

An Explicit $\bar{\partial}$ -Integration Formula for Weighted Homogeneous Varieties

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1. Introduction

As is well known, solving the $\bar{\partial}$ -equation forms a main part of complex analysis; but it also has deep consequences for algebraic geometry, partial differential equations, and other areas. In general, it is not easy to solve the $\bar{\partial}$ -equation. The existence of solutions depends mainly on the geometry of the variety on which the equation is considered. There is a vast literature about this subject on smooth manifolds, both in books and papers (see e.g. [10; 11; 12]), but the theory on singular varieties has been developed only recently.

Let Σ be a singular subvariety of the space \mathbb{C}^n , and let λ be a bounded $\bar{\partial}$ -closed differential form on the regular part of Σ . Fornæss, Gavosto, and Ruppenthal have proposed a general technique for solving the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on the regular part of Σ , a technique they have successfully applied to varieties of the form $\{z^m = w_1^{k_1} \cdots w_n^{k_n}\} \subset \mathbb{C}^{n+1}$; see [6; 9; 14]. They exploit the fact that such a variety can be considered as an m -sheeted analytic covering of the complex space \mathbb{C}^n . The $\bar{\partial}$ -equation is then projected into \mathbb{C}^n by the use of symmetric combinations, and it is solved there with certain weights. The form g is constructed from the pull-back of a finite set of previous solutions. There is a certain chance for this strategy to work in general varieties, because any locally irreducible complex space can be locally represented as a finitely sheeted analytic covering over a complex number space.

On the other hand, Acosta, Solís, and Zeron have developed an alternative technique for solving the $\bar{\partial}$ -equation (if λ is bounded) on any kind of singular quotient variety embedded in \mathbb{C}^m and generated by a finite group of unitary matrices, such as hypersurfaces in \mathbb{C}^3 with only a rational double point singularity; see [1; 2; 18]. They use the quotient structure in order to pull back the $\bar{\partial}$ -equation into a complex space \mathbb{C}^n and to solve the original equation by using symmetric combinations. This strategy has the drawback that not all varieties are quotient ones.

In both of these approaches, the main strategy is to transfer the problem into some nonsingular complex space, to solve the $\bar{\partial}$ -equation in this well-known situation, and then to carry over the solution into the singular variety Σ . The main objective

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of this paper is to present and analyze an explicit formula for calculating solutions g to the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on the regular part of the original variety Σ , where Σ is a weighted homogeneous variety and λ is a $\bar{\partial}$ -closed $(0, 1)$ -differential form with compact support. We analyze the weighted homogeneous varieties because they are a main model for classifying the singular subvarieties of \mathbb{C}^n . A detailed analysis of the weighted homogeneous varieties may be found in [4] (see Sec. 4 of Chap. 2 and Apx. B).

DEFINITION 1. Let $\beta \in \mathbb{Z}^n$ be a fixed integer vector with strictly positive entries $\beta_k \geq 1$. A polynomial $Q(z)$ holomorphic on \mathbb{C}^n is said to be *weighted homogeneous* of degree $d \geq 1$ with respect to β if the following equality holds for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$:

$$Q(s^\beta * z) = s^d Q(z) \tag{1}$$

with the action

$$s^\beta * (z_1, z_2, \dots, z_n) := (s^{\beta_1} z_1, s^{\beta_2} z_2, \dots, s^{\beta_n} z_n). \tag{2}$$

An algebraic subvariety Σ in \mathbb{C}^n is said to be weighted homogeneous with respect to β if Σ is the zero locus of a finite number of weighted homogeneous polynomials $Q_k(z)$ of (possibly different) degrees $d_k \geq 1$ but all of them with respect to the same fixed vector β .

Let $\Sigma \subset \mathbb{C}^n$ be any subvariety. We use the following notation throughout this paper. The regular part $\Sigma^* = \Sigma_{\text{reg}}$ is the complex manifold consisting of the regular points of Σ , and it is always endowed with the induced metric such that Σ^* is a Hermitian submanifold in \mathbb{C}^n with corresponding volume element dV_Σ and induced norm $|\cdot|_\Sigma$ on the Grassmannian $\Lambda T^* \Sigma^*$. Thus, any Borel-measurable $(0, 1)$ -form λ on Σ^* admits a representation $\lambda = \sum_k f_k d\bar{z}_k$, where the coefficients f_k are Borel-measurable functions on Σ^* that satisfy the inequality $|f_k(w)| \leq |\lambda(w)|_\Sigma$ for all points $w \in \Sigma^*$ and indexes $1 \leq k \leq n$. Notice that such a representation is by no means unique (see [14, Lemma 2.2.1] for a more detailed treatment of this point). We also introduce the L^2 -norm of a measurable (p, q) -form \aleph on an open set $U \subset \Sigma^*$ via the formula

$$\|\aleph\|_{L^2_{p,q}(U)} := \left(\int_U |\aleph|_\Sigma^2 dV_\Sigma \right)^{1/2}.$$

We can now present the main result of this paper. We will assume that the $\bar{\partial}$ -differentials are calculated in the sense of distributions, for we work with Borel-measurable functions.

THEOREM 2 (Main). *Let Σ be a weighted homogeneous subvariety of \mathbb{C}^n with respect to a given vector $\beta \in \mathbb{Z}^n$, where $n \geq 2$ and all entries $\beta_k \geq 1$. Consider a $(0, 1)$ -form λ given by $\sum_k f_k d\bar{z}_k$, where the coefficients f_k are all Borel-measurable functions in Σ and where z_1, \dots, z_n are the Cartesian coordinates of \mathbb{C}^n . Let $\rho \in \mathbb{C}$ be fixed. The following function is well-defined for almost all $z \in \Sigma$ if the form λ is essentially bounded and has compact support in Σ :*

$$g_\rho(z) := \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta * z) \frac{\overline{(w^{\beta_k} z_k)} dw \wedge d\bar{w}}{\bar{w}(w - \rho)}. \tag{3}$$

If Σ^* is the regular part of Σ and if λ is $\bar{\partial}$ -closed on $\Sigma^* \setminus \{0\}$, then the function g_0 is holomorphic on $\Sigma^* \setminus \{0\}$ and the function g_1 is a solution of the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g_1$ on $\Sigma^* \setminus \{0\}$.

Note that the origin of \mathbb{C}^n is in general a singular point of Σ (according to Definition 1) so that $\Sigma^* \setminus \{0\}$ coincides with Σ^* . Theorem 2 is proved in Section 2 of this paper. The main idea of the proof is to show that $\lambda = \bar{\partial}g_1$ when λ is $\bar{\partial}$ -closed. This is a local statement, so we cover Σ by charts that we call *generalized cones*. When we blow up these cones to complex manifolds, we realize that the integral formula (3) looks essentially like the inhomogeneous Cauchy–Pompeiu integral formula in one complex variable (see (15)), so we can deduce the statement of Theorem 2 from some classical results.

The functions g_ρ defined in (3) have many interesting properties. For example, it easily follows that

$$g_1(z) - g_0(z) = \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta * z) \frac{\overline{(w^{\beta_k} z_k)} dw \wedge d\bar{w}}{|w|^2(w - 1)}. \tag{4}$$

The differential $\bar{\partial}[g_1 - g_0] = \lambda$ on $\Sigma^* \setminus \{0\}$ when λ is $\bar{\partial}$ -closed; moreover, the value $[g_1 - g_0](0)$ vanishes. The change of variables $w = su$ in (3) yields some useful identities: for every point $z \in \Sigma$ and number $s \neq 0$ in \mathbb{C} ,

$$g_s(z) = g_1(s^\beta * z) \quad \text{and} \quad g_0(z) = g_0(s^\beta * z). \tag{5}$$

On the other hand, recalling some main principles of the proof, we also deduce anisotropic Hölder estimates for the $\bar{\partial}$ -equation in the case where Σ is a homogeneous variety having only one singularity at the origin. The main idea is to use the regularity properties of the Cauchy–Pompeiu formula. We obviously need to specify the metric on Σ : given a pair of points z and w in Σ , we define $\text{dist}_\Sigma(z, w)$ to be the infimum of the length of piecewise smooth curves connecting z and w inside Σ . It is clear that such curves exist in this situation and that the length of each curve can be measured in the regular part Σ^* or in the ambient space \mathbb{C}^n ; but both measures coincide, for Σ^* carries the induced norm. The main result of Section 3 is the following estimate.

