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On Families of Rational Curves in the Hilbert Square of a Surface

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(with an Appendix by Edoardo Sernesi)

1. Introduction

For any smooth surface S, the Hilbert scheme $S^{[n]}$ parameterizing 0-dimensional length-n subschemes of S is a smooth $2n$-dimensional variety whose inner geometry is naturally related to that of S. For instance, if $\Delta \subset S^{[n]}$ is the exceptional divisor—that is, the exceptional locus of the Hilbert–Chow morphism $\mu\colon S^{[n]} \to \mathrm{Sym}^n(S)$—then irreducible (possibly singular) rational curves not contained in Δ roughly correspond to irreducible (possibly singular) curves on S with a $\mathfrak{g}^1_{n'}$ on their normalizations for some $n' \leq n$ (see Section 2.1 for the precise correspondence when $n = 2$). One of the features of this paper is to show how ideas and techniques from one of the two sides of the correspondence make it possible to shed light on problems naturally arising on the other side.

If S is a $K3$ surface then $S^{[n]}$ is a *hyperkähler manifold* (cf. [5]), and rational curves play a fundamental role in the study of the (birational) geometry of $S^{[n]}$. Indeed, a result due to Huybrechts [32] and Boucksom [11] implies in particular that these curves govern the ample cone of $S^{[n]}$. The presence of a $\mathbb{P}^n \subset S^{[n]}$ gives rise to a birational map (the so-called *Mukai flop* [41]) to another hyperkähler manifold and, for $n = 2$, all birational maps between hyperkähler 4-folds factor through a sequence of Mukai flops [12; 30; 62; 63]. Moreover, as shown by Huybrechts [32], uniruled divisors allow us to describe the birational Kähler cone of $S^{[n]}$. For hyperkähler 4-folds that are deformation equivalent to the Hilbert square of a $K3$ surface, a conjectural description of the Mori cone and of the numerical and geometric properties of the rational curves that are extremal in the Mori cone has been proposed by Hassett and Tschinkel [24] (and partly confirmed in [25]).

The scope of this paper, and the structure of it as well, is twofold: we first devise general methods and tools to study families of curves with hyperelliptic normalizations on a surface S (Sections 2–4). Then we apply these to obtain concrete results in the case of $K3$ surfaces (Sections 5–7).

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1.1. Families of Singular Curves with Hyperelliptic Normalizations

The main question we address in the first part of the paper is whether there exists an upper bound on the dimension of families of irreducible curves on a projective surface with hyperelliptic normalizations. One easily sees that, if the canonical system of the surface is birational, then no curve with hyperelliptic normalization can move (cf. e.g. [34]). On the other hand, any surface S admitting a (generically) $2:1$ map onto a rational surface R carries families of arbitrarily high dimensions of curves on S having hyperelliptic normalizations. Nevertheless, for a large class of surfaces, we derive the following geometric consequence on the family when its dimension is greater than 2.

Theorem 4.6′. *Let S be a smooth projective surface with $p_g(S) > 0$. Let V be a reduced and irreducible scheme parameterizing a flat family of irreducible curves on S with hyperelliptic normalizations (of genus ≥ 2) such that* $\dim(V) \geq 3$. *Then the algebraic equivalence class $[C]$ of the curves parameterized by V has a decomposition $[C] = [D_1] + [D_2]$ into algebraically moving classes such that the point parameterizing $D_1 + D_2$ lies in the closure $\bar{V}$ of V in the component of the Hilbert scheme of S containing V. Moreover, the rational curves in $S^{[2]}$ corresponding to the irreducible curves parameterized by V cover only a (rational) surface $R \subset S^{[2]}$.*

In fact, we prove a stronger result (Theorem 4.6) that relates the decomposition $[C] = [D_1] + [D_2]$ to the $\mathfrak{g}^1_2$ on the normalizations of the curves parameterized by V. This additional point will be crucial in our application of this result. An immediate corollary is a simple dimension bound under natural additional hypotheses on V (Corollary 4.7).

The proof of Theorem 4.6 illustrates well the rich interplay between the properties of curves on S and those of subvarieties of $S^{[2]}$. It relies on two ingredients. First, by a suitable version of Mumford's theorem on 0-cycles on surfaces with $p_g > 0$ (cf. Corollaries 3.2 and 3.4), the family of rational curves in $S^{[2]}$ associated to the irreducible curves on S with hyperelliptic normalizations can cover only a surface if $\dim(V) \geq 3$. Then, by Mori's bend-and-break technique (see Lemma 2.10), we produce a reducible member in $S^{[2]}$. From this, in Proposition 4.3 we produce a decomposition of the curves on S into algebraically moving classes.

One application of Theorem 4.6 is a Reider-like result for families of singular curves with hyperelliptic normalizations obtained in [34], where also more examples of such families are given. In the rest of this paper, we focus on $K3$ surfaces and, in particular, apply Theorem 4.6 to show the following result.

Theorem 5.2. *Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 4$. Then the family of nodal curves in $|H|$ of geometric genus 3 with hyperelliptic normalizations is nonempty, and each of its irreducible components is 2-dimensional.*

It is well known that there exist finitely many (nodal) rational curves, a 1-parameter family of (nodal) elliptic curves, and a 2-dimensional family of (nodal) curves of geometric genus 2 in $|H|$ (see Section 5). Every such family yields in a natural way a 2-dimensional family of irreducible rational curves in $S^{[2]}$ (Section 2). Therefore, Theorem 5.2 is the first nontrivial existence result about curves with hyperelliptic normalizations on general $K3$ surfaces of any polarization, and consequently about rational curves in $S^{[2]}$. Also note that, by a result of Ran [47], the *expected dimension* of a family of rational curves in a symplectic 4-fold—whence a posteriori also of a family of curves with hyperelliptic normalizations lying on a $K3$ surface—is 2 (Lemma 5.1).

The proof of Theorem 5.2 takes the entire Section 5 and relies on a general principle of constructing curves with hyperelliptic normalizations on general $K3$ surfaces that is outlined in Proposition 5.11. First construct a marked $K3$ surface (S_0, H_0) of genus p such that $|H_0|$ contains a family of dimension ≤ 2 of nodal (possibly reducible) curves with the property that a desingularization of some $\delta > 0$ of the nodes is a limit of a hyperelliptic curve in the moduli space $\bar{\mathcal{M}}_{p-\delta}$ of stable curves of genus $p - \delta$ and such that this family is not contained in a higher-dimensional such family. Then consider the parameter space $\mathcal{W}_{p,\delta}$ of pairs $((S, H), C)$, where (S, H) is a smooth, primitively marked $K3$ surface of genus p and $C \in |H|$ is a nodal curve with at least δ nodes. Now map (the local branches of) $\mathcal{W}_{p,\delta}$ into $\bar{\mathcal{M}}_{p-\delta}$ by partially normalizing the curves at δ of the nodes and mapping them to their respective classes. By construction, the image of this map intersects the hyperelliptic locus $\bar{\mathcal{H}}_{p-\delta} \subset \bar{\mathcal{M}}_{p-\delta}$. A dimension count then shows that the dimension of the parameter space $\mathcal{I} \subset \mathcal{W}_{p,\delta}$ consisting of $((S, H), C)$ such that a desingularization of some $\delta > 0$ of the nodes of C is a limit of a hyperelliptic curve is at least 21. Now the dominance on the 19-dimensional moduli space of primitively marked $K3$ surfaces of genus p follows because the dimension of the special family on S_0 did not exceed 2.

The technical difficulties in the proof of Proposition 5.11 arise mostly because the curves in the special family on S_0 may be reducible. Hence we need to partially desingularize families of nodal curves, and the tool for this is provided in the Appendix by E. Sernesi. Moreover, we need a careful study of the Severi varieties of *reducible* nodal curves on $K3$ surfaces, and here we use results of Tannenbaum [56].

Given Proposition 5.11, the proof of Theorem 5.2 is then accomplished by constructing a suitable (S_0, H_0) in Proposition 5.19 with $|H_0|$ containing a desired 2-dimensional family of special curves, with $\delta = p - 3$, and then showing that the curves in the special family on S_0 in fact deform to curves with *precisely* δ nodes on the general S in Lemma 5.20. Showing that the special family on S_0 is not contained in a family of higher-dimensional curves with the same property is quite delicate, and it is here that we use the full version of Theorem 4.6.

We also show (Corollary 5.3) that the associated rational curves in $S^{[2]}$ cover a 3-fold.

1.2. Results on the Mori Cone of $S^{[2]}$

Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 2$. Then $N_1(S^{[2]})_{\mathbb{R}} \simeq \mathbb{R}[Y] \oplus \mathbb{R}[\mathbb{P}^1_\Delta]$, where $\mathbb{P}^1_\Delta$ is the class of a rational curve in the ruling of the exceptional divisor $\Delta \subset S^{[2]}$ and where $Y := \{\xi \in S^{[2]} \mid \operatorname{Supp}(\xi) = \{p_0, y\}$ with $p_0 \in S$ and $y \in C \in |H|\}$, where p_0 and C are chosen. One has that $\mathbb{P}^1_\Delta$ lies on the boundary of the Mori cone; by the result of Huybrechts and Boucksom mentioned previously, if the Mori cone is closed then also the other boundary is generated by the class of a rational curve. Notice that the conjecture of Hassett and Tschinkel [24] on the properties of these extremal classes is still open even in the case of the Hilbert square of a general $K3$ surface. It therefore seems useful to obtain more information on the Mori cone and to find examples where the particular classes pointed out by Hassett and Tschinkel appear.

If now $C \in |mH|$ is an irreducible curve with hyperelliptic normalization, let $g_0(C) \geq p_g(C)$ be the arithmetic genus of the *minimal* partial desingularization of C that carries the $\mathfrak{g}^1_2$ (see Section 2.1 and Section 6.2). By the unicity of the $\mathfrak{g}^1_2$, C defines a unique irreducible rational curve $R_C \subset S^{[2]}$ with class $R_C \sim_{\text{alg}} mY - \left(\frac{g_0(C)+1}{2}\right)\mathbb{P}^1_\Delta$; see (6.11). Thus, the higher is $g_0(C)$ (or $p_g(C)$) and the lower is m, the closer is R_C to the boundary of the Mori cone. This motivates the search for curves on S that have hyperelliptic normalizations of high geometric genus and thus are "unexpected" from Brill–Noether theory.

If $X \sim_{\text{alg}} aY - b\mathbb{P}^1_\Delta$ is an irreducible curve in $S^{[2]}$ with $a, b \neq 0$, then we define a/b to be the *slope* of the curve. Describing the Mori cone $\operatorname{NE}(S^{[2]})$ amounts to computing

$$\operatorname{slope}(\operatorname{NE}(S^{[2]})) := \inf\{\operatorname{slope}(X) \mid X \text{ is an irreducible curve in } S^{[2]}\},$$

and, if the Mori cone is closed, then $\operatorname{slope}(\operatorname{NE}(S^{[2]})) = \operatorname{slope}_{\text{rat}}(\operatorname{NE}(S^{[2]}))$, where

$$\operatorname{slope}_{\text{rat}}(\operatorname{NE}(S^{[2]})) := \inf\{\operatorname{slope}(X) \mid X \text{ is an irreducible } \textit{rational} \text{ curve in } S^{[2]}\}.$$

(See Sections 6.1–6.3 for further details.) Combining various results, we obtain five bounds of a different nature on the slope of effective 1-cycles in the Hilbert square $S^{[2]}$ of a $K3$ with $\operatorname{Pic}(S) = \mathbb{Z}[H]$.

(1) If $X \in N_1(S^{[2]})_{\mathbb{Z}}$ with $X \sim_{\text{alg}} Y - k\mathbb{P}^1_\Delta$, then $k \leq \frac{p_a(H)+4}{4}$; that is, $\operatorname{slope}(X) \geq \frac{4}{p_a(H)+4}$ (cf. Theorem 6.18, which is related to the "singular Brill–Noether invariant" introduced in [21]).

(2) $\operatorname{slope}(\operatorname{NE}(S^{[2]})) \leq \sqrt{2/(p_a - 1)}$ (cf. Theorem 6.21, which is related to Seshadri constants).

In Section 7 we give a couple of existence results of a different type than Theorem 5.2: in Propositions 7.2 and 7.7 we find *general* primitively polarized $K3$ surfaces (S, H) of infinitely many degrees such that $S^{[2]}$ contains either a $\mathbb{P}^2$ (shown to us by B. Hassett) or a 3-fold birational to a $\mathbb{P}^1$-bundle over a $K3$. We also find the classes corresponding to the lines and fibers, respectively, and the geometric genus of the corresponding curves on S with hyperelliptic normalizations. The lines and the fibers are interesting because, according to the conjecture of Hassett and Tschinkel [24], they should generate an extremal ray of $\operatorname{NE}(S^{[2]})$. As a by-product of these constructions, we also obtain:

(3) $\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]})) \leq \mathrm{slope}(\text{line in a } \mathbb{P}^2) = \frac{2}{2n-9}$ if $p_a(H) = n^2 - 9n + 20$ for some $n \geq 6$;

(4) $\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]})) \leq \mathrm{slope}(\text{fiber of a } \mathbb{P}^1\text{-bundle}) = \frac{1}{d}$ if $p_a(H) = d^2$ for some $d \geq 2$.

Moreover, Proposition 7.2 also shows the sharpness of Theorem 4.6 even in the case of a surface with Picard number 1. In fact, the $(3m-1)$-dimensional family of rational curves in $\mathcal{O}_{\mathbb{P}^2}(m)$ gives rise to a $(3m-1)$-dimensional family of curves with hyperelliptic normalizations in $|mH|$.

The idea of the proofs of Propositions 7.2 and 7.7 is to start with a special quartic surface $S_0 \subset \mathbb{P}^3$ such that $S_0^{[2]}$ contains a $\mathbb{P}^2$ or a 3-fold birational to a $\mathbb{P}^1$-bundle over itself; perform the standard involution on $S_0^{[2]}$ to produce a new such surface; and then deform $S_0^{[2]}$, keeping the new one by maintaining a suitable polarization on the surface that is different from $\mathcal{O}_{S_0}(1)$. Here we use results from [24] and [58] on deformations of symplectic 4-folds.

Finally, we remark that combining Theorem 4.6 with the deformation-theoretic argument of Proposition 5.11 yields the following general procedure for deforming (even reducible) rational curves on the Hilbert square of a *special* $K3$ to the general one.

(5) Let (S_0, H_0) be a primitively marked $K3$ surface. Suppose $|H_0|$ contains a maximal family of (possibly reducible) curves with the property that some partial desingularization is a limit of a smooth hyperelliptic curve of genus p_g. Suppose further that this family is 2-dimensional (apply Theorem 4.6). Then these curves deform to *irreducible* curves with hyperelliptic normalizations on the general $K3$ surface (S, H). Hence also the associated rational curves R_0 in $S_0^{[2]}$ deform to $S^{[2]}$. In particular, $\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]})) \leq \mathrm{slope}(R_0) = \frac{2}{p_g+1}$.

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2. Rational Curves in $S^{[2]}$

Let S be a smooth projective surface. In this section we gather some basic results that will be needed in the rest of the paper. We first describe the natural correspondence between rational curves in $S^{[2]}$ and curves on S with rational elliptic or hyperelliptic normalizations. Then, in Section 2.2, we apply Mori's bend-and-break technique to rational curves in $\mathrm{Sym}^2(S)$ covering a surface.

Recall that we have the natural Hilbert–Chow morphism $\mu\colon S^{[2]} \to \mathrm{Sym}^2(S)$ that resolves $\mathrm{Sing}(\mathrm{Sym}^2(S)) \simeq S$. The μ-exceptional divisor $\Delta \subset S^{[2]}$ is a $\mathbb{P}^1$-bundle over S. The Hilbert–Chow morphism gives an obvious one-to-one correspondence between irreducible curves in $S^{[2]}$ not contained in Δ and irreducible curves in $\mathrm{Sym}^2(S)$ not contained in $\mathrm{Sing}(\mathrm{Sym}^2(S))$. We will therefore often switch back and forth between working on $S^{[2]}$ and on $\mathrm{Sym}^2(S)$.

2.1. Irreducible Rational Curves in $S^{[2]}$ and Curves on S

Let $T \subset S \times S^{[2]}$ be the incidence variety with projections $p_2\colon T \to S^{[2]}$ and $p_S\colon T \to S$. Then p_2 is finite of degree 2 and branched along $\Delta \subset S^{[2]}$. (In particular, T is as smooth as Δ is.)

Let $X \subset S^{[2]}$ be an irreducible rational curve not contained in Δ. We will now see how X is equivalent to one of three sets of data on S.

Let $\nu_X\colon \tilde{X} \simeq \mathbb{P}^1 \to X$ be the normalization and set $X' := p_2^{-1}(X) \subset T$. By the universal property of blowing up, we obtain the commutative square

$$\begin{tikzcd} \tilde{C}_X \arrow[r, "f"] \arrow[d, "\tilde{\nu}_X"'] & \tilde{X} \simeq \mathbb{P}^1 \arrow[d, "\nu_X"] \\ X' \arrow[r, "p_2|_{X'}"] & X, \end{tikzcd} \tag{2.1}$$

defining the curve $\tilde{C}_X$ as well as $\tilde{\nu}_X$ and f. In particular, $\tilde{\nu}_X$ is birational and $\tilde{C}_X$ admits a $\mathfrak{g}^1_2$ (i.e., a $2:1$ morphism onto $\mathbb{P}^1$ that is given by f) but may be singular or even reducible. Set $\tilde{\nu} := p_S|_{X'} \circ \tilde{\nu}_X\colon \tilde{C}_X \to S$.

