Michigan Math. J. 58 (2009)

A Generalization of the Griffiths' Theorem on Rational Integrals, II

Alexandru Dimca, Morihiko Saito, & Lorenz Wotzlaw

0. Introduction

Let $X = \mathbf{P}^n$, and let $Y \subset X$ be a hypersurface defined by a reduced polynomial f of degree d. Set $U = X \setminus Y$. Let F and P denote, respectively, the global Hodge and pole order filtrations on the cohomology $H^n(U, \mathbf{C})$ (see [5; 6]). Locally it is easy to calculate the difference between these two filtrations at least in the case of isolated weighted homogeneous singularities; see (1.3.2) in the next section. However, this is quite nontrivial globally (i.e., on the cohomology). It is important to know when the two filtrations coincide globally, since the Hodge filtration and especially the Kodaira–Spencer map can be calculated rather easily if they coincide (see [9, Thm. 4.5]). It is known that they are different if Y has bad singularities (see [7] and also [9, 2.5]). In case the singularities consist of ordinary double points, however, it was unclear whether they still differ globally. They coincide for $n = 2$ in this case [7; 9], but the calculation for the case $n > 2$ is quite complicated in general. In this paper we prove the following result.

Theorem 1. *Assume $d = 3$ with $n \geq 5$ or $d = 4$ with $n \geq 3$. Set $m = [n/2]$, and assume that $1 + (n+1)/d \leq p \leq n - m$. Then, for a sufficiently general singular hypersurface Y, we have $F^p \neq P^p$ on $H^n(U, \mathbf{C})$.*

Here a *sufficiently general* singular hypersurface is one that corresponds to a point of a certain (sufficiently small) nonempty Zariski-open subset of $D \setminus \operatorname{Sing} D$, where D is the parameter space of singular hypersurfaces of degree d in $\mathbf{P}^n$; see Section 3.6. In particular, $\operatorname{Sing} Y$ consists of one ordinary double point. It is unclear whether the two filtrations differ whenever $\operatorname{Sing} Y$ consists of one ordinary double point. According to Theorem 1, the formula for the Kodaira–Spencer map in [9, Thm. 4.5] is effective only for $p > n - m$ in the ordinary double point case. By Theorem 2, however, we can show a similar formula in the ordinary point case that is valid also for $p \leq n - m$; see Corollary 4.5. In the case of n odd, we can also use the self-duality for the calculation of the Kodaira–Spencer map; see Remark 3.9(ii).

Received January 15, 2008. Revision received September 16, 2008.
The first author is grateful to ASSMS, Government College University, Lahore, Pakistan, where part of the work on this paper was done.

Theorem 1 implies that $F^p \neq P^p$ on $H_Y^{n+1}(X, \mathbf{C})$ by the long exact sequence associated with local cohomology. If n is odd and Y has only ordinary double points as singularities, then Y is a $\mathbf{Q}$-homology manifold and so $H^{n-1}(Y, \mathbf{Q})$ coincides with the intersection cohomology $\mathrm{IH}^{n-1}(Y, \mathbf{Q})$ (see [2; 10]) and also with the local cohomology $H_Y^{n+1}(X, \mathbf{Q})(1)$. In particular, they have a pure Hodge structure in this case. If $n = 3$ then we cannot directly calculate F^1 on $\mathrm{IH}^2(Y, \mathbf{Q})$, but this can be obtained from F^2 if we can calculate the intersection pairing. For example, if $(n, d) = (3, 4)$ then Y is a singular $K3$ surface—that is, its blow-up along the singular points is a smooth $K3$ surface, and there is a lot of work on the lattice and the intersection pairing.

Let $R = \mathbf{C}[x_0, \dots, x_n]$ with $x_0, \dots, x_n$ the coordinates of $\mathbf{C}^{n+1}$. Let $J \subset R$ be the Jacobian ideal of f (i.e., generated by $f_j := \partial f / \partial x_j$) and let I be the ideal generated by homogeneous functions vanishing at the singular points of Y. Let R_k denote the degree-k part of R, and similarly for I_k and so forth. Set $q = n - p$, $m = [n/2]$, and $I^j = R$ for $j \leq 0$. Assume that $\operatorname{Sing} Y$ consists of ordinary double points. Then Wotzlaw [23, 6.5] proposed the following.

CONJECTURE 1. $\operatorname{Gr}_F^p H^n(U, \mathbf{C}) = (I^{q-m+1}/I^{q-m}J)_{(q+1)d-n-1}$.

This is a generalization of the Griffiths' theorem on rational integrals [12], but it is quite different from the one in [9]. Indeed, the formula in [9, Thm. 1] is for the case of general singularities, and it is not easy to calculate concrete examples because of torsion and the inductive limit, which create infinite-dimensional vector spaces and so make explicit calculations quite difficult. Conjecture 1 is much more explicit and algebraic (or ring theoretic). In the ordinary double point case it is much easier to calculate concrete examples using Conjecture 1. The relation between these two generalizations of the Griffiths' theorem is unclear, since the results of [9] imply only that $\operatorname{Gr}_F^p H^n(U, \mathbf{C})$ is a quotient of $(I^{q-m+1}/fI^{q-m})_{(q+1)d-n-1}$.

The original argument in [23] was essentially correct for $p \geq n - m$ (using [9; 19; 20]). Actually, Conjecture 1 holds for such p in the case of general singularities by modifying m and I appropriately; see Theorem 2.2. In the case $p < n - m$, however, there are some difficulties: among others, the coincidence of the Hodge and pole order filtrations—which is not true, as is shown in Theorem 1—was used (in fact, this problem was rather extensively studied there using the theory of logarithmic forms for strongly quasi-homogeneous singularities; see e.g. a remark after Theorem 3.14 in [23]). For other difficulties, see (2.3.1) and (2.3.4) in Section 2.

Let $\mathcal{I} \subset \mathcal{O}_X$ be the reduced ideal of $\operatorname{Sing} Y \subset X$. Set $I_k^{(i)} = \Gamma(X, \mathcal{I}^i(k))$ and $I^{(i)} = \bigoplus_k I_k^{(i)}$. The difference between I^i and $I^{(i)}$ is one of the main problems; see the remarks after (2.3.1). We have by definition the exact sequences

$$0 \to I_k^{(i)} \to R_k \xrightarrow{\beta_k^{(i)}} \bigoplus\nolimits_{y \in \operatorname{Sing} Y} \mathcal{O}_{X,y}/\mathfrak{m}_{X,y}^i, \tag{0.1}$$

choosing a trivialization of $\mathcal{O}_{X,y}(k)$, where $\mathfrak{m}_{X,y} = \mathcal{I}_y$ is the maximal ideal of $\mathcal{O}_{X,y}$. In this paper we prove a variant of Conjecture 1 as follows.

Theorem 2. *Assume that the singular points are ordinary double points. For* $q = n - p > m = [n/2]$, *we have canonical isomorphisms*

$$\mathrm{Gr}_F^p H^n(U, \mathbf{C}) = (I^{(q-m+1)}/I^{(q-m)}J)_{(q+1)d-n-1}$$
$$= (I^{(q-m+2)}/(I^{(q-m+2)} \cap I^{(q-m)}J))_{(q+1)d-n-1}, \qquad (0.2)$$

if the following condition is satisfied (*notation as in* (0.1)):

(A) $\beta_k^{(i)}$ *is surjective for* $(k, i) = (qd - n, q - m + 1)$ *and* $(qd - n - 1, q - m)$.

Moreover, condition (A) *is satisfied if*

(B) *for* $e = m(d-1) - p$, *the image of the singular points by the e-fold Veronese embedding consists of linearly independent points.*

Note that $(I^{(q-m)}J)_{(q+1)d-n-1} = \sum_{j=0}^{n} f_j I^{(q-m)}_{(q+1)d-n-d}$. Condition (B) means that, for each singular point y, there is a hypersurface of degree e containing the singular points other than y but not y; see (2.3.5). In order to satisfy (B), there should hold at least the inequality $|\mathrm{Sing}\, Y| \le \binom{e+n}{n}$. By [22], this is always satisfied for n even. For n odd, however, this is not necessarily satisfied—for example, if $n = 3$, $d = 4$, $q = 2$, and Y is a Kummer surface with 16 ordinary double points where condition (A) is not satisfied either but Conjecture 1 seems to hold. There seem to be some examples for which condition (A) is satisfied but (B) is not; see Examples 4.7. The proof of Theorem 2 uses the theory of Brieskorn modules [3] in the ordinary double point case by restricting to a neighborhood of each singular point; see Section 4.

In a special case, we can deduce the following from Theorem 2 and Lemma 2.5.

Corollary 1. *Conjecture 1 is true if the singular points consist of ordinary double points and are linearly independent points in* $\mathbf{P}^n$ (*in particular, in this case* $|\mathrm{Sing}\, Y| \le n + 1$).

In general, Conjecture 1 is still open.

The rest of the paper proceeds as follows. In Section 1, we review some basic facts from the theory of Hodge and pole order filtrations for a hypersurface of a smooth variety. In Section 2 we study the case of hypersurfaces of projective spaces, and in Section 3 we prove Theorem 1 by constructing examples explicitly. In Section 4, we prove Theorem 2 and Corollary 1 after reviewing some basic facts about Brieskorn modules in the ordinary double point case.

1. Hodge and Pole Order Filtrations

1.1. Let X be a proper smooth complex algebraic variety of dimension $n \ge 2$, and let Y be a reduced divisor on X. Set $U = X \setminus Y$. Let $\mathcal{O}_X(*Y)$ be the localization of the structure sheaf $\mathcal{O}_X$ along Y. We have the Hodge filtration F on $\mathcal{O}_X(*Y)$. This is uniquely determined by using the relation with the V-filtration of Kashiwara [14] and Malgrange [16] (see [17]). Moreover, F induces the Hodge filtration F^p of $H^j(U,\mathbf{C})$ by taking the jth cohomology group of the subcomplex $F^p\,\mathrm{DR}(\mathcal{O}_X(*Y))$ defined by

$$F_{-p}\mathcal{O}_X(*Y) \to \cdots \to F_{n-p}\mathcal{O}_X(*Y) \otimes \Omega_X^n. \tag{1.1.1}$$

Indeed, this is reduced to the normal crossing case by using a resolution of singularities together with the stability of mixed Hodge modules by the direct image under a proper morphism. In this case the Hodge filtration F on $\mathcal{O}_X(*Y)$ is given by using the sum of the pole orders along the irreducible components, and the assertion follows from [4] (as is well known).

Let P be the pole order filtration on $\mathcal{O}_X(*Y)$ (see [6]); in other words,

$$P_i\mathcal{O}_X(*Y) = \begin{cases} \mathcal{O}_X((i+1)Y) & \text{if } i \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the pole order filtration in [4, II, (3.12.2)] is by the sum of the orders of poles along the irreducible components in the normal crossing case, and it actually coincides with our Hodge filtration F on the de Rham complex.