THEOREM 3 (Hölder). *In the situation of Theorem 2, suppose that Σ is homogeneous (a cone) and has only one singularity at the origin of \mathbb{C}^n , so that each entry $\beta_k = 1$ in Definition 1. Moreover, assume that the support of the form λ is contained in a ball B_R of radius $R > 0$ and center at the origin. Then, for each parameter $0 < \theta < 1$, there exists a constant $C_\Sigma(R, \theta) > 0$ that does not depend on λ and such that the following inequality holds for the function g_1 given in (3) and almost all points z and w in the intersection $B_R \cap \Sigma$:*

$$|g_1(z) - g_1(w)| \leq C_\Sigma(R, \theta) \cdot \text{dist}_\Sigma(z, w)^\theta \cdot \|\lambda\|_\infty. \tag{6}$$

Here $\|\lambda\|_\infty$ denotes the essential supremum of $|\lambda(\cdot)|_\Sigma$ on Σ ; recall that λ is bounded and has compact support. We should mention that Theorem 3 is a significant improvement of the known results about Hölder regularity. Consider for example $\{z^2 = w_1 w_2\} \subset \mathbb{C}^3$. For this variety, Acosta, Fornæss, Gavosto, Solis, and Zeron were able to prove the statement of Theorem 3 only for $\theta < 1/2$ (see [1; 2; 6]); Ruppenthal [14] showed the same statement for $\theta = 1/2$. Previous outcomes are all far away from the result of Theorem 3. This theorem is proved in Section 3, where we consider the difference $g = g_1 - g_0$ (see (4)) instead of g_1 . We can make this reduction because $\Sigma \setminus \{0\}$ is a homogeneous complex manifold and so the holomorphic function g_0 in (3) is, in fact, constant: equation (5) implies that $g_0(z)$ is equal to $g_0(sz)$ for every $s \neq 0$ in \mathbb{C} (recall that each entry $\beta_k = 1$), so that g_0 is constant on all the complex lines \mathbb{C}^* of $\Sigma \setminus \{0\}$ passing through the origin. It follows that g_0 is constant on Σ because it is holomorphic, and thus constant, on the compact projective manifold $\tilde{\Sigma}$ associated to Σ in $\mathbb{C}\mathbb{P}^{n-1}$.

Finally, similar techniques and a slight modification of equation (3) can also be used to produce a $\bar{\partial}$ -solution operator with L^2 -estimates on homogeneous subvarieties with arbitrary singular locus as follows.

THEOREM 4 (L^2 -estimates). *Let Σ be a pure d -dimensional homogeneous subvariety of \mathbb{C}^n , where $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Consider a $(0, 1)$ -form λ given by $\sum_k f_k d\bar{z}_k$, where the coefficients f_k are all square integrable functions on Σ and where z_1, \dots, z_n are the Cartesian coordinates of \mathbb{C}^n . Then the following function is well-defined for almost all $z \in \Sigma$ whenever the form λ has compact support on Σ :*

$$g(z) := \sum_{k=1}^n \frac{1}{2\pi i} \int_{w \in \mathbb{C}} f_k(wz) \frac{w^{d-1} \bar{z}_k dw \wedge d\bar{w}}{w-1}. \tag{7}$$

If Σ^ is the regular part of Σ and if λ is $\bar{\partial}$ -closed on $\Sigma^* \setminus \{0\}$, then the function g is a solution of the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g$ on $\Sigma^* \setminus \{0\}$. Finally, if we assume that the support of λ is contained in an open ball B_R of radius $R > 0$ and center at the origin, then there exists a constant $C_\Sigma(R, 2) > 0$ that does not depend on λ and such that*

$$\|g\|_{L^2(\Sigma \cap B_R)} \leq C_\Sigma(R, 2) \cdot \|\lambda\|_{L^2_{0,1}(\Sigma)}. \tag{8}$$

We prove this theorem in Section 4. The proof is based on an analysis of the behavior of norms under blowing up the origin and on the L^2 -regularity of the Cauchy–Pompeiu formula. We should mention that—to our knowledge—an L^2 -solution operator for forms with compact support is known only for isolated singularities, so that Theorem 4 is a new result; see [8, Prop. 3.1].

The obstructions to solving the $\bar{\partial}$ -equation with L^2 -estimates on subvarieties of \mathbb{C}^n are not completely understood in general. An L^2 -solution operator (for forms with noncompact support) is known only in the case where Σ is a complete intersection (more precisely, a Cohen–Macaulay space) of pure dimension ≥ 3 with only isolated singularities. This operator was built by Fornæss, Øvrelid, and Vassiliadou in [8] via an extension theorem for $\bar{\partial}$ -cohomology groups originally

presented by Scheja [17]. Usually, the L^2 -results come with some obstructions to the solvability of the $\bar{\partial}$ -equation. Different situations have been analyzed in the works of Diederich, Fornæss, Øvrelid, Ruppenthal, and Vassiliadou: it is shown that the $\bar{\partial}$ -equation is solvable with L^2 -estimates for forms lying in a closed subspace of finite codimension of the vector space of all the $\bar{\partial}$ -closed L^2 -forms *provided* the variety has only isolated singularities [3; 5; 8; 13; 15]. Moreover, in [7] the $\bar{\partial}$ -equation is solved locally with some weighted L^2 -estimates for forms that vanish to a sufficiently high order on the (arbitrary) singular locus of the given varieties.

There is a second line of research about the $\bar{\partial}$ -operator on complex projective varieties. Although that area clearly has much in common with the topic of $\bar{\partial}$ -equations on analytic subvarieties of \mathbb{C}^n , it is a somewhat different theory because of the strong global tools that cannot be used in the (local) situation of Stein spaces (owing to the lack of compactness).

Since the estimates in Theorems 3 and 4 are given only for homogeneous varieties, in Section 5 we propose a useful technique for generalizing the estimates in Theorems 3 and 4 so as to consider weighted homogeneous subvarieties instead of homogeneous ones. We do not elaborate on this in detail because it is more or less straightforward and it is not clear whether the results would be optimal in that case.

2. Proof of Main Theorem

Let $\{Q_k\}$ be the set of polynomials on \mathbb{C}^n that defines the algebraic variety Σ as its zero locus. The definition of weighted homogeneous varieties implies that the polynomials $Q_k(z)$ are all weighted homogeneous with respect to the same fixed vector β . Equation (1) automatically yields that every point $s^\beta * z$ lies in Σ for all $s \in \mathbb{C}$ and $z \in \Sigma$, so each coefficient $f_k(\cdot)$ in equations (3) and (4) is well-evaluated in Σ . Moreover, fixing any point $z \in \Sigma$, the given hypotheses imply that the following Borel-measurable functions are all bounded and have compact support in \mathbb{C} :

$$w \mapsto f_k(w^\beta * z).$$

Hence, the function $g_\rho(z)$ in (3) is well-defined for almost all $z \in \Sigma$. We shall prove that $g_1(z)$ is a solution of the equation $\bar{\partial}g_1 = \lambda$ if the $(0, 1)$ -form λ is $\bar{\partial}$ -closed. We may suppose, without loss of generality and because of the given hypotheses, that the regular part of Σ does not contain the origin. Let $\xi \neq 0$ be any fixed point in the regular part of Σ . We may suppose by simplicity that the first entry $\xi_1 \neq 0$, and so we define the following mapping $\eta: \mathbb{C}^n \rightarrow \mathbb{C}^n$ and variety Y :

$$\begin{aligned} \eta(y) &:= (y_1/\xi_1)^\beta * (\xi_1, y_2, y_3, \dots, y_n) \quad \text{for } y \in \mathbb{C}^n, \\ Y &:= \{\hat{y} \in \mathbb{C}^{n-1} : Q_k(\xi_1, \hat{y}) = 0 \ \forall k\}. \end{aligned} \tag{9}$$