Assume first that $\tilde{C}_X$ is irreducible. We set $C_X := \tilde{\nu}(\tilde{C}_X) \subset S$. Since $X \not\subset \Delta$, it follows that C_X is a curve. Since $\tilde{C}_X$ carries a $\mathfrak{g}^1_2$, it is easily seen that also the normalization of C_X does—that is, C_X has rational elliptic or hyperelliptic normalization. Moreover, it is easily seen that $\tilde{\nu}\colon \tilde{C}_X \to C_X$ is generically of degree 1. Indeed, for general $x \in C_X$, since $x \notin p_S(p_2^{-1}(\Delta))$ we can write $(p_S|_{X'})^{-1}(x) = \{(x, x+y_1), \dots, (x, x+y_n)\}$, where $n := \deg \tilde{\nu}$. By the definition of p_2 and since $X' = p_2^{-1}(X)$, we must have that each $(y_i, x+y_i) \in X'$ for $i = 1, \dots, n$ and that each couple $((x, x+y_i), (y_i, x+y_i))$ is the push-down by $\tilde{\nu}_X$ of an element of the $\mathfrak{g}^1_2$ on $\tilde{C}_X$. Hence, each couple (x, y_i) is the push-down by the normalization morphism of an element of the induced $\mathfrak{g}^1_2$ on the normalization of C_X. Since x was chosen to be general, $x \notin \mathrm{Sing}(C_X)$; hence we must have $n = 1$ as claimed.

In particular, by construction we know that $\tilde{\nu}\colon \tilde{C}_X \to C_X$ is a partial desingularization of C_X; in fact, it is the *minimal* partial desingularization of C_X carrying the $\mathfrak{g}^1_2$ in question (which is unique if $p_g(C_X) \geq 2$). We have therefore obtained:

(I) the data of an irreducible curve $C_X \subset S$ together with a partial normalization $\tilde{\nu}\colon \tilde{C}_X \to C_X$ with a $\mathfrak{g}^1_2$ on $\tilde{C}_X$ (unique, if $p_g(C_X) \geq 2$) such that $\tilde{\nu}$ is minimal with respect to the existence of the $\mathfrak{g}^1_2$.

Next we treat the case where $\tilde{C}_X$ is reducible. In this case, it must consist of two irreducible smooth rational components, $\tilde{C}_X = \tilde{C}_{X,1} \cup \tilde{C}_{X,2}$, that are identified by f.

If $\tilde{\nu}$ does not contract any of the components, set $C_{X,i} := \tilde{\nu}(\tilde{C}_{X,i}) \subset S$ and $n_{X,i} := \deg \tilde{\nu}|_{\tilde{C}_{X,i}}$ for $i = 1, 2$. We therefore obtain:

(II) the data of a curve $C_X = n_{X,1}C_{X,1} + n_{X,2}C_{X,2} \subset S$ with $n_{X,i} \in \mathbb{N}$ and $C_{X,i}$ an irreducible rational curve; a morphism $\tilde{\nu}\colon \tilde{C}_X = \tilde{C}_{X,1} \cup \tilde{C}_{X,2} \to C_{X,1} \cup C_{X,2}$ (resp. $\tilde{\nu}\colon \tilde{C}_X \to C_{X,1}$ if $C_{X,1} = C_{X,2}$) that is $n_{X,i} : 1$ on each component and where $\tilde{C}_{X,i}$ is the normalization of $C_{X,i}$; and an identification morphism $f\colon \tilde{C}_{X,1} \cup \tilde{C}_{X,2} \simeq \mathbb{P}^1 \cup \mathbb{P}^1 \to \mathbb{P}^1$.

If $\tilde{\nu}$ contracts one of the two components of $\tilde{C}_X$, say $\tilde{C}_{X,2}$, to a point $x_X \in S$ (it is easily seen that it cannot contract both), then $\mu(X) \subset \mathrm{Sym}^2(S)$ is of the form $\{x_X + C_X\}$ for an irreducible curve $C_X \subset S$, which is necessarily rational. It is easily seen that $C_X = \tilde{\nu}(\tilde{C}_{X,1})$ and $\deg \tilde{\nu}|_{\tilde{C}_{X,1}} = 1$, so we obtain:

(III) the data of an irreducible rational curve $C_X \subset S$ together with a point $x_X \in S$.

Note that in (I)–(III) the support of the curve C_X on S is simply

$$\mathrm{Supp}(C_X) = \text{1-dimensional part of } \{x \in S \mid x \in \mathrm{Supp}(\xi) \text{ for some } \xi \in X\}, \quad (2.2)$$

and the set is already purely 1-dimensional except in (III) with $x_X \notin C$.

Conversely, from the data (I), (II), or (III) one can recover an irreducible rational curve in $S^{[2]}$ that is not contained in Δ. Indeed, in (I) (resp. (II)) the $\mathfrak{g}^1_2$ on $\tilde{C}_X$ (resp., the identification f) induces a $\mathbb{P}^1 \subset \mathrm{Sym}^2(\tilde{C}_X)$, and this is mapped to an irreducible rational curve in $\mathrm{Sym}^2(S)$ by the natural composed morphism

$$\mathrm{Sym}^2(\tilde{C}_X) \xrightarrow{\tilde{\nu}^{(2)}} \mathrm{Sym}^2(C_X) \hookrightarrow \mathrm{Sym}^2(S).$$

The irreducible rational curve $X \subset S^{[2]}$ is the strict transform by μ of this curve. In (III), $X \subset S^{[2]}$ is the strict transform by μ of $\{x_X + C_X\} \subset \mathrm{Sym}^2(S)$.

We see that the data (III) correspond precisely to rational curves of type $\{x_0 + C\} \subset \mathrm{Sym}^2(S)$, where $x_0 \in S$ is a point and $C \subset S$ is an irreducible rational curve. Moreover, it is easily seen that the data (II) correspond precisely to the images by $\alpha\colon \tilde{C}_1 \times \tilde{C}_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \to C_1 + C_2 \subset \mathrm{Sym}^2(S)$ (resp. $\alpha\colon \mathrm{Sym}^2(\tilde{C}) \simeq \mathbb{P}^2 \to \mathrm{Sym}^2(C) \subset \mathrm{Sym}^2(S)$) of irreducible rational curves in $|n_1F_1 + n_2F_2|$ for $n_1, n_2 \in \mathbb{N}$ (resp. $|nF|$ for an integer $n \geq 2$), where $\mathrm{Pic}(\tilde{C}_1 \times \tilde{C}_2) \simeq \mathbb{Z}[F_1] \oplus \mathbb{Z}[F_2]$ (resp. $\mathrm{Pic}(\mathrm{Sym}^2(\tilde{C})) \simeq \mathbb{Z}[F]$) and C_1, C_2 (resp. C) are irreducible rational curves on S and where the tilde ($\tilde{\ }$) denotes normalization. However, data of type (II) will not be studied in this paper, where the focus is on data of type (I) and (III)—mostly the former.

Observe that (a) an irreducible rational curve $X \subset \mathrm{Sym}^2(S)$ arising from rational (resp. elliptic) curves C as in (I) moves in $\mathrm{Sym}^2(C)$, which is a surface birational to $\mathbb{P}^2$ (resp. an elliptic ruled surface), and (b) a curve $X \subset \mathrm{Sym}^2(S)$ of the form $\{x_X + C\}$ moves in the 3-fold $\{S + C\}$, which is birational to a $\mathbb{P}^1$-bundle over S and contains $\mathrm{Sym}^2(C)$.

At the same time, it is well known that if $\mathrm{kod}(S) \geq 0$ then rational curves on S do not move and elliptic curves move in at most 1-dimensional families. This follows, for instance, from the following general result (that we will later need in the case $p_g = 2$).

Lemma 2.3. *Let S be a smooth projective surface, with* $\mathrm{kod}(S) \geq 0$, *containing an n-dimensional irreducible family of irreducible curves of geometric genus p_g. Then $n \leq p_g$ and, if equality occurs, either the family consists of a single smooth rational curve*; *or* $\mathrm{kod}(S) \leq 1$ *and* $n \leq 1$; *or* $\mathrm{kod}(S) = 0$.

Proof. This is "folklore". For a proof, see [34]. □

As a consequence, if $\mathrm{kod}(S) \geq 0$ then rational curves in $\mathrm{Sym}^2(S)$ arising from rational or elliptic curves on S move in families of dimension at most 2 in $\mathrm{Sym}^2(S)$.

On the other hand, irreducible rational curves $X \subset \mathrm{Sym}^2(S)$ arising from curves on S with hyperelliptic normalizations of geometric genus $p_g \geq 2$ (necessarily of type (I)) move in a family whose dimension equals that of the family of curves with hyperelliptic normalizations in which $C \subset S$ moves (by unicity of the $\mathfrak{g}^1_2$). Apart from some special cases, it is easy to see that the converse is also true. The proof of this is straightforward and is left to the reader.

Lemma 2.4. *Let $\{X_b\}_{b\in B}$ be a 1-dimensional irreducible family of irreducible rational curves in* $\mathrm{Sym}^2(S)$ *covering a* (*dense subset of a*) *proper, reduced and irreducible surface $Y \subset$* $\mathrm{Sym}^2(S)$ *that does not coincide with* $\mathrm{Sing}(\mathrm{Sym}^2(S)) \cong S$.

Then, with notation as before, $C = C_{X_b}$ in S for every $b \in B$ if and only if either $Y =$ $\mathrm{Sym}^2(C_0)$, *with either $C_0 \subset S$ an irreducible rational curve and $C \equiv nC_0$ for $n \geq 1$ or $C_0 = C \subset S$ an irreducible elliptic curve*; *or $Y = C + C' :=$* $\{p + p' \mid p \in C, p' \in C'\}$, *with C an irreducible rational curve and $C' \subset S$ any irreducible curve*; *or $Y = C_1 + C_2$, with $C_1, C_2 \subset S$ irreducible rational curves and $C = n_1C_1 + n_2C_2$ for $n_1, n_2 \in \mathbb{N}$.*

We note that, by Lemma 2.3, also the rational curves in $\mathrm{Sym}^2(S)$ arising from singular curves of geometric genus 2 on S move in at most 2-dimensional families. We will show that, under some additional hypotheses, this is a general phenomenon. We will focus our attention on curves with hyperelliptic normalizations (of genus $p_g \geq 2$) in Sections 4–7.

2.2. *Bend-and-Break in* $\mathrm{Sym}^2(S)$

Let $V \subseteq \mathrm{Hom}(\mathbb{P}^1, \mathrm{Sym}^2(S))$ be a reduced and irreducible subscheme (not necessarily complete). We consider the universal map

$$\mathcal{P}_V := \mathbb{P}^1 \times V \xrightarrow{\Phi_V} \mathrm{Sym}^2(S) \tag{2.5}$$

and assume that the following two conditions hold:

$$\text{for any } v \in V,\ \Phi_V(\mathbb{P}^1 \times v) \nsubseteq \mathrm{Sing}(\mathrm{Sym}^2(S)) \simeq S; \tag{2.6}$$

$$\text{the natural map } \mathcal{P}_V \longrightarrow \mathrm{Rat}(\mathrm{Sym}^2(S)) \text{ defined by } \Phi_V \text{ is generically finite.} \tag{2.7}$$

Here $\mathrm{Rat}(\mathrm{Sym}^2(S))$ is the union of the components of $\mathrm{Hilb}(\mathrm{Sym}^2(S))$ whose general points correspond to reduced connected curves with rational components [16, 5.6]. (This simply means that V induces a flat family of rational curves in $\mathrm{Sym}^2(S)$ of dimension $\dim(V)$.) Set

$$R_V := \overline{\mathrm{im}(\Phi_V)}, \tag{2.8}$$

the Zariski closure of $\mathrm{im}(\Phi_V)$ in $\mathrm{Sym}^2(S)$. It is the (irreducible) uniruled subvariety of $\mathrm{Sym}^2(S)$ covered by the curves parameterized by V. In the language of [36, Def. 2.3], R_V is the closure of the *locus* of the family Φ_V. Note that, by (2.7), $\dim(R_V) \geq 2$ if $\dim(V) \geq 1$. Moreover (see e.g. [23, Prop. 2.1]),

$$\dim(R_V) \leq 3 \quad \text{if } \mathrm{kod}(S) \geq 0. \tag{2.9}$$

When R_V is a surface, using Mori's bend-and-break technique yields the following result. In the statement we emphasize that the breaking can be made in such a way that, for general $\xi, \eta \in R_V$, two components of the reducible (or nonreduced) member at the border of the family pass through ξ and η, respectively. This will be central in our applications (Proposition 4.3 and Section 5, where we prove Theorem 5.2). We give the proof because we could not find in the literature precisely the statement we need.

LEMMA 2.10. *Assume that* $\dim(V) \geq 3$ *and* $\dim(R_V) = 2$. *Let* ξ *and* η *be any two distinct general points of* R_V. *Then there is a curve* $Y_{\xi,\eta}$ *in* R_V *such that* $Y_{\xi,\eta}$ *is algebraically equivalent to* $(\Phi_V)_*(\mathbb{P}^1_v)$ *and either*

(a) *there is an irreducible nonreduced component of* $Y_{\xi,\eta}$ *containing* ξ *and* η; *or*
(b) *there are two distinct, irreducible components of* $Y_{\xi,\eta}$ *containing* ξ *and* η, *respectively.*

Proof. Since $\dim(V) \geq 3$ by assumption, by (2.7) we can choose a 1-dimensional smooth subscheme $B = B_{\xi,\eta} \subset V$ parameterizing curves in V such that $(\Phi_V)_*(\mathbb{P}^1 \times v)$ contains both ξ and η for every $v \in B$. We thus have the family of rational curves

$$\Phi_B := (\Phi_V)|_B : \mathbb{P}^1 \times B \longrightarrow R_V, \tag{2.11}$$

together with two marked (distinct) points $x, y \in \mathbb{P}^1$ such that $\Phi_B(x \times B) = \xi$ and $\Phi_B(y \times B) = \eta$ and such that each $\Phi_B(\mathbb{P}^1 \times v)$ is nonconstant for any $v \in B$; in particular, $\Phi_B(\mathbb{P}^1 \times B)$ is a surface.

As in the proofs of [37, Lemma 1.9] and [36, Cor. II.5.5], let $\bar{B}$ be any smooth compactification of B. Consider the surface $\mathbb{P}^1 \times \bar{B}$. Let $0 \in \bar{B}$ denote a point at the boundary, $\mathbb{P}^1_0$ the fiber over 0 of the projection onto the second factor, and $x_0, y_0 \in \mathbb{P}^1_0 \subset \mathbb{P}^1 \times \bar{B}$ the corresponding marked points. By the rigidity lemma [37, Lemma 1.6], Φ_B cannot be defined at the point x_0, as in the proof of [37, Cor. 1.7], and the same argument works for y_0.

Therefore, to resolve the indeterminacies of the rational map $\Phi_B : \mathbb{P}^1 \times \bar{B} \dashrightarrow R_V$, we must at least blow up $\mathbb{P}^1 \times \bar{B}$ at the points x_0 and y_0. Now let W be the blow-up of $\mathbb{P}^1 \times \bar{B}$ such that $\bar{\Phi}_B : W \to R_V$ is an extension of Φ_B; that is, suppose we have the commutative diagram

$$\begin{tikzcd} W \arrow[d, "\pi"'] \arrow[dr, "\bar{\Phi}_B"] & \\ \mathbb{P}^1 \times \bar{B} \arrow[r, dashed, "\Phi_B"] & R_V. \end{tikzcd}$$

Let $E_{x_0} := \pi^{-1}(x_0)$ and $E_{y_0} := \pi^{-1}(y_0)$. Observe that neither of these can be contracted by $\bar{\Phi}_B$, for otherwise Φ_B itself would be defined at x_0 or y_0.

As a result, the curve $\bar{\Phi}_B(E_{x_0})$ has an irreducible component Γ_ξ containing ξ and the curve $\bar{\Phi}_B(E_{y_0})$ has an irreducible component Γ_η containing η. By construction, $\Gamma_\xi + \Gamma_\eta \subseteq \bar{\Phi}_{B*}(\pi^{-1}(\mathbb{P}^1 \times 0))$, and $\bar{\Phi}_{B*}(\pi^{-1}(\mathbb{P}^1 \times 0))$ is the desired curve $Y_{\xi,\eta}$. The two cases (a) and (b) occur as $\Gamma_\xi = \Gamma_\eta$ or $\Gamma_\xi \neq \Gamma_\eta$, respectively. □

3. Rationally Equivalent 0-Cycles on Surfaces with $p_g > 0$

In this section we extend to the singular case a consequence of Mumford's result [43, Cor., p. 203] for 0-cycles on surfaces with $p_g > 0$ and reformulate the results in terms of rational quotients.

3.1. Mumford's Theorem

The main result of this subsection, which we prove in detail for the reader's convenience, relies on the following generalization of Mumford's result (see [59, Chap. 22] for a detailed account).

Theorem 3.1 (cf. [59, Prop. 22.24]). *Let T and Y be smooth projective varieties, and let $Z \subset Y \times T$ be a cycle of codimension equal to* $\dim(T)$. *Suppose there exists a subvariety $T' \subset T$ of dimension k_0 such that, for all $y \in Y$, the 0-cycle Z_y is rationally equivalent in T to a cycle supported on T'.*

Then, for all $k > k_0$ and all $\eta \in H^0(T, \Omega_T^k)$,

$$[Z]^*\eta = 0 \text{ in } H^0(Y, \Omega_Y^k),$$

where, as is customary, $[Z]^*\eta$ denotes the differential form induced on Y by the correspondence Z.

Combining this theorem with Mumford's original "symplectic" argument, we obtain the following.

Corollary 3.2. *Let S be a smooth, irreducible projective surface with $p_g(S) > 0$ and let $\Sigma \subset S^{[n]}$ be a reduced, irreducible (possibly singular) complete subscheme such that $\mu(\Sigma) \not\subset$* $\mathrm{Sing}(\mathrm{Sym}^n(S))$, *where $\mu\colon S^{[n]} \to$* $\mathrm{Sym}^n(S)$ *is the Hilbert–Chow morphism. If there exists a subvariety $\Gamma \subset$* $\mathrm{Sym}^n(S)$ *such that* $\dim(\Gamma) \leq 1$, $\Gamma \not\subset \mathrm{Sing}(\mathrm{Sym}^n(S))$, *and all the 0-cycles parameterized by $\mu(\Sigma)$ are rationally equivalent to 0-cycles supported on Γ, then* $\dim(\Sigma) \leq n$.

Proof. Let $\pi : \tilde{\Sigma} \to \Sigma \subset S^{[n]}$ be the desingularization morphism of Σ. Let $Z = \Lambda_\pi \subset \tilde{\Sigma} \times S^{[n]}$ be the graph of π. Then $Z \cong \tilde{\Sigma}$, so that $\mathrm{codim}(Z) = \dim(S^{[n]})$

as in Theorem 3.1. By assumption, $\mu(\Sigma)$ parameterizes 0-cycles of length n on S that are all rationally equivalent to 0-cycles supported on Γ with $\dim(\Gamma) \le 1$. Since $\mu(\Sigma)$ is not contained in $\mathrm{Sing}(\mathrm{Sym}^n(S))$ by assumption, it follows that $\mu|_\Sigma \colon \Sigma \to \mu(\Sigma)$ is birational. If Γ' denotes the strict transform of Γ under μ, then $\dim(\Gamma') \le 1$.

