot S = 0$ if and only if $S = R_i$. Consider the divisor

$$H := (H_0 \cdot C_1)H_1 + (H_1 \cdot C_0)(-E_1 \cdot C_1)H_0 + (H_0 \cdot C_1)(H_1 \cdot C_0)E_1.$$

It is easy to see that $H \cdot R_0 = H \cdot R_1 = 0$ and that $H \cdot S > 0$ for every extremal ray $S \neq R_0, R_1$. Then $R_0 + R_1$ is a face of NE(X).

Let's show that E_1 is Fano. If $\gamma \in \overline{\mathrm{NE}}(E_1)$ is nonzero, then

$$-K_{E_1} \cdot \gamma = -(K_X + E_1)|_{E_1} \cdot \gamma = -(K_X + E_1) \cdot i_*(\gamma),$$

where $i \colon E_1 \hookrightarrow X$ is the inclusion. First of all we observe that $i_*(\gamma)$ is nonzero. Indeed, if A is an ample divisor on X, then

$$A \cdot i_*(\gamma) = A|_{E_1} \cdot \gamma > 0.$$

Moreover, $i_*(\overline{\mathrm{NE}}(E_1)) \subseteq \mathrm{NE}(X)$, so that $i_*(\gamma) \in R_0 + R_1$ and hence $E_1 \cdot i_*(\gamma) < 0$. This gives $-K_{E_1} \cdot \gamma > 0$.

The restriction $\varphi_1|_{E_1} \colon E_1 \to \varphi_1(E_1)$ is surjective with connected fibers. Since $\varphi_1(E_1)$ is dominated by fibers of $\varphi_0|_{E_1}$, it is a rational curve, and $\varphi_1|_{E_1}$ induces a Mori contraction

$$\varphi \colon E_1 \longrightarrow \mathbb{P}^1$$

that does not contract the fibers of $\varphi_0|_{E_1}$. Then $E_1 \cong W \times \mathbb{P}^1$ by the following lemma.

LEMMA 4.9.   *Let E be a smooth variety and let $\pi \colon E \to W$ be a smooth morphism with fiber $\mathbb{P}^r$. Suppose that E has a Mori contraction $\phi \colon E \to \mathbb{P}^r$ that is finite on fibers of π. Then $E \cong W \times \mathbb{P}^r$.*

We postpone the proof of Lemma 4.9 and carry on with the proof of Proposition 4.8.

Let R_2 be an extremal ray of $\mathrm{NE}(X)$ with $E_1 \cdot R_2 > 0$. Then R_2 is different from R_0 and R_1, so that $R_2 \not\subset \mathcal{N}_1(E_1, X)$, and φ_2 is finite on E_1 (notice that necessarily $\rho_X \geq 3$). If φ_2 is of fiber type, then it is a conic bundle and we have $\rho_{Y_2} = 2$ and $\rho_X = 3$.

Suppose that φ_2 is birational. Then it is of type $(n-1, n-2)^{\mathrm{sm}}$, so Y_2 is smooth and φ_2 is the blow-up of $A := \varphi_2(E_2) \subset \varphi_2(E_1) \subset Y_2$. We set $D := \varphi_2(E_1)$. Notice that $\varphi_2(E_1 \cap E_2) = \varphi_2(E_2)$ and that $C \cdot E_2 \geq 0$ for every curve $C \subset E_1$. Since $\varphi_2^*(-K_{Y_2}) = -K_X + E_2$, using the projection formula shows us that Y_2 is Fano.

Let $\psi \colon Y_2 \to Z$ be an elementary contraction such that $D \cdot \mathrm{NE}(\psi) > 0$, as in Remark 2.5. If ψ is of fiber type, then $\rho_Z \leq 2$ and $\rho_X \leq 4$. Assume that ψ is birational (Figure 6). Then ψ must be finite on D, because

$$\mathcal{N}_1(D, Y_2) \cap \mathrm{NE}(Y_2) = (\varphi_2)_*(\mathcal{N}_1(E_1, X) \cap \mathrm{NE}(X))$$
$$= (\varphi_2)_*(R_0) + (\varphi_2)_*(R_1). \tag{4.9}$$

If ψ were not finite on D, then we should have $\mathrm{NE}(\psi) = (\varphi_2)_*(R_0)$ or $\mathrm{NE}(\psi) = (\varphi_2)_*(R_1)$; in both cases $\mathrm{Exc}(\psi) = D$, which contradicts $D \cdot \mathrm{NE}(\psi) > 0$. Thus Z is smooth and ψ is of type $(n-1, n-2)^{\mathrm{sm}}$.
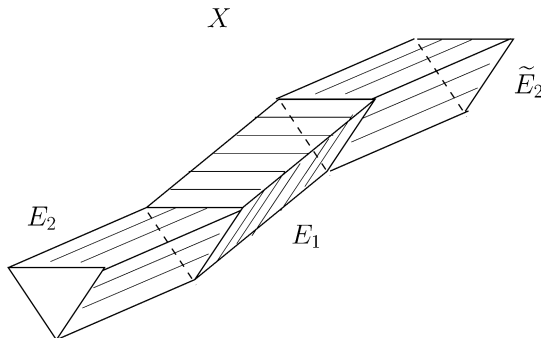

