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Distance-Regular Graphs of q-Racah Type and the q-Tetrahedron Algebra

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In memory of Donald Higman

1. Introduction

In [20], Hartwig and the second author gave a presentation of the three-point $\mathfrak{sl}_2$ loop algebra via generators and relations. To obtain this presentation they defined a Lie algebra $\boxtimes$ by generators and relations and then displayed an isomorphism from $\boxtimes$ to the three-point $\mathfrak{sl}_2$ loop algebra. The algebra $\boxtimes$ is called the *tetrahedron* algebra [20, Def. 1.1]. In [24] we introduced a q-deformation $\boxtimes_q$ of $\boxtimes$ called the q-tetrahedron algebra. In [24] and [25] we described the finite-dimensional irreducible $\boxtimes_q$-modules. In [26, Sec. 4] we displayed four homomorphisms into $\boxtimes_q$ from the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. In [26, Sec. 12] we found a homomorphism from $\boxtimes_q$ into the subconstituent algebra of a distance-regular graph that is self-dual with classical parameters. In this paper we do something similar for a distance-regular graph that is said to have q-Racah type. This type is described as follows. Let Γ denote a distance-regular graph with diameter $D \geq 3$ (See Section 4 for formal definitions). We say that Γ has *q-Racah type* whenever Γ has a Q-polynomial structure with eigenvalue sequence $\{\theta_i\}_{i=0}^D$ and dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^D$ that satisfy, for $0 \leq i \leq D$,

$$\theta_i = \eta + uq^{2i-D} + vq^{D-2i} \quad \text{and}$$
$$\theta^*_i = \eta^* + u^*q^{2i-D} + v^*q^{D-2i},$$

where q, u, v, u^*, v^* are nonzero and $q^{2i} \neq 1$ for $1 \leq i \leq D$. Assume that Γ has q-Racah type.

Fix a vertex x of Γ and let $T = T(x)$ denote the corresponding subconstituent algebra [32, Def. 3.3]. Recall that T is generated by the adjacency matrix A and the dual adjacency matrix $A^* = A^*(x)$ [32, Def. 3.10]. An irreducible T-module W is called *thin* whenever the intersection of W with each eigenspace of A and each eigenspace of A^* has dimension at most 1 [32, Def. 3.5]. Assuming that each irreducible T-module is thin, we display invertible central elements Φ and Ψ of T and a homomorphism $\vartheta : \boxtimes_q \to T$ such that

$$A = \eta I + u\Phi\Psi^{-1}\vartheta(x_{01}) + v\Psi\Phi^{-1}\vartheta(x_{12}) \quad \text{and}$$
$$A^* = \eta^* I + u^*\Phi\Psi\vartheta(x_{23}) + v^*\Psi^{-1}\Phi^{-1}\vartheta(x_{30}),$$

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where the x_{ij} are the standard generators of $\boxtimes_q$. It follows that T is generated by the image $\vartheta(\boxtimes_q)$ together with Φ and Ψ. In particular, T is generated by $\vartheta(\boxtimes_q)$ together with the center $Z(T)$. Our result settles [26, Conj. 13.10] for the case in which every irreducible T-module is thin.

The paper is organized as follows. In Section 2 we recall the definition of $\boxtimes_q$, and in Section 3 we describe how $\boxtimes_q$ is related to $U_q(\widehat{\mathfrak{sl}}_2)$. In Section 4 we recall the basic theory of a distance-regular graph Γ, focusing on the Q-polynomial property and the subconstituent algebra. In Section 5 we discuss the split decomposition of Γ, and in Section 6 we give our main results.

Throughout the paper, $\mathbb{C}$ denotes the field of complex numbers.

2. The q-Tetrahedron Algebra $\boxtimes_q$

In this section we recall the q-tetrahedron algebra. We fix a nonzero scalar $q \in \mathbb{C}$ such that $q^2 \neq 1$ and define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n = 0, 1, 2, \ldots.$$

We let $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ denote the cyclic group of order 4.

DEFINITION 2.1 [24, Def. 10.1]. Let $\boxtimes_q$ denote the unital associative $\mathbb{C}$-algebra that has generators

$$\{x_{ij} \mid i, j \in \mathbb{Z}_4,\ j - i = 1 \text{ or } j - i = 2\}$$

and the following relations.