We can apply Theorem 3.1 with $Z = Y = \tilde{\Sigma}$, $T = S^{[n]}$, and $T' = \Gamma'$. Thus, for each $k > 1$ and each $\eta \in H^0(\Omega^k_{S^{[n]}})$, we have $[Z]^*\eta = 0$ in $H^0(\tilde{\Sigma}, \Omega^k_{\tilde{\Sigma}})$.

Let $\omega \in H^0(S, K_S)$ be a nonzero 2-form on S. As in [43, Cor.], we define

$$\omega^{(n)} := \sum_{i=1}^{n} p_i^*(\omega) \in H^0(S^n, \Omega^2_{S^n}),$$

where S^n is the nth Cartesian product and p_i is the natural projection onto the ith factor, $1 \le i \le n$. The form $\omega^{(n)}$ is $\mathrm{Sym}(n)$-invariant and, since μ is surjective, this induces a canonical 2-form $\omega_\mu^{[n]} \in H^0(S^{[n]}, \Omega^2_{S^{[n]}})$ (see [43, Sec. 1], where $\omega_\mu^{[n]} = \eta_\mu$ in the notation there). From what we have observed here, $[Z]^*(\omega_\mu^{[n]}) = 0$ as a form in $H^0(\tilde{\Sigma}, \Omega^2_{\tilde{\Sigma}})$. Consider

$$(\mathrm{Sym}^n(S))_0 := \left\{ \xi = \sum_{i=1}^{n} x_i \;\middle|\; x_i \ne x_j,\ 1 \le i \ne j \le n, \text{ and } \omega(x_i) \in \Omega^2_{S,x_i} \text{ is not } 0 \right\}.$$

Then $(\mathrm{Sym}^n(S))_0 \subset \mathrm{Sym}^n(S)$ is an open dense subscheme that is isomorphic to its preimage via μ in $S^{[n]}$. For each $\xi \in (\mathrm{Sym}^n(S))_0$, ξ is a smooth point and

$$\pi_n \colon S^n \longrightarrow \mathrm{Sym}^n(S)$$

is étale over ξ. Thus, the 2-form $\omega^{(n)} \in H^0(S^n, \Omega^2_{S^n})$ is nondegenerate on the open subset $(S^n)_0$ of points in the preimage of $(\mathrm{Sym}^n(S))_0$; in other words, it defines a nondegenerate skew-symmetric form on the tangent space of $(S^n)_0$.

Let $\pi_n^0 := \pi_n|_{(S^n)_0}$. Since $\pi_n^0 \colon (S^n)_0 \to (\mathrm{Sym}^n(S))_0$ is étale, there exists a 2-form

$$\omega_0^{(n)} \in H^0((\mathrm{Sym}^n(S))_0, \Omega^2_{(\mathrm{Sym}^n(S))_0})$$

such that $\omega^{(n)} = \pi_n^*(\omega_0^{(n)})$ and $\omega_0^{(n)}$ is also nondegenerate. Therefore, the maximal isotropic subspaces of $\omega_0^{(n)}(\xi)$ are n-dimensional.

Now $\Sigma \subset S^{[n]}$ and $\Sigma \cap \mu^{-1}((\mathrm{Sym}^n(S))_0) \ne \emptyset$, since $\mu(\Sigma) \not\subset \mathrm{Sing}(\mathrm{Sym}^n(S))$ by assumption. Since Σ is reduced, let $\xi \in \Sigma \cap \mu^{-1}((\mathrm{Sym}^n(S))_0)$ be a smooth point. Then, since $\Sigma_{\mathrm{smooth}} = \pi^{-1}(\Sigma_{\mathrm{smooth}})$, by abuse of notation we still denote by $\xi \in \tilde{\Sigma}$ the corresponding point. We know that $[Z]^*\omega_\mu^{[n]}(\xi) = 0$ in the tangent space $T_\xi(\tilde{\Sigma})$. Since

$$\xi \in \Sigma_{\mathrm{smooth}} \cap \mu^{-1}((\mathrm{Sym}^n(S))_0) \subset (\mathrm{Sym}^n(S))_0,$$

it follows that $[Z]^*(\omega_\mu^{[n]}) = \omega_0^{(n)}|_{\Sigma_{\mathrm{smooth}} \cap \mu^{-1}((\mathrm{Sym}^n(S))_0)}$. This implies $\dim(\Sigma) \le n$. $\square$

3.2. The Property RCC and Rational Quotients

Recall that a variety T (not necessarily proper or smooth) is said to be *rationally chain connected* (RCC) if, for each pair of very general points $t_1, t_2 \in T$, there exists a connected curve $\Lambda \subset T$ such that $t_1, t_2 \in \Lambda$ and each irreducible component of Λ is rational (see [36]). Furthermore, by [16, Rem. 4.21(2)], if T is proper and RCC then *each pair* of points can be joined by a connected chain of rational curves.

Also recall that, for any smooth variety T, there exists a variety Q, called the *rational quotient of* T, together with a rational map

$$f : T \dashrightarrow Q \tag{3.3}$$

whose very general fibers are equivalence classes under the RCC-equivalence relation (see e.g. [16, Thm. 5.13] or [36, IV, Thm. 5.4]).

In this language, an equivalent statement of Corollary 3.2 is as follows.

COROLLARY 3.4. *Let S be a smooth projective surface with $p_g(S) > 0$. If $Y \subset S^{[n]}$ is a complete subvariety of dimension $> n$ not contained in* $\mathrm{Exc}(\mu)$, *then any desingularization of Y has a rational quotient of dimension ≥ 2.*

Proof. Let $\tilde{Y}$ be any desingularization of Y and let Q be its rational quotient. Up to resolving the indeterminacies of $f : \tilde{Y} \dashrightarrow Q$, we may assume that f is a proper morphism whose very general fiber is a RCC-equivalence class; thus, in particular, *each* fiber is RCC (see [36, Thm. 3.5.3]).

If $\dim(Q) = 0$, it follows that $\tilde{Y}$ (so also Y) is RCC, contradicting Corollary 3.2.

If $\dim(Q) = 1$, then cutting $\tilde{Y}$ with $\dim(Y) - 1$ general very ample divisors results in a curve Γ' that intersects every fiber of f. Every point of $\tilde{Y}$ is connected by a chain of rational curves to some point on Γ'. We thus obtain a contradiction by Corollary 3.2 (with Γ the image of Γ' in $\mathrm{Sym}^2(S)$). □

Let now R_V be the variety covered by a family of rational curves in $\mathrm{Sym}^2(S)$ parameterized by V, as defined in (2.8); let $\tilde{R}_V$ be any desingularization of R_V; and let Q_V be the rational quotient of $\tilde{R}_V$. Of course, $\dim(Q_V) \leq \dim(R_V) - 1$ because R_V is uniruled by construction.

LEMMA 3.5. *If* $\dim(V) \geq \dim(R_V)$, *then* $\dim(Q_V) \leq \dim(R_V) - 2$ (*for any desingularization $\tilde{R}_V$ of R_V*). *In particular, if* $\dim(V) \geq 2$ *and* $\dim(R_V) = 2$, *then any desingularization of R_V is a rational surface.*

Proof. With notation as in Section 2.2, we have $\dim(\mathcal{P}_V) \geq \dim(R_V) + 1$ and so the general fiber of Φ_V is at least 1-dimensional (cf. (2.5)). This means that if ξ is a general point of R_V then there exists a family of rational curves in R_V, passing through ξ, of dimension ≥ 1. Of course, the same is true for a general point of $\tilde{R}_V$. Thus, the very general fiber of f in (3.3) has dimension ≥ 2, whence $\dim(Q_V) \leq \dim(R_V) - 2$. The last statement follows because any smooth surface that is RCC is rational (cf. [36, IV.3.3.5]). □

Combining Corollary 3.4 and Lemma 3.5, we have the following statement.

PROPOSITION 3.6. *If* $p_g(S) > 0$ *and* $\dim(V) \geq 2$, *then either*

(i) R_V *is a surface with rational desingularization*; *or*

(ii) $\dim(V) = 2$, R_V *is a 3-fold, and any desingularization of* R_V *has a 2-dimensional rational quotient.*

Proof. By (2.9), $\dim(R_V) = 2$ or 3. If $\dim(R_V) = 2$ then (i) holds by Lemma 3.5; if $\dim(R_V) = 3$ then $\dim(Q_V) = 2$ by Corollary 3.4. Hence $\dim(V) = 2$ by Lemma 3.5 and so (ii) holds. □

REMARK 3.7. Let S be a smooth projective surface with $p_g(S) > 0$ and let $Y \subset S^{[2]}$ be a uniruled 3-fold that is different from $\mathrm{Exc}(\mu)$, where $\mu\colon S^{[2]} \to \mathrm{Sym}^2(S)$ is the Hilbert–Chow morphism.

Take a covering family $\{C_v\}_{v \in V}$ of rational curves on Y. By Corollary 3.4, the family must be 2-dimensional (see Lemma 3.5). Then the curves in the covering family yield, via the correspondence described in Section 2.1, curves on S with rational elliptic or hyperelliptic normalizations, and the correspondence is one-to-one in the hyperelliptic case. We therefore see that we must be in one of the following cases:

(a) S contains an irreducible rational curve Γ and
$$Y = \{\xi \in S^{[2]} \mid \mathrm{Supp}(\xi) \cap \Gamma \neq \emptyset\};$$

(b) S contains a 1-dimensional irreducible family $\{E\}_{v \in V}$ of irreducible elliptic curves and
$$Y = \overline{\{\xi \in E_v^{[2]}\}_{v \in V}};$$

(c) S contains a 2-dimensional, irreducible family of irreducible curves with hyperelliptic normalizations that is not contained in a higher-dimensional irreducible family, and Y is the locus covered by the corresponding rational curves in $S^{[2]}$.

(Note that case (b) can occur only for $\mathrm{kod}(S) \leq 1$, by Lemma 2.3, and that case (c) can occur only when $|K_S|$ is not birational. The latter fact is easy to show; see e.g. [34].)

In the case of $K3$ surfaces, uniruled divisors play a particularly important role [32, Sec. 5]. Cases (a)–(c) occur on a general projective $K3$ surface with a polarization of genus ≥ 6. In fact, cases (a) and (b) occur on any projective $K3$ surface, which necessarily contains a 1-dimensional family of irreducible elliptic curves and a 0-dimensional family of rational curves (by a well-known theorem of Mumford; see the proof in [39, pp. 351–352] or [2, pp. 365–367]). Case (c) occurs on a general primitively polarized $K3$ surface of genus $p \geq 6$ (by Corollary 5.3, to follow) with a family of curves of geometric genus 3. In addition to this, in Proposition 7.7 we will see that there is another 3-fold as in (c) arising from curves of geometric genus > 3 in the hyperplane linear system on general projective $K3$ surfaces of infinitely many degrees.

Moreover, there is not a one-to-one correspondence between families as in (a)–(c) and uniruled 3-folds in $S^{[2]}$. In fact, in Proposition 7.2 we will see that, in the hyperplane linear systems on general $K3$ surfaces of infinitely many degrees, there is a 2-dimensional family of curves with hyperelliptic normalizations, as in (c), whose associated rational curves cover only a $\mathbb{P}^2$ in $S^{[2]}$.

4. Families of Curves with Hyperelliptic Normalizations

The purpose of this section is to study the dimension of families of curves on a smooth projective surface S with hyperelliptic normalizations.

It is not difficult to see that if $|K_S|$ is birational then the dimension of such a family is forced to be 0 (see e.g. [34]). At the same time it is easy to find obvious examples of surfaces, even with $p_g(S) > 0$, that include large families of curves with hyperelliptic normalizations—namely, surfaces admitting a finite $2:1$ map onto a rational surface (see e.g. [10; 26; 27; 28; 29; 49; 52; 54]). In these cases one can pull back the families of rational curves on the rational surface to obtain families of curves on S with hyperelliptic normalizations of arbitrarily high dimensions. Moreover, in Proposition 7.2 we will see that, for infinitely many degrees, even a general, primitively polarized $K3$ surface (S, H) contains a $\mathbb{P}^2$ in its Hilbert square, which is not contained in Δ (but the surface is not a double cover of a $\mathbb{P}^2$, by generality). Therefore, by the correspondence in Section 2.1, S contains large families of curves with hyperelliptic normalizations. One can see that, in all these examples of large families, the algebraic equivalence class of the members breaks into nontrivial effective decompositions. For example, in the $K3$ case of Proposition 7.2, we will see that the curves in $|\mathcal{O}_{\mathbb{P}^2}(n)|$ in $\mathbb{P}^2 \subset S^{[2]}$ correspond to curves in $|nH|$. In this section we will see, with the help of Lemma 2.10, that this is a general phenomenon.

Toward this end, let V be a reduced and irreducible scheme parameterizing a flat family of curves on S all of constant geometric genus $p_g \geq 2$ and with hyperelliptic normalizations. Let $\varphi\colon \mathcal{C} \to V$ be the universal family. After normalizing $\mathcal{C}$ we obtain, possibly restricting to an open dense subscheme of V, a flat family $\tilde{\varphi}\colon \tilde{\mathcal{C}} \to V$ of smooth hyperelliptic curves of genus $p_g \geq 2$ (cf. [57, Thm. 1.3.2]). Let $\omega_{\tilde{\mathcal{C}}/V}$ be the relative dualizing sheaf. As in [38, Thm. 5.5(iv)], consider the morphism $\gamma : \tilde{\mathcal{C}} \to \mathbb{P}(\tilde{\varphi}_*(\omega_{\tilde{\mathcal{C}}/V}))$ over V. This morphism is finite and of relative degree 2 onto its image, which we denote by $\mathcal{P}_V$. We thus obtain a universal family $\psi : \mathcal{P}_V \to V$ of rational curves mapping to $\mathrm{Sym}^2(S)$, as in (2.5), that satisfies (2.6) and (2.7). (Strictly speaking, (2.5) denoted a universal family of *maps,* whereas it now denotes a universal family of *curves.*) To summarize, recalling (2.8), we have

$$\begin{tikzcd} & \tilde{\mathcal{C}} \arrow[r, "\gamma"] \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\varphi}"'] & \mathcal{P}_V \arrow[r, "\Phi_V"] \arrow[d, "\psi"] & R_V \\ S & & V. & \end{tikzcd} \tag{4.1}$$

Also note that (4.1) is compatible with the correspondence of case (I) from Section 2.1 in the sense that, for general $v \in V$, we have (using the same notation as in Section 2.1)

$$\begin{gathered}\pi(\tilde{\varphi}^{-1}(v)) = p_S(p_2^{-1}X_v) = (p_S)_*(p_2^{-1}X_v) = C_{X_v} \\ \text{with } X_v = \mu_*^{-1}\big(\Phi_V(\psi^{-1}(v))\big) \subset S^{[2]},\end{gathered} \tag{4.2}$$

where μ is the Hilbert–Chow morphism (in particular, p_S and p_2 are the first and second projections, respectively, from the incidence variety $T \subset S \times S^{[2]}$). Note that the second equality in (4.2) follows because p_S is generically one-to-one on the curves in question, as we saw in Section 2.1. This will be central in our proof of the next result. We now apply Lemma 2.10 to "break" the curves on S.

PROPOSITION 4.3. *Let S be a smooth projective surface, and let V and R_V be as before. Assume that* $\dim(V) \geq 3$ *and* $\dim(R_V) = 2$, *and let* $[C]$ *be the algebraic equivalence class of the members parameterized by V.*

Then there is a decomposition into two effective, algebraically moving classes

$$[C] = [D_1] + [D_2]$$

such that, for general $\xi, \eta \in R_V$, there exist effective divisors $D_1' \sim_{\text{alg}} D_1$ and $D_2' \sim_{\text{alg}} D_2$ with $\xi \subset D_1'$ and $\eta \subset D_2'$ and $[D_1' + D_2'] \in \bar{V}$, where $\bar{V}$ is the closure of V in the component of the Hilbert scheme of S containing V.

Proof. For general $\xi, \eta \in R_V$ supported at two distinct points on S, let $B = B_{\xi,\eta} \subset V$ be as in the proof of Lemma 2.10 and let $\bar{B}$ be any smooth compactification of B. By abuse of notation, we will consider ξ and η as being points in $S^{[2]}$. By (the proof of) Lemma 2.10 and using the Hilbert–Chow morphism, there is a flat family $\{X_b\}_{b\in\bar{B}}$ of curves in the surface $\mu_*^{-1}(R_V) \subset S^{[2]}$ (where μ is the Hilbert–Chow morphism as usual) parameterized by $\bar{B}$ and such that, for general $b \in B$, X_b is an irreducible rational curve and

$$C_{X_b} = (p_S)_*(p_2^{-1}(X_b)) = \pi(\tilde{\varphi}^{-1}(b)), \tag{4.4}$$

with notation as in Section 2.1 (cf. (4.2)). In particular, $\{C_{X_b}\}_{b\in B}$ is a 1-dimensional nontrivial subfamily of the family $\{C_{X_v}\}_{v\in V}$ given by V. Moreover, for some $b_0 \in \bar{B} \setminus B$ we have $X_{b_0} \supseteq Y_\xi + Y_\eta$, where Y_ξ and Y_η are irreducible rational curves (possibly coinciding) such that $\xi \in Y_\xi$ and $\eta \in Y_\eta$. Also note that $Y_\xi, Y_\eta \not\subset \Delta \subset S^{[2]}$.

