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Some Results on the Second Gaussian Map for Curves

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1. Introduction

The first Gaussian map for the canonical series has been intensively studied. It has been shown that, for a general curve of genus different from 9 and ≤ 10, the first Gaussian map is injective, while for genus ≥ 10 and different from 11 it is surjective [7; 9; 20]. In [21] it is proved that if a curve lies on a $K3$ surface then the first Gaussian map cannot be surjective, and it is known (see [18]) that the general curve of genus 11 lies on a $K3$ surface.

In this paper we study some properties of the second Gaussian map

$$\mu_2 \colon I_2(K_X) \to H^0(X, 4K_X).$$

Our geometrical motivation comes from its relation with the curvature of the moduli space M_g of curves of genus g endowed with the Siegel metric induced by the period map $j \colon M_g \to A_g$, which we started to analyze in [10]. There the curvature is computed using the formula for the associated second fundamental form given in [11]. In particular, in [11] it is proved that the second fundamental form lifts the second Gaussian map μ_2, as stated in an unpublished paper of Green and Griffiths (cf. [15]).

In [10, Cor. (3.8)] we give a formula for the holomorphic sectional curvature of M_g along the a Schiffer variation ξ_P, for P a point on the curve X, in terms of the holomorphic sectional curvature of A_g and the second Gaussian map.

The relation of the second Gaussian map with curvature properties of M_g in A_g suggests that its rank could give information on the geometry of M_g. Note that surjectivity can be expected for a general curve of genus at least 18. Recall that M_g is uniruled for $g \leq 15$, has Kodaira dimension at least 2 for $g = 23$, and is of general type for all other values of $g \geq 22$.

Along these lines, in this paper we exhibit infinitely many examples of curves lying on the product of two curves with surjective second Gaussian map. Other examples of curves whose second Gaussian map is surjective were given in [4] for complete intersections. Both classes of examples generalize constructions given by Wahl [21; 22] for the first Gaussian map.

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We are also able to determine the rank of μ_2 on the hyperelliptic and trigonal loci. More precisely: for any hyperelliptic curve of genus $g \geq 3$, we show that $\mathrm{rk}(\mu_2) = 2g - 5$ and that its image has the Weierstrass points as base points. For any trigonal (non-hyperelliptic) curve of genus $g \geq 8$, we show that $\mathrm{rk}(\mu_2) = 4g - 18$ and that its image has the ramification points of the g^1_3 as base points. Finally, we prove that for any non-hyperelliptic, non-trigonal curve of genus $g \geq 5$, the image of μ_2 has no base points.

In [10] we apply these results to the holomorphic sectional curvature of M_g. In particular, along a Schiffer variation ξ_P, the holomorphic sectional curvature $H(\xi_P)$ of M_g is strictly smaller than the holomorphic sectional curvature of A_g for a non-trigonal, non-hyperelliptic curve [10, (4.4)]. Instead, if P is either a Weierstrass point of a hyperelliptic curve or a ramification point of the g^1_3 on a trigonal curve, then the holomorphic sectional curvature $H(\xi_P)$ is equal to the holomorphic sectional curvature of A_g, which equals -1 [10, (4.4) and (5.3)].

The computations are based on the observation that, for a quadric Q of rank at most 4, $\mu_2(Q)$ can be written as the product of the first Gaussian maps associated to sections of the two adjoint line bundles L and $K \otimes L^{-1}$, which define the quadric. As a first straightforward consequence we show that, for any curve, the rank of μ_2 is greater or equal to $g - 3$.

In order to study the trigonal case, we use the related results of [6] and [13] on the first Gaussian map for trigonal curves. A crucial step in determining the rank of the first (and hence the second) Gaussian map on the trigonal locus is the observation that a trigonal curve lies on a rational normal scroll. A natural question is to understand whether restrictions on the rank of μ_2 can be obtained if a curve lies on a special surface, as it happens for the first Gaussian map and $K3$ surfaces, or if it occurs in a nontrivial linear system of a surface (see [19] for related results). We intend to continue our investigations on the rank properties of μ_2 for general curves and for curves on surfaces in the near future.

The paper is organized as follows. In Section 2 we study the second Gaussian map, on quadrics of rank ≤ 4, in terms of the first Gaussian maps associated to sections of the two adjoint line bundles L and $K \otimes L^{-1}$, which define the quadric. In Section 3 we show a class of infinitely many examples of curves with surjective second Gaussian map, and in Section 4 we determine the rank of μ_2 for hyperelliptic and trigonal curves.

In Section 5 we prove injectivity of μ_2 for general curves of genus at ≤ 6 by specialization on trigonal curves and on smooth plane quintics. Finally, in Section 6 we study the global generation of the image of μ_2.

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2. The Second Gaussian Map

We first recall the definition of the Gaussian maps (cf. [23]). Let X be a smooth projective curve, let $S := X \times X$, and let $\Delta \subset S$ be the diagonal. Let L be a line bundle on X and let $L_S := p_1^*(L) \otimes p_2^*(L)$, where $p_i\colon S \to X$ are the natural projections. Consider the restriction map

$$\tilde{\mu}_{n,L}\colon H^0(S, L_S(-n\Delta)) \to H^0(\Delta, L_S(-n\Delta)|_\Delta).$$

Notice that, since $\mathcal{O}(\Delta)|_\Delta \cong T_X$,

$$H^0(\Delta, L_S(-n\Delta)|_\Delta) \cong H^0(X, 2L \otimes nK_X).$$

In the case $L = K_X$ we have $I_2(K_X) \subset H^0(S, K_S(-2\Delta))$, so we can define the second Gaussian map

$$\mu_2\colon I_2(K_X) \to H^0(X, 4K_X)$$

as the restriction $\tilde{\mu}_{2,K}|_{I_2(K_X)}$.