If Y is smooth, then $F_i = P_i$ on $\mathcal{O}_X(*Y)$ (see also [11; 12]). So in the general case we obtain

$$F_i \subset P_i \ \text{ on } \mathcal{O}_X(*Y).$$

Let h be a local defining equation of Y at $y \in Y$, let $b_{h,y}(s)$ be the b-function of h, and let $\tilde{\alpha}_{Y,y}$ be the smallest root of $b_{h,y}(-s)/(1-s)$. Then by [18] we have

$$F_i = P_i \ \text{ on } \mathcal{O}_{X,y}(*Y) \quad \text{if } i \leq \tilde{\alpha}_{Y,y} - 1. \tag{1.1.2}$$

If y is an ordinary double point, then $b_{h,y}(s) = (s+1)(s+n/2)$ and hence $\tilde{\alpha}_{Y,y} = n/2$, as is well known. Note that (1.1.2) was first obtained by Deligne at least for the case of h a homogenous polynomial of degree r with an isolated singularity (where $\tilde{\alpha}_{Y,y} = n/r$) (see e.g. [18, Rem. 4.6]).

As a corollary of (1.1.2) we have

$$F^p = P^p \ \text{ on } H^j(U,\mathbf{C}) \quad \text{if } p \geq j - \tilde{\alpha}_Y + 1, \tag{1.1.3}$$

where $\tilde{\alpha}_Y = \min\{\tilde{\alpha}_{Y,y} \mid y \in \operatorname{Sing} Y\}$. Indeed, P on $H^j(U,\mathbf{C})$ is defined by the image of the jth cohomology group of the complex $P^p\,\mathrm{DR}(\mathcal{O}_X(*Y))$ as in (1.1.1) with F replaced by P, and this coincides with the image of the cohomology group of the subcomplex $\sigma_{\leq j}P^p\,\mathrm{DR}(\mathcal{O}_X(*Y))$, where $\sigma_{\leq j}$ is the filtration “bête” in [5]:

$$\sigma_{\leq j}P^p\,\mathrm{DR}(\mathcal{O}_X(*Y)) = [P_{-p}\mathcal{O}_X(*Y) \to \cdots \to P_{j-p}\mathcal{O}_X(*Y) \otimes \Omega_X^j]. \tag{1.1.4}$$

Indeed, the kth cohomology group of the quotient complex of $P^p\,\mathrm{DR}(\mathcal{O}_X(*Y))$ by (1.1.4) vanishes for $k \leq j$.

If $j = n = 3$ or 4 and if $\operatorname{Sing} Y$ consists of ordinary double points as in Theorem 1, then $\tilde{\alpha}_Y = n/2$, $n - m = 2$, and the equality in (1.1.3) holds for $p \neq n - m$.

1.2. Local Cohomology. Because $H^j(X, \mathrm{DR}(\mathcal{O}_X(*Y)/\mathcal{O}_X)) = H_Y^{j+1}(X,\mathbf{C})$, we derive the Hodge and pole order filtrations on $H_Y^{j+1}(X,\mathbf{C})$ in a similar way. Moreover, we have the compatibility of the long exact sequence

$$\cdots \to H^j(X,\mathbf{C}) \to H^j(U,\mathbf{C}) \to H^{j+1}_Y(X,\mathbf{C}) \to H^{j+1}(X,\mathbf{C}) \to \cdots, \quad (1.2.1)$$

with the pole order filtration (i.e., it is exact after taking P^p) if $X = \mathbf{P}^n$.

Indeed, we have a short exact sequence

$$0 \to P_i\mathcal{O}_X \to P_i\mathcal{O}_X(*Y) \to P_i(\mathcal{O}_X(*Y)/\mathcal{O}_X) \to 0, \quad (1.2.2)$$

where the filtration P on $\mathcal{O}_X$ and $\mathcal{O}_X(*Y)/\mathcal{O}_X$ are, respectively, the induced and quotient filtrations. This induces the long exact sequence

$$H^j(P^p\,\mathrm{DR}(\mathcal{O}_X)) \xrightarrow{\alpha_j} H^j(P^p\,\mathrm{DR}(\mathcal{O}_X(*Y))) \xrightarrow{\beta_j} H^j(P^p\,\mathrm{DR}(\mathcal{O}_X(*Y)/\mathcal{O}_X)),$$

where the cohomology group is taken over X and where the filtration P on $\mathrm{DR}(\mathcal{O}_X)$ and $\mathrm{DR}(\mathcal{O}_X(*Y)/\mathcal{O}_X))$ is defined as in (1.1.1) with F replaced by P. Since $X = \mathbf{P}^n$, the restriction morphism $H^j(X,\mathbf{C}) \to H^j(U,\mathbf{C})$ vanishes for $j \neq 0$ and the long exact sequence splits into a family of short exact sequences. This implies that $\alpha_j = 0$ for $j \neq 0$, using $F = P$ on $\mathrm{DR}(\mathcal{O}_X)$, because α_j with P replaced by F vanishes by the strictness of the Hodge filtration F on $\mathbf{R}\Gamma(X,\mathrm{DR}(\mathcal{O}_X(*Y)))$. So the assertion follows from the snake lemma and using the strictness of $F = P$ on $\mathbf{R}\Gamma(X,\mathrm{DR}(\mathcal{O}_X))$.

1.3. Semi–Weighted Homogeneous Case. Assume Y has only isolated singularities that are locally semi–weighted homogeneous. In other words, Y is analytically locally defined by a holomorphic function $h = \sum_{\alpha\ge 1} h_\alpha$, where (a) the h_α for $\alpha \in \mathbf{Q}$ are weighted homogeneous polynomials of degree α with respect to some local coordinates $x_1,\dots,x_n$ around $y \in \mathrm{Sing}\,Y$ and some positive weights $w_1,\dots,w_n$ and (b) $h_1^{-1}(0)$ (and hence Y) has an isolated singularity at y. In this case, it is well known that

$$\tilde{\alpha}_{Y,y} = \sum_i w_i \quad (1.3.1)$$

by Kashiwara's unpublished work (this also follows from [15] together with [3]).

Let $\mathcal{O}_{X,y}^{\ge\beta}$ be the ideal of $\mathcal{O}_{X,y}$ generated by $\prod_i x_i^{\nu_i}$ with $\sum_i w_i\nu_i \ge \beta - \tilde{\alpha}_{Y,y}$. Let $\mathcal{D}_X$ be the sheaf of linear differential operators with the filtration F by the order of differential operators. Put $k_0 = [n - \tilde{\alpha}_{Y,y}] - 1$. Then by [19] we have

$$\begin{aligned} F_p(\mathcal{O}_{X,y}(*Y)) &= \sum_{k\ge 0} F_{p-k}\mathcal{D}_{X,y}(\mathcal{O}_{X,y}^{\ge k+1}h^{-k-1}) \\ &= \sum_{k=0}^{k_0} F_{p-k}\mathcal{D}_{X,y}(\mathcal{O}_{X,y}^{\ge k+1}h^{-k-1}). \end{aligned} \quad (1.3.2)$$

If $w_i = 1/b$ for any i with $b \in \mathbf{N}$, then (1.3.2) implies for $p = m := [\tilde{\alpha}_{Y,y}]$ that

$$F_m(\mathcal{O}_{X,y}(*Y)) = \mathcal{O}_{X,y}^{\ge m+1}h^{-m-1}. \quad (1.3.3)$$

This does not hold in general (e.g., if the weights are $\frac{1}{3}, \frac{1}{3}, \frac{1}{2}$ with $n = 3$).

1.4. Ordinary Double Point Case. Assume that $\mathrm{Sing}\,Y$ consists of ordinary double points. Then $b_{h,y} = (s+1)(s+n/2)$ and hence $\tilde{\alpha}_{Y,y} = n/2$, as is well known (see also (1.3.1)). Set $m = [n/2]$. Then $k_0 = m-1$ and $\mathcal{O}_{X,y}^{\ge k+1} = \mathcal{O}_{X,y}$ for $k \le k_0$. Hence (1.3.2) becomes

$$F_p(\mathcal{O}_{X,y}(*Y)) = F_{p-m+1}\mathcal{D}_{X,y}(\mathcal{O}_{X,y}h^{-m}) \quad \text{if } p \geq m-1, \tag{1.4.1}$$

where $F_p(\mathcal{O}_{X,y}(*Y)) = P_p(\mathcal{O}_{X,y}(*Y))$ if $p < m-1$.

This implies the following lemma, which is compatible with (1.1.2) and was conjectured by Wotzlaw (see [20]).

1.5. LEMMA. *With the preceding notation and assumption, we have*

$$F_p(\mathcal{O}_X(*Y)) = \mathcal{I}^{p-m+1}\mathcal{O}_X((p+1)Y) \quad \textit{for } p \geq 0, \tag{1.5.1}$$

where $\mathcal{I}$ is the reduced ideal of Sing $Y \subset X$ *and $\mathcal{I}^{p-m+1} = \mathcal{O}_X$ for $p \leq m-1$.*

Proof. We reproduce here an argument in [20]. By (1.4.1) it is enough to show the following by increasing induction on $p \geq 0$:

$$F_p\mathcal{D}_{X,y}h^{-m} = \mathcal{I}_y^p h^{-m-p}. \tag{1.5.2}$$

Here $\mathcal{I}$ is the maximal ideal at y, and we may assume $h = \sum_{i=1}^n x_i^2$ (using GAGA if necessary). We must show by increasing induction on $p \geq 0$ that

$$u = x^\nu h^{-m-p} \in F_p\mathcal{D}_{X,y}h^{-m} \quad \text{if } |\nu| = p, \tag{1.5.3}$$

where $x^\nu = \prod_i x_i^{\nu_i}$ for $\nu = (\nu_1, \ldots, \nu_n) \in \mathbf{N}^n$. Here we may assume that $\nu_i \neq 1$ for any i and $p > 1$, because otherwise the assertion is easy. Then we have $x^\nu = x_i^2 x^\mu$ for some i, and

$$\partial_i(x_i x^\mu h^{-(m+p-1)}) = ((\mu_i+1)h - (m+p-1)x_i h_i)x^\mu h^{-m-p}.$$

Adding this over i yields (1.5.3), because $|\mu| + n - 2(m+p-1) \neq 0$. So (1.5.2) and hence (1.5.1) follow. □

2. Projective Hypersurface Case

2.1. HODGE FILTRATION. With the notation of Section 1.1, assume that $X = \mathbf{P}^n$ with $n \geq 2$. Then by [9, Prop. 2.2] we have

$$H^k(X, F_p\mathcal{O}_X(*Y)) = 0 \quad \text{for } k > 0. \tag{2.1.1}$$

As a corollary, $F^pH^j(U, \mathbf{C})$ is given by the jth cohomology of the complex

$$\Gamma(X, F_{-p}\mathcal{O}_X(*Y)) \to \cdots \to \Gamma(X, F_{n-p}\mathcal{O}_X(*Y) \otimes \Omega_X^n).$$

Let $R = \mathbf{C}[x_0, \ldots, x_n]$, where $x_0, \ldots, x_n$ are the coordinates of $\mathbf{C}^{n+1}$. Let J be the ideal of R generated by $f_i := \partial f/\partial x_i$ $(0 \leq i \leq n)$. Let R_k denote the degree-k part of R so that $R = \bigoplus_k R_k$, and similarly for $J_k, \ldots$. Let

$$\xi = \frac{1}{d}\sum_i x_i \frac{\partial}{\partial x_i},$$

so that $\xi f = f$. Let ι_ξ denote the interior product by ξ. Let Ω^j be the vector space of global algebraic (i.e., polynomial) j-forms on $\mathbf{C}^{n+1}$, and let $\Omega^j[f^{-1}]_k$ be the degree-k part of $\Omega^j[f^{-1}]$, where the degrees of x_i and $\mathrm{d}x_i$ are 1. Then

$$\iota_\xi(\Omega^{j+1}[f^{-1}]_0) = \Gamma(U, \Omega_U^j). \tag{2.1.2}$$

This is compatible with the differential d up to a sign because

$$\iota_\xi \circ \mathrm{d} + \mathrm{d} \circ \iota_\xi = L_\xi,$$

where L_ξ is the Lie derivation and $L_\xi \eta = (k/d)\eta$ for $\eta \in (\Omega^j[f^{-1}])_k$. For $g \in R$ we have

$$\mathrm{d}(gf^{-k}\omega_i) = (-1)^i(f\partial_i g - kgf_i)f^{-k-1}\omega, \tag{2.1.3}$$

where $\omega = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$ and $\omega_i = \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$.