Pulling back to the incidence variety $T \subset S \times S^{[2]}$, we obtain a flat family $\{X_b' := p_2^{-1}(X_b)\}_{b\in\bar{B}}$ of curves in T such that

$$X_{b_0}' := p_2^{-1}(X_{b_0}) \supseteq p_2^{-1}(Y_\xi) + p_2^{-1}(Y_\eta) =: Y_\xi' + Y_\eta'. \tag{4.5}$$

Observe that the family $\{X_b'\}_{b\in\bar{B}}$ is a family of curves in the incidence variety $T_0 \subset S \times \mu_*^{-1}(R_V)$, which is a surface contained in T. By (4.4), p_S maps this family to a family of curves covering (an open dense subset of) S, so we see that $(p_S)|_{T_0}$ is surjective and, in particular, generically finite. Thus, choosing ξ and η general enough, we can make sure they lie outside of the images by p_2 of the finitely many curves contracted by $(p_S)|_{T_0}$. Hence $p_2^{-1}(Y_\xi)$ and $p_2^{-1}(Y_\eta)$ are not contracted by p_S.

Therefore, recalling (4.4) and (4.5) and letting $b' \in B$ be a general point, we obtain

$$C \sim_{\text{alg}} (p_S)_* X'_{b'} \sim_{\text{alg}} (p_S)_* X'_{b_0} \supseteq (p_S)_* Y'_\xi + (p_S)_* Y'_\eta \supseteq D_\xi + D_\eta,$$

where $D_\xi := p_S(p_2^{-1} Y_\xi)$ and $D_\eta := p_S(p_2^{-1} Y_\eta)$.

By construction we have $D_\xi \supset \xi$ and $D_\eta \supset \eta$, viewing ξ and η as length-2 subschemes of S. (Note that D_ξ and D_η are not necessarily distinct.) Possibly after adding additional components to D_ξ and D_η, we can assume that

$$C \sim_{\text{alg}} (p_S)_* X'_{b'} = D_\xi + D_\eta,$$

where D_ξ and D_η are not necessarily reduced and irreducible. Since this construction can be repeated for general $\xi, \eta \in R_V$ and since the set $\{x \in S \mid x \in \operatorname{Supp}(\xi)$ for some $\xi \in R_V\}$ is dense in S (because the curves parameterized by V cover the whole surface S), it follows that the obtained curves D_ξ and D_η must move in an algebraic system of dimension at least 1.

By construction, $D_\xi + D_\eta$ lies in the border of the family $\varphi\colon \mathcal{C} \to V$ of curves on S; as such, $[D_\xi + D_\eta]$ lies in the closure of V in the component of the Hilbert scheme of S containing V. Moreover, because the number of such decompositions is finite (since S is projective and since the divisors are effective), we can find one decomposition $[C] = [D_1] + [D_2]$ holding for general $\xi, \eta \in R_V$. □

The next two results are immediate consequences.

Theorem 4.6. *Let S be a smooth projective surface with $p_g(S) > 0$. Then the following conditions are equivalent*:

(i) *$S^{[2]}$ contains an irreducible surface R with rational desingularization such that $R \neq \mu_*^{-1}(C_1 + C_2)$, $\mu_*^{-1}(\operatorname{Sym}^2(C))$ for rational curves $C, C_1, C_2 \subset S$, and $R \not\subset \operatorname{Exc}(\mu)$, where $\mu\colon S^{[2]} \to \operatorname{Sym}^2(S)$ is the Hilbert–Chow morphism*;

(ii) *S contains a flat family of irreducible curves with hyperelliptic normalizations of geometric genus $p_g \geq 3$ that is parameterized by a reduced and irreducible scheme V such that $\dim(V) \geq 3$.*

Furthermore, if either of these two conditions holds then:

(a) *the rational curves in $S^{[2]}$ that correspond to the irreducible curves parameterized by V cover only the surface R in $S^{[2]}$; and*

(b) *the algebraic equivalence class $[C]$ of the curves parameterized by V has an effective decomposition $[C] = [D_1] + [D_2]$ into algebraically moving classes such that, for general $\xi, \eta \in R$, there are effective divisors $D'_1 \sim_{\text{alg}} D_1$ and $D'_2 \sim_{\text{alg}} D_2$ such that $\xi \subset D'_1$ and $\eta \subset D'_2$ and such that the point parameterizing $D'_1 + D'_2$ lies in the closure $\bar{V}$ of V in the component of the Hilbert scheme of S containing V.*

Proof. Assume (ii) holds. Then, by Proposition 3.6, $R_V \subset \operatorname{Sym}^2(S)$ is a surface with rational desingularization, so that (i) holds.

Assume now that (i) holds. Then R carries a family of rational curves of dimension $n \geq 3$. By Lemma 2.4 and the assumptions in (i), this yields an n-dimensional family of curves on S that have rational elliptic or hyperelliptic normalizations. Hence (ii) follows from Lemma 2.3.

Finally, assume that (i) and (ii) both hold. Then (a) follows from Proposition 3.6 again, where R is the proper transform via μ of the surface R_V therein; (b) then follows from Proposition 4.3. □

COROLLARY 4.7. *Let S be a smooth projective surface with $p_g(S) > 0$, and let V be a reduced irreducible scheme parameterizing a flat family of irreducible curves with hyperelliptic normalizations of geometric genus ≥ 2. Denote by $[C]$ the algebraic equivalence class of the members of V.*

If $[C]$ has no decomposition into effective, algebraically moving classes, then $\dim(V) \leq 2$.

In particular, Corollary 4.7 holds when, for example, $NS(S) = \mathbb{Z}[C]$.

The examples with the double covers of smooth rational surfaces, together with the result in Proposition 7.2 mentioned previously, show that the results above are natural.

The statement in Theorem 4.6(b) shows that the length-2 0-dimensional schemes on the curves in the family corresponding to the elements of the $\mathfrak{g}^1_2$ on their normalization are, in fact, "generically cut out" by moving divisors in a fixed algebraic decomposition of the class of the members in the family. This recalls the well-known results of Reider [48] and their generalizations [8; 9]. In fact, Theorem 4.6(b) can be used to prove a Reider-like result involving the arithmetic and geometric genera of the curves in the family (cf. [34]). Moreover, the precise statement in Theorem 4.6(b) will be crucial in the next section, where we will use degeneration methods to prove existence of curves with hyperelliptic normalizations.

5. Nodal Curves of Geometric Genus 3 with Hyperelliptic Normalizations on $K3$ Surfaces

In the rest of the paper we will focus on the existence of curves with "Brill–Noether special" hyperelliptic normalizations (i.e., of geometric genera > 2), and in this section we show that Theorem 4.6(b) is particularly suitable for proving existence results by degeneration arguments.

To do this and to discuss some consequences on $S^{[2]}$, in the sequel we focus on $K3$ surfaces, which were actually one of our original motivations for this work.

We start with the following observation, which combines a result of Ran (mentioned in the Introduction) with results from the previous section.

LEMMA 5.1. *Let S be a smooth, projective $K3$ surface, and let L be a globally generated line bundle of sectional genus $p \geq 2$ on S. Let $|L|^{\text{hyper}} \subseteq |L|$ be the subscheme parameterizing irreducible curves in $|L|$ with hyperelliptic normalizations. Then any irreducible component of $|L|^{\text{hyper}}$ has dimension ≥ 2, with equality holding if L has no decomposition into moving classes.*

Proof. Any n-dimensional component of $|L|^{\text{hyper}}$ yields an n-dimensional family of irreducible rational curves in $S^{[2]}$. By [47, Cor. 5.1], we have $n \geq 2$. The last statement follows from Corollary 4.7. □

The main aim of this section is to apply Theorem 4.6(b) to prove the following statement.

THEOREM 5.2. *Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 4$. Then the family of nodal curves in $|H|$ of geometric genus 3 and with hyperelliptic normalizations is nonempty, and each of its irreducible components is 2-dimensional.*

In [21] we studied which linear series may appear on normalizations of irreducible curves on $K3$ surfaces. To do so, we introduced a singular Brill–Noether number $\rho_{\text{sing}}(p_a, r, d, p_g)$—whose negativity, when $\text{Pic}(S) \simeq \mathbb{Z}[H]$, ensures the nonexistence of curves in $|H|$—with $p_a = p_a(H)$ and of geometric genus p_g, whose normalizations admit a $\mathfrak{g}^r_d$ (we will return to this in Section 6.3). Moreover, [21, Exs. 2.8 and 2.10] give examples of nodal curves with hyperelliptic normalizations of geometric genus 3 and arithmetic genus 4 or 5. Theorem 5.2 shows that this is a general phenomenon. The proof will be given in the balance of this section. We will also determine the dimension of the locus covered in $S^{[2]}$ by the rational curves associated to curves in a component of the family, as follows.

COROLLARY 5.3. *Let (S, H) be a general, smooth, primitively polarized $K3$ surface of genus $p = p_a(H) \geq 6$. Then the subscheme of $|H|$ parameterizing nodal curves of geometric genus 3 with hyperelliptic normalizations contains a 2-dimensional component V such that* $\dim(R_V) = 3$.

This corollary shows in particular that all three cases in Remark 3.7 occur on a general $K3$ surface. In Sections 6.2 and 6.3 we will compute the classes of the corresponding rational curves in $S^{[2]}$ (see (6.25)) and also discuss some of the consequences of Theorem 5.2 on the Mori cone of $S^{[2]}$.

Before starting on the proof of Theorem 5.2, we recall the following convention. For any smooth surface S, any line bundle L on S such that $|L|$ contains smooth irreducible curves of genus $p := p_a(L)$, and any positive integer $\delta \leq p$, one denotes by $V_{|L|,\delta}$ the locally closed and functorially defined subscheme of $|L|$ parameterizing the universal family of irreducible curves in $|L|$ having δ nodes as the only singularities and, consequently, having geometric genus $p_g := p - \delta$. These are classically called *Severi varieties* of irreducible δ-nodal curves on S in $|L|$.

It is now well known—as a direct consequence of Mumford's theorem on the existence of nodal rational curves on $K3$ surfaces (see [2] or [39]) and standard results on Severi varieties—that if (S, H) is a general, primitively polarized $K3$ surface of genus $p \geq 3$, then the Severi variety $V_{|H|,\delta}$ is nonempty and *regular*; in other words, it is smooth and of the expected dimension $p - \delta$ for each $\delta \leq p$ ([56, Lemma 2.4, Thm. 2.6]; see also [15; 20]).

The regularity property follows from the fact that, since by definition $V_{|L|,\delta}$ parameterizes irreducible curves, the nodes of these curves impose independent conditions on $|L|$ [15; 20; 56, Rem. 2.7]. In terms of equisingular deformation theory, this implies that suitable obstructions are zero to some locally trivial deformations. In other words, it implies that $V_{|L|,\delta'} \subset \bar{V}_{|L|,\delta}$ for any $\delta' > \delta$ (see [53, Anhang F], [60], and [51, Thm. 4.7.18] for $\mathbb{P}^2$ and [56, Sec. 3] for $K3$ surfaces). Furthermore, if $[C] \in V_{|L|,\delta+k}$ for $k > 0$ is a general point of an irreducible component, then the fact that the nodes impose independent conditions enables a clear description of what $\bar{V}_{|L|,\delta}$ looks like locally around the point $[C]$: it is the union of $\binom{\delta+k}{\delta}$ smooth branches through $[C]$, where each branch corresponds to a choice of δ "marked" (or "assigned") nodes among the $\delta + k$ nodes of C and where these branches intersect transversally at $[C]$; moreover, the other k "unassigned" nodes of C disappear when one deforms $[C]$ in the corresponding branch of $\bar{V}_{|L|,\delta}$ (see [53, Anhang F], [60], and [50, Sec. 1] for $\mathbb{P}^2$ and [56, Sec. 3] for $K3$ surfaces).

The situation is slightly different for *reducible* nodal curves in $|L|$. Since they appear in the proof of Theorem 5.2, we must also take care of this case. Toward that end, we define the "degenerated" version of $V_{|L|,\delta}$ as

$$W_{|L|,\delta} := \{C \in |L| \mid C, \text{ not necessarily irreducible, has only nodes as singularities and } \textit{at least } \delta \text{ nodes}\}. \tag{5.4}$$

For the same reasons given before, $W_{|L|,\delta}$ is a locally closed subscheme of $|L|$. Note that

$$W_{|L|,\delta} = \bigcup_{\delta' \geq \delta} V_{|L|,\delta'} \quad \text{if all the curves in } |L| \text{ are irreducible,} \tag{5.5}$$

which is a partial compactification of $V_{|L|,\delta}$.

Let $[C] \in W_{|L|,\delta}$. Choosing any subset $\{p_1, \dots, p_\delta\}$ of δ of its nodes, one obtains a *pointed curve* $(C; p_1, \dots, p_\delta)$, where $p_1, \dots, p_\delta$ are also called the *marked* (or *assigned*) nodes of C (see [56, Defs. 3.1(ii) and 3.6(i)]).

Recall that there exists an algebraic scheme, which we denote by

$$\mathcal{B}(C; p_1, p_2, \dots, p_\delta) \tag{5.6}$$

and is locally closed in $|L|$, representing the functor of infinitesimal deformations of C in $|L|$ that preserve the marked nodes—that is, the functor of locally trivial infinitesimal deformations of the pointed curve $(C; p_1, \dots, p_\delta)$ (cf. [56, Prop. 3.3], where we have identified the schemes therein with their projections into the linear system $|L|$). In other words, $\mathcal{B}(C; p_1, p_2, \dots, p_\delta)$ is the *local branch* of $W_{|L|,\delta}$ around $[C] \in W_{|L|,\delta}$ corresponding to the choice of the δ marked nodes.

THEOREM 5.7 (cf. [56, Thm. 3.8]). *Let $(C; p_1, \dots, p_\delta)$ be as before. Assume that the general element of $|L|$ is a smooth irreducible curve and that the partial normalization of C at the δ marked nodes $p_1, \dots, p_\delta$ is a connected curve.*

Then $\mathcal{B}(C; p_1, p_2, \dots, p_\delta)$ is smooth at the point $[(C; p_1, p_2, \dots, p_\delta)]$ *of dimension* $\dim(|L|) - \delta$.

Proof. This follows from [56, Thm. 3.8] since, by our assumptions, the pointed curve $(C; p_1, \dots, p_\delta)$ is *virtually connected* in the language of [56, Def. 3.6]. □

For the proof of Theorem 5.2 we need to recall other fundamental facts. We first define, for any globally generated line bundle L of sectional genus $p := p_a(L) \geq 2$ on a $K3$ surface S and for any integer δ such that $0 < \delta \leq p - 2$, the locus in the Severi variety $V_{|L|,\delta}$:

$$V^{\text{hyper}}_{|L|,\delta} := \{C \in V_{|L|,\delta} \mid \text{its normalization is hyperelliptic}\}. \tag{5.8}$$

Observe in particular that, for any $p \geq 3$, one always has $V^{\text{hyper}}_{|L|,p-2} = V_{|L|,p-2} \neq \emptyset$ and, by regularity of $V_{|L|,p-2}$, this is smooth and of dimension 2.

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g, which is quasi-projective of dimension $3g - 3$ for $g \geq 2$. Denote by $\bar{\mathcal{M}}_g$ its Deligne–Mumford compactification. Then $\bar{\mathcal{M}}_g$ is the moduli space of stable genus-g curves. Let $\mathcal{H}_g \subset \mathcal{M}_g$ denote the locus of hyperelliptic curves, which is known to be an irreducible variety of dimension $2g - 1$ (see e.g., [1]), and let $\bar{\mathcal{H}}_g \subset \bar{\mathcal{M}}_g$ be its compactification.

Moreover, recall from [22, Def. (3.158)] that a nodal curve C (not necessarily irreducible) is *stably equivalent* to a stable curve C' if C' is obtained from C by contracting to a point all smooth rational components of C meeting the other components in only one or two points.

As before, we define the degenerated version of $V^{\text{hyper}}_{|L|,\delta}$ by

$$\begin{aligned} W^{\text{hyper}}_{|L|,\delta} := \{C \in W_{|L|,\delta} \mid\ & \text{there exists a desingularization } \tilde{C} \text{ of } \delta \text{ of the} \\ & \text{nodes of } C \text{ such that } \tilde{C} \text{ is stably equivalent to} \\ & \text{a (stable) curve } C' \text{ with } [C'] \in \bar{\mathcal{H}}_{p_a(L)-\delta}\}. \end{aligned} \tag{5.9}$$

Note that, by definition, any such $\tilde{C}$ is connected. Similarly as in (5.5), we have

$$W^{\text{hyper}}_{|L|,\delta} = \bigcup_{\delta' \geq \delta} V^{\text{hyper}}_{|L|,\delta} \text{ if all the curves in } |L| \text{ are irreducible.} \tag{5.10}$$

Theorem 5.2 will be a direct consequence of the next three results: Proposition 5.11, Proposition 5.19, and Lemma 5.20. The central degeneration argument is given as follows.

Proposition 5.11. *Let $p \geq 3$ and $\delta \leq p - 2$ be positive integers. Assume there exists a smooth $K3$ surface S_0 with a globally generated, primitive line bundle H_0 on S_0 with $p_a(H_0) = p$ and such that $W^{\text{hyper}}_{|H_0|,\delta}(S_0) \neq \emptyset$ and* $\dim(W^{\text{hyper}}_{|H_0|,\delta}(S_0)) \leq 2$.

Then, on the general, primitively marked $K3$ surface (S, H) of genus p, it follows that $W^{\text{hyper}}_{|H|,\delta}(S)$ is nonempty and equidimensional of dimension 2.

Proof. Let $\mathcal{B}_p$ be the moduli space of primitively marked $K3$ surfaces of genus p. It is well known that $\mathcal{B}_p$ is smooth and irreducible of dimension 19 (see e.g. [2, Thm. VIII 7.3 and p. 366]. We let $b_0 = [(S_0, H_0)] \in \mathcal{B}_p$. Similarly as in [5], consider the *scheme of pairs*

$$\mathcal{W}_{p,\delta} := \{(S, C) \mid [(S, H)] \in \mathcal{B}_p \text{ and } [C] \in W_{|H|,\delta}(S)\} \tag{5.12}$$

and the natural projection

$$\pi : \mathcal{W}_{p,\delta} \longrightarrow \mathcal{B}_p. \tag{5.13}$$

(That $\mathcal{W}_{p,\delta}$ is a scheme—in fact, a locally closed scheme—follows from the proof of Mumford's theorem on the existence of nodal rational curves [39, pp. 351–352; 2, pp. 365–367].)