The action $s^\beta * z$ was given in (2). We have that $\eta(\xi) = \xi$ and that the following identities hold for all $s \in \mathbb{C}$ and $\hat{y} \in \mathbb{C}^{n-1}$ (recall equation (1) and the fact that Σ is the zero locus of the polynomials $\{Q_k\}$):

$$\begin{aligned}
 Q_k(\eta(s, \hat{y})) &= (s/\xi_1)^{dk} Q_k(\xi_1, \hat{y}) \quad \text{and so} \\
 \eta(\mathbb{C}^* \times Y) &= \{z \in \Sigma : z_1 \neq 0\}.
 \end{aligned}
 \tag{10}$$

The symbol \mathbb{C}^* stands for $\mathbb{C} \setminus \{0\}$. The mapping η is locally a biholomorphism whenever the first entry $y_1 \neq 0$. Whence the point ξ lies in the regular part of the variety $\mathbb{C} \times Y$, because $\xi = \eta(\xi)$ also lies in the regular part of Σ and $\xi_1 \neq 0$. Thus, we can find a biholomorphism

$$\pi = (\pi_2, \dots, \pi_n): U \rightarrow Y \subset \mathbb{C}^{n-1},$$

defined from an open domain U in \mathbb{C}^{d-1} onto an open set in the regular part of Y , such that $\pi(\zeta)$ is equal to (ξ_2, \dots, ξ_n) for some $\zeta \in U$. Consider the following holomorphic mapping defined for all points $s \in \mathbb{C}$ and $x \in U$:

$$\Pi(s, x) := s^\beta * (\xi_1, \pi(x)) = \eta(s\xi_1, \pi(x)) \in \Sigma.
 \tag{11}$$

The image $\Pi(\mathbb{C} \times U)$ will be called a generalized cone from now on. Notice that $\Pi(\mathbb{C}^* \times U)$ lies in the regular part of Σ , for $\pi(U)$ is contained in the regular part of Y . The mapping $\Pi(s, x)$ is locally a biholomorphism whenever $s \neq 0$ because η is also a local biholomorphism for $y_1 \neq 0$. Finally, the image $\Pi(1, \zeta)$ is equal to ξ . Hence, recalling the differential form λ and the function g_1 defined in (3), we need only prove that the pull-back $\Pi^*\lambda$ is equal to $\bar{\partial}(g_1 \circ \Pi) = \bar{\partial}\Pi^*g_1$ inside $\mathbb{C}^* \times U$ in order to conclude that the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g_1$ holds in a neighborhood of ξ . Consider the following identity obtained by inserting (2) and (11) into (3) (we define $\pi_1(x) \equiv \xi_1$):

$$g_1(\Pi(s, x)) = \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{\mathbb{C}} f_k(\Pi(ws, x)) \frac{\overline{(ws)^{\beta_k} \pi_k(x)} dw \wedge d\bar{w}}{\bar{w}(w-1)}.
 \tag{12}$$

The given hypotheses on λ yield that the pull-back $\Pi^*\lambda$ is $\bar{\partial}$ -closed and bounded in $\mathbb{C}^* \times U$, so it is also bounded and $\bar{\partial}$ -closed in $\mathbb{C} \times U$ (see [14, Lemma 4.3.2] or [18, Lemma (2.2)]). We can use equations (2) and (11) to calculate $\Pi^*\lambda$ when λ is given by $\sum_k f_k d\bar{z}_k$:

$$\Pi^*\lambda = F_0(s, x)d\bar{s} + \sum_{j \geq 1} F_j(s, x)d\bar{x}_j,$$

where

$$F_0(s, x) = \sum_{k=1}^n f_k(\Pi(s, x)) \beta_k s^{\beta_k-1} \overline{\pi_k(x)},
 \tag{13}$$

$$F_j(s, x) = \sum_{k=1}^n f_k(\Pi(s, x)) \left[s^{\beta_k} \frac{\partial \pi_k}{\partial x_j} \right].
 \tag{14}$$

Recall that $\pi_1(x) \equiv \xi_1$. Equation (11) and the fact that λ has compact support on Σ also imply that the previous function $F_0(s, x)$ has compact support on every complex line $\mathbb{C} \times \{x\}$ for all $x \in U$. Hence, the Cauchy–Pompeiu integral (applied to $\Pi^*\lambda$) solves the $\bar{\partial}$ -equation $\Pi^*\lambda = \bar{\partial}G$ in the product $\mathbb{C} \times U$ if we define

$$G(s, x) := \frac{1}{2\pi i} \int_{u \in \mathbb{C}} \frac{F_0(u, x)}{u - s} du \wedge d\bar{u} \tag{15}$$

for $s \in \mathbb{C}$ and $x \in U$. Finally, equations (12) and (15) are identical when $s \neq 0$, for we only need to apply the change of variables $u = ws$. Thus, the differential $\bar{\partial}\Pi^*g_1$ (resp. $\bar{\partial}g_1$) is equal to the form $\Pi^*\lambda$ (resp. λ) in the product $\mathbb{C}^* \times U$ (resp. an open neighborhood of ξ); therefore, the $\bar{\partial}$ -equation $\lambda = \bar{\partial}g_1$ holds in the regular part of Σ because $\xi \neq 0$ was chosen in an arbitrary way in the regular part of Σ .

We conclude this section by showing that the function g_0 in (3) is holomorphic on $\Sigma^* \setminus \{0\}$. The previous condition is equivalent to proving that Π^*g_0 is holomorphic on $\mathbb{C}^* \times U$. We easily have that Π^*g_0 is constant with respect to the first entry $s \in \mathbb{C}^*$, because

$$\begin{aligned} g_0(\Pi(s, x)) &= \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{\mathbb{C}} f_k(\Pi(ws, x)) \frac{\overline{(ws)^{\beta_k} \pi_k(x)}}{|w|^2} dw \wedge d\bar{w} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}} F_0(u, x) \frac{du \wedge d\bar{u}}{u}. \end{aligned}$$

Here we have made the change of variables $u = ws$ and have used equation (13). We may also calculate the derivatives with respect to \bar{x}_j , using equation (14) and the fact that $\bar{\partial}\lambda = 0$:

$$\begin{aligned} \frac{\partial g_0(\Pi(s, x))}{\partial \bar{x}_j} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial F_0(u, x)}{\partial \bar{x}_j} \cdot \frac{du \wedge d\bar{u}}{u} \\ &= \frac{-1}{2\pi i} \int_{\mathbb{C}} \frac{\partial F_j(u, x)}{\partial \bar{u}} \cdot \frac{du \wedge d\bar{u}}{u} = F_j(0, x) = 0. \end{aligned}$$

Previous derivations are all calculated in the sense of distributions. Nevertheless, the fact that they all vanish is sufficient to assure that Π^*g_0 is holomorphic with respect to x ; see [10, p. 12]. Hence, the function g_0 is holomorphic on a neighborhood of the arbitrary point $\xi \neq 0$ in Σ^* because Π^*g_0 is also constant with respect to the first entry s .

3. Hölder Estimates

In this section, we will prove anisotropic Hölder estimates on the subvariety $\Sigma \subset \mathbb{C}^n$ in the particular case when Σ is homogeneous (a cone) and has only one isolated singularity at the origin (Lemma 5). These estimates lead easily to optimal Hölder estimates on such varieties (Theorem 3). We will later show in Section 5 how we can use previous results in order to deduce Hölder estimates on weighted homogeneous varieties with an isolated singularity as well.