As before, we fix a basis $\{\omega_i\}$ of $H^0(K_X)$. In local coordinates, $\omega_i = f_i(z)dz$. Let $Q \in I_2(K_X)$ and $Q = \sum_{i,j} a_{ij}\omega_i \otimes \omega_j$, and recall that $\sum_{i,j} a_{ij} f_i f_j \equiv 0$; since the $a_{i,j}$ are symmetric, we also have $\sum_{i,j} a_{ij} f_i' f_j \equiv 0$. The local expression of $\mu_2(Q)$ is

$$\mu_2(Q) = \sum_{i,j} a_{ij} f_i'' f_j (dz)^4 = -\sum_{i,j} a_{ij} f_i' f_j' (dz)^4. \tag{1}$$

We also recall the definition of the first Gaussian map (cf. [22])

$$\mu_{1,L}\colon \Lambda^2 H^0(L) \to H^0(2L \otimes K_X)$$

as the restriction of $\tilde{\mu}_{1,L}$ to $\Lambda^2 H^0(L) \subset H^0(S, L_S(-\Delta))$. In local coordinates, if $s_0, s_1 \in H^0(L)$ with $s_i = g_i l$, where l is a local section of L, then

$$\mu_{1,L}(s_0 \wedge s_1) = (g_0 g_1' - g_1 g_0') l^2 dz.$$

Moreover, we have

$$\operatorname{div}(\mu_{1,L}(s_0 \wedge s_1)) = 2F + R, \tag{2}$$

where F is the base locus of $|\langle s_0, s_1\rangle| \subset |H^0(L)|$ and R is the ramification divisor of the induced morphism (see e.g. [9; 23]).

Remark 2.1. Recall that there is the following bijection:

$$\begin{aligned} &\{[Q] \in \mathbb{P}(I_2(K_X)) \mid \operatorname{rk}(Q) \le 4\} \\ &\qquad \longleftrightarrow \{\{L, K_X - L, V, W\} \mid V \subset H^0(L), \dim V = 2, \\ &\qquad\qquad\qquad\qquad W \subset H^0(K_X - L), \dim W = 2\}. \end{aligned}$$

Here $\operatorname{rk}(Q) = 3$ if and only if $2L = K_X$ and $V = W$ (see e.g. [3, p. 261]).

Lemma 2.2. *If a quadric Q of rank at most 4 corresponds to $\{L, K_X - L, V, W\}$ and if $V = \langle s_0, s_1\rangle$ and $W = \langle t_0, t_1\rangle$, then*

$$\mu_2(Q) = \mu_{1,L}(s_0 \wedge s_1)\mu_{1,K-L}(t_0 \wedge t_1).$$

In particular, $\mu_2(Q) \neq 0$.

Proof. By construction, $Q = (s_0t_0) \otimes (s_1t_1) - (s_0t_1) \otimes (s_1t_0) \in I_2(K_X)$. Locally we have $s_i = g_i l$, where l is a local section of L, and $t_i = h_i l^{-1} dz$, so

$$\begin{aligned}\mu_2(Q) &= -((g_0h_0)'(g_1h_1)' - (g_0h_1)'(h_0g_1)')(dz)^4 \\ &= (g_0g_1' - g_1g_0')(h_0h_1' - h_1h_0')(l^2dz)((l^{-1}dz)^2dz) \\ &= \mu_{1,L}(s_0 \wedge s_1)\mu_{1,K-L}(t_0 \wedge t_1). \qquad \square\end{aligned}$$

REMARK 2.3. Recall that, by a theorem of M. Green ([14]; see also [3, p. 255]), for a non-hyperelliptic smooth curve of genus $g \geq 4$ it follows that I_2 is generated by quadrics of rank ≤ 4.

We now make an easy linear algebra remark that will be useful in the sequel.

REMARK 2.4. Let $X \subset \mathbb{P}^n = \mathbb{P}(V)$ be a projective variety. Let $f: V \to W$ be a linear map, $\dim(W) = m + 1$, and let $\bar{f}: \mathbb{P}(V) = \mathbb{P}^n \dashrightarrow \mathbb{P}(W) = \mathbb{P}^m$ be the corresponding projection. Let K be the kernel of f, and assume that $\mathbb{P}(K) \cap X = \emptyset$ (i.e., $\bar{f}|_X$ is a morphism). This clearly implies that $\dim(X) + \dim(K) - 1 \leq n - 1$ or, equivalently, that $\mathrm{rk}(f) \geq \dim(X) + 1$.

Consider the rational map

$$\bar{\mu}_2 : \mathbb{P}(I_2(K_X)) \dashrightarrow \mathbb{P}(H^0(4K_X)), \quad [Q] \mapsto [\mu_2(Q)].$$

Let $\Gamma = \{[Q] \in \mathbb{P}(I_2(K_X)) \mid \mathrm{rk}(Q) \leq 4\}$; then, by Lemma 2.2, the restriction of this map to Γ is a morphism.

PROPOSITION 2.5. *For any curve of genus $g \geq 4$,*

$$\mathrm{rk}(\mu_2) \geq \dim \Gamma + 1 \geq g - 3.$$

Proof. Since $\bar{\mu}_2|_\Gamma$ is a morphism, by Remark 2.4 we know that $\mathrm{rk}(\mu_2) \geq \dim \Gamma + 1$. Denote by $\mathcal{W} \subset W^1_{g-1}$ the subset of line bundles $L \in W^1_{g-1}$ such that $h^0(L) = 2$; $\mathcal{W}$ is a nonempty open subset of W^1_{g-1} of dimension $\geq g - 4$ (cf. e.g. [3]). If we set $\mathcal{Y} := \mathcal{W}/\langle\tau\rangle$, where τ is the involution that maps L to $K_X - L$, then we can identify $\mathcal{Y}$ with a subset of Γ. In fact, given a line bundle $L \in \mathcal{W}$, the set $\{L, K - L, H^0(L), H^0(K - L)\}$ determines a quadric of rank ≤ 4, as we saw in Remark 2.1.

Therefore, $\dim(\Gamma) \geq \dim(\mathcal{W}) \geq g - 4$ and so $\mathrm{rk}(\mu_2) \geq g - 3$. $\square$

3. Surjectivity

In this section we give a class of examples of curves contained in the product of two curves for which the second Gaussian map is surjective, as Wahl does in [22, Thm. 4.11] for the first Gaussian map. Other examples of curves whose second Gaussian map is surjective have been obtained by Ballico and Fontanari [4] in the case of complete intersections, generalizing Wahl's result [21] on the first Gaussian map for complete intersections.