Let

$$m = [\tilde{\alpha}_Y].$$

For $q \in \mathbf{N}$, let $\mathcal{I}_{(q)}$ be the ideal of $\mathcal{O}_X$ such that

$$F_q(\mathcal{O}_X(*Y)) = \mathcal{I}_{(q)}\mathcal{O}_X((q+1)Y). \tag{2.1.4}$$

Then $\mathcal{I}_{(q)} = \mathcal{O}_X$ for $q < m$ by (1.1.2). Let

$$I_k = \Gamma(X, \mathcal{I}_{(m)}(k)) \subset R_k, \qquad I = \bigoplus_{k \in \mathbf{N}} I_k \subset R.$$

Taking local coordinates $y_0, \ldots, y_n$ of $\mathbf{C}^{n+1} \setminus \{0\}$ such that $\partial/\partial y_0 = \xi$, we get

$$\iota_\xi(I\Omega^{n+1}) = \mathrm{Im}\, \iota_\xi \cap I\Omega^n \tag{2.1.5}$$

by using the injectivity of

$$\iota_\xi : \Omega^{n+1}[f^{-1}] \to \Omega^n[f^{-1}].$$

We can also argue that, for $g \in R_k$, we have $g \in I_k$ if and only if $x_i g \in I_{k+1}$ for any $i \in [0, n]$. (This follows from the definition of I.)

Observe that $m = \tilde{\alpha}_Y = +\infty$ if Y is smooth and that $\tilde{\alpha}_{Y,y} = \sum_{i=0}^n w_i$ if Y is analytically locally defined by a semi–weighted homogeneous function h with weights $w_0, \ldots, w_n$ at $y \in \mathrm{Sing}\, Y$; see (1.3.1).

From (2.1.1)–(2.1.5) we can deduce a generalization of a theorem of Griffiths [12] as follows (here no condition on the singularities of Y is assumed).

2.2. Theorem. *With notation as before (e.g., $m = [\tilde{\alpha}_Y]$), we have*

$$\mathrm{Gr}_F^{n-q} H^n(U, \mathbf{C}) = \begin{cases} (R/J)_{(q+1)d-n-1} & \text{if } q < m, \\ (I/J)_{(q+1)d-n-1} & \text{if } q = m. \end{cases} \tag{2.2.1}$$

Proof. Since $f \in J$, the assertion immediately follows from (2.1.1)–(2.1.5). □

2.3. Ordinary Double Point Case. Assume Sing Y consists of ordinary double points so that $m = [\tilde{\alpha}_Y] = [n/2]$ as in Section 1.4. Then $\mathcal{I}_{(m)}$ in (2.1.4) coincides with the (reduced) ideal $\mathcal{I}$ of Sing $Y \subset X$ by (1.5.1). Without our assumption on the singularities, this claim does not hold; see (1.3.3). Using (2.1.1) and (1.5.1), Wotzlaw obtained (2.2.1) in this case (i.e., Conjecture 1 for $p \geq n - m$); see [23, 6.5].

Let

$$I_k^{(i)} = \Gamma(\mathbf{P}^n, \mathcal{I}^i(k)) \subset R_k, \qquad I^{(i)} = \bigoplus_k I_k^{(i)} \subset R.$$

Then $(I^i)_k \subset I_k^{(i)} \subset R_k$, but it is not clear whether

$$(I^i)_k = I_k^{(i)}. \tag{2.3.1}$$

Note that (2.3.1) holds for $k \gg 0$, because the restriction to Spec $R \setminus \{0\}$ of the sheaf corresponding to I^i coincides with that for $I^{(i)}$. However, (2.3.1) for an arbitrary k does not hold in general if $q \geq 2$. For example, let $f = xyz(x+y+z)$ with $n = 2$. In this case there is no hypersurface of degree ≤ 2 passing through all the six singular points of Y (i.e., $I_i = 0$ for $i \leq 2$), so $g \in I_4^{(2)} \neq (I^2)_4 = 0$. See also Section 2.4.

Choosing a section of $\mathcal{O}_X(1)$ that does not vanish at $y \in \operatorname{Sing} Y$, we can trivialize $\mathcal{O}_{X,y}(k)$ so that we get exact sequences

$$0 \longrightarrow I_k^{(i+1)} \longrightarrow I_k^{(i)} \xrightarrow{\gamma_k^{(i)}} \bigoplus_{y \in \operatorname{Sing} Y} \frac{\mathfrak{m}_{X,y}^i}{\mathfrak{m}_{X,y}^{i+1}}, \tag{2.3.2}$$

where $\mathfrak{m}_{X,y} = \mathcal{I}_y$ is the maximal ideal of $\mathcal{O}_{X,y}$. Let

$$I_k^{(i),(y)} = \operatorname{Ker}\left(\gamma_k^{(i)} : I_k^{(i)} \to \bigoplus_{y' \in \operatorname{Sing} Y \setminus \{y\}} \frac{\mathfrak{m}_{X,y'}^i}{\mathfrak{m}_{X,y'}^{i+1}}\right).$$

If $\gamma_k^{(i)}$ is surjective, then we have the surjectivity of

$$\gamma_k^{(i),(y)} : I_k^{(i),(y)} \to \mathfrak{m}_{X,y}^i / \mathfrak{m}_{X,y}^{i+1}, \tag{2.3.3}$$

where $\gamma_k^{(i),(y)}$ is the restriction of $\gamma_k^{(i)}$.

By (1.5.1) and (2.1.2), we have an injection

$$\iota_\xi((I^{(j-p-m+1)}\Omega^{j+1})_{(j-p+1)d} f^{-(j-p+1)}) \hookrightarrow \Gamma(U, F_{j-p}\mathcal{O}_X(*Y) \otimes \Omega_X^j).$$

Here $(I^{(i)}\Omega^j)_k = I_{k-j}^{(i)} \otimes_{\mathbf{C}} (\Omega^j)_j$ because $\Omega^j = R \otimes_{\mathbf{C}} (\Omega^j)_j$.

One of the main problems is whether the preceding injection is surjective—that is, does

$$\iota_\xi(I^{(i')}\Omega^{j+1})_{k'} = \operatorname{Im} \iota_\xi \cap (I^{(i')}\Omega^j)_{k'}, \tag{2.3.4}$$

where $i' = j - p - m + 1$ and $k' = (j - p + 1)d$. Note that (2.3.4) for $j = n$ holds by the same argument as in the proof of (2.1.5). However, (2.3.4) for $j < n$ does not hold—for example, when $i' = k' - j$ (without assuming that i', k' are as before).

In Sections 2.6–2.8 we will show that (2.3.4) is closely related to the surjectivity of (2.3.3) and also to the following:

(2.3.5) for each $y \in \operatorname{Sing} Y$, there is a $g_{(y)} \in \Gamma(X, \mathcal{O}_X(e))$ such that $y \notin g_{(y)}^{-1}(0)$ and $\operatorname{Sing} Y \setminus \{y\} \subset g_{(y)}^{-1}(0)$, where e is a given positive integer.

This condition is satisfied for any $e' > e$ if it is satisfied for e. (Indeed, it is enough to replace $g_{(y)}$ with $h_{(y)}g_{(y)}$, where $h_{(y)}$ is any section of $\mathcal{O}_X(e' - e)$ such that $y \notin h_{(y)}^{-1}(0)$.) Condition (2.3.5) means that the images of the singular points by the e-fold Veronese embedding $i_{(e)}$ in Section 3.6 correspond to linearly independent vectors in the affine space.

2.4. Linearly Independent Case. Assume that the singular points correspond to linearly independent vectors in $\mathbf{C}^{n+1}$. Replacing the coordinates if necessary, we may assume that $\operatorname{Sing} Y = \{P_0, \dots, P_s\}$, where $s \in [0, n]$ and the P_i are defined by the ith unit vector of $\mathbf{C}^{n+1}$. In this case $I^{(i)} \subset R$ is a monomial ideal, and for a monomial $x^\nu := \prod_j x_j^{\nu_j}$ we have

$$x^\nu \in I^{(i)} \iff x^\nu|_{x_j=1} \in \mathfrak{m}_j^i \text{ for each } j \in [0, s], \tag{2.4.1}$$

where $\mathfrak{m}_j$ is the maximal ideal generated by x_l $(l \neq j)$. Let $\Gamma^{(i)} \subset \mathbf{N}^{n+1}$ such that

$$I^{(i)} = \sum_{\nu \in \Gamma^{(i)}} \mathbf{C} x^\nu.$$

Set $|\nu|_{(j)} = \sum_{k \neq j} \nu_k$. Then

$$\Gamma^{(i)} = \{\nu \in \mathbf{N}^{n+1} \mid |\nu|_{(j)} \geq i \ (j \in [0, s])\}. \tag{2.4.2}$$

If $|\nu| = k$, then the condition $|\nu|_{(j)} \geq i$ is equivalent to $\nu_j \leq k - i$. If $i = 1$, then I is generated by x_j for $j > s$ and by $x_j x_l$ for $j, l \in [0, s]$ with $j \neq l$.

In the case $s = 0$, we have

$$I^{(i)} = I^i \quad \text{for any } i \geq 1 \quad \text{if} \quad |\operatorname{Sing} Y| = 1. \tag{2.4.3}$$

Assume $s = n$ for simplicity. Then I is generated by $x_i x_j$ for $i \neq j$, and $I^{(2)}$ is generated by $x_i^2 x_j^2$ for $i \neq j$ and by $x_i x_j x_l$ for i, j, l mutually different. So we get $I_k^{(2)} = (I^2)_k$ for $k \geq 4$, but $I_3^{(2)} \neq (I^2)_3 = 0$.

More generally, we have the following statement.