Note that for general $[(S_b, H_b)] = b \in \mathcal{B}_p$ we have

$$\pi^{-1}(b) = \bigcup_{\delta' \geq \delta} V_{|H_b|,\delta'}(S_b)$$

by (5.5) (since $\mathrm{Pic}(S_b) \simeq \mathbb{Z}[H_b]$), so $\pi^{-1}(b)$ is nonempty, equidimensional, and of dimension $g := p - \delta$ by the regularity property recalled previously. In particular, π is dominant. Observe that $\mathcal{W}_{p,\delta}$ is singular in codimension 1, so it is not normal.

For brevity, let $\mathcal{W} := \mathcal{W}_{p,\delta}$ and let $\mathcal{C} \xrightarrow{f} \mathcal{W}$ be the universal curve. As in Theorem A.1(i) and (ii) (see the Appendix), there exists a commutative diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "f'"'] & \mathcal{C} \arrow[d, "f"] \\ \mathcal{W}_{(\delta)} \arrow[r, "\alpha"] & \mathcal{W}, \end{tikzcd}$$

where α is a finite unramified morphism defining a marking of all the δ-tuples of nodes of the fibers of f (cf. Theorem A.1 with $V = \mathcal{W}$ and $E_{(\delta)} = \mathcal{W}_{(\delta)}$). More precisely: in the notation of Theorem A.1, if for $w \in \mathcal{W}$ the curve $\mathcal{C}(w)$ has $\delta + \tau$ nodes with $\tau \in \mathbb{Z}^+$, then $\alpha^{-1}(w)$ consists of $\binom{\delta+\tau}{\delta}$ elements because any $\eta_w \in \alpha^{-1}(w)$ parameterizes an unordered marked δ-tuple of the $\delta + \tau$ nodes of $\mathcal{C}(w)$.

Let $\eta_w \in \mathcal{W}_{(\delta)}$. Then η_w is represented by a pointed curve $(C; p_1, p_2, \ldots, p_\delta)$, where $(S, C) \in \mathcal{W}$ and where $p_1, p_2, \ldots, p_\delta$ are δ marked nodes on C.

Let $\mathcal{W}(S, H)$ (resp. $\mathcal{W}_{(\delta)}(S, H)$) be the fiber of π (resp. of $\alpha \circ \pi$) over $[(S, H)] \in \mathcal{B}_p$, and let

$$\alpha(S, H) \colon \mathcal{W}_{(\delta)}(S, H) \longrightarrow \mathcal{W}(S, H)$$

be the induced morphism. For $\eta_w \in \mathcal{W}_{(\delta)}(S, H)$ as before, we have

$$T_{[\eta_w]}(\mathcal{W}_{(\delta)}(S, H)) \cong T_{[(C; p_1, p_2, \ldots, p_\delta)]}(\mathcal{B}(C; p_1, p_2, \ldots, p_\delta)), \tag{5.14}$$

where $\mathcal{B}(C; p_1, p_2, \ldots, p_\delta)$ is as in (5.6). Indeed, because α is finite and unramified, $\alpha(S, H)$ is also. Therefore, it suffices to consider the image of the differential $d\alpha(S, H)_{[\eta_w]}$. This image is given by first-order deformations of C in S (equivalently, in $|H|$) that are locally trivial at the δ marked nodes; these are precisely given by $T_{[(C; p_1, p_2, \ldots, p_\delta)]}(\mathcal{B}(C; p_1, p_2, \ldots, p_\delta))$ [56, Rem. 3.5].

Let $\widetilde{\mathcal{W}}_{(\delta)}$ be the smooth locus of $\mathcal{W}_{(\delta)}$. From Theorem 5.7 and (5.14), together with the fact that $\mathcal{B}_p$ is smooth, it follows that $\widetilde{\mathcal{W}}_{(\delta)}$ contains all the pairs (S, C) with δ marked nodes on C such that $|C|$ is globally generated (i.e., its general element is a smooth irreducible curve), and the partial normalization of C at these marked nodes is a connected curve. More precisely, by the proof of Mumford's theorem (see [39] or [2]), any irreducible component of $\mathcal{W}_{(\delta)}$ has dimension $\geq 19 + p - \delta = 19 + g$; furthermore, by (5.14) we have $\dim(T_{[\eta_w]}(\mathcal{W}_{(\delta)}(S,H))) = g$, where η_w represents (S,C) where C has the δ marked nodes. It also follows that $\mathcal{W}_{(\delta)}$ is smooth and of dimension $19+g$ at these points.

If we restrict $\mathcal{C}'$ to $\widetilde{\mathcal{W}}_{(\delta)}$ then, by parts (iv) and (v) of Theorem A.1, we have a commutative diagram

$$\begin{tikzcd} \tilde{\mathcal{C}} \arrow[r] \arrow[d, "\tilde{f}"'] & \mathcal{C} \arrow[d, "f"] \\ \widetilde{\mathcal{W}}_{(\delta)} \arrow[r, "\tilde{\alpha}"] & \mathcal{W}, \end{tikzcd}$$

where $\tilde{\alpha} = \alpha|_{\widetilde{\mathcal{W}}_{(\delta)}}$ and where $\tilde{f}$ is the flat family of partial normalizations at δ nodes of the curves parameterized by $\alpha(\widetilde{\mathcal{W}}_{(\delta)})$ (in the notation of Theorem A.1, $\tilde{f} = \bar{f}$ in (v) and $\tilde{\mathcal{C}} = \bar{\mathcal{C}}$ in (iii) and (iv)).

There is an obvious rational map

$$\widetilde{\mathcal{W}}_{(\delta)} \overset{c}{\dashrightarrow} \bar{\mathcal{M}}_g$$

defined on the open dense subscheme $\widetilde{\mathcal{W}}^0_{(\delta)} \subset \widetilde{\mathcal{W}}_{(\delta)}$ such that, for $\eta_w \in \widetilde{\mathcal{W}}^0_{(\delta)}$, $\tilde{\mathcal{C}}(\eta_w)$ is stably equivalent to a stable curve of genus g.

Set $\psi := c|_{\widetilde{\mathcal{W}}^0_{(\delta)}}$. By definition, for any $\eta_w \in \widetilde{\mathcal{W}}^0_{(\delta)}$, the map ψ contracts all possible smooth rational components of $\tilde{\mathcal{C}}(\eta_w)$ meeting the other components in only one or two points and also maps the resulting stable curve into its equivalence class in $\bar{\mathcal{M}}_g$.

Pick any $C_0 \in W^{\text{hyper}}_{|H_0|,\delta}(S_0)$ and let $w_0 = [(S_0, C_0)] \in \mathcal{W}$ be the corresponding point. Now $|H_0|$ is globally generated, and the normalization of C_0 at some δ nodes satisfying the conditions in (5.9) is a connected curve. Therefore, letting $\eta_{w_0} \in \alpha^{-1}(w_0)$ be the point that corresponds to marking these δ nodes, we have that $\eta_{w_0} \in \widetilde{\mathcal{W}}^0_{(\delta)}$ and the map c is defined at η_{w_0}.

Let $\tilde{\mathcal{V}} \subseteq \widetilde{\mathcal{W}}^0_{(\delta)}$ be the irreducible component containing η_{w_0}; then, as already proved, $\dim(\tilde{\mathcal{V}}) = 19 + g$.

By assumption, $\psi(\tilde{\mathcal{V}}) \cap \bar{\mathcal{H}}_g \neq \emptyset$. Hence, for any irreducible component $\mathcal{K} \subseteq \psi(\tilde{\mathcal{V}}) \cap \bar{\mathcal{H}}_g$,

$$\dim(\mathcal{K}) \geq \dim(\psi(\tilde{\mathcal{V}})) + \dim(\bar{\mathcal{H}}_g) - \dim(\bar{\mathcal{M}}_g) = \dim(\psi(\tilde{\mathcal{V}})) + 2 - g. \tag{5.15}$$

Pick any $\mathcal{K}$ containing $\psi(\eta_{w_0})$ and let $\mathcal{I} \subseteq \psi^{-1}|_{\tilde{\mathcal{V}}}(\mathcal{K})$ be any irreducible component containing η_{w_0}. Since the general fiber of $\psi|_{\tilde{\mathcal{V}}}$ has dimension

$$\dim(\tilde{\mathcal{V}}) - \dim(\psi(\tilde{\mathcal{V}})) = 19 + g - \dim(\psi(\tilde{\mathcal{V}})),$$

from (5.15) it follows that

$$\begin{aligned} \dim(\mathcal{I}) &= \dim(\mathcal{K}) + 19 + g - \dim(\psi(\tilde{\mathcal{V}})) \\ &\geq \dim(\psi(\tilde{\mathcal{V}})) + 2 - g + 19 + g - \dim(\psi(\tilde{\mathcal{V}})) = 21. \end{aligned} \tag{5.16}$$

Consider now

$$\pi \circ (\tilde{\alpha}|_{\mathcal{I}}) \colon \mathcal{I} \longrightarrow \mathcal{B}_p. \tag{5.17}$$

By assumption, the fiber over $b_0 = [(S_0, H_0)]$ is at most 2-dimensional, so we may conclude from (5.16) that $\pi \circ (\tilde{\alpha}|_{\mathcal{I}})$ is dominant, that all the fibers are precisely 2-dimensional, and that $\dim(\mathcal{I}) = 21$. This shows that $W^{\text{hyper}}_{|H|,\delta} \neq \emptyset$ for general $[(S, H)] \in \mathcal{B}_p$, and Lemma 5.1 implies that any irreducible component of $W^{\text{hyper}}_{|H|,\delta}(S)$ has dimension 2. □

REMARK 5.18. In particular, Lemma 5.1, Proposition 5.11 and [21, Exs. 2.8 and 2.10] prove Theorem 5.2 for $p = 4$ and 5.

We next construct the desired special primitively marked $K3$ surface.

PROPOSITION 5.19. *Let $d \geq 2$ and $k \geq 1$ be integers. Then there exists a $K3$ surface S_0 with*

$$\mathrm{Pic}(S_0) = \mathbb{Z}[E] \oplus \mathbb{Z}[F] \oplus \mathbb{Z}[R],$$

with intersection matrix

$$\begin{bmatrix} E^2 & E.F & E.R \\ F.E & F^2 & F.R \\ R.E & R.F & R^2 \end{bmatrix} = \begin{bmatrix} 0 & d & k \\ d & 0 & k \\ k & k & -2 \end{bmatrix},$$

and such that the following conditions are satisfied:

(a) *$|E|$ and $|F|$ are elliptic pencils*;
(b) *R is a smooth, irreducible rational curve*;
(c) *$H_0 := E + F + R$ is globally generated—in particular, the general member of $|H_0|$ is a smooth irreducible curve of arithmetic genus $p := 2k + d$*;
(d) *the only effective decompositions of H_0 are*

$$H_0 \sim E + F + R \sim (E + F) + R \sim (E + R) + F \sim (F + R) + E.$$

Proof. Since the lattice has signature $(1, 2)$, a result of Nikulin [44] (see also [40, Cor. 2.9(i)]) shows the existence of a $K3$ surface S_0 with that as Picard lattice. Performing Picard–Lefschetz reflections on the lattice, we can assume that H_0 is nef by [2, VIII, Prop. 3.9]. Straightforward calculations on the Picard lattice rule out the existence of effective divisors Γ satisfying $\Gamma^2 = -2$ and $\Gamma.E < 0$ or satisfying $\Gamma.F < 0$ or satisfying $\Gamma^2 = 0$ and $\Gamma.H_0 = 1$. Hence (a) and (c) follow from [49, Prop. 2.6 and (2.7)]. One may similarly compute that if $\Gamma > 0$, $\Gamma^2 = -2$, and $\Gamma.R < 0$, then $\Gamma = R$; this proves (b).

Similarly, (d) is proved by direct calculations using the nefness of E, F, and H_0 and by recalling that, according to Riemann–Roch and Serre duality, a divisor D on a $K3$ surface is effective and irreducible only if $D^2 \geq -2$ and $D.N > 0$ for some nef divisor N. □

The following result, together with (5.10) and Proposition 5.11, now concludes the proof of Theorem 5.2 and Corollary 5.3. Given Remark 5.18, we need only consider $p \geq 6$.

LEMMA 5.20. *Let $p \geq 6$ be an integer. There exists a smooth $K3$ surface S_0 with a globally generated, primitive line bundle H_0 on S_0 with $p = p_a(H_0)$ such that*:

(a) $W^{\text{hyper}}_{|H_0|,\,p-3}(S_0) \neq \emptyset$;
(b) $\dim(W^{\text{hyper}}_{|H_0|,\,p-3}(S_0)) = 2$;
(c) *there exists a component of* $W^{\text{hyper}}_{|H_0|,\,p-3}(S_0)$ *whose general member deforms to a curve* $[C_t] \in V^{\text{hyper}}_{|H_t|,\,p-3}(S_t)$ *for general* $[(S_t, H_t)] \in \mathcal{B}_p$;
(d) *for general* $[(S_t, H_t)] \in \mathcal{B}_p$, *the 2-dimensional irreducible component* $V_t \subseteq V^{\text{hyper}}_{|H_t|,\,p-3}(S_t)$ *given by* (c) *satisfies* $\dim(R_{V_t}) = 3$ (*with notation as in Section 2.2*).

Proof. Set $k = 1$ if p is even and $k = 2$ if p is odd, and let $d := p - 2k \geq 2$. Consider the marked $K3$ surface (S_0, H_0) in Proposition 5.19.

We will consider two general smooth elliptic curves $E_0 \in |E|$ and $F_0 \in |F|$ and curves of the form

$$C_0 := E_0 \cup F_0 \cup R,$$

with transversal intersections and a desingularization

$$\tilde{C}_0 = \tilde{E}_0 \cup \tilde{F}_0 \cup \tilde{R} \to C_0 \tag{5.21}$$

of the $\delta := p - 3 = d + 2k - 3$ nodes marked in Figure 1—that is, all *but one* of each of the intersection points $E_0 \cap F_0$, $E_0 \cap R$, and $F_0 \cap R$. Then $[C_0] \in W^{\text{hyper}}_{|H_0|,\,p-3}$, since $\tilde{C}_0$ is stably equivalent to a union of two smooth elliptic curves intersecting in two points [22, Exer. (3.162)], proving (a). The closure of the family we have constructed is clearly isomorphic to $|E| \times |F| \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and so is 2-dimensional. Denote by $W_0 \subset W^{\text{hyper}}_{|H_0|,\,p-3}$ this 2-dimensional subscheme.

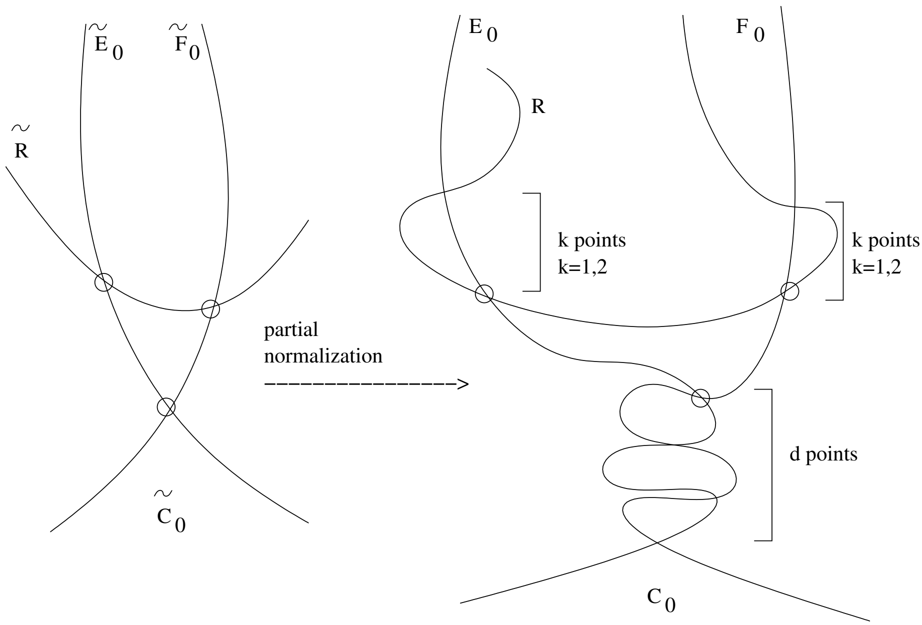

Figure 1 The curves C_0 and $\tilde{C}_0$

We will now show that any irreducible component W of $W^{\text{hyper}}_{|H_0|,\, p-3}$ has dimension ≤ 2.

A central observation, which will be used together with Theorem 4.6(b), is that—given our choices of k—we have

$$E.H_0 = F.H_0 = d + k = p - k \text{ is odd.} \tag{5.22}$$

We start by considering families of reducible curves. These are all classified in Proposition 5.19(d).

If the general element in W is of the form $D \cup R$ for $D \in |E + F|$, then in order for a partial desingularization $\tilde{D} \cup \tilde{E}$ to be (degenerated) hyperelliptic we must have $\deg(\tilde{D} \cap \tilde{R}) = 2$, so we must desingularize $2(k-1)$ of the intersection points of $D \cap R$. Finally, since $p_a(\tilde{D} \cup \tilde{R}) = 3$, we must have $p_a(\tilde{D}) = 2$. Therefore $W \subseteq W_D \times \{R\} \simeq W_D$, where $W_D \subset |D|$ is a subfamily of irreducible curves of geometric genus ≤ 2. It follows from Lemma 2.3 that $\dim(W) \leq \dim(W_D) \leq 2$.

If the general element in W is of the form $D \cup E$ for $D \in |F + R|$, then in order for a partial desingularization $\tilde{D} \cup \tilde{R}$ to be (degenerated) hyperelliptic we must have $\deg(\tilde{D} \cap \tilde{E}) = 2$. If the projection $W \to |E|$ is dominant then this means that $\mathfrak{g}^1_2(\tilde{D}) \subseteq |f^*E||_{\tilde{D}}$, where $f \colon \tilde{S} \to S$ denotes the composition of blow-ups of S that induces the partial desingularization $\tilde{D} \cup \tilde{R} \to D \cup R$. But this would mean that $|f^*E||_{\tilde{D}}$, which is base point free on $\tilde{D}$, is composed with the $\mathfrak{g}^1_2(\tilde{D})$—a contradiction because $\deg(\mathcal{O}_{\tilde{D}}(f^*E)) = E.D = E.H_0$ is odd by (5.22). Therefore, the projection $W \to |E|$ is not dominant, whence $\dim(W) \leq \dim(|D|) = \frac{1}{2}D^2 + 1 = k \leq 2$, as desired. By symmetry, the case where the general element in W is of the form $D \cup F$ (for $D \in |E + R|$) is treated in the same way.