Let C_1, C_2 be two smooth curves of respective genera g_1, g_2; denote by $K_i = K_{C_i}$, $i = 1, 2$; and choose D_i divisors on C_i of degree d_i, $i = 1, 2$. Let $X = C_1 \times C_2$, and let $C \in |p_1{}^*(D_1) \otimes p_2{}^*(D_2)|$ be a smooth curve, where p_i is the projection from $C_1 \times C_2$ on C_i and $K_X(C) = p_1{}^*(K_1(D_1)) \otimes p_2{}^*(K_2(D_2))$.

THEOREM 3.1. *If $g_1, g_2 \geq 2$ with $d_i \geq 2g_i + 5$ for $i = 1, 2$, or if $g_1 \geq 2$ and $g_2 = 1$ with $d_1 \geq 2g_1 + 5$ and $d_2 \geq 7$, or if $g_2 = 0$ with $d_2 \geq 7$ and $d_2(g_1 - 1) > 2d_1 \geq 4g_1 + 10$, then μ_{2,K_C} is surjective for a smooth curve $C \in |{p_1}^*D_1 \otimes {p_2}^*D_2|$.*

Therefore, under these assumptions and for the general curve of genus $g = 1 + (g_2 - 1)d_1 + (g_1 - 1)d_2 + d_1 d_2$, the second Gaussian map is surjective.

Proof. Denote by $I_2(K_X(C))$ the kernel of the multiplication map

$$S^2 H^0(K_X(C)) \to H^0(K_X^2(2C)).$$

Let $\mu^X_{2,K_X(C)} \colon I_2(K_X(C)) \to H^0(S^2\Omega^1_X \otimes K_X^2(2C))$ be the second Gaussian map of the line bundle $K_X(C)$ on the surface X. We have the following commutative diagram:

$$\begin{tikzcd}
I_2(K_X(C)) \arrow[r, "\mu^X_{2,K_X(C)}"] \arrow[dd, "g"] & H^0(S^2\Omega^1_X \otimes K_X^2(2C)) \arrow[dr, "p_1"] & \\
& & H^0(S^2\Omega^1_X|_C \otimes K_C^2), \arrow[dl, "p_2"] \\
I_2(K_C) \arrow[r, "\mu_2"'] & H^0(K_C^4) &
\end{tikzcd}$$

where p_1 is the restriction map and p_2 is the map that comes from the conormal extension.

We will prove that p_1, p_2 and $\mu^X_{2,K_X(C)}$ are surjective. From this we clearly obtain the surjectivity of μ_2.

We want to show that $H^1(\Omega^1_X|_C \otimes K_C^2(-C)) = 0$, from which the surjectivity of p_2 will follow. We have

$$\begin{aligned} H^1(\Omega^1_X|_C \otimes K_C^2(-C)) = {} & H^1\big(C, \mathcal{O}_C({p_1}^*(K_1^3(D_1)) \otimes {p_2}^*(K_2^2(D_2)))\big) \\ & \oplus H^1\big(C, \mathcal{O}_C({p_1}^*(K_1^2(D_1)) \otimes {p_2}^*(K_2^3(D_2)))\big), \end{aligned}$$

so it is sufficient to check that, under our assumptions,

$$\mathcal{O}_C({p_1}^*(K_1^3(D_1)) \otimes {p_2}^*(K_2^2(D_2))) \quad \text{and} \quad \mathcal{O}_C({p_1}^*(K_1^2(D_1)) \otimes {p_2}^*(K_2^3(D_2)))$$

both have degree greater than $2g(C) - 2 = d_1(2g_2 - 2 + d_2) + d_2(2g_1 - 2 + d_1)$.

Let us now consider the map p_1. We have

$$\begin{aligned} S^2\Omega^1_X \otimes K_X^2(C) = {} & ({p_1}^*(K_1^4(D_1)) \otimes {p_2}^*(K_2^2(D_2))) \\ & \oplus ({p_1}^*(K_1^2(D_1)) \otimes {p_2}^*(K_2^4(D_2))) \\ & \oplus ({p_1}^*(K_1^3(D_1)) \otimes {p_2}^*(K_2^3(D_2))); \end{aligned}$$

thus, by Künneth, if $g_i \geq 1$ for $i = 1, 2$ or if $g_2 = 0$ with $d_2 \geq 7$ and $g_1 \geq 1$, then $H^1(S^2\Omega^1_X \otimes K_X^2(C)) = 0$. Hence p_1 is surjective.

We want now to show that $\mu^X_{2,K_X(C)}$ is surjective. Observe that

$$S^2H^0(K_X(C)) = (S^2H^0(K_1(D_1)) \otimes S^2H^0(K_2(D_2))) \\ \oplus (\Lambda^2H^0(K_1(D_1)) \otimes \Lambda^2H^0(K_2(D_2))),$$

so we have

$$I_2(K_X(C)) = \{(I_2(K_1(D_1)) \otimes S^2H^0(K_2(D_2))) \\ + (S^2H^0(K_1(D_1)) \otimes I_2(K_2(D_2)))\} \\ \oplus (\Lambda^2H^0(K_1(D_1)) \otimes \Lambda^2H^0(K_2(D_2))).$$