The given hypotheses imply that $\Sigma \setminus \{0\}$ is a homogeneous complex manifold in \mathbb{C}^n . Consider the holomorphic function g_0 defined on $\Sigma \setminus \{0\}$ by (3) with $\rho = 0$. Equation (5) yields that $g_0(z)$ is equal to $g_0(sz)$ for every $s \neq 0$ in \mathbb{C} because all entries $\beta_k = 1$, so that g_0 is constant on all the complex lines \mathbb{C}^* of $\Sigma \setminus \{0\}$ passing through the origin. Hence, the function g_0 is constant on Σ because it

is well-defined, holomorphic, and constant on the compact projective manifold $\tilde{\Sigma}$ associated to Σ in $\mathbb{C}\mathbb{P}^{n-1}$. Previous facts imply that we just need to show the Hölder estimate (6) for the function defined in (4):

$$g(z) := g_1(z) - g_0 = \sum_{k=1}^n \frac{1}{2\pi i} \int_{t \in \mathbb{C}} f_k(tz) \frac{\bar{z}_k dt \wedge d\bar{t}}{t(t-1)}. \tag{16}$$

We shall soon see that it is easier to work with this function $g(z)$ than with $g_1(z)$. In particular, note that $g(0) = 0$.

We divide the proof of Theorem 3 into four parts. First, we reduce the problem to working on suitable cones that cover the original subvariety Σ . We then calculate Hölder estimates for any pair of points in the same line through the origin, followed by estimates for any two points that lie in the same slice of a given suitable cone. Finally, we combine both kinds of estimates to deduce general anisotropic Hölder estimates (Lemma 5), thereby proving the statement of Theorem 3.

3.1. Reduction to Suitable Cones

Consider the compact link K obtained by intersecting Σ with the unit sphere bB of radius \sqrt{n} and center at the origin in \mathbb{C}^n . Note that every point $\xi \in K$ has at least one coordinate with absolute value $|\xi_k| \geq 1$. Our arguments will follow those used in proving the Main Theorem.

Thus, given any point $\xi \in K$, we construct a generalized cone that contains it. For example, if the first entry $|\xi_1| \geq 1$, we build the subvariety Y_ξ as in (9). Then we consider a biholomorphism π_ξ defined from an open set $U_\xi \subset \mathbb{C}^m$ into a neighborhood of (ξ_2, \dots, ξ_n) in Y_ξ as well as the mapping Π_ξ defined as in (11) from $\mathbb{C} \times U_\xi$ into Σ . We also restrict the domain of Π_ξ to a smaller set $\mathbb{C} \times U''_\xi$, where $U''_\xi \Subset U'_\xi \Subset U_\xi$, the open set U'_ξ is smoothly bounded, and $\pi_\xi(U''_\xi)$ is an open neighborhood of (ξ_2, \dots, ξ_n) in Y_ξ . We also assume that both U'_ξ and U''_ξ are simply connected. The generalized cone $\Pi_\xi(\mathbb{C} \times U''_\xi)$ obviously contains ξ , as we wanted. We proceed in a similar way for any other entry $|\xi_k| \geq 1$.

Now, since the link K is compact, we may choose finitely many (let us say N) points ξ^1, \dots, ξ^N in K such that K itself is covered by their associated generalized cones $C_j := \Pi_{\xi^j}(\mathbb{C} \times U''_{\xi^j})$. We assert that the analytic set Σ is covered by the cones C_j . Let z be any point in $\Sigma \setminus \{0\}$. It is easy to deduce the existence of $s \in \mathbb{C}^*$ such that $s^\beta * z$ lies in K ; thus there exists an index $1 \leq j \leq N$ such that $s^\beta * z$ also lies in C_j . We may suppose that the first entry $|\xi_1^j| \geq 1$ and that Π_{ξ^j} is given as in (11). Hence, there is a pair (t, x) in the Cartesian product $\mathbb{C}^* \times U''_{\xi^j}$ with

$$\begin{aligned} s^\beta * z &= \Pi_{\xi^j}(t, x) = t^\beta * (\xi_1^j, \pi_{\xi^j}(x)) \quad \text{and so} \\ z &= (t/s)^\beta * (\xi_1^j, \pi_{\xi^j}(x)) = \Pi_{\xi^j}(t/s, x). \end{aligned}$$

The previous identity shows that the entire analytic set Σ is covered by the N generalized cones C_1, \dots, C_N . On the other hand, in order to prove the Hölder continuity of (6), we take a fixed parameter $0 < \theta < 1$ and a pair of points z and w in

the intersection of Σ with the open ball B_R of radius $R > 0$ and center at the origin in \mathbb{C}^n . We want to show that there is a constant $C_\Sigma(\theta) > 0$ that is not dependent on z or w and such that

$$|g(z) - g(w)| \leq C_\Sigma(\theta) \cdot \text{dist}_\Sigma(z, w)^\theta \cdot \|\lambda\|_\infty. \tag{17}$$

The first step is to show that we need only verify the previous Hölder inequality if the points z and w are both contained in $B_R \cap C_j$, where C_j is a unique generalized cone defined as in the preceding paragraphs. Let $\varepsilon > 0$ be a given parameter. The definition of $\text{dist}_\Sigma(z, w)$ implies the existence of a piecewise smooth curve $\gamma_\varepsilon : [0, 1] \rightarrow \Sigma$ joining z and w (i.e., $\gamma_\varepsilon(0) = z$ and $\gamma_\varepsilon(1) = w$) such that

$$\text{length}(\gamma_\varepsilon) = \int_0^1 \|\gamma'(t)\| dt \leq \text{dist}_\Sigma(z, w) + \varepsilon.$$

The image of γ_ε is completely contained in $B_R \cap \Sigma$ because Σ is homogeneous (a cone). So now we are done if the points z and w are both contained in the same generalized cone C_j . Otherwise, we run over the curve γ_ε from z to w , picking up a finite set $\{z_k\}$ inside $\gamma_\varepsilon \subset B_R$ such that the initial point $z_0 = z$, the final point $z_N = w$, two consecutive elements z_j and z_{j+1} lie in the same generalized cone, and no three arbitrary elements of $\{z_k\}$ lie in the same generalized cone. In particular we may also suppose, without loss of generality, that $z_0 = z$ is in C_1 , the final point $z_N = w$ is in C_N , and any other point z_j is in the intersection $C_j \cap C_{j+1}$ for every index $1 \leq j < N$. Thus, two consecutive points z_{j-1} and z_j lie in the same generalized cone $C_j \cap B_R$ for each index $1 \leq j \leq N$. Assume for the moment that there exist constants $C_\Sigma^j(\theta) > 0$ such that

$$|g(z_{j-1}) - g(z_j)| \leq C_\Sigma^j(\theta) \cdot \text{dist}_\Sigma(z_{j-1}, z_j)^\theta \cdot \|\lambda\|_\infty$$

for all $1 \leq j \leq N$. Then it follows that

$$\begin{aligned} |g(z) - g(w)| &\leq \sum_{j=1}^N |g(z_{j-1}) - g(z_j)| \leq \sum_{j=1}^N C_\Sigma^j(\theta) \text{dist}_\Sigma(z_{j-1}, z_j)^\theta \|\lambda\|_\infty \\ &\leq C_\Sigma(\theta) \cdot [\text{dist}_\Sigma(z, w) + \varepsilon]^\theta \cdot \|\lambda\|_\infty, \end{aligned}$$

where we have chosen $C_\Sigma(\theta) = \sum_j C_\Sigma^j(\theta)$. Since the previous inequality holds for all $\varepsilon > 0$, it follows that we only need to prove that the Hölder estimate (17) holds under the assumption that z and w are both contained in the intersection of a unique generalized cone C_j with the open ball B_R of radius $R > 0$ and center at the origin in \mathbb{C}^n . Moreover, we can suppose without loss of generality that C_j is indeed the generalized cone given in (11).