Finally, we have to consider the case of a family $W \subseteq |H_0|$ of irreducible curves. Assume $\dim(W) \geq 3$, and let C be a general curve parameterized by W. Then, by Theorem 4.6(b), there exists an effective decomposition into moving classes $H_0 \sim M + N$ such that

$$\mathfrak{g}^1_2(\tilde{C}) \subseteq |f^*M||_{\tilde{C}},\ |f^*N||_{\tilde{C}},$$

where $f \colon \tilde{S} \to S$ denotes the succession of blow-ups of S that induces the normalization $\tilde{C} \to C$. From Proposition 5.19(d) we see that we must have

$$\mathfrak{g}^1_2(\tilde{C}) \subseteq |f^*E||_{\tilde{C}} \quad \text{or} \quad \mathfrak{g}^1_2(\tilde{C}) \subseteq |f^*F||_{\tilde{C}},$$

which means that either $|f^*E||_{\tilde{C}}$ or $|f^*F||_{\tilde{C}}$ is composed with the $\mathfrak{g}^1_2(\tilde{C})$—again a contradiction, since both have odd degree by (5.22). We have therefore proved (b).

To prove (c), we will show that any $[C_0] \in W^{\text{hyper}}_{|H_0|,\, p-3}$ (in the 2-dimensional irreducible component W_0 considered before) deforms to a curve $[C_t] \in W^{\text{hyper}}_{|H_t|,\, p-3}(S_t)$, for general $[(S_t, H_t)] \in \mathcal{B}_p$, that has precisely $\delta = p - 3$ nodes (cf. (5.10)).

Toward this end, denote by $\mathcal{S} \to \mathcal{B}_p$ the universal family of $K3$ surfaces, let $\tilde{f} \colon \tilde{\mathcal{C}} \to \widetilde{\mathcal{W}}_{(\delta)}$ and $\mathcal{I} \subset \widetilde{\mathcal{W}}_{(\delta)}$ be as in the proof of Proposition 5.11, and let $\varphi \colon \tilde{\mathcal{C}}_{\mathcal{I}} \to \mathcal{I}$ be the restriction of $\tilde{f}$.

Because the fiber over $[(S_0, H_0)]$ of $\mathcal{I} \to \mathcal{B}_p$ as in (5.17) contains an open dense subset of $\mathbb{P}^1 \times \mathbb{P}^1$, we can find a smooth irreducible curve $B \subset \mathcal{I}$ satisfying: For

$x \in B$ general, $\varphi^{-1}(x)$ is a (partial) desingularization of $\delta = p - 3$ of the nodes of a curve in $W_{|H_t|,\delta}(S_t)$ (cf. (5.4)) for general $[(S_t, H_t)] \in \mathcal{B}_p$ and $\varphi^{-1}(x) \in \bar{\mathcal{H}}_3 \subset \bar{\mathcal{M}}_3$; moreover, B contains a point $x_0 \in \mathcal{I}$ such that $\varphi^{-1}(x_0)$ is $\tilde{C}_0$, as in (5.21), for C_0 general in W_0.

Let $\varphi_B \colon \tilde{\mathcal{C}}_B \to B$ be the induced universal curve. The dualizing sheaf of $\varphi_B^{-1}(x_0) = \tilde{C}_0$ is globally generated (since each component intersects the others in two points) and so, possibly after replacing B with an open neighborhood of x_0, we actually have a morphism $\gamma_B \colon \tilde{\mathcal{C}}_B \to \mathbb{P}(\tilde{\varphi}_*(\omega_{\tilde{\mathcal{C}}/B}))$ over B that is 2 : 1 on the general fiber $\varphi_B^{-1}(x)$, that contracts the rational component $\tilde{R}$ of $\varphi_B^{-1}(x_0)$, and that maps each of the two elliptic curves $\tilde{E}_0$ and $\tilde{F}_0$ 2 : 1 onto (different) $\mathbb{P}^1$s (cf. (5.21) and Figure 1).

Let $\nu \colon \tilde{\mathcal{C}}'_B \to \tilde{\mathcal{C}}_B$ be the normalization, and let

$$\tilde{\mathcal{C}}'_B \xrightarrow{\ \gamma_1\ } \tilde{\mathcal{C}}''_B \xrightarrow{\ \gamma_2\ } \mathbb{P}(\tilde{\varphi}_*(\omega_{\tilde{\mathcal{C}}_B/B}))$$

be the Stein factorization of $\gamma_B \circ \nu$. In particular, γ_2 is finite of degree 2 onto its image. Moreover, $\nu \circ \varphi_B \colon \tilde{\mathcal{C}}'_B \to B$ is a flat family whose general fiber $(\nu \circ \varphi_B)^{-1}(x)$ is a desingularization of $\varphi_B^{-1}(x) \in \tilde{\mathcal{C}}_B$. Let p_g be the geometric genus of this general fiber.

Let $\mathcal{D} \subset \tilde{\mathcal{C}}'_B$ be the strict transform via γ_1 of the closure of the branch divisor of γ_2 on the smooth locus of $\tilde{\mathcal{C}}''_B$. By Riemann–Hurwitz, for general $x \in B$ we have $\mathcal{D}.\varphi_B^{-1}(x) = 2p_g + 2$ whereas $\mathcal{D}.\varphi_B^{-1}(x_0) \geq 8$, because the curve $\gamma_1(\varphi_B^{-1}(x_0))$ contains two smooth elliptic curves and each is mapped 2 : 1 by γ_2 onto (different) $\mathbb{P}^1$s. This implies $p_g = 3$. Since for general $x \in B$ we have $p_g \leq p_a(\varphi_B^{-1}(x)) = p - \delta = 3$, we find that $\varphi_B^{-1}(x)$ is smooth. This means that the general curve in $W_{|H_t|,\delta}(S_t)$, for $(S_t, H_t) \in \mathcal{B}_p$ general, has precisely $\delta = p - 3$ nodes; this proves (c).

To prove (d), again we consider the morphism (up to possibly restricting $\mathcal{I}$ as before)

$$\gamma_{\mathcal{I}} \colon \mathcal{C}_{\mathcal{I}} \longrightarrow \mathbb{P}(\varphi_*(\omega_{\mathcal{C}_{\mathcal{I}}/\mathcal{I}}))$$

over $\mathcal{I}$; except for some possible contractions of rational components in special fibers over $\mathcal{I}$, $\gamma_{\mathcal{I}}$ is relatively 2 : 1 onto its image. We have a natural morphism $h \colon \mathcal{C}_{\mathcal{I}} \to \mathcal{S}$ that induces a natural map

$$\Phi \colon \operatorname{im}(\gamma_{\mathcal{I}}) \dashrightarrow \operatorname{Sym}^2(\mathcal{S}),$$

whose domain has nonempty intersection with every fiber over $\mathcal{B}_p$.

Let $\mathcal{R} := \overline{\operatorname{im}(\Phi)}$. Then $\mathcal{R} \cap \operatorname{Sym}^2(S_t) = R_{V_t}$ for general $[(S_t, H_t)] \in \mathcal{B}_p$. One easily sees that

$$\{\operatorname{Sym}^2(E')\}_{E' \in |E|} \cup \{\operatorname{Sym}^2(F')\}_{F' \in |F|} \subseteq \overline{\mathcal{R} \cap \operatorname{Sym}^2(S_0)}.$$

Since the two varieties on the left are 3-folds, it follows that $\dim(\Phi^{-1}(\xi_0)) = 0$ for general $\xi_0 \in \mathcal{R} \cap \operatorname{Sym}^2(S_0) \subset \mathcal{R}$. Thus, for general $\xi \in \mathcal{R}$, we have $\dim(\Phi^{-1}(\xi)) = 0$ and so $\dim(\mathcal{R}) = \dim(\mathcal{C}_{\mathcal{I}}) = \dim(\mathcal{I}) + 1 = 22$, whence $\dim(R_{V_t}) = 22 - \dim(\mathcal{B}_p) = 3$. $\square$

REMARK 5.23. For general $[(S_t, H_t)] \in \mathcal{B}_p$, the curves obtained in the last proof have $\delta = p - 3$ *nonneutral* nodes (cf. [21, Sec. 3]). In fact, a desingularization of less than $p - 3$ nodes of C_t admits no $\mathfrak{g}^1_2$, since a desingularization of fewer than $p - 3$ nodes of C_0 is clearly not stably equivalent to a curve in the hyperelliptic locus $\bar{\mathcal{H}}_3 \subset \bar{\mathcal{M}}_3$.

6. On the Mori Cone of the Hilbert Square of a $K3$ Surface

In this section we first summarize central results on the Hilbert square of a $K3$ surface and show how to compute the class of a rational curve in $S^{[2]}$. Then we discuss the relations between the existence of curves on S and the slope of the Mori cone of $S^{[2]}$—that is, the cone of effective classes in $N_1(S^{[2]}) = N_1(S^{[2]})_{\mathbb{R}}$. In particular, we show how to deduce the different bounds described in Section 1.2. Finally, we discuss the relation between the existence of a curve on S with given singular Brill–Noether number and the slope of the Mori cone of $S^{[2]}$.

6.1. Preliminaries on $S^{[2]}$ for a $K3$ Surface

Recall that for any smooth surface S we have

$$H^2(S^{[2]}, \mathbb{Z}) \simeq H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\mathfrak{e}, \tag{6.1}$$

where $\Delta := 2\mathfrak{e}$ is the class of the divisor parameterizing 0-dimensional subschemes supported on a single point (see [5]). So we may identify a class in $H^2(S, \mathbb{Z})$ with its image in $H^2(S^{[2]}, \mathbb{Z})$. When S is a $K3$ surface, the cohomology group $H^2(S^{[2]}, \mathbb{Z})$ is endowed with a quadratic form q, called the *Beauville–Bogomolov form,* such that (a) its restriction to $H^2(S, \mathbb{Z})$ is simply the cup product on S, (b) the two factors $H^2(S, \mathbb{Z})$ and $\mathbb{Z}\mathfrak{e}$ are orthogonal with respect to this form, and (c) $q(\mathfrak{e}) = -2$. The decomposition (6.1) induces an isomorphism

$$\mathrm{Pic}(S^{[2]}) \simeq \mathrm{Pic}(S) \oplus \mathbb{Z}[\mathfrak{e}], \tag{6.2}$$

and each divisor D on S corresponds to the divisor on $S^{[2]}$ (by abuse of notation, also indicated by D) consisting of length-2 subschemes with some support on D.

Given a primitive class $\alpha \in H_2(S^{[2]}, \mathbb{Z})$, there exists a unique class $w_\alpha \in H^2(S^{[2]}, \mathbb{Q})$ such that $\alpha.v = q(w_\alpha, v)$ for all $v \in H^2(S^{[2]}, \mathbb{Z})$, and we set

$$q(\alpha) := q(w_\alpha). \tag{6.3}$$

We denote by $\rho_\alpha \in H^2(S^{[2]}, \mathbb{Z})$ the corresponding primitive $(1,1)$-class such that $\rho_\alpha = c w_\alpha$ for some $c > 0$ (see [24] for details).

If now $\mathrm{Pic}(S) = \mathbb{Z}[H]$, then the Néron–Severi group of $S^{[2]}$ has rank 2. We may take as generators of $N_1(S^{[2]})_{\mathbb{R}}$ the class $\mathbb{P}^1_\Delta$ of a rational curve in the ruling of the exceptional divisor $\Delta \subset S^{[2]}$ and the class of the curve in $S^{[2]}$ defined as

$$\{\xi \in S^{[2]} \mid \mathrm{Supp}(\xi) = \{p_0, y\},\ y \in Y\},$$

where Y is a curve in $|H|$ and p_0 is a fixed point on S. By abuse of notation, we still denote the class of the curve in $S^{[2]}$ by Y. Note that we always have that

$$\mathbb{P}^1_\Delta \text{ lies on the boundary of the Mori cone.} \tag{6.4}$$

The curve $\mathbb{P}^1_\Delta$ is contracted by the Hilbert–Chow morphism $S^{[2]} \to \mathrm{Sym}^2(S)$, so the pull-back of an ample divisor on $\mathrm{Sym}^2(S)$ is nef but is zero along $\mathbb{P}^1_\Delta$.

Hence, by (6.4), describing the Mori cone $\mathrm{NE}(S^{[2]})$ amounts to computing

$$\mathrm{slope}(\mathrm{NE}(S^{[2]})) := \inf\left\{ \tfrac{a}{b} \;\middle|\; aY - b\mathbb{P}^1_\Delta \in N_1(S^{[2]}) \text{ is effective, } a, b \in \mathbb{Q}^+ \right\}. \tag{6.5}$$

We will also call the (possibly infinite) number a/b associated to an irreducible curve $X \sim_{\mathrm{alg}} aY - b\mathbb{P}^1_\Delta$ ($a > 0$, $b \geq 0$) the *slope of the curve* X and denote it by $\mathrm{slope}(X)$. Thus, the smaller is $\mathrm{slope}(X)$, the nearer is X to the boundary of $\mathrm{NE}(S^{[2]})$.

By a general result due to Huybrechts [32, Prop. 3.2] and Boucksom [11], a divisor D on $S^{[2]}$ is ample if and only if $q(D) > 0$ and $D.R > 0$ for any (possibly singular) rational curve $R \subset S^{[2]}$. As a consequence, if the Mori cone is closed then the boundary (which remains to be determined) is generated by the class of a rational curve (the other boundary is generated by $\mathbb{P}^1_\Delta$, by (6.4)). This would mean that $\mathrm{slope}(\mathrm{NE}(S^{[2]})) = \mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]}))$, where

$$\begin{aligned}\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]})) := \inf\Big\{ \tfrac{a}{b} \;\Big|\; & aY - b\mathbb{P}^1_\Delta \in N_1(S^{[2]}) \\ & \text{is the class of a rational curve, } a, b \in \mathbb{Q}^+ \Big\}.\end{aligned} \tag{6.6}$$

(A priori, one has only $\mathrm{slope}(\mathrm{NE}(S^{[2]})) \leq \mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]}))$.)

Hassett and Tschinkel proposed in [24] a conjectural description of the Mori cone of hyperkähler 4-folds that is deformation equivalent to the Hilbert square of a $K3$ surface (cf. [24, Conj. 3.1]) as well as of the numerical and geometric properties of the rational curves that are extremal in the Mori cone (cf. [24, Conj. 3.6]). We refer the reader to [25], where one implication of [24, Conj. 3.1] is proved.

6.2. *The Classes of Rational Curves in $S^{[2]}$*

Assume that $\mathrm{Pic}(S) = \mathbb{Z}[H]$ with $p_a(H) = p_a \geq 2$. Let $X \subset S^{[2]}$ be an irreducible rational curve. Let $C_X \subset S$ be the corresponding curve as in Section 2.1, and assume that $C_X \in |mH|$ with $m \geq 1$ (in particular, $m \geq 2$ if we are in case (II)). We can write

$$X \sim_{\mathrm{alg}} a_1 Y + a_2 \mathbb{P}^1_\Delta.$$

Since $X.H = m(2p_a - 2)$, $Y.H = 2p_a - 2$, and $\mathbb{P}^1_\Delta.H = 0$ (by the very definition of H as a divisor in $S^{[2]}$) and since $Y.\mathfrak{e} = 0$ and $\mathbb{P}^1_\Delta.\mathfrak{e} = -2$, we obtain

$$X \sim_{\mathrm{alg}} mY - \left(\frac{g_0(X)+1}{2}\right)\mathbb{P}^1_\Delta, \tag{6.7}$$

where $g_0(X) := X.\mathfrak{e} - 1$.

To compute $g_0(X)$, consider the diagram (2.1). Since $\nu_X^*\mathcal{O}_X(\Delta) \simeq (\nu_X^*\mathcal{O}_X(\mathfrak{e}))^{\otimes 2}$, the double cover f is defined by $\nu_X^*\mathcal{O}_X(\Delta)$. By Riemann–Hurwitz we therefore have

$$g_0(X) = p_a(\tilde{C}_X). \tag{6.8}$$

Note that in (II) and (III) of the correspondence in Section 2.1, $X.\mathfrak{e} = g_0(X) + 1$ is precisely the length of the intersection scheme $\tilde{C}_{X,1} \cap \tilde{C}_{X,2}$, where $\tilde{C}_X = \tilde{C}_{X,1} \cup \tilde{C}_{X,2}$. In (III), since $\tilde{\nu}\colon \tilde{C}_X \to S$ contracts one of the two components of $\tilde{C}_X$ to a point $x_X \in S$, we obtain

$$g_0(X) = \mathrm{mult}_{x_X}(C_X) - 1 \quad (\text{if } C_X \text{ is of type (III)}). \tag{6.9}$$

One can check that, for all divisors D in $S^{[2]}$, one has $X.D = q(w_X, D)$ with

$$w_X := mH - \left(\frac{g_0(X)+1}{2}\right)\mathfrak{e} \in H^2(S^{[2]}, \mathbb{Q}). \tag{6.10}$$

In particular, $2w_X \in H^2(S^{[2]}, \mathbb{Z})$.

From (6.5) and (6.7) we see that searching for irreducible rational curves in (or at least "near") the boundary of the Mori cone of $S^{[2]}$, or with negative square $q(X)$, amounts to searching for irreducible curves in $|mH|$ with (partial) hyperelliptic normalizations of high genus (case (I)), or for irreducible rational curves in $|mH|$ with high multiplicity at a point (case (III)), or for irreducible rational curves on S with some correspondence between some coverings of their normalizations (case (II)). Also, we should search for curves with as low m as possible. Now $m \geq 2$ in case (II), as remarked before. Moreover, by a result of Chen [14, Thm. 1.1], any rational curve in $|H|$ on a general S is nodal (the same is also conjectured for rational curves in $|mH|$ for $m > 1$; see [13, Conj. 1.2]), so that $g_0(X) \leq 1$ if C_X is of type (III) in these cases, by (6.9). Hence, we see that the most natural candidates are irreducible curves in $|H|$ with hyperelliptic normalizations.