Since

$$H^0(S^2\Omega^1_X \otimes K_X^2(2C)) = (H^0(C_1, K_1^4(2D_1)) \otimes H^0(C_2, K_2^2(2D_2))) \\ \oplus (H^0(C_1, K_1^2(2D_1)) \otimes H^0(C_2, K_2^4(2D_2))) \\ \oplus (H^0(C_1, K_1^3(2D_1)) \otimes H^0(C_2, K_2^3(2D_2))),$$

one can easily check that $\mu^X_{2,K_X(C)}\colon I_2(K_X(C)) \to H^0(S^2\Omega^1_X \otimes K_X^2(2C))$ is the sum of the three following maps:

$$\mu_{2,K_1(D_1)} \otimes m_2\colon I_2(K_1(D_1)) \otimes S^2H^0(K_2(D_2)) \\ \to H^0(K_1^4(2D_1)) \otimes H^0(K_2^2(2D_2)),$$

$$n_2 \otimes \mu_{2,K_2(D_2)}\colon S^2H^0(K_1(D_1)) \otimes I_2(K_2(D_2)) \\ \to H^0(K_1^2(2D_1)) \otimes H^0(K_2^4(2D_2)),$$

$$\mu_{1,K_1(D_1)} \otimes \mu_{1,K_2(D_2)}\colon \Lambda^2(H^0(K_1(D_1)) \otimes \Lambda^2H^0(K_2(D_2)) \\ \to H^0(K_1^3(2D_1)) \otimes H^0(K_2^3(2D_2));$$

here m_2 and n_2 are the multiplication maps. Now we apply [5, Thm. (1.7)] to the line bundles $L_i := K_i(D_i)$ on the curves C_i, $i = 1, 2$, to obtain that if $\deg(L_i) =: l_i$ satisfies $2l_i \geq 3(2g_i + 2) + 2g_i - 1$ then both μ_{2,L_i} and μ_{1,L_i} are surjective. Therefore, if $d_i \geq 2g_i + 5$, then μ_{2,L_i} and μ_{1,L_i} are surjective; hence $\mu^X_{2,K_X(C)}$ is surjective, and this concludes the proof. □

REMARK 3.2. The example of lowest genus of a smooth curve $C \in |p_1{}^*D_1 \otimes p_2{}^*D_2|$ with surjective second Gaussian map is 71, obtained by choosing $g_1 = 2$, $g_2 = 1$, $d_1 = 9$, and $d_2 = 7$.

4. Hyperelliptic and Trigonal Curves

Assume now that X is either a hyperelliptic curve of genus ≥ 3 or a trigonal curve of genus $g \geq 4$. Let $|F|$ denote the g^1_2 in the hyperelliptic case or the g^1_3 in the trigonal case. Let $\phi_F\colon X \to \mathbb{P}^1$ be the induced morphism and let $\nu\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^{g-1}$ be the Veronese embedding, so that (in the hyperelliptic case) $\phi_K = \nu \circ \phi_F$, where ϕ_K is the canonical map. Observe that, in the hyperelliptic case, the hyperelliptic involution τ acts as $-\mathrm{Id}$ on $H^0(K_X)$; we thus have an exact sequence

$$0 \to I_2(K_X) \to S^2(H^0(K_X)) \to H^0(2K_X)^+ \to 0,$$

where $H^0(2K_X)^+$ denotes the τ-invariant part of $H^0(2K_X)$ whose dimension is $(2g-1)$ and where $I_2(K_X)$ is the vector space of the quadrics containing the rational normal curve.

Set $L := K_X - F$. Fix a basis $\{x, y\}$ of $H^0(F)$ and a basis $\{t_1, \ldots, t_r\}$ of $H^0(L)$, both in the hyperelliptic and in the trigonal case. We have a linear map

$$\psi\colon \Lambda^2(H^0(L)) \to I_2, \quad t_i \wedge t_j \mapsto Q_{ij} = xt_i \odot yt_j - xt_j \odot yt_i.$$

We recall that, in both cases, the linear map $\psi\colon \Lambda^2(H^0(L)) \to I_2$ is an isomorphism, which can be easily checked or found in [1].

LEMMA 4.1. *Let X be either a hyperelliptic curve of genus ≥ 3 or a trigonal curve of genus $g \geq 4$, and let $q_1, \ldots, q_l$ be the ramification points of either the g^1_2 or the g^1_3. Then*

$$\mu_2(Q) = \mu_{1,F}(x \wedge y)\mu_{1,L}(\psi^{-1}(Q))$$

for any quadric Q of rank 4. In particular, the image of μ_2 is contained in $H^0(4K_X - (q_1 + \cdots + q_l))$ and $\mathrm{rk}(\mu_2) = \mathrm{rk}(\mu_{1,L})$.

Proof. The first statement is straightforward. So we have

$$\begin{aligned}\mathrm{div}(\mu_2(Q)) &= \mathrm{div}(\mu_{1,F}(x \wedge y)) + \mathrm{div}(\mu_{1,L}(\psi^{-1}(Q)))\\ &= q_1 + \cdots + q_l + \mathrm{div}(\mu_{1,L}(\psi^{-1}(Q))).\end{aligned}$$

Therefore, $\mu_2(Q)(q_i) = 0$ for all $i = 1, \ldots, l$. □

PROPOSITION 4.2. *Let X be a hyperelliptic curve of genus $g \geq 3$. Then the rank of μ_2 is $2g-5$.*

Proof. Given a hyperelliptic curve of genus g with equation $y^2 = f(x)$, where f has degree $2g+2$ and only simple roots, a basis of $H^0(K_X)$ is $\{\omega_i = x^i \frac{dx}{y} \mid 0 \leq i \leq g-1\}$. Let $|F|$ be the g^1_2 on X, and assume that $F = \phi_F^{-1}(0) =: p_1 + p_2$.

Set $L = K_X - F = K_X - p_1 - p_2$, let $H^0(L) \subset H^0(K_X)$, and let

$$\begin{aligned}\mu_{1,L}\colon \Lambda^2 H^0(L) &= \Lambda^2 H^0(K_X - p_1 - p_2)\\ &\to H^0(2L + K_X) = H^0(3K_X - 2p_1 - 2p_2)\end{aligned}$$

be the first Gaussian map of L; then

$$\mu_{1,L} = \mu_{1,K}|_{\Lambda^2 H^0(K_X - p_1 - p_2)}.$$

By Lemma 4.1, the rank of μ_2 is equal to the rank of $\mu_{1,L}$. As shown in [9],

$$\mu_{1,K}(\omega_i \wedge \omega_j) = (i-j)\frac{x^{i+j-1}}{y^2}(dx)^3, \quad 0 \leq i < j \leq g-1,$$

so there are exactly $2g-3$ distinct powers of x.