(i) For $i, j \in \mathbb{Z}_4$ such that $j - i = 2$,

$$x_{ij} x_{ji} = 1.$$

(ii) For $h, i, j \in \mathbb{Z}_4$ such that the pair $(i - h, j - i)$ is one of $(1,1)$, $(1,2)$, and $(2,1)$,

$$\frac{q x_{hi} x_{ij} - q^{-1} x_{ij} x_{hi}}{q - q^{-1}} = 1.$$

(iii) For $h, i, j, k \in \mathbb{Z}_4$ such that $i - h = j - i = k - j = 1$,

$$x_{hi}^3 x_{jk} - [3]_q x_{hi}^2 x_{jk} x_{hi} + [3]_q x_{hi} x_{jk} x_{hi}^2 - x_{jk} x_{hi}^3 = 0. \tag{1}$$

We call $\boxtimes_q$ the *q-tetrahedron algebra* or "q-tet" for short. We refer to the x_{ij} as the *standard generators* for $\boxtimes_q$.

NOTE 2.2. The equations (1) are the cubic q-Serre relations [29, p. 10].

We make some observations as follows.

LEMMA 2.3 [24, Lemma 6.3]. *There exists a $\mathbb{C}$-algebra automorphism ϱ of $\boxtimes_q$ that sends each generator x_{ij} to $x_{i+1,j+1}$. Moreover, $\varrho^4 = 1$.*

LEMMA 2.4 [24, Lemma 6.5]. *There exists a $\mathbb{C}$-algebra automorphism of $\boxtimes_q$ that sends each generator x_{ij} to $-x_{ij}$.*

3. The Quantum Affine Algebra $U_q(\widehat{\mathfrak{sl}}_2)$

In this section we consider how $\boxtimes_q$ is related to the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. We start with a definition.

DEFINITION 3.1 [7, p. 266]. The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is the unital associative $\mathbb{C}$-algebra with generators $K_i^{\pm 1}$ and $e_i^{\pm}$, $i \in \{0,1\}$, and the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1;$$
$$K_0 K_1 = K_1 K_0;$$
$$K_i e_i^{\pm} K_i^{-1} = q^{\pm 2} e_i^{\pm};$$
$$K_i e_j^{\pm} K_i^{-1} = q^{\mp 2} e_j^{\pm}, \quad i \neq j;$$
$$[e_i^{+}, e_i^{-}] = \frac{K_i - K_i^{-1}}{q - q^{-1}};$$
$$[e_0^{\pm}, e_1^{\mp}] = 0;$$
$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \quad i \neq j.$$

The following presentation of $U_q(\widehat{\mathfrak{sl}}_2)$ will be useful.

PROPOSITION 3.2 [23, Thm. 2.1; 38]. *The quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is isomorphic to the unital associative $\mathbb{C}$-algebra with generators $x_i^{\pm 1}, y_i, z_i$, $i \in \{0,1\}$, and the following relations*:

$$x_i x_i^{-1} = x_i^{-1} x_i = 1;$$
$$x_0 x_1 \ \textit{is central};$$
$$\frac{q x_i y_i - q^{-1} y_i x_i}{q - q^{-1}} = 1;$$
$$\frac{q y_i z_i - q^{-1} z_i y_i}{q - q^{-1}} = 1;$$
$$\frac{q z_i x_i - q^{-1} x_i z_i}{q - q^{-1}} = 1;$$
$$\frac{q z_i y_j - q^{-1} y_j z_i}{q - q^{-1}} = x_0^{-1} x_1^{-1}, \quad i \neq j;$$
$$y_i^3 y_j - [3]_q y_i^2 y_j y_i + [3]_q y_i y_j y_i^2 - y_j y_i^3 = 0, \quad i \neq j;$$
$$z_i^3 z_j - [3]_q z_i^2 z_j z_i + [3]_q z_i z_j z_i^2 - z_j z_i^3 = 0, \quad i \neq j.$$

An isomorphism with the presentation in Definition 3.1 is given by:

$$x_i^{\pm 1} \mapsto K_i^{\pm 1};$$
$$y_i \mapsto K_i^{-1} + e_i^{-};$$
$$z_i \mapsto K_i^{-1} - K_i^{-1} e_i^{+} q (q - q^{-1})^2.$$

The inverse of this isomorphism is given by:

$$K_i^{\pm 1} \mapsto x_i^{\pm 1};$$
$$e_i^- \mapsto y_i - x_i^{-1};$$
$$e_i^+ \mapsto (1 - x_i z_i) q^{-1} (q - q^{-1})^{-2}.$$

THEOREM 3.3 [26, Prop. 4.3]. *For $i \in \mathbb{Z}_4$ there exists a $\mathbb{C}$-algebra homomorphism from $U_q(\widehat{\mathfrak{sl}}_2)$ to $\boxtimes_q$ that sends*

$$x_1 \mapsto x_{i,i+2}, \quad x_1^{-1} \mapsto x_{i+2,i}, \quad y_1 \mapsto x_{i+2,i+3}, \quad z_1 \mapsto x_{i+3,i},$$
$$x_0 \mapsto x_{i+2,i}, \quad x_0^{-1} \mapsto x_{i,i+2}, \quad y_0 \mapsto x_{i,i+1}, \quad z_0 \mapsto x_{i+1,i+2}.$$