2.5. Lemma. *Assume that the singular points of Y correspond to linearly independent vectors in $\mathbf{C}^{n+1}$. Then*

$$(I^i)_k = I_k^{(i)} \quad \text{if } k \geq 2i. \tag{2.5.1}$$

Proof. We may assume that $i \geq 2$ and $s \neq 0$ by (2.4.3). With the notation of Section 2.4, any $x^\nu \in I_k^{(i)}$ is divisible either by x_j with $j > s$ or by $x_j x_l$ with $j, l \in [0, s]$ $(j \neq l)$. (Indeed, otherwise $x^\nu = x_j^k$ for some $j \in [0, s]$, but $x_j^k \notin I^{(i)}$.) So we can proceed by increasing induction on i, applying the inductive hypothesis to the case where i and k are replaced by $i - 1$ and $k - 2$, respectively. □

2.6. Lemma. *Assume that* $\operatorname{Sing} Y$ *consists of ordinary double points and that* (2.3.5) *is satisfied for $e = k - i(d-1)$. Then $\gamma_k^{(i)}$ in* (2.3.2) *is surjective and so we have a short exact sequence*

$$0 \longrightarrow I_k^{(i+1)} \longrightarrow I_k^{(i)} \xrightarrow{\gamma_k^{(i)}} \bigoplus_{y \in \operatorname{Sing} Y} \frac{\mathfrak{m}_{X,y}^i}{\mathfrak{m}_{X,y}^{i+1}} \longrightarrow 0, \tag{2.6.1}$$

where $\mathfrak{m}_{X,y} = \mathcal{I}_y$ is the maximal ideal of $\mathcal{O}_{X,y}$.

Proof. For each $y \in \operatorname{Sing} Y$, the $f_j \in I_{d-1}$ for $j \in [0, n]$ generate $\mathcal{I}_y = \mathfrak{m}_{X,y}$, and hence the $g_{(y)} \prod_j f_j^{\nu_j}$ for $|\nu| = i$ generate $\mathfrak{m}_{X,y}^i / \mathfrak{m}_{X,y}^{i+1}$. So the assertion follows. □

2.7. REMARKS. (i) The morphism $\beta_k^{(j)}$ in (0.1) is surjective if and only if $\gamma_k^{(i)}$ in (2.3.2) is surjective for any $i \in [0, j-1]$. Thus Lemma 2.6 shows that condition (B) in Theorem 2 implies (A), since $qd - n - (q-m)(d-1) = m(d-1) - p$ and $d \geq 2$.

(ii) Let $g = \sum_{|\nu|=k} a_\nu x^\nu \in R_k$. Then $g \in I_k^{(i)}$ if and only if

$$(\partial^\mu g)(y) = 0 \quad \text{for any } \; y \in \operatorname{Sing} Y \; \text{ and } \; \mu \in \mathbf{N}^{n+1} \; \text{ with } \; |\mu| = i-1, \quad (2.7.1)$$

where $\partial^\mu g = \prod_{i=0}^n \partial_i^{\mu_i} g$. Let $M = \binom{i-1+n}{n} |\operatorname{Sing} Y|$ and $N = \binom{k+n}{n}$. The a_ν are viewed as coordinates of $\mathbf{C}^N$ parameterizing the homogeneous polynomials of degree k, and (2.7.1) gives M linear relations among the a_ν defining the subspace $I_k^{(i)} \subset R_k$. Hence β_k^i is surjective if and only if these M relations are linearly independent (i.e., iff the corresponding matrix of size (M, N) has rank M).

2.8. PROPOSITION. *Assume that* Sing Y *consists of ordinary double points. Then* (2.3.4) *with* $j = n-1$ *holds if* $\gamma_k^{(i)}$ *in* (2.3.2) *is surjective for* $k = k' - n - 1$ *and any* $i \in [0, i'-1]$.

Proof. By increasing filtration on $i' > 0$, it is enough to show that

$$\iota_\xi(\eta) \in \iota_\xi(I^{(i')}\Omega_n)_{k'} \quad \text{if } \; \eta \in (I^{(i'-1)}\Omega_n)_{k'} \; \text{ with } \; \iota_\xi(\eta) \in (I^{(i')}\Omega^{n-1})_{k'}. \quad (2.8.1)$$

For each $y \in \operatorname{Sing} Y$, take coordinates $x_0^{(y)}, \dots, x_n^{(y)}$ such that $y = (1, 0, \dots, 0)$. With the notation of Lemma (2.6), set $k = k' - n - 1$. Then, in the notation of (2.3.3), the hypothesis of the proposition implies the surjectivity of

$$\gamma_k^{(i'-1),(y)} \colon I_k^{(i'-1),(y)} \to \mathfrak{m}_{X,y}^{i'-1} / \mathfrak{m}_{X,y}^{i'}.$$

So we may replace η with $\sum_y x_0^{(y)} \eta^{(y)}$, where

$$\eta^{(y)} \in I_k^{(i'-1),(y)} \otimes_{\mathbf{C}} (\Omega^n)_n \quad \text{with} \quad \gamma_{k+1}^{(i'-1)}(\eta) = \sum_y \gamma_{k+1}^{(i'-1),(y)}(x_0^{(y)} \eta^{(y)}).$$

Then, for the proof of (2.8.1) we may assume that

$$\eta \in x_0^{(y)} I_k^{(i'-1),(y)} \otimes_{\mathbf{C}} (\Omega^n)_n \quad \text{for some } \; y \in \operatorname{Sing} Y,$$

because $g|_{X \setminus \{y\}}$ is a section of $\mathcal{I}^{(i')}(k)|_{X \setminus \{y\}}$ for any $g \in I_k^{(i'-1),(y)}$.

Let $\omega^{(y)} = \mathrm{d}x_0^{(y)} \wedge \cdots \wedge \mathrm{d}x_n^{(y)}$ and $\omega_j^{(y)} = \mathrm{d}x_0^{(y)} \wedge \cdots \wedge \widehat{\mathrm{d}x_j^{(y)}} \wedge \cdots \wedge \mathrm{d}x_n^{(y)}$. Then

$$\eta = \sum_{j=0}^n x_0^{(y)} h_j^{(y)} \omega_j^{(y)} \quad \text{with} \quad h_j^{(y)} \in I_k^{(i'-1),(y)}.$$

Calculating modulo $I^{(i')}\Omega^{n-1}$ the coefficient of

$$\mathrm{d}x_1^{(y)} \wedge \cdots \wedge \widehat{\mathrm{d}x_j^{(y)}} \wedge \cdots \wedge \mathrm{d}x_n^{(y)} \; \text{ in } \; \iota_\xi\left(\sum_{j=0}^n x_0^{(y)} h_j^{(y)} \omega_j^{(y)}\right),$$

which belongs to $I^{(i')}\Omega^{n-1}$ by the hypothesis of (2.8.1), we see that $h_j^{(y)} \in I_k^{(i')}$ for $j \neq 0$. Then we may assume $h_j^{(y)} = 0$ for $j \neq 0$, so that

$$\eta = x_0^{(y)} h_0^{(y)} \omega_0^{(y)}.$$

By the definition of $I_k^{(i'-1),(y)}$, we have

$$x_j^{(y)} h_0^{(y)} \omega_j^{(y)} \in I^{(i')}\Omega^n \quad \text{for } j \neq 0,$$

and

$$\sum_{j=0}^{n} (-1)^j \iota_\xi (x_j^{(y)} h_0^{(y)} \omega_j^{(y)}) = \iota_\xi(\iota_\xi(h_0^{(y)} \omega^{(y)})) = 0.$$

Hence the assertion follows. $\square$

3. Proof of Theorem 1

3.1. Problem. Assume that $X = \mathbf{P}^n$ and that $\operatorname{Sing} Y$ consists of ordinary double points. One of the main problems in generalizing a theorem of Griffiths [12] is whether the following equality holds:

$$\begin{gathered} F^p H^n(U, \mathbf{C}) = P^p H^n(U, \mathbf{C}), \quad \text{that is,} \\ \operatorname{Im}\big(\iota_\xi((\Omega^{n+1})_{(q+1)d} f^{-(q+1)}) \to H^n(U, \mathbf{C})\big) \subset F^p H^n(U, \mathbf{C}), \end{gathered} \tag{3.1.1}$$

where $q = n - p$. This was rather extensively studied in [23] (see e.g. a remark after Theorem 3.14 there). We show that (3.1.1) does not hold in general; see Sections 3.7 and 3.8. This implies that the isomorphism in Conjecture 1 for $p < n - m$ (i.e., $q > m$) cannot be deduced by the method indicated there.

3.2. Proposition. *Let X and Y be as before. Assume that $q = n - p > m$ and that $F^{p+1} = P^{p+1}$ on $H^n(U, \mathbf{C})$. Then $\operatorname{Gr}_F^p H^n(U, \mathbf{C})$ is a subquotient of $(I/J)_{(q+1)d-n-1}$.*

Proof. From Section 2.1 we know that $H^n(U, \mathbf{C})$ is the cokernel of

$$\mathrm{d}\colon \Gamma(X, \Omega_X^{n-1}(*Y)) \to \Gamma(X, \Omega_X^n(*Y))$$

and that $P^p H^n(U, \mathbf{C})$ is the image of $\Gamma(X, (\mathcal{O}_X((q+1)Y)) \otimes \Omega_X^n)$, and similarly for F.

Let $\mathcal{I}$ be the reduced ideal of $\operatorname{Sing} Y \subset X$ and let $I_k = \Gamma(X, \mathcal{I}(k)) \subset R_k$. By assumption together with Lemma 1.5, it follows that

$$F_q(\mathcal{O}_Z(*Y)) \subset \mathcal{I}\mathcal{O}_X((q+1)Y) \quad \text{and} \quad F^{p+1} = P^{p+1} \text{ on } H^n(U, \mathbf{C}). \tag{3.2.1}$$

Thus we obtain a commutative diagram

$$\begin{tikzcd} \Gamma(X, F_{q-1}\mathcal{O}_X \otimes \Omega_X^{n-1}) \oplus \Gamma(X, F_{q-1}\mathcal{O}_X \otimes \Omega_X^n) \arrow[r] \arrow[d] & \Gamma(X, F_q\mathcal{O}_X \otimes \Omega_X^n) \arrow[d] \\ \Gamma(X, \Omega_X^{n-1}(qY)) \oplus \Gamma(X, \Omega_X^n(qY)) \arrow[r, "\phi"] & \Gamma(X, \mathcal{I}\Omega_X^n((q+1)Y)). \end{tikzcd}$$

By (2.1.2) and (2.1.3) together with the inclusion $Rf \subset J$, we have

$$\operatorname{Coker}\phi = (I/J)_{(q+1)d-n-1}. \tag{3.2.2}$$

Thus the assertion is reduced to

$$\operatorname{Gr}_F^p H^n(U,\mathbf{C}) \text{ is a subquotient of } \operatorname{Coker}\phi. \tag{3.2.3}$$