We now determine what the assumptions on λ would imply for $\Pi^*\lambda$. Recall the given hypotheses: the subvariety Σ is homogeneous and has only one isolated singularity at the origin of \mathbb{C}^n , so that each entry $\beta_k = 1$ in Definition 1. We fix a point ξ in the link $K \subset \Sigma$ and assume that its first entry $|\xi_1| \geq 1$. The subvariety Y is then given in (9), and the biholomorphism π is defined from an open set

$U \subset \mathbb{C}^m$ into a neighborhood of (ξ_2, \dots, ξ_n) in Y . Let λ be a $(0, 1)$ -form as in the hypotheses of Theorem 2. We may easily calculate the pull-back $\Pi^*\lambda$, with the mapping Π given in (11) for all $s \in \mathbb{C}$ and $x \in U$, as

$$\Pi(s, x) = s^{(1, \dots, 1)} * (\xi_1, \pi(x)) = (s\xi_1, s\pi(x)) \in \Sigma. \tag{18}$$

The pull-back $\Pi^*\lambda = F_0(s, x)d\bar{s} + \sum_j F_j d\bar{x}_j$ satisfies:

$$F_0(s, x) = \sum_{k=1}^n f_k(\Pi(s, x))\overline{\pi_k(x)}, \quad \pi_1(x) \equiv \xi_1; \tag{19}$$

$$F_j(s, x) = \sum_{k=2}^n f_k(\Pi(s, x)) \left[s \frac{\partial \pi_k}{\partial x_j} \right].$$

The hypotheses of Theorem 3 yield that the support of every f_k is contained in a ball of radius $R > 0$ and center at the origin. Whence equation (18) and $|\xi_1| \geq 1$ imply that each function $F_k(s, x)$ vanishes whenever $|s| > R$.

Consider a pair of simply connected open sets in \mathbb{C}^m such that U' is smoothly bounded, $U'' \Subset U' \Subset U$, and $\pi(U'')$ is an open neighborhood of (ξ_2, \dots, ξ_n) in Y . The biholomorphism π has a Jacobian (determinant) that is bounded from above and below (away from zero) in the compact closure \bar{U}' . Hence there exists a constant $D_1 > 0$ such that the following identities hold for every point (s, x) in $\mathbb{C} \times \bar{U}'$ and each index $1 \leq j \leq m$:

$$|F_0(s, x)| \leq D_1 \cdot \|\lambda\|_\infty; \tag{20}$$

$$|F_j(s, x)| \leq D_1 \cdot |s| \cdot \|\lambda\|_\infty. \tag{21}$$

We can now show that the Hölder estimate (17) holds for all points z and w in the intersection of the generalized cone $\Pi(\mathbb{C} \times U'')$ with the ball B_R , so we conclude that the same estimate holds on $B_R \cap \Sigma$.

3.2. Hölder Estimates for Points in the Same Line

Fix the parameter $0 < \theta < 1$. We shall analyze two different cases. First, we assume there exist a point $x \in U''$ and two complex numbers s and s' such that $z = \Pi(s, x)$ and $w = \Pi(s', x)$. In this case we say that z and w lie in the same complex line. Equation (18) and the fact that $|\xi_1| \geq 1$ together yield that $|s|$ is bounded:

$$|s| \leq |s\xi_1| \leq \|z\| < R. \tag{22}$$

Define the new function $G(s, x)$ as the pull-back of (16):

$$G(s, x) := g(\Pi(s, x)) = g_1(\Pi(s, x)) - g_0. \tag{23}$$

We easily have that $\bar{\partial}G = \Pi^*\lambda$ on $\mathbb{C}^* \times U$ because $\bar{\partial}g_1 = \lambda$ and Π is a biholomorphism; see the conclusions of Theorem 2. We may calculate $G(s, x)$ via equations (18)–(19) and the change of variables $u = st$:

$$\begin{aligned} G(s, x) &= \sum_{k=1}^n \frac{1}{2\pi i} \int_{t \in \mathbb{C}} f_k(\Pi(st, x)) \frac{\overline{s\pi_k(x)} dt \wedge d\bar{t}}{t(t-1)} \\ &= \frac{s}{2\pi i} \int_{|u| \leq R} F_0(u, x) \frac{du \wedge d\bar{u}}{u(u-s)}. \end{aligned}$$

Recall that $F_0(u, x)$ vanishes whenever $|u| > R$. It is then easy to see that $G(s, x)$ is bounded in $\mathbb{C} \times U$ and that $G(0, x) = 0$. Hence, [14, Lemma 4.3.2] implies that $\bar{\partial}G$ is equal to $\Pi^*\lambda$ in $\mathbb{C} \times U$. We also have that

$$\begin{aligned} |g(z) - g(w)| &= |G(s, x) - G(s', x)| \\ &= \frac{1}{2\pi} \left| \int_{|u| \leq R} F_0(u, x) \left(\frac{1}{u - s} - \frac{1}{u - s'} \right) du \wedge d\bar{u} \right|. \end{aligned}$$

It is well known that there exists a constant $D_2(R, \theta) > 0$, depending only on the radius $R > 0$ and the parameter θ , such that

$$|g(z) - g(w)| \leq D_2(R, \theta) |s - s'|^\theta D_1 \|\lambda\|_\infty. \tag{24}$$

Note that we have used (20), and see [14, chap. 6.1] for a (more general) version of inequality (24). The analysis in the previous paragraphs shows that (17) holds in the first case. Besides, since both $g(0)$ and $G(0, x)$ vanish, we also obtain the following useful estimate:

$$|G(s, x)| = |g(z)| \leq D_2(R, \theta) D_1 |s|^\theta \|\lambda\|_\infty. \tag{25}$$

3.3. Hölder Estimates for Points in the Same Slice

Let z and \hat{w} be a pair of points in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball B_R . Assume that there exist a complex number $s \neq 0$ and a pair of points x and x' in the open set U'' such that $z = \Pi(s, x)$ and $\hat{w} = \Pi(s, x')$. We say, in this case, that z and \hat{w} lie in the same slice. By a unitary change of coordinates that does not destroy the inequality (21), we may assume that the entries of x and x' are all equal, with the possible exception of the first one. In other words, we may assume that both x and x' lie in the complex line $L := \mathbb{C} \times \{(x_2, \dots, x_m)\}$. Recall that the differential $\bar{\partial}G$ is equal to $\Pi^*\lambda$ in the open set $\mathbb{C} \times U$, according to definition (23) and the paragraphs that follow it. We can therefore evaluate $g(z)$ via the inhomogeneous Cauchy–Pompeiu formula on the line L :

$$\begin{aligned} g(z) = G(s, x) &= \frac{1}{2\pi i} \int_{L \cap U'} F_1(s, t, x_2, \dots, x_m) \frac{dt \wedge d\bar{t}}{t - x_1} \\ &\quad + \frac{1}{2\pi i} \int_{L \cap bU'} G(s, t, x_2, \dots, x_m) \frac{dt}{t - x_1}, \end{aligned}$$

because x is in $L \cap U''$ and $U'' \Subset U'$. We introduce some notation in order to simplify the analysis. The symbols $I_1(s, x)$ and $I_2(s, x)$ will denote the preceding integrals on the set $L \cap U'$ and the boundary $L \cap bU'$, respectively. In particular, we have

$$g(\hat{w}) = G(s, x') = I_1(s, x') + I_2(s, x').$$

Recall that x and x' are both in $L \cap U''$ and that the difference $x - x'$ is equal to the vector $(x_1 - x'_1, 0, \dots, 0)$. Inequality (21) implies the existence of a constant $D_3(\theta) > 0$, depending only on the diameter of U' and the parameter θ , such that

$$|I_1(s, x) - I_1(s, x')| \leq D_3(\theta) |x_1 - x'_1|^\theta D_1 |s| \|\lambda\|_\infty. \tag{26}$$