Proof. Compare the defining relations for $U_q(\widehat{\mathfrak{sl}}_2)$ given in Proposition 3.2 with the relations in Definition 2.1. □

4. Distance-Regular Graphs: Preliminaries

We now turn our attention to distance-regular graphs. After a brief review of the basic definitions we recall the Q-polynomial property and the subconstituent algebra. For more information we refer the reader to [1; 3; 19; 32].

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitean inner product $\langle \cdot, \cdot \rangle$ that satisfies $\langle u, v \rangle = u^t \bar{v}$ for $u, v \in V$, where t denotes transpose and $\bar{\ }$ denotes complex conjugation. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe that $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. For an integer $k \geq 0$ we say that Γ is *regular with valency k* whenever each vertex of Γ is adjacent to exactly k distinct vertices of Γ. We say that Γ is *distance-regular* whenever, for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h = |\{z \in X \mid \partial(x, z) = i, \ \partial(z, y) = j\}|$$

is independent of x and y. The p_{ij}^h are called the *intersection numbers* of Γ. We abbreviate $c_i = p_{1,i-1}^i$ $(1 \leq i \leq D)$, $b_i = p_{1,i+1}^i$ $(0 \leq i \leq D-1)$, and $a_i = p_{1i}^i$ $(0 \leq i \leq D)$.

For the rest of this paper we assume Γ is distance-regular; to avoid trivialities we always assume $D \geq 3$. Note that Γ is regular with valency $k = b_0$. Moreover, $k = c_i + a_i + b_i$ for $0 \leq i \leq D$, where $c_0 = 0$ and $b_D = 0$.

We mention a fact for later use. By the triangle inequality, for $0 \le h, i, j \le D$ we have $p^h_{ij} = 0$ (resp. $p^h_{ij} \ne 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two.

We recall the Bose–Mesner algebra of Γ. For $0 \le i \le D$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (x, y)-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \ne i \end{cases} \qquad (x, y \in X).$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A = A_1$ and call this the *adjacency matrix* of Γ. We observe the following: (i) $A_0 = I$; (ii) $\sum_{i=0}^{D} A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (iv) $A_i^t = A_i$ $(0 \le i \le D)$; and (v) $A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h$ $(0 \le i, j \le D)$, where I (resp. J) denotes the identity matrix (resp. all-1 matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts, we find that $\{A_i\}_{i=0}^{D}$ is a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$, called the *Bose–Mesner algebra* of Γ. It turns out that A generates M [1, p. 190]. By [3, p. 45], M has a second basis $\{E_i\}_{i=0}^{D}$ such that: (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (iv) $E_i^t = E_i$ $(0 \le i \le D)$; and (v) $E_i E_j = \delta_{ij} E_i$ $(0 \le i, j \le D)$. We call $\{E_i\}_{i=0}^{D}$ the *primitive idempotents* of Γ.

We recall the eigenvalues of Γ. Since $\{E_i\}_{i=0}^{D}$ form a basis for M, there exist complex scalars $\{\theta_i\}_{i=0}^{D}$ such that $A = \sum_{i=0}^{D} \theta_i E_i$. Observe that $AE_i = E_i A = \theta_i E_i$ for $0 \le i \le D$. By [1, p. 197] the scalars $\{\theta_i\}_{i=0}^{D}$ are in $\mathbb{R}$. Observe that $\{\theta_i\}_{i=0}^{D}$ are mutually distinct because A generates M. We call θ_i the *eigenvalue* of Γ associated with E_i $(0 \le i \le D)$. Observe that

$$V = E_0V + E_1V + \cdots + E_DV \quad \text{(orthogonal direct sum)}.$$

For $0 \le i \le D$, the space E_iV is the eigenspace of A associated with θ_i.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij} A_i$ for $0 \le i, j \le D$, so M is closed under $\circ$. Thus there exist complex scalars q^h_{ij} $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \quad (0 \le i, j \le D).$$

By [2, p. 170], q^h_{ij} is real and nonnegative for $0 \le h, i, j \le D$. The q^h_{ij} are called the *Krein parameters* of Γ. The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents) if, for $0 \le h, i, j \le D$, we have $q^h_{ij} = 0$ (resp. $q^h_{ij} \ne 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two [3, p. 235]. See [4; 5; 6; 8; 11; 14; 15; 30] for background information on the Q-polynomial property. From now on we assume Γ is Q-polynomial with respect to $\{E_i\}_{i=0}^{D}$. We call the sequence $\{\theta_i\}_{i=0}^{D}$ the *eigenvalue sequence* for this Q-polynomial structure.