By the foregoing results, an irreducible curve $C \in |mH|$ with hyperelliptic normalization defines (by the unicity of the $\mathfrak{g}^1_2$) a unique irreducible rational curve $X = R_C \subset S^{[2]}$ with class

$$R_C \sim_{\mathrm{alg}} mY - \left(\frac{g_0(C)+1}{2}\right)\mathbb{P}^1_\Delta, \tag{6.11}$$

where $g_0(C) := g_0(R_C)$ is well-defined as

$$g_0(C) := \text{the arithmetic genus of a minimal partial desingularization of } C \text{ admitting a } \mathfrak{g}^1_2. \tag{6.12}$$

(For example, if C is nodal then we simply take the desingularization of the *non-neutral nodes* of C; cf. [21, Sec. 3]). From (6.5) we then get

$$\mathrm{slope}(\mathrm{NE}(S^{[2]})) \leq \frac{2m}{g_0(C)+1} \leq \frac{2m}{p_g(C)+1} \quad \text{if there exists a } C \in |mH| \text{ with hyperelliptic normalization} \tag{6.13}$$

and, by (6.3) and (6.10),

$$q(R_C) = 2m^2(p_a - 1) - \frac{(g_0(C)+1)^2}{2} \le 2m^2(p_a - 1) - \frac{(p_g(C)+1)^2}{2}. \quad (6.14)$$

In particular, the higher is $g_0(C)$ (or $p_g(C)$)—and thus the more "unexpected" is the curve on S from a Brill–Noether theory point of view—the lower is the bound on the slope of $\mathrm{NE}(S^{[2]})$ and the more negative is the square $q(R_C)$ in $S^{[2]}$.

6.3. The Invariant ρ_{sing}, Seshadri Constants, Curves with Hyperelliptic Normalizations, and the Slope of the Mori Cone

In [21] we introduced a singular Brill–Noether invariant,

$$\rho_{\mathrm{sing}}(p_a, r, d, g) := \rho(g, r, d) + p_a - g, \quad (6.15)$$

in order to study linear series on the normalization of singular curves. Precisely, we proved the following result.

Theorem 6.16. *Let S be a $K3$ surface such that $\mathrm{Pic}(S) \simeq \mathbb{Z}[H]$ with $p_a := p_a(H) \ge 2$. Let $C \in |H|$ and let $\tilde{C} \to C$ be a partial normalization of C such that $g := p_a(\tilde{C})$. Then, if $\rho_{\mathrm{sing}}(p_a, r, d, g) < 0$, it follows that $\tilde{C}$ carries no $\mathfrak{g}^r_d$.*

Proof. One easily sees that the proof of [21, Thm. 1] also holds for a *partial* normalization of C. □

For $r = 1$ and $d = 2$, we have

$$\rho_{\mathrm{sing}}(p_a, 1, 2, g) < 0 \iff g > \frac{p_a + 2}{2}. \quad (6.17)$$

In particular, a consequence of Theorem 6.16 is the following.

Theorem 6.18. *Let S be a smooth, projective $K3$ surface with $\mathrm{Pic}(S) \simeq \mathbb{Z}[H]$, and let $p_a := p_a(H) \ge 2$. Let Y and $\mathbb{P}^1_\Delta$ be the generators of $N_1(S^{[2]})$ with notation as in Section 6.1. Then, for $X \subset S^{[2]}$ an effective 1-cycle such that $X \sim_{\mathrm{alg}} Y - k\mathbb{P}^1_\Delta$, we have $k \le \frac{p_a+4}{4}$.*

Proof. We can assume that X is an irreducible curve. Then, precisely as in the case of a rational curve, X corresponds either (a) to the data of an irreducible curve $C \in |H|$ on S with a partial normalization $\tilde{C}$ admitting a $2:1$ morphism onto the normalization $\tilde{X}$ of X or (b) to the data of an irreducible curve $C \in |H|$ on S together with a point $x_0 := x_X \in S$. (The case corresponding to (II) in Section 2.1 does not occur because the coefficient of Y is 1, just as in the case of a rational X explained before.)

In the latter case $\mu(X) = \{x_0 + C\} \subset \mathrm{Sym}^2(S)$, where $\mu\colon S^{[2]} \to \mathrm{Sym}^2(S)$ is the Hilbert–Chow morphism as usual, and one easily computes $k = \frac{1}{2}\,\mathrm{mult}_{x_0}(C)$ as in the rational case discussed previously. Since clearly $\mathrm{mult}_{x_0}(C) \le 2$ if $p_a = 2$ and $\mathrm{mult}_{x_0}(C) \le 3$ if $p_a = 3$, we have $k \le \frac{p_a+4}{4}$ in these two cases. If $p_a \ge 4$ then—since $\dim|H| - 3 - (p_a - 4) = 1$ and since being singular at a given point imposes at most three independent conditions on $|H|$—we can find an irreducible curve $C' \in |H|$ that is different from C, is singular at x_0, and passes through at least $p_a - 4$ points of C. Therefore,

$$\begin{aligned} 2p_a - 2 = H^2 = C'.C &\geq \operatorname{mult}_{x_0}(C') \cdot \operatorname{mult}_{x_0}(C) + p_a - 4 \\ &\geq 2\operatorname{mult}_{x_0}(C) + p_a - 4, \end{aligned}$$

whence $\operatorname{mult}_{x_0}(C) \leq (p_a + 2)/2$ and so $k \leq (p_a + 2)/4$.

So in the first case, precisely as in the rational case before,

$$k = \frac{p_a(\tilde{C}) + 1}{2} - p_g(X) \tag{6.19}$$

from Riemann–Hurwitz. By Brill–Noether theory on $\tilde{X}$, it follows that $\tilde{C}$ carries a $\mathfrak{g}^1_d$ with

$$d \leq 2\left\lfloor \frac{p_g(X) + 3}{2} \right\rfloor.$$

By Theorem 6.16 we have $\rho_{\text{sing}}(p_a(C), 1, d, p_a(\tilde{C})) \geq 0$, and therefore $p_a(\tilde{C}) \leq d - 1 + p_a(C)/2$. The desired result now follows. $\square$

By the proof of Theorem 6.18 we see that, if $C \in |mH|$ is an irreducible curve and if $x_0 \in C$, then the class of the corresponding curve $\mu_*^{-1}\{x_0 + C\} \subset S^{[2]}$ is given by $mY - \frac{1}{2}\operatorname{mult}_{x_0}(C)\mathbb{P}^1_\Delta$. Hence

$$\begin{aligned} \operatorname{slope}(\operatorname{NE}(S^{[2]})) &\leq \inf_{m\in\mathbb{N}}\left(\inf_{C\in|mH|}\left(\inf_{x\in C}\frac{2m}{\operatorname{mult}_x(C)}\right)\right) \\ &= \inf_{m\in\mathbb{N}}\frac{2}{H^2}\left(\inf_{C\in|mH|}\left(\inf_{x\in C}\frac{C.H}{\operatorname{mult}_x(C)}\right)\right). \end{aligned}$$

It follows that

$$\operatorname{slope}(\operatorname{NE}(S^{[2]})) \leq \frac{\varepsilon(H)}{p_a - 1}, \tag{6.20}$$

where

$$\varepsilon(H) := \inf_{x\in S}\left(\inf_{C\ni x}\frac{C.H}{\operatorname{mult}_x(C)}\right)$$

(and the infimum is taken over all irreducible curves $C \subset S$ passing through x) is the (*global*) *Seshadri constant* of H (cf. [4; 17, Sec. 6; 18]). These constants are very difficult to compute; for instance, the only cases where they have been computed on general, primitively polarized $K3$ surfaces are when $H^2 = \alpha^2$, where $\varepsilon(H) = \alpha$ (cf. [3; 33]). However, it is well known that $\varepsilon(H) \leq \sqrt{H^2}$ on any surface (see e.g. [55, Rem. 1]). Hence, by (6.20) we obtain our next theorem.

THEOREM 6.21. *Let* (S, H) *be a primitively polarized* $K3$ *surface of genus* $p_a := p_a(H) \geq 2$ *such that* $\operatorname{Pic}(S) \simeq \mathbb{Z}[H]$. *Then* (*cf.* (6.5))

$$\operatorname{slope}(\operatorname{NE}(S^{[2]})) \leq \frac{\varepsilon(H)}{p_a - 1} \leq \sqrt{\frac{2}{p_a - 1}}. \tag{6.22}$$

In particular, (6.22) shows that there is no lower bound on the slope of the Mori cone of $S^{[2]}$ of $K3$ surfaces as the degree of the polarization tends to infinity; that is,

$$\inf\{\mathrm{slope}(\mathrm{NE}(S^{[2]})) \mid S \text{ is a projective } K3 \text{ surface}\} = 0. \tag{6.23}$$

The same fact about $\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]}))$ will follow from (7.4) and (7.9) to follow.

Note that by [33] one always has $\varepsilon(H) \geq \lfloor\sqrt{H^2}\rfloor - 2/\lfloor\sqrt{H^2}\rfloor + 1$ under the hypotheses of Theorem 6.21. It follows that

$$\frac{\varepsilon(H)}{p_a - 1} \geq \frac{\lfloor\sqrt{2p_a - 2}\rfloor - 2/3}{p_a - 1} \quad \text{for } (S, H) \text{ as in Theorem 6.21,} \tag{6.24}$$

showing that there is a natural limit to how good a bound one can obtain on the $\mathrm{slope}(\mathrm{NE}(S^{[2]}))$ by using Seshadri constants.

The bound in (6.22) is not (necessarily) obtained by rational curves in $S^{[2]}$. However, the presence of $p_g(X)$ in (6.19) suggests that better bounds will be obtained by rational curves in $S^{[2]}$. (Of course, if the Mori cone is closed then the bound will indeed be obtained by rational curves, as explained at the end of Section 6.1.) In fact, in Propositions 7.2 and 7.7 the bound (6.22) will be improved, for infinitely many values of H^2, by rational curves.

We now return to the study of irreducible *rational* curves in $S^{[2]}$ and to the $\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]}))$.

Given Theorem 6.16 and (6.17), a natural question to ask is whether there exist singular curves in $|H|$ with hyperelliptic normalizations of geometric genus p_g for $3 \leq p_g \leq \frac{p_a+2}{2}$. It is natural to try to construct such curves with at most nodes as singularities, since then one has better control of their deformations and their parameter spaces (the Severi varieties considered in Section 5). After the positive answer given to the above existence problem (with nodal curves) for the specific values $p_g = 3$ and $p_a = 4, 5$ in [21, Exs. 2.8 and 2.10], Theorem 5.2 gives the first examples to our knowledge of positive answers for primitively polarized $K3$ surfaces of *any* degree.

In Remark 5.23 we showed that $p_g(C) = g_0(C) = 3$ for these constructed curves $C \in |H|$ (cf. (6.12)), so that the classes of the associated rational curves $R_C \subset S^{[2]}$ are, using (6.10),

$$w_{R_C} = H - 2\mathfrak{e}, \tag{6.25}$$

with

$$q(w_{R_C}) = q(R_C) = 2p - 10 \geq -2.$$

Moreover, given (6.13), Theorem 5.2 yields the following (cf. (6.6)).

Corollary 6.26. *Let (S, H) be a general, primitively polarized $K3$ surface of genus $p_a(H) \geq 4$. Then*

$$\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]})) \leq \frac{1}{2}. \tag{6.27}$$

Note that the existence of nodal curves of geometric genus 2 in $|H|$—which was already known and, as explained at the beginning of Section 5, follows from the nonemptiness of the Severi varieties on general $K3$ surfaces—leads to the less good bound of $\frac{2}{3}$. Hence (6.27) is, again as far as we know, the first "nontrivial" bound on the slope of rational curves holding for *all* degrees of the polarization.

As already mentioned, for infinitely many degrees of the polarization we will actually improve this bound in Propositions 7.2 and 7.7.

REMARK 6.28. We do not know whether there will always be components in $|H|^{\text{hyper}}$ (whenever nonempty) of singular curves with hyperelliptic normalizations such that the singularities of the general member are as nice as possible—that is, all nodes and all nonneutral [21, Sec. 3].

7. $\mathbb{P}^2$s and 3-Folds Birational to $\mathbb{P}^1$-Bundles in the Hilbert Squares of $K3$ Surfaces

In this section we give examples of *general,* primitively polarized $K3$ surfaces (S, H), of infinitely many degrees, such that $S^{[2]}$ contains either a $\mathbb{P}^2$ or a 3-fold birational to a $\mathbb{P}^1$-bundle, thus showing both possibilities described in Proposition 3.6.

The examples are similar to Voisin's constructions in [58, Sec. 3]. The idea is to start with a smooth quartic surface S_0 such that $S_0^{[2]}$ contains an "obvious" $\mathbb{P}^2$ or a 3-fold birational to a $\mathbb{P}^1$-bundle over S_0; use the involution on the quartic to produce another such $\mathbb{P}^2$ or uniruled 3-fold; and then deform S_0, keeping the latter one but losing the former in the Hilbert square.

We remark that the question of existence of $\mathbb{P}^2$s in $S^{[2]}$ when S is $K3$ is an interesting problem because of the following fact. A $\mathbb{P}^2$ in $S^{[2]}$ gives rise to a birational map from $S^{[2]}$ onto another hyperkähler 4-fold; conversely, any birational transformation $X \dashrightarrow X'$ between projective symplectic 4-folds can be factorized into a finite sequence of Mukai flops [41, Thm. 0.7] by [62, Thm. 2] (see also [12; 30; 63]). Therefore, in the case of a $K3$ surface, if $S^{[2]}$ contains no $\mathbb{P}^2$s then $S^{[2]}$ admits no other birational model than itself. Also, uniruled divisors govern the birational Kähler cone of a hyperkähler manifold X [32].

7.1. $\mathbb{P}^2$s in $S^{[2]}$

The first nontrivial case, the case of degree 10, is particularly easy.

EXAMPLE 7.1 (Hassett). Let $S \subset \mathbb{P}^6$ be a general $K3$ surface of degree 10. By [42] the surface S is a complete intersection $S = G \cap T \cap Q$, where $G :=$ Grass$(2, 5)$ is the Grassmannian of lines in $\mathbb{P}^4$ embedded in $\mathbb{P}^9$ by its Plücker embedding, T is a general 6-dimensional linear subspace of $\mathbb{P}^9$, and Q is a hyperquadric in $\mathbb{P}^9$. Set $Y := G \cap T$. Then Y is a Fano 3-fold of index 2. Let $F(Y)$ be its variety of lines. It is classically known (see e.g. [19] for a modern proof) that $F(Y) \cong \mathbb{P}^2$. Then we may embed this plane in $S^{[2]}$ by mapping the point corresponding to a line $[\ell]$ to $\ell \cap Q$. By generality, S does not contain any line, so that this map is a morphism.

The construction behind the following result, which generalizes the previous example, was shown to us by B. Hassett.

PROPOSITION 7.2. *Let (S, H) be a general, primitively polarized $K3$ surface of degree $H^2 = 2(n^2 - 9n + 19)$ for $n \geq 6$. Then $S^{[2]}$ contains a $\mathbb{P}^2$.*

The class $w_\ell \in H^2(S^{[2]}, \mathbb{Q})$ corresponding to a line $\ell \subset \mathbb{P}^2$ is

$$w_\ell = H - \frac{2n-9}{2}\mathfrak{e}; \tag{7.3}$$

in particular,

$$\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]})) \leq \frac{2}{2n-9}. \tag{7.4}$$

Moreover, the curves $C \subset S$ with hyperelliptic normalizations associated to the lines $\ell \subset \mathbb{P}^2 \subset S^{[2]}$ lie in $|H|$ and have geometric genus $p_g = 2n - 10$, and $\rho_{\mathrm{sing}}(p_a(C), 1, 2, p_g) = n(n-13) + 42 \geq 0$.

Proof. Consider the lattice $\mathbb{Z}F \oplus \mathbb{Z}G$ with intersection matrix

$$\begin{bmatrix} F^2 & F.G \\ G.F & G^2 \end{bmatrix} = \begin{bmatrix} 2 & n \\ n & 4 \end{bmatrix}, \quad n \geq 6.$$

Because its signature is $(1,1)$, it follows from a result of Nikulin [44] (see also [40, Cor. 2.9(i)]) that there is an algebraic $K3$ surface S_0 with the given Picard lattice. Performing Picard–Lefschetz reflections on the lattice, we can assume that G is nef by [2, VIII, Prop. 3.9]. By Riemann–Roch and Serre duality, we have $G > 0$ and $F > 0$. Straightforward computations on the Picard lattice rule out the existence of divisors Γ satisfying $\Gamma^2 = -2$ and $\Gamma.F \leq 0$ or $\Gamma.G \leq 1$ or satisfying $\Gamma^2 = 0$ and $\Gamma.F = 1$ or $\Gamma.G = 1, 2$. By [49] it follows that both $|F|$ and $|G|$ are base point free, that $\varphi_{|F|}\colon S_0 \to \mathbb{P}^2$ is a double cover, and that $\varphi_{|G|}\colon S_0 \to \mathbb{P}^3$ is an embedding onto a smooth quartic not containing lines. As explained in Section 4, $S_0^{[2]}$ contains a $\mathbb{P}^2$ arising from the double cover.

If ℓ_0 is a line on the $\mathbb{P}^2$ then the corresponding class in $H^2(S_0^{[2]}, \mathbb{Q})$ is $w_{\ell_0} = 2F - 3\mathfrak{e}$, which coincides with the corresponding integral class ρ_{ℓ_0} (cf. [24, Ex. 5.1]).