Taking the image of the diagram by the canonical morphism to $H^n(U,\mathbf{C})$ and then adding the cokernels, we get

$$\begin{array}{ccccccc}
F^{p+1}H^n(U,\mathbf{C}) & \longrightarrow & F^pH^n(U,\mathbf{C}) & \longrightarrow & \operatorname{Gr}_F^p H^n(U,\mathbf{C}) & \longrightarrow & 0 \\
\| & & \downarrow\cap & & \downarrow\cap & & \\
P^{p+1}H^n(U,\mathbf{C}) & \xrightarrow{\bar\phi} & P^pH^n(U,\mathbf{C}) & \longrightarrow & \operatorname{Coker}\bar\phi & \longrightarrow & 0,
\end{array}$$

where the image of $d\Gamma(X,\Omega_X^{n-1}(qY))$ in $H^n(U,\mathbf{C})$ vanishes (considering the case $q=\infty$). Moreover, $\operatorname{Coker}\bar\phi$ is a quotient of $\operatorname{Coker}\phi$ by the snake lemma. So the assertion follows. □

3.3. HODGE NUMBERS OF SMOOTH HYPERSURFACES. Define integers $C(n+1, d,i)$ by

$$(t+\cdots+t^{d-1})^{n+1} = \sum_{i=n+1}^{(n+1)(d-1)} C(n+1,d,i)t^i, \tag{3.3.1}$$

so that

$$C(n+1,d,i) = C(n+1,d,(n+1)d-i),$$

where $C(n+1,d,i)=0$ unless $i\in[n+1,(n+1)(d-1)]$. This is the Poincaré polynomial of the graded vector space

$$\Omega^{n+1}/dg\wedge\Omega^n$$

if g is a homogeneous polynomial of degree d with an isolated singularity at the origin (e.g., if $g=\sum_i x_i^d$). For the hypersurface $Z'\subset X$ defined by g, we have (by Griffiths [12])

$$C(n+1,d,pd) = \dim\operatorname{Gr}_F^{n-p} H^{n-1}_{\mathrm{prim}}(Z',\mathbf{C}) \quad \text{for } p\in[1,n], \tag{3.3.2}$$

where $H^{n-1}_{\mathrm{prim}}(Z',\mathbf{C})$ denotes the primitive part.

3.4. ISOLATED SINGULARITY CASE. Assume that Y has only isolated singularities and that

$$n-p > q_0 := \max\{q \mid \operatorname{Gr}_F^q H^{n-1}(F_y,\mathbf{C})\neq 0 \text{ for some } y\in\operatorname{Sing}Y\}, \tag{3.4.1}$$

where F is the Hodge filtration on the vanishing cohomology $H^{n-1}(F_y,\mathbf{C})$ at $y\in\operatorname{Sing}Y$ (see [21]). Here F_y denotes the Milnor fiber around y. If $\operatorname{Sing}Y$ consist of ordinary double points then $q_0=m:=[n/2]$; see (3.5.1). In general, we have $q_0\geq(n-1)/2$ by the Hodge symmetry.

Under the preceding assumptions, we have

$$\dim\operatorname{Gr}_F^p H^n(U,\mathbf{C}) = \dim\operatorname{Gr}_F^p H_Y^{n+1}(X,\mathbf{C}) = C(n+1,d,pd). \tag{3.4.2}$$

Indeed, there is a perfect pairing of mixed Hodge structures

$$H_Y^{n+1}(X,\mathbf{Q}) \times H^{n-1}(Y,\mathbf{Q}) \to \mathbf{Q}(-n), \tag{3.4.3}$$

and condition (3.4.1) (together with $q_0 \geq (n-1)/2$) and (3.3.2) imply

$$\dim \mathrm{Gr}_F^{n-p} H^{n-1}(Y,\mathbf{C}) = C(n+1,d,pd). \tag{3.4.4}$$

The last assertion is reduced to the case of a smooth hypersurface by taking a 1-parameter deformation $Z_t =: \{f + tg = 0\}$ $(t \in \Delta)$ of $Y = Z_0$ whose general fibers Z_t and total space Z are smooth (we assume that the hypersurface $\{g = 0\}$ does not meet $\mathrm{Sing}\, Y$). Here we use also the exact sequence of mixed Hodge structures

$$\begin{aligned} 0 \to H^{n-1}(Y) \to H^{n-1}(Z_\infty) & \\ \xrightarrow{\rho} \bigoplus H^{n-1}(F_y) \to H^n(Y) \to H^n(Z_\infty) \to 0 & \end{aligned} \tag{3.4.5}$$

(see also [8, 1.9]), where $H^{n-1}(Z_\infty)$ denotes the limit mixed Hodge structure. Observe that $\mathrm{Gr}_F^{n-p} H^{n-1}(Z_\infty,\mathbf{C}) = \mathrm{Gr}_F^{n-p} H^{n-1}_{\mathrm{prim}}(Z_\infty,\mathbf{C})$ because $n - p > (n-1)/2$.

3.5. Remark. Assume that the singularities of Y are ordinary double points. Since the weight filtration on the unipotent (resp. non-unipotent) monodromy part of $H^{n-1}(F_y,\mathbf{Q})$ has the symmetry with center n (resp. $n-1$) by definition (see [21]) and since the monodromy on the vanishing cycles is $(-1)^n$, it follows that

$$H^{n-1}(F_y,\mathbf{Q}) = \mathbf{Q}(-m), \tag{3.5.1}$$

where $m = [n/2]$. In particular, ρ in (3.4.5) is surjective for n odd (considering the monodromy), and by the preceding argument we have

$$|\mathrm{Sing}\, Y| \leq C(n+1,d,(m+1)d) \quad \text{if } n = 2m+1. \tag{3.5.2}$$

This is related to [22]. Note that ρ can be nonsurjective if n is even and $\mathrm{Sing}\, Y$ consists of sufficiently many ordinary double points. Indeed, the Betti number $b_n(Y)$ may depend on the position of the singularities (see e.g. [7, Thm. (4.5), p. 208]). In [7] the position of singularities enters via the dimension of $I_{md-2m-1}$ (where $n = 2m$). The proof of [7, Thm. (4.5)] uses an exact sequence

$$P^{m+1}H^n(\mathbf{P}^n \setminus Y) \to \bigoplus_{y \in \mathrm{Sing}\, Y} H^n(B_y \setminus Y) \to H_0^n(Y)(-1) \to 0, \tag{3.5.3}$$

where $B_y \subset \mathbf{P}^n$ is a sufficiently small ball with center y. Here $H_0^n(Y)$ denotes the primitive cohomology defined by $\mathrm{Coker}(H^n(\mathbf{P}^n) \to H^n(Y))$. Note that

$$H^n(B_y \setminus Y) = \mathrm{Coker}(N\colon H^{n-1}(F_y) \to H^{n-1}(F_y)(-1)) = \mathbf{Q}(-m-1).$$

Using (3.4.5), (3.4.3), (1.2.1) and (2.2.1), we have also

$$\begin{aligned} \dim \mathrm{Ker}\, \rho &= \dim \mathrm{Gr}_F^m H^{n-1}(Y,\mathbf{C}) = \dim \mathrm{Gr}_F^m H_Y^{n+1}(X,\mathbf{C}) \\ &= \dim \mathrm{Gr}_F^m H^n(U,\mathbf{C}) = \dim(I/J)_{(m+1)d-2m-1}. \end{aligned}$$

If n is even and $d = 2$, then $H^{n-1}_{\mathrm{prim}}(Z_\infty,\mathbf{C}) = 0$ and ρ vanishes. If n is even, $d \geq 3$, and $\mathrm{Sing}\, Y$ consists of one ordinary double point, then $H^{n-1}_{\mathrm{prim}}(Z_\infty,\mathbf{C}) \neq 0$

and ρ is surjective (because ρ is nonzero by the theory of vanishing cycles). Hence $b_n(Y) = 1$ (for more general singularities, see [7, Thm. (4.17), p. 214]); thus, at the level of topology, nothing surprising may occur.

3.6. DISCRIMINANT. Let $i_{(d)}\colon X = \mathbf{P}^n \to \mathcal{P} = \mathbf{P}^N$ be the d-fold Veronese embedding defined by the line bundle $\mathcal{O}_X(d)$ (i.e., by using the monomials x^ν of degree d), where $N = \binom{n+d}{n} - 1$. Let $\mathcal{P}^\vee$ be the dual projective space of $\mathcal{P}$ parameterizing the hyperplanes of $\mathcal{P}$. Let $\mathcal{H} \subset \mathcal{P} \times \mathcal{P}^\vee$ be the universal hyperplane whose intersection with $\mathcal{P} \times \{z\}$ is the hyperplane corresponding to $z \in \mathcal{P}^\vee$. Let D be the discriminant of the projection

$$pr\colon (i_{(d)}(X) \times \mathcal{P}^\vee) \cap \mathcal{H} \to \mathcal{P}^\vee.$$

This is called the *dual variety* of $X \subset \mathcal{P}$. It is well known that D is irreducible (because D is the image of a $\mathbf{P}^{N-n-1}$-bundle over X corresponding to the hyperplanes that are tangent to X). By the theory of Lefschetz pencils, it is also known that Sing Y consists of one ordinary double point if and only if it corresponds to a smooth point of D.

3.7. PROOF OF THEOREM 1. By Section 3.6 it is enough to show that

$$F^{p+1} \neq P^{p+1} \quad \text{on } H^n(X \setminus Y, \mathbf{C})$$

for one hypersurface Y whose singularities consist of one ordinary double point, assuming $(n+1)/d \le p < n - m$ (i.e., $m < q \le n - (n+1)/d$). Indeed, $F^{-\infty}/F^{p+1}$ defines a vector bundle on the parameter space of hypersurfaces Y whose singularities consist of one ordinary double point; and, in the notation of Proposition 3.2, $gf^{-q}\omega$ for $g \in R_{qd-n-1}$ defines a section of this bundle when f varies. Because P^{p+1} is generated by these sections where $q = n - p$, the subset defined by the condition $P^{p+1}/F^{p+1} \neq 0$ is a Zariski-open subset.

Let

$$f = \sum_{i=1}^n \frac{x_i^d}{d} - x_0^{d-2} \sum_{i=1}^n \frac{x_i^2}{2},$$

so that

$$f_0 = -\frac{1}{2}\sum_{i=1}^3 x_i^2, \;\; f_i = x_i^2 - x_0 x_i \;\; (1 \le i \le 3) \quad \text{if } d = 3;$$

$$f_0 = -\sum_{i=1}^4 x_0 x_i^2, \;\; f_i = x_i^3 - x_0^2 x_i \;\; (1 \le i \le 4) \quad \text{if } d = 4.$$

Here I is generated by $x_1, \dots, x_n$ so that $R/I = \mathbf{C}[x_0]$. By assumption, (3.2.1) and (3.4.1) are satisfied (in particular, $q > n/2 > p$). Moreover,

$$C(n+1, d, pd) \neq 0;$$

see (3.3.1) for $C(n+1, d, k)$. The assumptions imply also that

$$p \ge 2, \;\; n \ge 5 \quad \text{if } d = 3,$$
$$p \ge 1, \;\; n \ge 3 \quad \text{if } d = 4.$$

Since $q > n/2$, we obtain

$$r := (q+1)d - n - 1 > d.$$

We will show that

$$\dim(I/J)_r < C(n+1, d, pd), \tag{3.7.1}$$

contradicting Proposition 3.2 and (3.4.2).