Then clearly a basis of $H^0(K_X - p_1 - p_2)$ is given by $\{x^i \frac{dx}{y}, i > 0\}$. We want to compute the dimension of the span of $\{\mu_{1,K}(\omega_i \wedge \omega_j), 0 < i < j \leq g-1\}$. Observe that $l := i + j - 1 = 0$ if and only if $i = 0$ and $j = 1$; $l = 1$ if and only if $i = 0$ and $j = 2$. But if $l \geq 2$, then $l = i + j - 1$ also for some $i, j > 0$.

Therefore, $\mathrm{rk}(\mu_{1,L}) = \mathrm{rk}(\mu_{1,K}) - 2 = 2g - 5$. □

Assume now that X is a non-hyperelliptic trigonal curve of genus $g \geq 4$. Let $|F|$ be the g^1_3 on X and assume that $F = p_1 + p_2 + p_3$, $p_i \in X$. Let us denote by $L = K_X - F = K_X - p_1 - p_2 - p_3$, $\deg(L) = 2g-5$, and $h^0(L) = g-2$. So $H^0(L) \subset H^0(K_X)$ and $\mu_{1,L} = \mu_{1,K}|_{\Lambda^2 H^0(K_X - p_1 - p_2 - p_3)}$. In [9] it is proved that, for the general trigonal curve of genus $g \geq 4$, $\dim(\operatorname{coker}(\mu_{1,K})) = g+5$; moreover, specific examples of trigonal curves (whose genera are all equal to 1 modulo 3) such that the corank of $\mu_{1,K}$ is $g+5$ are exhibited. Using results of [13], in [6] Brawner proved that $\dim(\operatorname{coker}(\mu_{1,K})) = g+5$ for any trigonal curve of genus $g \geq 4$.

We shall now compute the rank of μ_2 for trigonal curves. By Lemma 4.1 it suffices to compute $\operatorname{rk}(\mu_{1,L})$, which we shall do following the computation used in [13] and [6] for $\mu_{1,K}$.

Recall that a canonically embedded trigonal curve of genus g lies on a rational normal scroll $S_{k,l}$, where $k \leq l$ and $l + k = g - 2$; here k is the Maroni invariant, which is bounded by

$$\frac{g-4}{3} \leq k \leq \frac{g-2}{2} \tag{3}$$

(cf. [17]).

The surface $S_{k,l}$ is isomorphic to $\mathbf{F}_n$ with $n = l - k$. Let us denote by H the hyperplane section and by R the fiber of the ruling; set $B \equiv H - lR$. We have

$$\begin{gathered} H^2 = g-2, \qquad B^2 = -n, \\ C \equiv 3H - (g-4)R, \\ K_S \equiv -2H + (g-4)R; \end{gathered}$$

consequently,

$$K_S + C - R \equiv H - R \equiv B + (l-1)R$$

and

$$(K_S + C - R)|_C \equiv L.$$

THEOREM 4.3. *For any trigonal curve C of genus $g \geq 8$, the rank of μ_2 is $4g-18$. Hence, for the general curve of genus $g \geq 8$, μ_2 has rank $\geq 4g-18$.*

Proof. As in [13, (2.1)], we have the following commutative diagram involving the first Gaussian map $\mu^S_{1,H-R}$ for the scroll $S := S_{k,l}$:

$$\begin{array}{ccc} \Lambda^2 H^0(S, \mathcal{O}_S(K_S + C - R)) & \xrightarrow{\mu^S_{1,H-R}} & H^0(S, \Omega^1_S(2K_S + 2C - 2R)) \\ \Big\downarrow{\scriptstyle \mathrm{Res}} & & \Big\downarrow{\scriptstyle \gamma'} \\ \Lambda^2 H^0(C, K_C - F)) & \xrightarrow{\mu_{1,L}} & H^0(C, 3K_C - 2F). \end{array}$$

We will prove that the map $\mu^S_{1,H-R}$ is surjective, that γ' is injective, and that Res is surjective. This implies that $\operatorname{rk}(\mu_{1,L}) = h^0(S, \Omega^1_S(2K_S + 2C - 2R)) = h^0(S, \Omega^1_S(2B + 2(l-1)R))$.

Observe that, by the bound (3) of the Maroni invariant, $k \geq 2$ for $g \geq 8$; hence the hypotheses of [13, Cor. (3.3.2)] are satisfied, so $h^0(S, \Omega^1_S(2B + 2(l-1)R)) = 4g - 18$. In fact, [13, Cor. (3.3.2)] asserts that $h^0(S, \Omega^1_S(rB + sR)) = 2rs - nr^2 - 2$ if $r \geq 1$ and $s \geq nr + 2$.

The surjectivity of $\mu^S_{1,H-R}$ follows by [13, Thm. (4.5)], which states that $\mu^S_{1,rB+sR}$ is surjective if $r \geq 0$ and $s \geq nr + 1$.

In [6, (3.4)] it is proved that the map

$$\gamma\colon H^0(S, \Omega^1_S(2H)) \to H^0(C, 3K_C)$$

is injective. Since γ' is the restriction of γ to $H^0(S, \Omega^1_S(2H - 2R))$, it follows that γ' is also injective.

We finally show that the restriction map

$$H^0(S, \mathcal{O}_S(H - R)) \to H^0(C, L)$$

is surjective. Consider the exact sequence

$$0 \to \mathcal{O}_S(H - R - C) \to \mathcal{O}_S(H - R) \to \mathcal{O}_C(H - R) \to 0.$$

An easy computation on the scroll shows that

$$H^1(S, \mathcal{O}_S(H - R - C)) = H^1(S, \mathcal{O}_S(-2H + (g-5)R)) = 0,$$

proving our assertion. □

5. Injectivity for Low Genus

We now give some examples of computations of the rank of μ_2 for genus ≤ 7, from which will follow that μ_2 is injective for the general curve of genus ≤ 6. Note that if $g(X) = 4$, then $I_2(K_X)$ has dimension 1 and so μ_2 is injective.