Since S_0 is a quartic surface not containg lines, $S_0^{[2]}$ admits an involution

$$\iota\colon S_0^{[2]} \to S_0^{[2]}; \quad \xi \mapsto (\ell_\xi \cap S_0) \setminus \xi,$$

by [6, Prop. 11], where ℓ_ξ is the line determined by ξ and where the "\" means that we take the residual subscheme. The corresponding involution on cohomology is given by

$$v \mapsto q(G - \mathfrak{e}, v) \cdot (G - \mathfrak{e}) - v$$

(cf. e.g. [45, (4.1.6)–(4.1.7)]). The involution sends the $\mathbb{P}^2$ into another $\mathbb{P}^2$, and the corresponding class associated to a line on it is

$$q(G - \mathfrak{e}, 2F - 3\mathfrak{e}) \cdot (G - \mathfrak{e}) - (2F - 3\mathfrak{e}) = 2((n-3)G - F) - (2n-9)\mathfrak{e}. \tag{7.5}$$

In order to obtain a general $K3$ with the desired property, we now deform $S_0^{[2]}$. More precisely, we consider a general deformation of $S_0^{[2]}$ such that (i) $\mathfrak{e}$ remains algebraic and (ii) $\iota(\mathbb{P}^2)$ is preserved. Deformations satisfying (i) form a countable union of hyperplanes in the deformation space of $S_0^{[2]}$, which is smooth and of dimension 21, and may be characterized as those of the form $S^{[2]}$, where S is a $K3$ surface (see [5, Thm. 6 and Rem. 2]). Deformations preserving $\iota(\mathbb{P}^2)$ can be characterized as those preserving the image in $H^2(S^{[2]},\mathbb{Z})$ of the class of the line in $\iota(\mathbb{P}^2)$ as an algebraic class (see [24, Thm. 4.1 and Cor. 4.2] or [58])—that is, using (7.5), those deformations keeping $H := (n-3)G - F \in \mathrm{Pic}(S_0^{[2]})$ or, equivalently, $H \in \mathrm{Pic}(S)$ by (6.2). Since $H^2 = [(n-3)G-F]^2 = 2(n^2-9n+19) \geq 2$ for $n \geq 6$ and since H is primitive, it follows from [35, Thm. 14] that those deformations form a divisor in the 20-dimensional space of deformations keeping $\mathfrak{e}$ algebraic.

We therefore obtain a 19-dimensional space of deformations of $S_0^{[2]}$ whose general member is $S^{[2]}$, where (S,H) is a general, primitively polarized (algebraic) $K3$ surface of degree $H^2 = 2(n^2-9n+19)$ for $n \geq 6$ and where $S^{[2]}$ contains a plane.

The class $w_\ell \in H^2(S^{[2]},\mathbb{Q})$ corresponding to the line ℓ is as in (7.3), yielding (7.4).

Because S is general, it does not contain smooth rational curves and so the $\mathbb{P}^2$ is not of the form $C^{[2]}$ for a smooth rational curve C on S. By Lemma 2.4, the lines in the $\mathbb{P}^2$ in $S^{[2]}$ give rise to a 2-dimensional family V of curves on S with hyperelliptic normalizations so that $R_V = \mu(\mathbb{P}^2)$, where $\mu\colon S^{[2]} \to \mathrm{Sym}^2(S)$ is the Hilbert–Chow morphism. By (7.3) we have $\ell.H = H^2$; hence, by the very definition of the divisor H in $H^2(S^{[2]},\mathbb{Z})$, the lines in the $\mathbb{P}^2$ correspond to curves $C \in |H|$. Comparing (6.10) and (7.3), we see that $g_0(C) = 2n-10$ (cf. (6.12)). Now we note that the general line in the $\mathbb{P}^2$ is not tangent to $\Delta = 2\mathfrak{e}$. (Indeed, this follows by deformation: in $S_0^{[2]}$ we have that $\iota(\mathbb{P}^2) \cap \Delta$ is a smooth plane sextic because there is a composite map $S_0 \to \mathbb{P}^2 \to \iota(\mathbb{P}^2)$ that is finite of degree 2 and hence ramified along a smooth sextic, since S_0 is a smooth $K3$.) We thus have $p_g(C) = 2n-10$, and we compute $\rho_{\mathrm{sing}} = n(n-13)+42 \geq 0$ (recall that $n \geq 6$). □

The examples provided by Proposition 7.2 are interesting in several respects. Observe first that $q(\ell) = -5/2$ (cf. (6.3)), in accordance with the prediction in [24, Conj. 3.6].

The proposition shows in particular that the correspondence in Remark 3.7 is not one-to-one, and it also shows that the case $\dim(V) = \dim(R_V) = 2$ of Proposition 3.6 actually occurs.

Furthermore, the result gives nontrivial examples of curves in $|H|$ with hyperelliptic normalizations, and it answers the hyperelliptic existence problem in the affirmative for $p_a = n^2-9n+20$ and $p_g = 2n-10$, $n \geq 6$. Moreover, (7.4) shows that there is no lower bound on $\mathrm{slope}_{\mathrm{rat}}(\mathrm{NE}(S^{[2]}))$ because the degree of the polarization tends to infinity. The same conclusion follows from (7.9) in Proposition 7.7. In fact, both the bounds (7.4) and (7.9) yield better bounds on $\mathrm{slope}(\mathrm{NE}(S^{[2]}))$ than does (6.22).

Finally, the conics on the $\mathbb{P}^2$ give a 5-dimensional family $V(2)$ of irreducible curves with hyperelliptic normalizations on S. Of course, this family has obvious nonintegral members that correspond to nonintegral conics. More generally: for any $m \geq 3$, the $(3m-1)$-dimensional family of nodal rational curves in $|\mathcal{O}_{\mathbb{P}^2}(m)|$ (cf. [15, Thm. 1.1]) yields corresponding families $V(m)$ of curves in $|mH|$ with hyperelliptic normalizations where $\dim V(m) = 3m - 1 \geq 5$ and $\dim(R_V) = 2$, showing in particular that the case $\dim(V) > \dim(R_V) = 2$ of Proposition 3.6 actually occurs.

In the case of the conics, we compute $p_g = 4n - 19$ as before; since $p_a(2H) = 4n^2 - 36n + 77$, we get $\rho_{\text{sing}} = 4n(n-11) + 117 \geq -3$ in these cases. This does not contradict [21, Thm. 1].

7.2. *3-folds Birational to* $\mathbb{P}^1$*-Bundles in* $S^{[2]}$

We start with an explicit example in the special case of a quartic surface.

Example 7.6. In the case of a general quartic S in $\mathbb{P}^3$, we can find a $\mathbb{P}^1$-bundle over S in $S^{[2]}$ that arises from the 2-dimensional family of hyperplane sections of geometric genus 2. In fact, taking the tangent plane through the general point of S, we obtain a nodal curve of geometric genus 2. We derive in this way a family V of nodal curves with hyperelliptic normalizations in the hyperplane linear system. This family is parameterized by an open subset of S, and the locus in $S^{[2]}$ covered by the associated rational curves is birational to a $\mathbb{P}^1$-bundle over this open subset. To see this, set $C_p := (S \cap T_pS)$ and let $\tilde{C}_p$ be the normalization of C_p. Note that the $\mathfrak{g}^1_2$ on $\tilde{C}_p$, as viewed on C_p, is given by the pencil of lines in T_pS through the node p. If, for two distinct points $p, q \in S$, the $\mathfrak{g}^1_2$ on $\tilde{C}_p$ and $\tilde{C}_q$ had two common points, say x and y (so that the map Φ_V in (2.5) sends $(p, x+y)$ and $(q, x+y)$ to the same point $x+y$ in $\operatorname{Sym}^2(S)$), then the line $T_pS \cap T_qS$, which is bitangent to S, would also pass through x and y. This is absurd, since $\deg(S) = 4$.

By (6.10), the class $w \in H^2(S^{[2]}, \mathbb{Q})$ corresponding to the curves of geometric genus 2 is $w = H - \frac{3}{2}\mathfrak{e}$, whence $q(w) = -1/2$ as predicted by [24, Conj. 3.6]. Moreover, performing the usual involution on the quartic, we send the constructed uniruled 3-fold to another one, with corresponding fiber class given by $\mathfrak{e}$, so that it simply is the $\mathbb{P}^1$-bundle Δ over S. This shows that also our original 3-fold was smooth and thus a $\mathbb{P}^1$-bundle over S.

We now give a series of examples of general $K3$ surfaces whose Hilbert squares contain 3-folds birational to $\mathbb{P}^1$-bundles.

Proposition 7.7. *Let (S, H) be a general, primitively polarized $K3$ surface of degree $H^2 = 2(d^2 - 1)$ for $d \geq 2$. Then $S^{[2]}$ contains a 3-fold birational to a $\mathbb{P}^1$-bundle over a $K3$ surface.*

The class $w_f \in H^2(S^{[2]}, \mathbb{Q})$ corresponding to a fiber is

$$w_f = H - d\mathfrak{e} \in H^2(S^{[2]}, \mathbb{Z}); \tag{7.8}$$

in particular,

$$\operatorname{slope}_{\text{rat}}(\operatorname{NE}(S^{[2]})) \leq \frac{1}{d}. \tag{7.9}$$

Moreover, the curves $C \subset S$ with hyperelliptic normalizations associated to the fibers of the 3-fold lie in $|H|$ and have geometric genus $p_g = 2d - 1$, and we have $\rho_{\text{sing}}(p_a(C), 1, 2, p_g) = d(d-4) + 4 \geq 0$.

Proof. This time we start with the lattice $\mathbb{Z}F \oplus \mathbb{Z}G$ with intersection matrix

$$\begin{bmatrix} F^2 & F.G \\ G.F & G^2 \end{bmatrix} = \begin{bmatrix} -2 & d \\ d & 4 \end{bmatrix}, \quad d \geq 2.$$

As in the proof of Proposition 7.2, one easily shows that there is an algebraic $K3$ surface S_0 with $\mathrm{Pic}(S_0) = \mathbb{Z}F \oplus \mathbb{Z}G$, that $\varphi_{|G|}\colon S_0 \to \mathbb{P}^3$ is an embedding onto a smooth quartic not containing lines, and that F is a smooth, irreducible rational curve.

We now consider the divisor $F \subset S_0^{[2]}$, defined as the length-2 schemes with some support along F. One easily sees that this is a 3-fold birational to a $\mathbb{P}^1$-bundle over S_0 and that the class in $H^2(S_0^{[2]}, \mathbb{Z})$ corresponding to the fibers f is $\rho_f = F$ (cf. [24, Ex. 4.6]).

The involution on the quartic sends this 3-fold to another 3-fold birational to a $\mathbb{P}^1$-bundle over S_0, and the corresponding class of the fibers is $dG - F - d\mathfrak{e}$. This latter 3-fold satisfies the conditions in [24, Thm. 4.1] by [24, Ex. 4.6], so that—as in the previous proposition—we can deform $S_0^{[2]}$, keeping $\mathfrak{e}$ algebraic and $H := dG - F$. The rest now follows as in the proof of Proposition 7.2. □

The square of the class of the fibers of the uniruled 3-folds constructed here is $q(f) = -2$, as predicted in [24, Conj. 3.6].

The obtained family V of curves on S with hyperelliptic normalizations has $\dim(V) = 2$ and $\dim(R_V) = 3$, showing that also this case of Proposition 3.6 actually occurs. This family gives nontrivial examples of curves in $|H|$ with hyperelliptic normalizations and gives a positive answer to the hyperelliptic existence problem for $p_a = 2(d^2 - 1)$ and $p_g = 2d - 1$ for every $d \geq 2$. Note that the case $d = 2$ is the one described in [21, Ex. 2.8].

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Appendix: Partial Desingularizations of Families of Nodal Curves

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In this appendix we show how to construct simultaneous partial desingularizations of families of nodal curves, generalizing the well-known procedure of simultaneous total desingularization described in [4].

We work over an algebraically closed field $\mathbf{k}$ of characteristic 0. For every morphism $X \to Y$ and every $y \in Y$, we denote by $X(y)$ the scheme-theoretic fiber of y.

THEOREM A.1. *Let*

$$f : \mathcal{C} \longrightarrow V$$

be a flat projective family of curves, with $\mathcal{C}$ and V algebraic schemes, such that all fibers have at most ordinary double points (nodes) as singularities. Let $\delta \geq 1$ be an integer. Then there is a commutative diagram,

$$\begin{tikzcd} D_\delta \arrow[r, hook] \arrow[dr, "q"'] & \mathcal{C}' \arrow[r] \arrow[d, "f'"] & \mathcal{C} \arrow[d, "f"] \\ & E_{(\delta)} \arrow[r, "\alpha"] & V, \end{tikzcd}$$

with the following properties.

(i) *α is finite and unramified, the square is Cartesian, and q is an étale cover of degree δ.*

(ii) *The left triangle defines a marking of all δ-tuples of nodes of fibers of f. In particular, f' parameterizes all curves of the family f having at least δ nodes and, for each $\eta \in E_{(\delta)}$, $D_\delta(\eta) \subset \mathcal{C}'(\eta)$ is a set of δ nodes of the curve $\mathcal{C}'(\eta)$.*

(iii) *The diagram is universal with respect to properties* (i) *and* (ii)*. Precisely, if*

$$\begin{tikzcd} \tilde{D} \arrow[r, hook] \arrow[dr, "\tilde{q}"'] & \tilde{E} \times_V \mathcal{C} \arrow[r] \arrow[d, "\tilde{f}"] & \mathcal{C} \arrow[d, "f"] \\ & \tilde{E} \arrow[r] & V \end{tikzcd}$$

is a diagram with properties analogous to those of (i) *and* (ii)*, then there is a unique factorization*

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$$\tilde{E} \xrightarrow{\varphi} E_{(\delta)} \xrightarrow{\alpha} V$$

such that $\tilde{q}$ and $\tilde{f}$ are obtained by pulling back q and f' by φ.

If, moreover, $E_{(\delta)}$ is normal, then the preceding diagram can be enlarged as follows:

$$\begin{tikzcd}
 & \bar{\mathcal{C}} \arrow[d, "\beta"] & \\
D_\delta \arrow[r, hook] \arrow[dr, "q"'] & \mathcal{C}' \arrow[r] \arrow[d, "f'"] & \mathcal{C} \arrow[d, "f"] \\
 & E_{(\delta)} \arrow[r, "\alpha"] & V,
\end{tikzcd}$$

where:

(iv) *β is a birational morphism such that, for each $\eta \in E_{(\delta)}$, the restriction*

$$\beta(\eta) \colon \bar{\mathcal{C}}(\eta) \longrightarrow \mathcal{C}'(\eta)$$

is the partial normalization at the nodes $D_\delta(\eta)$; *and*

(v) *the composition $\bar{f} := f' \circ \beta$ is flat.*

Proof. Consider the first relative cotangent sheaf $\mathcal{T}^1_{\mathcal{C}/V}$. Since all fibers of f are nodal, it follows that $\mathcal{T}^1_{\mathcal{C}/V}$ commutes with base change ([3, Lemma 4.7.5] or [5]) and thus on every fiber $\mathcal{C}(v)$, $v \in V$, it restricts to $\mathcal{T}^1_{\mathcal{C}(v)}$, which is the structure sheaf of the scheme of nodes of $\mathcal{C}(v)$. We therefore have

$$\mathcal{T}^1_{\mathcal{C}/V} = \mathcal{O}_E$$

for a closed subscheme $E \subset \mathcal{C}$ supported on the nodes of the fibers of f. Consider the composition

$$f_E \colon E \subset \mathcal{C} \xrightarrow{f} V.$$

By construction, it follows that f_E is finite and unramified. Now fix $\delta \geq 1$ and consider the fiber product

$$\underbrace{E \times_V \cdots \times_V E}_{\delta}.$$

Because f_E is finite and unramified, it follows from [2, Exp. 1, Prop. 3.1] and by induction on δ (see [3, Lemma 4.7.11(i)]) that we have a disjoint union decomposition

$$E \times_V \cdots \times_V E = \Delta \coprod E_\delta,$$

where Δ is the union of all the diagonals and E_δ consists of all the ordered δ-tuples of distinct points of E mapping to the same point of V. Moreover, the natural projection morphism

$$E_\delta \longrightarrow V$$

is finite and unramified.

There is a natural action of the symmetric group Σ_δ on E_δ that commutes with the projection to V. We denote the quotient E_δ/Σ_δ by $E_{(\delta)}$. Since the composition

$$E_\delta \longrightarrow E_{(\delta)} \longrightarrow V$$

is finite and unramified and since the first morphism is an étale cover, the morphism $\alpha\colon E_{(\delta)} \to V$ is finite and unramified. Note that if, for a closed point $v \in V$, $\mathcal{C}(v)$ has $\delta + t$ nodes as the only singularities with $t > 0$, then $\alpha^{-1}(v)$ has degree $\binom{\delta+t}{t}$. Now let

$$D_\delta = \{(\eta, e)\colon e \in \mathrm{Supp}(\eta)\} \subset E_{(\delta)} \times_V E.$$

Then the first projection defines the tautological family

$$\begin{tikzcd} D_\delta \arrow[d, "q"] & \subset & E_{(\delta)} \times_V E & \subset & E_{(\delta)} \times_V \mathcal{C} \\ E_{(\delta)}, \end{tikzcd} \tag{A.2}$$

which is an étale cover of degree δ. The fiber $D_\delta(\eta)$ is the δ-tuple parameterized by η for each $\eta \in E_{(\delta)}$. (If $\delta = 1$, then $E_{(1)} = E$ and $D_1 \subset E \times_V E$ is the diagonal.) We therefore have the following diagram:

$$\begin{tikzcd} D_\delta \arrow[r, hook] \arrow[dr, "q"'] & \mathcal{C}' \arrow[r] \arrow[d, "f'"] & \mathcal{C} \arrow[d, "f"] \\ & E_{(\delta)} \arrow[r, "\alpha"] & V, \end{tikzcd}$$

where $\mathcal{C}' = E_{(\delta)} \times_V \mathcal{C}$. The fibers of f' are all the curves of the family f having at least δ nodes. For each $\eta \in E_{(\delta)}$ the divisor $D_\delta(\eta) \subset \mathcal{C}'(\eta)$ marks the set of δ nodes parameterized by η. This proves parts (i) and (ii).

Part (iii) follows because (a) $\alpha\colon E_{(\delta)} \to V$ is the relative Hilbert scheme of degree δ of $f_E\colon E \to V$ and (b) the family (A2) is the universal family.

Assume that $E_{(\delta)}$ is normal. Then we can normalize $\mathcal{C}'$ locally around D_δ as in [4, Thm. 1.3.2] to obtain a birational morphism β that has the required properties (iv) and (v). $\square$

A typical example of the situation considered in the theorem is when V parameterizes a complete linear system of curves on an algebraic surface. If the morphism f_E is *self-transverse of codimension 1* (see [3, Def. 4.7.13]), then the Severi variety of irreducible δ-nodal curves is nonsingular and of codimension δ and also $E_{(\delta)}$ is nonsingular (see [3, Lemma 4.7.14]); hence the theorem applies and the simultaneous partial desingularization exists. This happens, for example, with the linear systems of plane curves [3, Prop. 4.7.17].

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