We can calculate similar estimates for I_2 as follows. Let $\delta > 0$ be the distance between the compact sets $\overline{U''}$ and bU' in \mathbb{C}^m . We obviously have that $\delta > 0$ because $U'' \Subset U'$. The following estimates are deduced from (25) and the mean value theorem, and the maximum is calculated over all v in $L \cap \overline{U''}$:

$$|I_2(s, x) - I_2(s, x')| \leq \frac{|x_1 - x'_1|}{2\pi} \max_v \left| \int_{L \cap bU'} \frac{G(s, t, x_2, \dots) dt}{(t - v)^2} \right| \tag{27}$$

$$\leq \frac{|x_1 - x'_1|}{2\pi} \cdot \frac{\text{length}(L \cap bU')}{\delta^2} D_2(R, \theta) D_1 |s|^\theta \|\lambda\|_\infty. \tag{28}$$

3.4. Hölder Estimates for Any Two Points

The estimates (24), (26), and (28) can be summarized in the following lemma. It is convenient to recall that the points x and x' are both contained in the bounded set $U'' \Subset \mathbb{C}^m$. Moreover, we also have that $|s| < R$ and $|s'| < R$ because z, w , and \hat{w} are all contained in the ball B_R ; recall the proof of (22).

LEMMA 5 (Anisotropic estimates). *In the situation of Theorem 2 and Theorem 3, consider the functions g and Π given in (16) and (18), respectively, and the bounded open set $U'' \Subset \mathbb{C}^m$ defined at the end of Section 3.1. Then, for every parameter $0 < \theta < 1$, there is a constant $D_4(R, \theta) > 0$ that does not depend on λ and such that the following statements hold for all the points $z = \Pi(s, x)$ and $w = \Pi(s', x')$ in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball B_R :*

- (i) $|g(z) - g(w)| \leq D_4(R, \theta) |s - s'|^\theta \|\lambda\|_\infty$ whenever $x = x'$ (i.e., z and w are in the same line); and
- (ii) $|g(z) - g(w)| \leq D_4(R, \theta) \|x - x'\|^\theta |s|^\theta \|\lambda\|_\infty$ whenever $s = s'$ (i.e., z and w are in the same slice).

It is now easy to prove that the Hölder estimates given in (6) and (17) hold for all points z and w that fulfil the assumptions of Lemma 5—namely, those points that lie in the intersection of the generalized cone $\Pi(\mathbb{C} \times U'')$ with the ball B_R . The definition of Π given in (18) allows to write down the following identities:

$$z = \Pi(s, x) = s(\xi_1, \pi(x)), \quad w = \Pi(s', x') = s'(\xi_1, \pi(x')). \tag{29}$$

Fix the point $z' := \Pi(s, x') = s(\xi_1, \pi(x'))$, which is in the same line as w and in the same slice as z . We can suppose without loss of generality that $z' \in B_R$ because z and w also lie in B_R . Otherwise, if the norm $\|z'\| \geq R$ then we only need to use $\Pi(s', x)$ instead. We can easily deduce the following estimate from (29) and the fact that $|\xi_1| \geq 1$:

$$|s - s'| \leq |s\xi_1 - s'\xi_1| \leq \|z - w\| \leq \text{dist}_\Sigma(z, w).$$

Recall that π is a biholomorphism whose Jacobian (determinant) is bounded from above and below (away from zero) on the compact set $\overline{U''}$. Hence, recalling (29), we can deduce the existence of a constant $D_5 > 0$, depending only on π and U'' , such that

$$\begin{aligned} \frac{|s| \cdot \|x - x'\|}{D_5} &\leq |s| \cdot \|\pi(x) - \pi(x')\| \leq \|z - z'\| \\ &\leq \|z - w\| + \|w - z'\| = \|z - w\| + |s - s'| \cdot \|(\xi_1, \pi(x'))\| \\ &\leq \text{dist}_\Sigma(z, w) \cdot [2 + \|\pi(x')\|]. \end{aligned}$$

Thus, there exists a constant $D_6 > 0$, depending only on π and U'' , such that the following identities hold for all the points $z = \Pi(s, x)$ and $w = \Pi(s', x')$ in the intersection of $\Pi(\mathbb{C} \times U'')$ with the ball B_R :

$$|s - s'| \leq D_6 \cdot \text{dist}_\Sigma(z, w), \quad |s| \cdot \|x - x'\| \leq D_6 \cdot \text{dist}_\Sigma(z, w).$$

Recall that z' is in the same line as w and in the same slice as z . Lemma 5 then yields that

$$\begin{aligned} |g(z) - g(w)| &\leq |g(z) - g(z')| + |g(z') - g(w)| \\ &\leq D_4(R, \theta)[|s|^\theta \cdot \|x - x'\|^\theta + |s - s'|^\theta] \|\lambda\|_\infty \\ &\leq D_4(R, \theta) 2D_6^\theta \text{dist}_\Sigma(z, w)^\theta \|\lambda\|_\infty. \end{aligned}$$

This completes the proof that the Hölder estimates given in (6) and (17) hold for all pairs of points z and w in the intersection of the ball B_R with the generalized cone $\Pi(\mathbb{C} \times U'')$. We can therefore conclude that the same Hölder estimates hold for arbitrary points z and w in $B_R \cap \Sigma$ (we need only recall the reduction argument of Section 3.1).

4. L^2 -Estimates

We prove Theorem 4 in this section. Recall that Σ is a pure d -dimensional homogeneous subvariety of \mathbb{C}^n with arbitrary singular locus, so that $n \geq 2$ and each entry $\beta_k = 1$ in Definition 1. Moreover, the $(0, 1)$ -form λ is given by $\sum_k f_k d\bar{z}_k$, where the coefficients f_k are all square-integrable functions in Σ . Assume that the support of λ is contained in the open ball B_R of radius $R > 0$ and center at the origin. We begin by showing that the function g given in (7) is indeed well-defined for almost all $z \in \Sigma$. In particular, we have that

$$g(z) = \sum_{k=1}^n \frac{H_k(z)}{2\pi i}, \quad H_k(z) := \int_{|w| \leq R/\|z\|} f_k(wz) \frac{w^d \bar{z}_k dw \wedge d\bar{w}}{w(w-1)}. \tag{30}$$

Hence, we need only show that the following integrals exist:

$$\int_{w \in \mathbb{C}} \int_{z \in \Sigma \cap B_R} \left| \frac{f_k(wz) w^d z_k}{w(w-1)} \right| dV_\Sigma dV_\mathbb{C} < \infty. \tag{31}$$

Recall that dV_Σ and $dV_\mathbb{C}$ are the respective volume forms on Σ and \mathbb{C} . A direct application of Fubini's theorem yields that the integrals in (30) are all well-defined for almost all z in $\Sigma \cap B_R$; they are also well-defined for almost all $z \in \Sigma$ because the radius R can be as large as we like. The fact that Σ is a pure $2d$ -real dimensional and homogeneous (cone) subvariety implies the existence of a constant $C_0 > 0$ such that the following equations hold for all $w \in \mathbb{C}$ and real $\rho > 0$:

$$\int_{z \in \Sigma \cap B_\rho} \|z\|^2 = C_0 \rho^{2d+2}, \quad \int_{z \in \Sigma} |f_k(wz)|^2 = \frac{\|f_k\|_{L^2(\Sigma)}^2}{|w|^{2d}}. \tag{32}$$