Figure 6   The case ψ birational

Lemma 3.11 states that $\text{Exc}(\psi) \cap A$ is a union of fibers of ψ, but ψ is finite on A and so $\text{Exc}(\psi) \cap A = \emptyset$. Hence the composition

$$\psi \circ \varphi_2 \colon X \longrightarrow Z$$

is just the blow-up of two disjoint subvarieties in Z. Set $\tilde{E}_2 := \varphi_2^{-1}(\text{Exc}(\psi))$, so that $\text{Exc}(\psi \circ \varphi_2) = E_2 \cup \tilde{E}_2$.

Let's show that $E_2 \cdot R_1 = \tilde{E}_2 \cdot R_1 = 0$. The intersection $E_1 \cap E_2$ has pure dimension $n - 2 \geq 2$, so $\varphi_1|_{E_1 \cap E_2} \colon E_1 \cap E_2 \to \varphi_1(E_1)$ has positive dimensional fibers. Take a curve C in one of these fibers: then $[C] \in R_1$ and $C \subset E_2$; hence $C \cap \tilde{E}_2 = \emptyset$ and so $\tilde{E}_2 \cdot R_1 = 0$. In the same way we see that $E_2 \cdot R_1 = 0$. Therefore both $E_1 \cap E_2$ and $E_1 \cap \tilde{E}_2$ are the union of finitely many fibers of φ_1.

We apply Remark 4.7 to R_1 and R_2, which allows us to deduce that $R_1 + R_2$ is a face of $\text{NE}(X)$ and that $S_1 := (\varphi_2)_*(R_1)$ is an extremal ray of $\text{NE}(Y_2)$ of type $(n - 1, 1)$ with exceptional divisor D.

By (4.9) we know that, apart from S_1, the other possible extremal ray contained in $\mathcal{N}_1(D, Y_2)$ is $(\varphi_2)_*(R_0)$. It is easy to see that $E_2 \cdot R_0 > 0$ and $D \cdot (\varphi_2)_*(R_0) \geq 0$. This shows that S_1 is the unique extremal ray of $\text{NE}(Y_2)$ having negative intersection with D, and Theorem 4.2 then yields $\rho_{Y_2} \leq 4$ and $\rho_X \leq 5$.

Recall that $\text{NE}(\psi)$ is a birational extremal ray of $\text{NE}(Y_2)$ with $D \cdot \text{NE}(\psi) > 0$ and $\text{NE}(\psi) \not\subset \mathcal{N}_1(D, Y_2)$. Moreover, $\tilde{E}_2 \cdot R_1 = 0$ in X yields $\text{Exc}(\psi) \cdot S_1 = 0$ in Y_2. Hence we can apply Remark 4.7 to Y_2, S_1, and $\text{NE}(\psi)$ as we did for X, R_1, and R_2. We deduce that Z is Fano, $\psi_*(S_1)$ is an extremal ray of type $(n - 1, 1)$ with exceptional divisor $\psi(D)$, and X is the blow-up of Z in two fibers of the associated contraction. Notice that $\psi \circ \varphi_2$ is finite and birational on E_1, so the normalization of $\psi(D)$ is $W \times \mathbb{P}^1$. $\qquad\square$

Proof of Lemma 4.9. We proceed similarly to the proof of [AW2, Lemma 1.2.2]. Let K_π be the relative canonical bundle of π. Let's show that $K_\pi \cdot C = 0$ for every curve C contracted by ϕ. Because ϕ is a Mori contraction, it is enough to show this when C is an irreducible rational curve. Then $\pi(C)$ is again an irreducible rational curve. Let $\nu \colon \mathbb{P}^1 \to W$ be the morphism given by the normalization of $\pi(C) \subset W$, and consider the fiber product of $\mathbb{P}^1$ and E over W:

$$
\begin{array}{ccc}
& \overset{\displaystyle \phi_C}{\overbrace{\hspace{4cm}}} & \\
E_C & \xrightarrow{\tilde{\nu}} \quad E \quad \xrightarrow{\phi} & \mathbb{P}^r \\
{\scriptstyle \pi_C}\downarrow & \downarrow{\scriptstyle \pi} & \\
\mathbb{P}^1 & \xrightarrow{\ \nu\ } \quad W. &
\end{array}
$$

Notice that $\pi_C \colon E_C \to \mathbb{P}^1$ is a $\mathbb{P}^r$-bundle and $K_{\pi_C} = \tilde{\nu}^*(K_\pi)$.