PROPOSITION 5.1. *For any trigonal curve X of genus 5, μ_2 is injective. Hence, for the general curve of genus 5, μ_2 is injective.*

Proof. For a curve of genus 5, the dimension of $I_2(K_X)$ is 3. Let us assume that X is trigonal. Then there exists a line bundle L on X such that $h^0(L) = 2$ with $\deg(L) = 3$, so that $h^0(K - L) = 3$. Let $\mathrm{Gr}(2, H^0(K - L))$ be the Grassmannian of the 2-dimensional subspaces in $H^0(K - L)$. To any $W \in \mathrm{Gr}(2, H^0(K - L))$ we associate the quadric Q_W of rank 4 corresponding to the set $\{L, K - L, H^0(L), W\}$ as in Lemma 2.2. Thus we have a morphism

$$\mathrm{Gr}(2, H^0(K - L)) \to \mathbb{P}(H^0(4K)), \quad W \mapsto \bar{\mu}_2(Q_W).$$

Then, by Remark 2.4, we have

$$\mathrm{rk}(\mu_2) \geq \dim(\mathrm{Gr}(2, H^0(K - L)) + 1 = 3. \qquad \square$$

THEOREM 5.2. *Let X be a smooth plane quintic; then the map μ_2 is injective and its image has no base points. Then, for the general curve of genus 6, μ_2 is injective.*

Proof. Because X has genus 6, the dimension of $I_2(K_X)$ is 6. We will find six quadrics of rank at most 4 such that their images under μ_2 are linearly independent. Observe that $K_X \equiv \mathcal{O}_X(2)$; hence $L := \mathcal{O}_X(1)$ is such that $2L \equiv K_X$. Let q_1, q_2, q_3 be distinct points of X, in general position, such that the tangent line r_i of X at q_i is a simple tangent, $i = 1, 2, 3$. Assume also that $P_{12} := r_1 \cap r_2$, $P_{13} := r_1 \cap r_3$, and $P_{23} := r_2 \cap r_3$ are in general position and do not lie on X. Denote by $\pi_{12}, \pi_{13}, \pi_{23}$ the respective projections $X \to \mathbb{P}^1$ from the points P_{12}, P_{13}, P_{23}. These three projections π_{ij} correspond to three pencils $V_{ij} \subset H^0(L)$ with $2L \equiv K$. Let R_{ij} (for $1 \le i < j \le 3$) be the ramification divisor of π_{ij}; then we have

$$R_{ij} = q_i + q_j + A_{ij}.$$

Observe that, by construction, $q_3 \notin A_{12}$; otherwise, P_{12} would lie on r_3 and we would have $P_{12} = P_{13} = P_{23}$. Analogously $q_1 \notin A_{23}$ and $q_2 \notin A_{13}$. Observe that, since r_2 and r_3 are simple tangents, it follows that $q_2, q_3 \notin A_{23}$ and hence there must exist a point $q_4 \in A_{23}$ that is different from q_2 and q_3. Notice that $q_4 \notin R_{12} \cup R_{13}$. In fact, by construction $P_{23} \in r_4$ (r_4 is the tangent line of X at q_4), hence $P_{23} = r_2 \cap r_3 = r_4 \cap r_2 = r_4 \cap r_3$. So if $q_4 \in R_{12}$ then $P_{12} \in r_4$; thus $P_{12} = r_4 \cap r_2 = P_{23}$, a contradiction. Analogously, if $q_4 \in R_{13}$ then we get $P_{12} = P_{23} = P_{13}$, which is impossible.

Define now the six quadrics in Γ by the following six sets as in Lemma 2.2:

$$Q_{ij,kl} \longleftrightarrow \{L, L = K - L, V_{ij}, V_{kl}\}$$

for $1 \le i < j \le 3$ and $1 \le k < l \le 3$. Then, by Lemma 2.2, we have

$$\begin{aligned}
\operatorname{div}(\mu_2(Q_{12,12})) &= 2R_{12} = 2q_1 + 2q_2 + 2A_{12},\\
\operatorname{div}(\mu_2(Q_{12,13})) &= R_{12} + R_{13} = 2q_1 + q_2 + q_3 + A_{12} + A_{13},\\
\operatorname{div}(\mu_2(Q_{12,23})) &= R_{12} + R_{23} = q_1 + 2q_2 + q_3 + A_{12} + A_{23},\\
\operatorname{div}(\mu_2(Q_{13,13})) &= 2R_{13} = 2q_1 + 2q_3 + 2A_{13},\\
\operatorname{div}(\mu_2(Q_{13,23})) &= R_{13} + R_{23} = q_1 + q_2 + 2q_3 + A_{13} + A_{23},\\
\operatorname{div}(\mu_2(Q_{23,23})) &= 2R_{23} = 2q_2 + 2q_3 + 2A_{23}.
\end{aligned}$$

Assume now that there exists a linear combination $\sum \lambda_{ij,kl}\mu_2(Q_{ij,kl}) = 0$. Then, evaluating in q_1 yields $\lambda_{23,23}\mu_2(Q_{23,23})(q_1) = 0$ and so $\lambda_{23,23} = 0$, since $q_1 \notin R_{23}$. Evaluating in q_2 yields $\lambda_{13,13}\mu_2(Q_{13,13})(q_2) = 0$; thus $\lambda_{13,13} = 0$ since $q_2 \notin R_{13}$. By evaluating in q_3 we obtain $\lambda_{12,12}\mu_2(Q_{12,12})(q_3) = 0$; thus $\lambda_{12,12} = 0$ since $q_3 \notin R_{12}$. We now evaluate in $q_4 \in A_{23}$ and find that $\lambda_{12,13}\mu_2(Q_{12,13})(q_4) = 0$. This implies $\lambda_{12,13} = 0$, since otherwise we would have $q_4 \in A_{12} + A_{13}$, which is impossible. Finally, we must have $\lambda_{12,23} = \lambda_{13,23} = 0$, for otherwise we would have $R_{12} = R_{13}$—a contradiction. This proves the injectivity of μ_2.