Recall that each $\|f_k\|_{L^2(\Sigma)}^2 < \infty$ because of the given hypotheses. Observe that we need not calculate the integral (31) in the Cartesian product of \mathbb{C} times $\Sigma \cap B_R$. We can simplify the calculations by integrating over the set Ξ defined next, because $f_k(wz) = 0$ whenever $\|wz\| \geq R$:

$$\Xi := \{(w, z) \in \mathbb{C} \times \Sigma : \|z\| < R, \|wz\| < R\}. \tag{33}$$

By use of (32), it follows easily that

$$\begin{aligned} \left\| \frac{f_k(wz)w^d}{|w^2 - w|^{2/3}} \right\|_{L^2(\Xi)}^2 &\leq \int_{w \in \mathbb{C}} \int_{z \in \Sigma} \frac{|f_k(wz)w^d|^2}{|w^2 - w|^{4/3}} \\ &\leq \|f_k\|_{L^2(\Sigma)}^2 \int_{w \in \mathbb{C}} \frac{1}{|w^2 - w|^{4/3}} < \infty \end{aligned} \tag{34}$$

and that

$$\begin{aligned} \left\| \frac{z_k}{|w^2 - w|^{1/3}} \right\|_{L^2(\Xi)}^2 &\leq \int_{w \in \mathbb{C}} \int_{\substack{z \in \Sigma \cap B_R \\ \|z\| < R/|w|}} \frac{\|z\|^2}{|w^2 - w|^{2/3}} \\ &\leq \int_{|w| \leq 1} \frac{C_0 R^{2d+2}}{|w^2 - w|^{2/3}} + \int_{|w| > 1} \frac{C_0 (R/|w|)^{2d+2}}{|w^2 - w|^{2/3}} < \infty. \end{aligned} \tag{35}$$

The last integral in the first line of (35) must be separated into two parts depending on the value of $|w|$. Then one must apply (32) with ρ equal to R or $R/|w|$, respectively.

Now the Cauchy–Schwartz inequality $\|ab\|_{L^1} \leq \|a\|_{L^2} \|b\|_{L^2}$ allows us to deduce (31) from the inequalities (34) and (35): we need only integrate over the set Ξ given in (33).

The next step is to show that g in (30) satisfies the differential equation $\bar{\partial}g = \lambda$ on the regular part Σ^* . We can simply follow, step by step, the proof presented in Section 2. The only difference is that we must use a weighted Cauchy–Pompeiu integral in (13) and (15), with $m = d - 1$ integer:

$$\mathcal{G}(s, x) := \frac{1}{2\pi i} \int_{u \in \mathbb{C}} \frac{F_0(u, x)}{u - s} \left[\frac{u^m}{s^m} \right] du \wedge d\bar{u}, \tag{36}$$

where

$$F_0(u, x) = \sum_{k=1}^n f_k(\Pi(u, x)) \overline{\pi_k(x)}.$$

Notice that $\Pi(u, x) = u(\xi_1, \pi(x))$ because each entry $\beta_k = 1$ in (2) and (11). We must prove that $u^m \Pi^* \lambda$ lies in $L^2_{0,1}(\mathbb{C} \times U)$ and is $\bar{\partial}$ -closed. It is easy to calculate the pull-back of the volume form dV_Σ :

$$\begin{aligned} \Pi^* dV_\Sigma &= \sum_{|I|=|J|=d} \beta_{I,J}(z) dz_I \wedge d\bar{z}_J \Big|_{z=u(\xi_1, \pi(x))} \\ &= \Theta(x) |u|^{2d-2} [du \wedge d\bar{u}] \wedge \bigwedge_{k=1}^{d-1} [dx_k \wedge d\bar{x}_k]. \end{aligned} \tag{37}$$

Recall that x lies in $U \subset \mathbb{C}^{d-1}$. Since Σ is a d -dimensional homogeneous (cone) subvariety of \mathbb{C}^n , it follows that the coefficients $\beta_{l,j}(z)$ are all invariant under the transformations $z \mapsto uz$ and so $\Theta(x)$ depends only on the values of $\pi(x)$ and all its partial derivatives (it is constant with respect to u). That Π is a biholomorphism from $\mathbb{C}^* \times U$ onto its image also implies that Θ cannot vanish. Hence, choosing a smaller set U if necessary, we can suppose that $|\Theta|$ is bounded from below by a constant $M > 0$. It is then easy to see that $u^m \Pi^* \lambda$ is $L^2_{0,1}$, because

$$M \int_{\mathbb{C} \times U} |u^m \Pi^* f_k|^2 dV_{\mathbb{C} \times U} \leq \int_{\Pi(\mathbb{C} \times U)} |f_k|^2 dV_{\Sigma} \leq \|\lambda\|_{L^2_{0,1}(\Sigma)}^2 < \infty.$$

Here we have used equation (37) with $m = d - 1$. Working as in Section 2, it follows from [14, Lemma 4.3.2] that $u^m \Pi^* \lambda$ is $\bar{\partial}$ -closed and that the differential $\bar{\partial}g$ is equal to $[s^m \Pi^* \lambda] / s^m = \Pi^* \lambda$. Hence, the function g given in (30) is a solution to $\bar{\partial}g = \lambda$ on Σ^* because $g(\Pi(s, x))$ is identically equal to (36) after setting $u = sw$ and $\pi_1(x) \equiv \xi_1$.

Finally, we must calculate the L^2 -norm of g in order to prove the L^2 -estimates (8). It is well known that the Cauchy–Pompeiu formula is an L^2 -bounded operator:

$$\int_{|t| < R} \left| \frac{1}{2\pi i} \int_{|u| < R} h(u) \frac{du \wedge \bar{d}\bar{u}}{u - t} \right|^2 dV_{\mathbb{C}}(t) \lesssim \int_{|t| < R} |h(t)|^2 dV_{\mathbb{C}}. \tag{38}$$

The reader may find a complete proof in [12] or [14], for example.

Let $\tilde{\Sigma}$ be the projective variety associated to Σ in the space $\mathbb{C}\mathbb{P}^{n-1}$; recall that Σ is a pure d -dimensional homogeneous subvariety of \mathbb{C}^n . We use the fact that any integral on Σ can be decomposed as a pair of nested integrals on \mathbb{C} and $\tilde{\Sigma}$; that is:

$$\int_{z \in \Sigma} \Phi(z) dV_{\Sigma}(z) = \int_{[z] \in \tilde{\Sigma}} \int_{t \in \mathbb{C}} \Phi(\dot{z}t) |t|^{2d-2} dV_{\mathbb{C}}(t) dV_{\tilde{\Sigma}}([z]),$$

where $\dot{z} \in \Sigma$ is any representative of $[z] \in \tilde{\Sigma}$ with $\|\dot{z}\| = 1$. It is easy to calculate each norm $\|H_k\|_{L^2(\Sigma)}$ in (30) with $m = d - 1$ and $u = wt$:

$$\begin{aligned} & \int_{z \in \Sigma \cap B_R} \left| \frac{1}{\bar{z}_k} \int_{|w| < R/\|z\|} \frac{f_k(wz)w^d}{w(w-1)} dV_{\mathbb{C}}(w) \right|^2 dV_{\Sigma} \\ & \leq \int_{[z] \in \tilde{\Sigma}} \int_{|t| < R} |t|^2 \left| \int_{|w| < R} \frac{f_k(w\dot{z}t)w^m}{w-1} dV_{\mathbb{C}}(w) \right|^2 |t|^{2m} dV_{\mathbb{C}} dV_{\tilde{\Sigma}} \\ & = \int_{[z] \in \tilde{\Sigma}} \int_{|t| < R} |t|^2 \left| \int_{|u| < R} \frac{f_k(u\dot{z})u^m}{(u-t)t^{m-1}} \cdot \frac{dV_{\mathbb{C}}(u)}{|t|^2} \right|^2 |t|^{2m} dV_{\mathbb{C}} dV_{\tilde{\Sigma}} \\ & = \int_{[z] \in \tilde{\Sigma}} \int_{|t| < R} \left| \int_{|u| < R} \frac{f_k(u\dot{z})u^m}{u-t} dV_{\mathbb{C}}(u) \right|^2 dV_{\mathbb{C}} dV_{\tilde{\Sigma}} \\ & \lesssim \int_{[z] \in \tilde{\Sigma}} \int_{|t| < R} |f_k(t\dot{z})t^m|^2 dV_{\mathbb{C}} dV_{\tilde{\Sigma}} \\ & = \int_{z \in \Sigma \cap B_R} |f_k(z)|^2 dV_{\Sigma} = \|f_k\|_{L^2(\Sigma)}^2 \leq \|\lambda\|_{L^2_{0,1}(\Sigma)}^2. \end{aligned}$$