Take $x^\nu = \prod_{i=0}^n x_i^{\nu_i} \in I_r$ with $\nu = (\nu_0, \dots, \nu_n) \in \mathbf{N}^{n+1}$, where

$$|\nu| := \sum_{i=0}^n \nu_i = r, \quad \nu_0 < r.$$

Using f_i for $i > 0$, we can replace x^ν with x^μ mod J_r (i.e., $x^\nu - x^\mu \in J_r$) so that

$$\mu_i \le d-2 \ \ (i > 0), \qquad \nu_i - \mu_i \in (d-2)\mathbf{Z}.$$

So we may assume that $\nu_i \le d-2$ for $i > 0$. Let $|\nu|' = \sum_{i=1}^n \nu_i$ and

$$s = \min\{s \in \mathbf{Z} \mid |\nu|' - s \in (d-2)\mathbf{Z},\ s \ge r - (d-2)\}.$$

We first show that if $|\nu|' < r - (d-2)$ (i.e., if $\nu_0 > d-2$) then

$$x^\nu = (-1)^{(|\nu|'-s)/(d-2)} \sum_\mu e_{\nu,\mu} x^\mu \bmod J_r, \tag{3.7.2}$$

where the summation is taken over μ such that $|\mu|' = s$ and $\mu_i \le d-2$ for $i > 0$ and where the $e_{\nu,\mu}$ are nonnegative numbers with $e_{\nu,\mu} \ne 0$ for some μ (for each ν). By decreasing induction on $|\nu|'$, it is enough to show (3.7.2) with the summation taken over b such that $|\mu|' = |\nu|' + (d-2)$ instead of $|\mu|' = s$. But this modified assertion follows from

$$x^\nu x_0^{-2} \sum_{i=1}^n x_i^2 \in J_r \quad \text{if } \ \nu_0 > d-2, \tag{3.7.3}$$

because for $i > 0$ we have (using f_i)

$$x^\nu = x^\nu x_0^{-2} x_i^2 \bmod J_r \quad \text{if } \ \nu_i > 0,\ \nu_0 \ge 2. \tag{3.7.4}$$

(For the last argument we need the assumption $d = 3$ or 4.)

Let V_r be the vector space with basis x^μ such that $|\mu| = r$ and $\mu_i \le d-2$ for $i \ge 0$. Let $V_{r,k}$ be the vector subspace of V_r generated by x^μ such that $\mu_0 = k$ (i.e., $|\mu|' = r - k$). Then the preceding argument implies that $(I/J)_r$ is spanned by $V_r = \sum_{k=0}^{d-2} V_{r,k}$ and, moreover, that $x_0^{r-2} \sum_{i=1}^n x_i^2 \in J_r$ gives a nontrivial relation in $V_{r,r-s}$. Thus we get (3.7.1); that is,

$$\dim(I/J)_r < \dim V_r = C(n+1, d, (q+1)d) = C(n+1, d, pd).$$

Therefore, the assertion follows.

3.8. Other Examples. (i) It is not easy to extend the argument just given to the case $d \ge 5$. Let $n = 4$, $d = 5$, and

$$f = x_0^3(x_1x_4 + x_2x_3) - \sum_{i=1}^{4} \frac{x_i^5}{5},$$

so that $f_0 = 3x_0^2(x_1x_4 + x_2x_3)$ and $f_i = x_0^3x_{5-i} - x_i^4$ $(1 \le i \le 4)$. Then $F^2H^4(U,\mathbf{C}) \neq P^2H^4(U,\mathbf{C})$ for this hypersurface and hence for a sufficiently general singular hypersurface.

(ii) In the previous examples, $\operatorname{Sing} Y$ consists of one point. Let $n = 3$ and $d = 4$; let

$$f = \sum_{0\le i<j\le 3} \frac{x_i^2x_j^2}{2}, \qquad f_i = x_i \sum_{k\neq i} x_k^2.$$

Then $\operatorname{Sing} Y$ consists of four points corresponding to the unit vectors of $\mathbf{C}^4$. For this hypersurface, $F^2H^3(U,\mathbf{C}) \neq P^2H^3(U,\mathbf{C})$.

3.9. REMARKS. (i) In [9, Thm. 4.5], two of the authors gave this formula for the Kodaira–Spencer map:

$$\operatorname{Gr}_F \nabla_\xi : \operatorname{Gr}_F^{p+1} H^n(U_s,\mathbf{C}) \to \operatorname{Gr}_F^p H^n(U_s,\mathbf{C}), \tag{3.9.1}$$

where $\{Y_s\}$ is an equisingular family of hypersurfaces (see [9]). When the Y_s have only ordinary double points, Theorem 1 implies that the formula is useful only for $p > n - m$. In this case, however, (3.9.1) is given by the multiplication by $-(n-p)(\xi f)_s$ for any p under the isomorphisms of Theorem 2 and Theorem 2.2; see Theorem 4.5.

(ii) In case n is odd, Y is a $\mathbf{Q}$-homology manifold and so

$$H^n(U_s,\mathbf{C}) = H^{n+1}_{Y_s}(X,\mathbf{C})_{\text{prim}} = H^{n-1}_{\text{prim}}(Y_s,\mathbf{C})(-1).$$

Then the Kodaira–Spencer map for $p \le n - m$ can be calculated using duality, because the horizontality of the canonical pairing on $H^{n-1}_{\text{prim}}(Y_s,\mathbf{C})$ implies that the Kodaira–Spencer map is self-dual up to a sign.

4. Proof of Theorem 2

4.1. BRIESKORN MODULES FOR ORDINARY DOUBLE POINTS. We first review some basic facts about algebraic Brieskorn modules. Let $z_1, \dots, z_n$ be the coordinates of $Z = \mathbf{C}^n$ and let $h = \sum_{i=1}^n z_i^2$. We denote by $(\Omega_Z^\bullet, \mathrm{d})$ the complex of algebraic differential forms on Z. Let $(A_h^\bullet, \mathrm{d})$ be the subcomplex defined by

$$A_h^i = \operatorname{Ker}(\mathrm{d}h\wedge : \Omega_Z^i \to \Omega_Z^{i+1}).$$

Since $(\Omega_Z^\bullet, \mathrm{d}h\wedge)$ is the Koszul complex associated to the regular sequence $h_i = 2z_i$ for $i \in [0,n]$, we have

$$H^i(\Omega_Z^\bullet, \mathrm{d}h\wedge) = 0 \quad \text{for } i \neq n. \tag{4.1.1}$$

This implies that the cohomology group $H^iA_h^\bullet$ is a left $\mathbf{C}[t]\langle\partial_t\rangle$-module for $i \neq n$ and that the action of ∂_t^{-1} is well-defined on the algebraic Brieskorn module

$$H^n A_h^\bullet = \Omega_Z^n / \mathrm{d}h \wedge \mathrm{d}\Omega_Z^{n-2}.$$

Here $\partial_t[\eta] = [\phi]$ for $\eta, \phi \in A_h^i$ if there is a $\sigma \in A_h^{i-1}$ such that

$$[\eta] = [\mathrm{d}h \wedge \sigma], \qquad [\phi] = [\mathrm{d}\sigma], \tag{4.1.2}$$

where $[\eta]$ denotes the class of η in $H^i A_h^\bullet$ (see [3]). The action of t is defined by the multiplication by h. We have the finiteness of $H^i A_h^\bullet$ over $\mathbf{C}[t]$ by using the canonical compactification of the morphism h. (The argument is essentially the same as in the analytic case in [3].) Then $H^i A_h^\bullet$ is t-torsion free for $i < n$, and by the theory of Milnor fibration it follows that

$$H^i A_h^\bullet = 0 \quad \text{for } i \neq 1, n. \tag{4.1.3}$$

We have the graded structure such that $\deg z_i = \deg \mathrm{d}z_i = 1$. This is compatible with d and $\mathrm{d}h\wedge$ (up to a shift of degree) and defines a graded structure on $H^n A_h^\bullet$. Let $H^n A_{h,k}^\bullet$ denote the degree-k part of $H^n A_h^\bullet$, so that

$$H^n A_h^\bullet = \bigoplus_{k \geq n} H^n A_{h,k}^\bullet.$$

Using the relation $\sum_i z_i h_i = 2h$ yields a well-known formula:

$$2t\partial_t[\phi] = (k-2)[\phi] \quad \text{for } [\phi] \in H^n A_{h,k}^\bullet. \tag{4.1.4}$$

This implies the t-torsion-freeness of $H^n A_h^\bullet$ (because we may assume $k \geq n$).

For $i = 1$, $H^1 A_h^\bullet$ is a free $\mathbf{C}[t]$-module of rank 1 generated by $[\mathrm{d}h]$. Since $A_h^0 = 0$, this implies that

$$H^1 A_h^\bullet = \mathbf{C}[h]\mathrm{d}h = \mathrm{Ker}(\mathrm{d}\colon A_h^1 \to A_h^2). \tag{4.1.5}$$

Define $D'_q\colon \Omega_Z^i \to \Omega_Z^{i+1}$ for $q \in \mathbf{Z}$ by

$$D'_q \eta = h\mathrm{d}\eta - q\mathrm{d}h \wedge \eta.$$

This is compatible with the graded structure up to the shift by $\deg h = 2$, and we have $D'_q \circ D'_{q-1} = 0$. We will denote by $\Omega_{Z,j}^i$ the degree-j part of Ω_Z^i.

The following lemma will be used in the proof of Theorem 2 in Section 4.3.

4.2. Lemma. *Assume that $j \neq 2q \neq 0$. Then the following statements hold.*

(i) $D'_q\colon \Omega_{Z,j}^{n-1} \to \Omega_{Z,j+2}^n$ *is surjective if* $j + 2 > n$.

(ii) $\mathrm{Im}(D'_{q-1}\colon \Omega_{Z,j-2}^{n-2} \to \Omega_{Z,j}^{n-1}) = \mathrm{Ker}(D'_q\colon \Omega_{Z,j}^{n-1} \to \Omega_{Z,j+2}^n)$ *if* $q \neq 1$.

Proof. Let $\phi \in \Omega_{Z,j+2}^n$. There is an $\eta \in \Omega_{Z,j}^{n-1}$ such that $\mathrm{d}h \wedge \eta = \phi$ since $j+2 > n$. Then (4.1.2) and (4.1.4) imply that

$$[D'_q \eta] = t\partial_t[\phi] - q[\phi] = (j/2 - q)[\phi].$$

So, replacing ϕ with $\phi - \alpha D'_q \eta$ where $\alpha = (j/2-q)^{-1}$, we may assume that $[\phi] = 0$ (i.e., $\phi \in \mathrm{d}h \wedge \mathrm{d}\Omega_Z^{n-2}$). Take $\sigma \in \Omega_{Z,j}^{n-2}$ such that $\phi = \mathrm{d}h \wedge \mathrm{d}\sigma$. Then $D'_q(-q^{-1}\mathrm{d}\sigma) = \phi$, and assertion (i) follows.