Set $\phi_C := \phi \circ \tilde{\nu}$. Then ϕ_C is surjective and its Stein factorization gives a contraction $\xi_C \colon E_C \to P$ that is finite on fibers of π_C and such that $\dim P = r$. This easily implies (e.g., using toric geometry) that $E_C \cong \mathbb{P}^1 \times \mathbb{P}^r$, $P \cong \mathbb{P}^r$, and ξ_C is the projection. Then $K_{\pi_C} = \xi_C^*(K_{\mathbb{P}^r})$.

Now set $\tilde{C} := \tilde{\nu}^{-1}(C) \subset E_C$. Since $\phi(C) = \{pt\}$, we have $\phi_C(\tilde{C}) = \{pt\}$ and hence $\xi_C(\tilde{C}) = \{pt\}$. Moreover, $\tilde{\nu}_*(\tilde{C}) = mC$ for some $m \in \mathbb{Z}_{\geq 1}$. Finally, we have

$$K_\pi \cdot C = \frac{1}{m} K_\pi \cdot \tilde{\nu}_*(\tilde{C}) = \frac{1}{m} \tilde{\nu}^*(K_\pi) \cdot \tilde{C} = \frac{1}{m} K_{\pi C} \cdot \tilde{C} = \frac{1}{m} \xi_C^*(K_{\mathbb{P}^r}) \cdot \tilde{C} = 0.$$

Let F be a general fiber of ϕ, and let d be the degree of the finite map $E \to W \times \mathbb{P}^r$ induced by π and ϕ. Then $g := \pi|_F : F \to W$ is finite of degree d. Since F is Fano, the numerical and linear equivalence for divisors in F coincide and therefore $(K_\pi)|_F \cong \mathcal{O}_F$. We get

$$K_F = (K_E)|_F = (K_\pi + \pi^* K_W)|_F = g^* K_W,$$

so that g is étale. Then W is Fano, too; in particular, it is simply connected. Thus g is an isomorphism and $d = 1$.

Proof of Corollary 1.2. The statement is a straightforward consequence of Theorem 1.1 and [Ta, Thm. 1.1]. □

Proof of Corollary 1.3. By Proposition 3.1, X cannot have elementary contractions of type $(3, 0)$. The possible values of ρ_X when X has a fiber type elementary contraction follow from [Ca2, Cor. 1.2], and they follow from Theorem 1.1 for type $(3, 1)$.

Concerning positive examples, any product of Del Pezzo surfaces $X \cong S_1 \times S_2$ with $\rho_{S_1} \geq 3$ has elementary contractions of type $(3, 2)$. In the toric case one can find examples of Fano 4-folds with small contractions up to Picard number 6; see [Ba] and [Ca2, Ex. 7.9]. □

EXAMPLE 4.10. It is not difficult to find examples of Fano varieties X, as in Theorem 1.1, with $\rho_X = 5$. For instance, in the toric case we know after [S] (and [Ba] for the 4-dimensional case) that there are exactly $n - 2$ possibilities for X, which can be obtained as follows.

Let a be an integer with $1 \leq a \leq n - 2$, and consider

$$Z := \mathbb{P}_{\mathbb{P}^{n-2} \times \mathbb{P}^1}(\mathcal{O}(0, 1) \oplus \mathcal{O}(a, 0)).$$

Then Z is Fano with $\rho_Z = 3$. The $\mathbb{P}^1$-bundle $Z \to \mathbb{P}^{n-2} \times \mathbb{P}^1$ has a section E_Z with normal bundle $\mathcal{N}_{E_Z/Z} \cong \mathcal{O}_{\mathbb{P}^{n-2} \times \mathbb{P}^1}(-a, 1)$, and Z has an extremal ray of type $(n - 1, 1)$ with exceptional divisor E_Z. Blowing up Z along $\mathbb{P}^{n-2} \times \{p_1, p_2\} \subset E_Z$ (where $p_1, p_2 \in \mathbb{P}^1$ are two distinct points) yields a toric Fano variety X with $\rho_X = 5$, where the proper transform $E \cong \mathbb{P}^{n-2} \times \mathbb{P}^1$ of E_Z has normal bundle $\mathcal{O}_{\mathbb{P}^{n-2} \times \mathbb{P}^1}(-a, -1)$. Finally, X has an extremal ray of type $(n - 1, 1)$ and one of type $(n - 1, n - 2)^{\text{sm}}$, both with exceptional divisor E.

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