Notice that if P is a base point of the image of μ_2, then the three projections π_{ij} must have a common ramification point, and by construction this is impossible.

□

If X is a trigonal curve of genus 7, then the argument of Proposition 5.1 yields $\operatorname{rk}(\mu_2) \ge \dim \operatorname{Gr}(2, H^0(K - L)) + 1 = 7$. We will now exhibit an example of a trigonal curve of genus 7 such that $\mathrm{rk}(\mu_2) = \mathrm{rk}(\mu_{1,L}) = 9$. Our example is the cyclic covering of $\mathbb{P}^1$, constructed in [9], whose affine equation is

$$y^3 = x^9 - 1.$$

In [9] the map $\mu_{1,K}$ is explicitly computed on the elements $\sigma_{ij} = x^i y^j \frac{dx}{y^2}$ for $0 \le j \le 1$ and $0 \le i \le 3(2-j) - 2$, which form a basis of $H^0(K_X)$. It is shown that the image of $\mu_{1,K}$ (of dimension 18) is spanned by the following three types of elements:

$$\mu_{1,K}(\sigma_{i0} \wedge \sigma_{k0}) = [(k-i)x^{i+k-1}y^{-4}] \cdot (dx)^3, \quad 0 \le i < k \le 4; \tag{4}$$

$$\mu_{1,K}(\sigma_{i0} \wedge \sigma_{k1}) = [(k-i)x^{i+k-1}y^{-3} + 3x^{i+k+8}y^{-6}] \cdot (dx)^3, \quad 0 \le i \le 4,\ 0 \le k \le 1; \tag{5}$$

$$\mu_{1,K}(\sigma_{01} \wedge \sigma_{11}) = y^{-2} \cdot (dx)^3. \tag{6}$$

The g^1_3 on our curve is the linear system $|F| = |p_1 + p_2 + p_3|$, where p_i can be chosen to be the points $(0, y_i)$ with $y_i^3 = -1$. Therefore, if $L = K_X - p_1 - p_2 - p_3$ then we can identify $H^0(L)$ with the subspace of $\langle \sigma_{ij} \rangle$ generated by the elements σ_{ij}, where $i > 0$. Since $\mu_{1,L} = \mu_{1,K}|_{\wedge^2 H^0(L)}$, one must compute the dimension of $\langle \mu_{1,K}(\sigma_{ij} \wedge \sigma_{kl}) \mid i, k > 0 \rangle$, which turns out to be 9.

6. Base Points

We will now show global generation of the image of μ_2 for curves that are neither hyperelliptic nor trigonal.

THEOREM 6.1. *Assume that X is a smooth curve, of genus $g \ge 5$, that is non-hyperelliptic and non-trigonal. Then, for any $P \in X$, there exists a quadric $Q \in I_2$ such that $\mu_2(Q)(P) \ne 0$. Equivalently,* $\mathrm{Im}(\mu_2) \cap H^0(4K_X - P) \ne \mathrm{Im}(\mu_2)$ *for all $P \in X$.*

Proof. We will show that, for any $P \in X$, there exists a quadric Q of rank 4 such that $\mu_2(Q)(P) \ne 0$. We recall that any component of the space $W^1_{g-1}(X)$ has dimension $\ge g - 4$, and in [12, Lemma (2.1.1)] it is proved that, if X is non-hyperelliptic, non-trigonal, and not isomorphic to a smooth plane quintic, then there exists a line bundle $L \in W^1_{g-1}$ such that both $|L|$ and $|K_X - L|$ are base point free. If X is a plane quintic, then by Theorem 5.2 we know that μ_2 has no base points. So we can assume that there exists a nonempty irreducible open subset $\mathcal{V}$ in $W^1_{g-1}(X)$ consisting of line bundles L such that $h^0(L) = 2$, $L \not\equiv K_X - L$, and both $|L|$ and $|K_X - L|$ are base point free. So the condition $\mu_2(Q)(P) = 0$ for the quadric associated to $|L|$ and $|K_X - L|$ says that P is a ramification point either for the morphism $\phi_{|L|}\colon X \to \mathbb{P}^1$ or for the morphism $\phi_{|K_X - L|}\colon X \to \mathbb{P}^1$.

We claim that there exists an $L \in \mathcal{V}$ such that P is at most a simple ramification point for both $\phi_{|L|}$ and $\phi_{|K_X-L|}$; that is, $h^0(L - 3P) = 0$ and $h^0(K - L - 3P) = 0$. In fact, assume for all $L \in \mathcal{V}$ that either $h^0(L - 3P) \ge 1$ or $h^0(K - L - 3P) \ge 1$. Consider two maps: $F_1\colon \mathrm{Sym}^{g-4}(X) \to \mathrm{Pic}^{g-1}(X)$ with $F_1(D) = D + 3P$; and

$F_2\colon \mathrm{Sym}^{g-4}(X) \to \mathrm{Pic}^{g-1}(X)$ with $F_2(D) = K_X - D - 3P$. Then $\mathcal{V}$ is contained in $\mathrm{Im}(F_1) \cup \mathrm{Im}(F_2)$ and, since they have the same dimension and since $\mathcal{V}$ is irreducible, it follows that either $\bar{\mathcal{V}} = \mathrm{Im}(F_1)$ or $\bar{\mathcal{V}} = \mathrm{Im}(F_2)$. This means that, for all $x_1, \dots, x_{g-4} \in X$, $h^0(x_1 + \cdots + x_{g-4} + 3P) = h^0(K_X - x_1 - \cdots - x_{g-4} - 3P) \geq 2$; but this is absurd, since $h^0(K_X - 3P) = g - 3$ because X is non-hyperelliptic and non-trigonal.