For assertion (ii), let $\eta \in \Omega_{Z,j}^{n-1}$ be such that $D'_q\eta = 0$. Set

$$\phi = q\mathrm{d}h \wedge \eta = h\mathrm{d}\eta.$$

Then $t\partial_t[\phi] = q[\phi]$ by (4.1.2) and so $[\phi] = 0$ by (4.1.4) (using $j \neq 2q$). Since $H^nA_h^\bullet$ is t-torsion-free, we have $[\mathrm{d}\eta] = 0$; that is, $\mathrm{d}\eta = \mathrm{d}h \wedge \mathrm{d}\sigma$ with $\sigma \in \Omega_{Z,j}^{n-2}$. Then

$$\mathrm{d}(D'_{q-1}\sigma) = q\mathrm{d}h \wedge \mathrm{d}\sigma = q\mathrm{d}\eta;$$

replacing η by $\eta - q^{-1}D'_{q-1}(\sigma)$, we may assume that $d\eta = 0$ and hence $\mathrm{d}h \wedge \eta = 0$.

If $n > 2$, then this together with (4.1.3) and (4.1.1) implies

$$\eta = \mathrm{d}\sigma' = -\mathrm{d}h \wedge \mathrm{d}\sigma'' \quad \text{with } \sigma' = \mathrm{d}h \wedge \sigma'' \in A_{h,j}^{n-2},\ \sigma'' \in \Omega_{h,j-2}^{n-3}$$

and hence $\eta = (q-1)^{-1}D'_{q-1}(\mathrm{d}\sigma'')$. Thus the assertion follows in this case.

For the case $n = 2$, by (4.1.5) we have $\eta = \beta h^i\mathrm{d}h$ with $\beta \in \mathbf{C}$ if j is even and positive (where $j = 2i + 2$) and $\eta = 0$ otherwise. If $j = 2i + 2$, then

$$D'_{q-1}h^i = (i - q + 1)h^i\mathrm{d}h$$

and $i - q + 1 \neq 0$ by $j \neq 2q$. The assertion follows. □

4.3. PROOF OF THEOREM 2. Let $q = n - p$, $i = q - m + 1$, and $k = (q + 1)d$ where $q > m$. By Lemma 2.6 and Remark 2.7(i), it is enough to treat the case where condition (A) is satisfied. With the notation of Section 2.3, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (I^{(i)}\Omega^n)_{k-d} \arrow[r] \arrow[d, "\psi'_a"] & (I^{(i-1)}\Omega^n)_{k-d} \arrow[r] \arrow[d, "\psi_a"] & \dfrac{(I^{(i-1)}\Omega^n)_{k-d}}{(I^{(i)}\Omega^n)_{k-d}} \arrow[r] \arrow[d, "\psi''_a"] & 0 \\
0 \arrow[r] & \dfrac{(I^{(i+1)}\Omega^{n+1})_k}{(fI^{(i-1)}\Omega^{n+1})_k} \arrow[r] & \dfrac{(I^{(i)}\Omega^{n+1})_k}{(fI^{(i-1)}\Omega^{n+1})_k} \arrow[r] & \dfrac{(I^{(i)}\Omega^{n+1})_k}{(I^{(i+1)}\Omega^{n+1})_k} \arrow[r] & 0,
\end{tikzcd}$$

where $\psi'_a, \psi_a, \psi''_a$ are induced by

$$D_q := \begin{cases} f\mathrm{d} - q\mathrm{d}f\wedge & \text{if } a = 1, \\ \mathrm{d}f\wedge & \text{if } a = 2. \end{cases}$$

Note that D_q is closely related to (2.1.3). Using coordinates $x_0, \dots, x_n$, we have

$$(I^{(i-1)}\Omega^n)_{k-d} = \bigoplus_{j=0}^{n} I_{k-n-d}^{(i-1)}\omega_j, \qquad (I^{(i)}\Omega^{n+1})_k = I_{k-n-1}^{(i)}\omega,$$

and so forth, where $\omega = \mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n$ and $\omega_j = \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_n$. Then we get

$$\operatorname{Coker}\psi_1 = \mathrm{Gr}_F^p H^n(U, \mathbf{C}), \qquad \operatorname{Coker}\psi_2 = (I^i/JI^{i-1})_k. \tag{4.3.1}$$

Indeed, the first isomorphism of (4.3.1) follows from Section 2.1 together with Proposition 2.8, and the second is trivial because $f \in J$. Observe that the assumption of Proposition 2.8 is satisfied by condition (A) for $(k, i) = (qd - n - 1, q - m)$, because $i' = q - m$ and $k' = qd$ in (2.3.4) with $j = n - 1$.

Because $\partial_j I^{(i)} \subset I^{(i-1)}$, we see that $f\mathrm{d}$ in ψ_1' vanishes, and hence

$$\operatorname{Coker} \psi_1' = \operatorname{Coker} \psi_2'.$$

We will show that ψ_a'' is surjective for $a = 1, 2$ by identifying it with

$$\bigoplus_{y \in \operatorname{Sing} Y} \bigoplus_{j=0}^{n} \left(\frac{\mathfrak{m}_{X,y}^{i-1}}{\mathfrak{m}_{X,y}^{i}} \right) \omega_j^{(y)} \to \bigoplus_{y \in \operatorname{Sing} Y} \left(\frac{\mathfrak{m}_{X,y}^{i}}{\mathfrak{m}_{X,y}^{i+1}} \right) \omega^{(y)}, \tag{4.3.2}$$

where $\omega_j^{(y)}$ and $\omega^{(y)}$ are associated to some coordinates $x_0^{(y)}, \dots, x_n^{(y)}$ depending on $y \in \operatorname{Sing} Y$. By the snake lemma, we then have an exact sequence

$$\operatorname{Ker} \psi_a'' \xrightarrow{\rho_a} \operatorname{Coker} \psi_a' \longrightarrow \operatorname{Coker} \psi_a \longrightarrow 0. \tag{4.3.3}$$

For $a = 1$ this implies the last isomorphism of the formula in Theorem 2. For the first isomorphism of the formula, we will further show that

$$\operatorname{Im} \rho_1 = \operatorname{Im} \rho_2. \tag{4.3.4}$$

We start with the proof of the surjectivity of ψ_a''. For each $y \in \operatorname{Sing} Y$, choose appropriate coordinates $x_0^{(y)}, \dots, x_n^{(y)}$ such that y is given by $(1, 0, \dots, 0)$ and

$$h(z_1^{(y)}, \dots, z_n^{(y)}) := \frac{f}{(x_0^{(y)})^d} = \sum_{j=1}^{n} (z_j^{(y)})^2 + \text{higher terms},$$

where $z_j^{(y)} = x_j^{(y)}/x_0^{(y)}$. (The last condition is satisfied by using a linear transformation of $z_1^{(y)}, \dots, z_n^{(y)}$.) We trivialize $\mathcal{O}_{X,y}(1)$ by using $x_0^{(y)}$. Then $\gamma_k^{(i)}$ in (2.3.2) is induced by substituting $x_0^{(y)} = 1$ and $x_j^{(y)} = z_j^{(y)}$ for $j > 0$. So $z_j^{(y)}$ is identified with $x_j^{(y)}/x_0^{(y)}$. Since $f_0/(x_0^{(y)})^{d-1} \in \mathfrak{m}_{X,y}^2$ and since $f_j/(x_0^{(y)})^{d-1} = 2z_j^{(y)}$ in $\mathfrak{m}_{X,y}/\mathfrak{m}_{X,y}^2$ for $j \neq 0$, we see that ψ_a'' is identified with (4.3.2). Indeed, the assertion is equivalent to the surjectivity of $\gamma_{k-n-d}^{(i-1)}$ and $\gamma_{k-n-1}^{(i)}$ in (2.3.2). But the first surjectivity follows from condition (A) for $(k, i) = (qd - n, q - m + 1)$, and the second is reduced to the first by using a commutative diagram as before together with the surjectivity of the morphism (4.3.2) induced by $df\wedge$; see (4.1.1). So ψ_a'' is identified with (4.3.2), and we also get the surjectivity of ψ_2''.

The morphisms (4.3.2) induced by D_q and $df\wedge$ are compatible with the direct sum over $y \in \operatorname{Sing} Y$ (using the pull-back by the surjection $\gamma_{k-n-d}^{(i-1),(y)}$; see (2.3.3)). Moreover, the restriction of ψ_1'',

$$\bigoplus_{j=1}^{n} \left(\frac{\mathfrak{m}_{X,y}^{i-1}}{\mathfrak{m}_{X,y}^{i}} \right) \omega_j^{(y)} \to \left(\frac{\mathfrak{m}_{X,y}^{i}}{\mathfrak{m}_{X,y}^{i+1}} \right) \omega^{(y)}, \tag{4.3.5}$$

is identified with D_q' in Lemma 4.2. Here $j := i + n - 2 \neq 2q$ since $q > m$. So ψ_1'' is also surjective, and the kernel of (4.3.5) does not contribute to $\operatorname{Im} \rho_1$ by using

$$D_{q-1} \colon (I^{i-2}\Omega^n / I^{i-1}\Omega^n)_{k-2d} \to (I^{i-1}\Omega^n / I^{i}\Omega^n)_{k-d}$$

because it lifts D_{q-1}' in Lemma 4.2 and satisfies $D_q \circ D_{q-1} = 0$. For $a = 2$, the kernel of (4.3.5) induced by $\mathrm{d}f\wedge$ does not contribute to $\operatorname{Im} \rho_2$ by a similar argument using (4.1.1).

Hence it is enough to consider the contribution to $\operatorname{Im}\rho_a$ of $(\mathfrak{m}_{X,y}^{i-1}/\mathfrak{m}_{X,y}^{i})\omega_0^{(y)}$, which is contained in the kernel of (4.3.2) for $a = 1, 2$. Because $\partial/\partial x_0^{(y)}$ preserves the maximal ideal of R generated by $x_j^{(y)}$ ($j \neq 0$), it does not contribute to $\operatorname{Im}\rho_1$ using the pull-back by the surjection $\gamma_{k-n-1}^{(i),(y)}$ in (2.3.3). Then the contributions to $\operatorname{Im}\rho_a$ for $a = 1, 2$ are both given by using the pull-back by the surjection $\gamma_{k-n-1}^{(i),(y)}$ together with the multiplication by $\partial f/\partial x_0^{(y)}$. Thus we obtain (4.3.4). (Note that the assertion (4.3.4) is independent of the choice of coordinates and that the isomorphism derived in the formula of Theorem 2 is well-defined.) This completes the proof of Theorem 2.

4.4. PROOF OF COROLLARY 1. Let $q = n - p$, $i = q - m + 1$, and $k = (q+1)d$. Since $q > m \geq 1$, the condition in (2.5.1) for $d \geq 3$ is satisfied when k and i in (2.5.1) are $k - n - 1$ and i or $k - n - d$ and $i - 1$ (i.e., we have $k - n - 1 \geq 2i$ and $k - n - d \geq 2i - 2$); see (2.4.3) for the case $d = 2$. Moreover, $m(d-1) - p > 0$ when $d \geq 3$, because $n - p > m \geq 1$. If $d = 2$, then $|\operatorname{Sing} Y| = 1$ and (2.3.5) is satisfied for $e = 0$. So the assertion follows from Theorem 2 and Lemma 2.5.