Here we have used equation (38) with $h(u) = f_k(u\dot{z})u^m$. This completes the proof of Theorem 4 because, by equation (30),

$$\|g\|_{L^2(\Sigma)} \leq \sum_{k=1}^n \frac{\|H_k\|_{L^2(\Sigma)}}{2\pi} \lesssim \|\lambda\|_{L^2_{0,1}(\Sigma)}.$$

5. Weighted Homogeneous Estimates

We want to close this paper by presenting a useful technique for generalizing the estimates given in Theorems 3 and 4 so as to consider weighted homogeneous subvarieties instead of cones. Let $X \subset \mathbb{C}^n$ be a weighted homogeneous subvariety with only one singularity at the origin and defined as the zero locus of a finite set of polynomials $\{Q_k\}$. Thus, the polynomials $Q_k(x)$ are all weighted homogeneous with respect to the same vector $\beta \in \mathbb{Z}^n$, and each entry $\beta_k \geq 1$. Define the following holomorphic mapping:

$$\Theta: \mathbb{C}^n \rightarrow \mathbb{C}^n \quad \text{with} \quad \Theta(z) = (z_1^{\beta_1}, z_2^{\beta_2}, \dots, z_n^{\beta_n}). \tag{39}$$

It is easy to see that each polynomial $Q_k(\Theta)$ is homogeneous, so the subvariety $\Sigma \subset \mathbb{C}^n$ defined as the zero locus of $\{Q_k(\Theta)\}$ is a cone. Moreover, since Θ is locally a biholomorphism in $\mathbb{C}^n \setminus \{0\}$, it follows that Σ has only one singularity at the origin as well. Consider a $(0, 1)$ -form \mathfrak{N} given by the sum $\sum_k f_k d\bar{x}_k$, where the coefficients f_k are all Borel-measurable functions in X and where x_1, \dots, x_n are the Cartesian coordinates of \mathbb{C}^n .

We may follow two different paths when solving the equation $\bar{\partial}h = \mathfrak{N}$. We may apply Theorem 2 whenever \mathfrak{N} is bounded and has compact support on X , thereby obtaining the solution

$$h(x) = \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(w^\beta * x) \frac{\overline{(w^{\beta_k} x_k)}}{\bar{w}(w-1)} dw \wedge d\bar{w}.$$

Otherwise, we may consider the pull-back $\Theta^*\mathfrak{N}$ and then apply Theorem 3 to solve the equation $\bar{\partial}g = \Theta^*\mathfrak{N}$ on Σ . We easily have that

$$\begin{aligned} \Theta^*\mathfrak{N} &= \sum_{k=1}^n f_k(\Theta(z)) \beta_k \bar{z}_k^{\beta_k-1} d\bar{z}_k \quad \text{and} \\ g(z) &= \sum_{k=1}^n \frac{\beta_k}{2\pi i} \int_{w \in \mathbb{C}} f_k(\Theta(wz)) \frac{\overline{(wz_k)}^{\beta_k}}{\bar{w}(w-1)} dw \wedge d\bar{w}. \end{aligned}$$

Both paths yield exactly the same solution because $g(z)$ is identically equal to $h(\Theta(z))$. Recall that $w^\beta * \Theta(z)$ is equal to $\Theta(wz)$ for all $w \in \mathbb{C}$ and $z \in \mathbb{C}^n$. Hence we may calculate the solution g and then use the Hölder estimates given in equation (6) to obtain

$$|g(z) - g(w)| \leq C_\Sigma(R, \theta) \cdot \text{dist}_\Sigma(z, w)^\theta \cdot \|\Theta^*\mathfrak{N}\|_\infty.$$

A final step is to *push forward* these estimates in order to deduce similar Hölder estimates for the solution h on X (see [18] for a detailed analysis of the procedure). On the other hand, we can use a similar procedure for L^2 -estimates; in that case, the subvariety X could have arbitrary singularities.

References

- [1] F. Acosta and E. S. Zeron, *Hölder estimates for the $\bar{\partial}$ -equation on surfaces with simple singularities*, Bol. Soc. Mat. Mexicana (3) 12 (2006), 193–204.
- [2] ———, *Hölder estimates for the $\bar{\partial}$ -equation on surfaces with singularities of the type E_6 and E_7* , Bol. Soc. Mat. Mexicana (3) 13 (2007), 73–86.
- [3] K. Diederich, J. E. Fornæss, and S. Vassiliadou, *Local L^2 results for $\bar{\partial}$ on a singular surface*, Math. Scand. 92 (2003), 269–294.
- [4] A. Dimca, *Singularities and topology of hypersurfaces*, Springer-Verlag, New York, 1992.
- [5] J. E. Fornæss, *L^2 results for $\bar{\partial}$ in a conic*, International symposium on complex analysis and related topics (Cuernavaca, 1996), Oper. Theory Adv. Appl., 114, pp. 67–72, Birkhäuser, Basel, 2000.
- [6] J. E. Fornæss and E. A. Gavosto, *The Cauchy Riemann equation on singular spaces*, Duke Math. J. 93 (1998), 453–477.
- [7] J. E. Fornæss, N. Øvrelid, and S. Vassiliadou, *Semiglobal results for $\bar{\partial}$ on a complex space with arbitrary singularities*, Proc. Amer. Math. Soc. 133 (2005), 2377–2386.
- [8] ———, *Local L^2 results for $\bar{\partial}$: The isolated singularities case*, Internat. J. Math. 16 (2005), 387–418.
- [9] E. A. Gavosto, *Hölder estimates for the $\bar{\partial}$ -equation in some domains of finite type*, J. Geom. Anal. 7 (1997), 593–609.
- [10] G. Henkin and J. Leiterer, *Theory of functions on complex manifolds*, Monogr. Math., 79, Birkhäuser, Basel, 1984.
- [11] ———, *Andreotti–Grauert theory by integral formulas*, Progr. Math., 74, Birkhäuser, Basel, 1988.
- [12] I. Lieb and J. Michel, *The Cauchy–Riemann complex. Integral formulae and Neumann problem*, Aspects Math., E34, Vieweg, Braunschweig, 2002.
- [13] N. Øvrelid and S. Vassiliadou, *Solving $\bar{\partial}$ on product singularities*, Complex Var. Elliptic Equ. 51 (2006), 225–237.
- [14] J. Ruppenthal, *Zur Regularität der Cauchy–Riemannschen Differentialgleichungen auf komplexen Räumen*, Bonner Math. Schriften, 380, Univ. Bonn, Bonn, 2006.
- [15] ———, *About the $\bar{\partial}$ -equation at isolated singularities with regular exceptional set*, Internat. J. Math. 20 (2009), 459–489.
- [16] ———, *The $\bar{\partial}$ -equation on homogeneous varieties with an isolated singularity*, Math. Z. 263 (2009).
- [17] G. Scheja, *Riemannsche Hebbarkeitssätze für Cohomologieklassen*, Math. Ann. 144 (1961), 345–360.
- [18] M. Solís and E. S. Zeron, *Hölder estimates for the $\bar{\partial}$ -equation on singular quotient varieties*, Bol. Soc. Mat. Mexicana (3) (to appear).

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