So assume that P is a simple ramification point for $\phi := \phi_{|L|}\colon X \to \mathbb{P}^1$, and let R be its ramification divisor. Consider the exact sequence

$$0 \longrightarrow T_X \xrightarrow{\phi_*} \phi^* T_{\mathbb{P}^1} \longrightarrow \mathcal{N}_\phi \longrightarrow 0.$$

Notice that $\phi^* T_{\mathbb{P}^1} = T_X(R)$, $\mathcal{N}_\phi = T_X(R)|_R$, and ϕ_* is the inclusion map of T_X in $T_X(R)$; hence our exact sequence is

$$0 \longrightarrow T_X \longrightarrow T_X(R) \longrightarrow T_X(R)|_R \longrightarrow 0.$$

The induced cohomology exact sequence is

$$0 \longrightarrow H^0(T_X(R)) \xrightarrow{i} H^0(T_X(R)|_R) \xrightarrow{\beta} H^1(T_X) \longrightarrow 0.$$

We recall (see e.g. [2]) that $H^0(T_X(R)) = H^0(\phi^* T_{\mathbb{P}^1})$ parameterizes the infinitesimal deformations of the morphism ϕ that do not move X, and it contains the 3-dimensional subspace $\phi^*(H^0(T_{\mathbb{P}^1}))$. Note that, since $H^1(T_X(R)) = 0$, any first-order deformation extends.

The strategy of the proof is to exhibit an infinitesimal deformation $\rho \in H^0(T_X(R))$ of ϕ such that $i(\rho) \notin H^0(T_X(R-P)|_{R-P}) \subset H^0(T_X(R)|_R)$ and ρ is not contained in $\phi^*(H^0(T_{\mathbb{P}^1}))$. The assumption that the ramification in P is simple implies that ρ extends to a deformation of ϕ such that the new map is no longer ramified in P.

Denote by $\xi_P \in H^1(T_X)$ a Schiffer variation in P—that is, by definition a generator of the subspace $\mathrm{Im}(H^0(T_X(P))|_P) \subset H^1(T_X)$. Since P is a ramification point,

$$\xi_P \in \beta(H^0(T_X(R)|_R)).$$

If we can prove that $\beta(H^0(T_X(R-P)|_{R-P})) \subset \beta(H^0(T_X(R)|_R))$ generates $H^1(T_X)$, then there exists an element $x \in H^0(T_X(R-P)|_{R-P})$ such that $\xi_P = \beta(x)$. So if $c_P \in H^0(T_X(P)|_P) \subset H^0(T_X(R)|_R)$ is such that $\beta(c_P) = \xi_P$, then the element $\eta = c_P - x \in H^0(T_X(R)|_R)$ maps to zero in $H^1(T_X)$; hence there exists an element $\theta \in H^0(T_X(R))$ such that $\eta = i(\theta)$ and $\eta \notin H^0(T_X(R-P)|_{R-P})$, since the coefficient of c_P in η is nonzero.

Therefore, we seek to prove the existence of an $L \in \mathcal{V}$ with simple ramification in P such that $\beta(H^0(T_X(R-P)|_{R-P}))$ generates $H^1(T_X)$. Assume to the contrary that, for any $L \in \mathcal{V}$, $\beta(H^0(T_X(R-P)|_{R-P}))$ lies on a hyperplane in $H^1(T_X)$—that is, there exists an element $\omega \in H^0(2K_X)$ such that $\omega(P_i) = 0$ for all $i \geq 2$ and $\mathrm{ord}_{P_i} \omega = n_i$, where $\sum_{i \geq 2} n_i P_i = R - P$. Then we have

$$2K_X \equiv \mathrm{div}(\omega) = P + \sum_{i \geq 2} n_i P_i - P + q \equiv K_X + 2L - P + q$$

for some $q \in X$. So for any L there exists a point $q \in X$ such that

$$2L \equiv K_X + P - q.$$

Since L varies in an open subset of W^1_{g-1}, which has dimension at least $g-4$, while q varies in X, it follows that if $g \geq 6$ then this cannot hold for all $L \in \mathcal{V}$.

If $g = 5$, we still get a contradiction by noting that the multiplication by 2 in the Jacobian restricts to a connected topological covering $\tilde{X}$ of the curve X of degree 2^{10} corresponding to the surjective homomorphism $\pi_1(X) \to H_1(X, \mathbb{Z}/2\mathbb{Z})$. Hence $\tilde{X}$ cannot coincide with a component of W^1_4 that is a 2-to-1 covering of a quintic plane curve ramified along at most ten points (cf. [3, p. 270]).

Finally, we show that we can choose the deformation outside $\phi^*(H^0(T_{\mathbb{P}^1}))$. Set $W = i(H^0(T_X(R)) \cap H^0(T_X(R-P)|_{R-P}))$. We have just proved that, for L general, $i(H^0(T_X(R))) \not\subset H^0(T_X(R-P)|_{R-P})$; thus $\dim(W) = h^0(T_X(R)) - 1 = g-2 \geq 3$ since $g \geq 5$. As a result, if $\{e_1, e_2, e_3\}$ are three linearly independent elements in W then the four elements $\{i(\theta) = \eta, \eta + e_1, \eta + e_2, \eta + e_3\}$ are linearly independent, since η is not contained in $H^0(T_X(R-P)|_{R-P})$. Hence there exists a deformation ρ with $i(\rho) \in \{i(\theta) = \eta, \eta + e_1, \eta + e_2, \eta + e_3\} \subset i(H^0(T_X(R)))$ that does not belong to the 3-dimensional subspace $i(\phi^*(H^0(T_{\mathbb{P}^1})))$, so ρ is the deformation we are looking for.

We have shown that if L does not belong to the curve γ given by the equation $2L \equiv K_X + P - q$ with $q \in X$, then we can deform L in such a way that P is no longer a ramification point. Analogously, if $L_1 := K_X - L$ does not belong to the curve γ, then we can deform $L_1 = K_X - L$ in such a way that P is no longer a ramification point of the corresponding morphism. So if we take $L \in \mathcal{V} - \gamma - \iota^{-1}(\gamma)$, where $\iota\colon W^1_{g-1} \to W^1_{g-1}$ is the involution sending L to $K_X - L$, we find deformations of L (then also of $K_X - L$) such that, for L' the deformed line bundle, P is not a ramification point of either $\phi_{|L'|}$ or $\phi_{|K_X - L'|}$. □

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