From Theorem 2 we can deduce the following.

4.5. COROLLARY. *Let Y_s be an equisingular family of hypersurfaces in $\mathbf{P}^n$ that are parameterized by a smooth variety S and whose singularities are ordinary double points. Assume condition* (A) *for q in Theorem 2 is satisfied for any $s \in S$ if $q > m$, and assume the same with q replaced by $q - 1$ if $q - 1 > m$. Set $U_s = \mathbf{P}^n \setminus Y_s$. Then, for a vector field θ on S, we have a commutative diagram*

$$\begin{array}{ccc}
\operatorname{Gr}_F^{n-q+1} H^n(U_s, \mathbf{P}) & \xrightarrow{\operatorname{Gr}_F \nabla_\theta} & \operatorname{Gr}_F^{n-q} H^n(U_s, \mathbf{P}) \\
\| & & \| \\
(I_s^{(q-m)}/I_s^{(q-m-1)}J_s)_{qd-n-1} & \xrightarrow{-q(\theta f)_s} & (I_s^{(q-m+1)}/I_s^{(q-m)}J_s)_{(q+1)d-n-1},
\end{array}$$

where the vertical isomorphisms are given by Theorem 2 and Theorem 2.2.

Proof. The action of θ on the relative de Rham cohomology can be calculated by $\iota_\theta \circ \mathrm{d}$. Hence the assertion follows from Theorem 2 (using $\iota_\xi \circ \mathrm{d} + \mathrm{d} \circ \iota_\xi = L_\xi$ and $\iota_\theta \circ \iota_\xi = -\iota_\xi \circ \iota_\theta$), because the cohomology class is represented using the first isomorphism in (0.2). $\square$

4.6. REMARKS. (i) By Varchenko [22] (conjectured by Arnold) and [13], we have

$$\begin{aligned}
|\operatorname{Sing} Y| &\leq \sum_{(n-2)/2+1<i\leq nd/2} C(n,d,i) = C(n+1, d, [nd/2]+1) \\
&= \sum_{i\geq 0} \binom{n+1}{i} \binom{[nd/2] - i(d-1)}{n} < \binom{[nd/2]}{n}, \qquad (4.6.1)
\end{aligned}$$

where $C(n,d,i)$ is as in (3.3.1). (This also follows from (3.5.2) applied to a hypersurface in $\mathbf{P}^{n+1}$ or $\mathbf{P}^{n+2}$ defined by $f + x_{n+1}^d$ or $f + x_{n+1}^d + x_{n+2}^d$.)

If n is even (i.e., if $n = 2m$), then (4.6.1) implies

$$\binom{e+n}{n} \geq \binom{md+1}{n} > \binom{[nd/2]}{n} > |\operatorname{Sing} Y| \quad \text{for } q \geq m+1, \qquad (4.6.2)$$

and it is possible that condition (B) in Theorem 2 is satisfied. However, if n is odd (i.e., if $n = 2m+1$), then (4.6.2) does not hold and condition (B) cannot be satisfied—for example, if $n = 3$, $q = 2$, and Y is a Kummer surface with sixteen ordinary double points where $d = 4$, $e = 2$, and $\binom{e+n}{n} = 10$ (see Example 4.7(ii)) or if Y has 65 ordinary double points with $d = 6$ as in [1] where $e = 4$ and $\binom{e+n}{n} = 35$.

(ii) For condition (A), we have matrices of size (M, N) with

$$M = \binom{i-1+n}{n} |\operatorname{Sing} Y| \quad \text{and} \quad N = \binom{k+n}{n},$$

where $(k, i) = (qd-n, q-m+1)$ and $(qd-n-1, q-m)$. See Remark 2.7(ii).

4.7. Examples. (i) Assume that $n = 3$, $d = 4$, and

$$f = \sum_{i=0}^{3} x_i^4 - \sum_{0 \leq i < j \leq 3} 2x_i^2 x_j^2 \quad \text{so that} \quad f_j = 4x_j\Big(x_j^2 - \sum_{i \neq j} x_i^2\Big).$$

This has twelve ordinary double points. Indeed, there are two singular points defined by $x_i^2 = x_j^2$ and $x_k^2 = 0$ ($k \in [0,3] \setminus \{i, j\}$) for each $\{i, j\} \subset [0,3]$ with $i \neq j$. Hence condition (B) cannot be satisfied for $q = 2$ because $12 > \binom{e+n}{n} = 10$ in the notation of Remark 4.6. However, condition (A) seems to be satisfied for $q = 2$ when, in the notation of Remark 4.6(ii), $(M, N) = (48, 56)$ and $(12, 35)$.

(ii) Assume that Y is a singular Kummer surface defined by

$$f = \sum_{i=0}^{3} x_i^4 - \sum_{0 \leq i < j \leq 3} x_i^2 x_j^2 \quad \text{so that} \quad f_j = 2x_j\Big(2x_j^2 - \sum_{i \neq j} x_i^2\Big).$$

This has sixteen ordinary double points. Indeed, there are four singular points defined by $x_k = 0$ and $x_i^2 = 1$ ($i \neq k$) for each $k = 0, \ldots, 3$. Condition (A) for $q = 2$ cannot be satisfied because (M, N) can be $(64, 56)$ in the notation of Remark 4.6(ii). However, it seems that

$$\dim I_8^{(2)} = \dim (I^2)_8 = \binom{8+3}{3} - 4|\operatorname{Sing} Y| = 101, \qquad \dim (IJ)_8 = 100;$$

hence at least a noncanonical isomorphism still holds in Conjecture 1 for $q = 2$.

(iii) It would be difficult to calculate the right-hand side of Conjecture 1 for the Barth surface [1], so we consider the case where Y is defined by

$$f = \Big(\sum_{i=0}^{3} x_i^2\Big)^3 - \sum_{i=0}^{3} x_i^6 \quad \text{so that} \quad f_j = 6x_j\Big(\Big(\sum_{i=0}^{3} x_i^2\Big)^2 - x_j^4\Big).$$

This has 52 ordinary double points. Indeed, there are four singular points defined by $x_i = 1$ and $x_k = 0$ ($k \neq i$) for $i = 0, \ldots, 3$ as well as four singular points defined by $x_i = 1$, $x_j = 0$, and $x_k^2 = -1$ ($k \in [0,3] \setminus \{i,j\}$) for each $(i,j) \in [0,3]^2 \setminus \{\text{diagonal}\}$. Condition (B) cannot be satisfied because $\binom{e+n}{n} = 35 < 52$, but it is not clear whether condition (A) is satisfied where $(M,N) = (208,220)$ and $(52,165)$. It seems that

$$\dim I_{14}^{(2)} = \dim(I^2)_{14} = \binom{14+3}{3} - 4|\mathrm{Sing}\, Y| = 472, \qquad \dim(IJ)_{14} = 462;$$

hence at least a noncanonical isomorphism still holds in Conjecture 1 for $q = 2$.

References

[1] W. Barth, *Two projective surfaces with many nodes, admitting the symmetries of the icosahedron,* J. Algebraic Geom. 5 (1996), 173–186.

[2] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers,* Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque 100 (1982), 5–171.

[3] E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen,* Manuscripta Math. 2 (1970), 103–161.

[4] P. Deligne, *Equations différentielles à points singuliers réguliers,* Lecture Notes in Math, 163, Springer-Verlag, Berlin, 1970.

[5] ———, *Théorie de Hodge I,* Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 425–430, Gauthier-Villars, Paris, 1971; II, Inst. Hautes Études Sci. Publ. Math. 40 (1971), 5–57; III, Inst. Hautes Études Sci. Publ. Math. 44 (1974), 5–77.

[6] P. Deligne and A. Dimca, *Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières,* Ann. Sci. Ecole Norm. Sup. (4) 23 (1990), 645–656.

[7] A. Dimca, *Singularities and topology of hypersurfaces,* Springer-Verlag, New York, 1992.

[8] ———, *Hodge numbers of hypersurfaces,* Abh. Math. Sem. Univ. Hamburg 66 (1996), 377–386.

[9] A. Dimca and M. Saito, *A generalization of Griffiths's theorem on rational integrals,* Duke Math. J. 135 (2006), 303–326.

[10] M. Goresky and R. MacPherson, *Intersection homology theory,* Topology 19 (1980), 135–162.

[11] M. Green, *Infinitesimal method in Hodge theory,* Algebraic cycles and Hodge theory (Torino, 1993), Lecture Notes in Math., 1594, pp. 1–92, Springer-Verlag, Berlin, 1994.

[12] P. Griffiths, *On the period of certain rational integrals I, II,* Ann. of Math. (2) 90 (1969), 460–541.

[13] O. A. Ivanov and N. Yu. Netsvetaev, *Estimates of the number of singularities of a complex hypersurface and related questions,* J. Math. Sci. (New York) 91 (1998), 3448–3455.

[14] M. Kashiwara, *Vanishing cycle sheaves and holonomic systems of differential equations,* Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., 1016, pp. 134–142, Springer-Verlag, Berlin, 1983.

[15] B. Malgrange, *Le polynôme de Bernstein d'une singularité isolée,* Lecture Notes in Math., 459, pp. 98–119, Springer-Verlag, Berlin, 1975.

[16] ———, *Polynôme de Bernstein–Sato et cohomologie évanescente,* Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque 101–102 (1983), 243–267.

[17] M. Saito, *Mixed Hodge modules,* Publ. Res. Inst. Math. Sci. 26 (1990), 221–333.
[18] ———, *On b-function, spectrum and rational singularity,* Math. Ann. 295 (1993), 51–74.
[19] ———, *On the Hodge filtration of Hodge modules,* RIMS preprint 1078, May 1996, math.AG/0504177.
[20] ———, *e-mail to L. Wotzlaw,* Jan. 27, 2005.
[21] J. H. M. Steenbrink, *Mixed Hodge structure on the vanishing cohomology,* Real and complex singularities (Proc. Nordic Summer School, Oslo, 1976), pp. 525–563, Sijthoff & Noordhoff, Alphen a/d Rijn, 1977.
[22] A. N. Varchenko, *Semicontinuity of the spectrum and an upper bound for the number of singular points of the projective hypersurface,* Dokl. Akad. Nauk SSSR 270 (1983), 1294–1297.
[23] L. Wotzlaw, *Intersection cohomology of hypersurfaces,* Ph.D. dissertation (preprint), Humboldt Universität zu Berlin, August 2006.

A. Dimca
Laboratoire J.A. Dieudonné
UMR du CNRS 6621
Université de Nice – Sophia Antipolis
06108 Nice Cedex 02
France

dimca@math.unice.fr

M. Saito
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606–8502
Japan

msaito@kurims.kyoto-u.ac.jp

L. Wotzlaw
Fachbereich Mathematik und Informatik II
Mathematisches Institut
Freie Universität Berlin
D-14195 Berlin
Germany

wotzlaw@math.fu-berlin.de