We recall the dual Bose–Mesner algebra of Γ. For the rest of this paper we fix a vertex $x \in X$. We view x as a "base vertex". For $0 \le i \le D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y, y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \quad (y \in X). \tag{2}$$

We call E_i^* the ith *dual idempotent* of Γ with respect to x [32, p. 378]. We observe that (i) $\sum_{i=0}^D E_i^* = I$; (ii) $\overline{E_i^*} = E_i^*$ $(0 \leq i \leq D)$; (iii) $E_i^{*t} = E_i^*$ $(0 \leq i \leq D)$; and (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \leq i, j \leq D)$. By these facts, $\{E_i^*\}_{i=0}^D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose–Mesner algebra* of Γ with respect to x [32, p. 378]. For $0 \leq i \leq D$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with (y,y)-entry $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. Then $\{A_i^*\}_{i=0}^D$ is a basis for M^* [32, p. 379]. Moreover, (i) $A_0^* = I$; (ii) $\overline{A_i^*} = A_i^*$ $(0 \leq i \leq D)$; (iii) $A_i^{*t} = A_i^*$ $(0 \leq i \leq D)$; and (iv) $A_i^* A_j^* = \sum_{h=0}^D q_{ij}^h A_h^*$ $(0 \leq i, j \leq D)$ [32, p. 379]. We call $\{A_i^*\}_{i=0}^D$ the *dual distance matrices* of Γ with respect to x. We abbreviate $A^* = A_1^*$ and call this the *dual adjacency matrix* of Γ with respect to x. The matrix A^* generates M^* [32, Lemma 3.11].

We recall the dual eigenvalues of Γ. Since $\{E_i^*\}_{i=0}^D$ form a basis for M^*, there exist complex scalars $\{\theta_i^*\}_{i=0}^D$ such that $A^* = \sum_{i=0}^D \theta_i^* E_i^*$. Observe that $A^* E_i^* = E_i^* A^* = \theta_i^* E_i^*$ for $0 \leq i \leq D$. By [32, Lemma 3.11] the scalars $\{\theta_i^*\}_{i=0}^D$ are in $\mathbb{R}$. The scalars $\{\theta_i^*\}_{i=0}^D$ are mutually distinct because A^* generates M^*. We call θ_i^* the *dual eigenvalue* of Γ associated with E_i^* $(0 \leq i \leq D)$. We call the sequence $\{\theta_i^*\}_{i=0}^D$ the *dual eigenvalue sequence* for the given Q-polynomial structure.

We recall the subconstituents of Γ. From (2) we find

$$E_i^* V = \mathrm{span}\{\hat{y} \mid y \in X, \ \partial(x,y) = i\} \quad (0 \leq i \leq D). \tag{3}$$

By (3) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V, we find

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(orthogonal direct sum).}$$

For $0 \leq i \leq D$, the space $E_i^* V$ is the eigenspace of A^* associated with θ_i^*. We call $E_i^* V$ the ith *subconstituent* of Γ with respect to x.

We recall the subconstituent algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [32, Def. 3.3]. Observe that T has finite dimension. Moreover, T is semisimple because it is closed under the conjugate transpose map [13, p. 157]. We note that A, A^* together generate T. By [32, Lemma 3.2], the following are relations in T. For $0 \leq h, i, j \leq D$,

$$E_h^* A_i E_j^* = 0 \quad \text{iff} \quad p_{ij}^h = 0; \tag{4}$$

$$E_h A_i^* E_j = 0 \quad \text{iff} \quad q_{ij}^h = 0. \tag{5}$$

See [9; 10; 12; 16; 17; 18; 21; 31; 32; 33; 34] for more information on the subconstituent algebra.

We recall the T-modules. By a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module and let W' denote a T-module contained in W. Then the orthogonal complement of W' in W is a T-module [18, p. 802]. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. In particular, V is an orthogonal direct sum of irreducible T-modules.

Let W denote an irreducible T-module. Observe that W is the direct sum of the nonzero spaces among $E_0^*W, \ldots, E_D^*W$. Similarly, W is the direct sum of the nonzero spaces among $E_0W, \ldots, E_DW$. By the *endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}$. By the *diameter* of W we mean $|\{i \mid 0 \le i \le D,\ E_i^*W \ne 0\}| - 1$. By the *dual endpoint* of W we mean $\min\{i \mid 0 \le i \le D,\ E_iW \ne 0\}$. By the *dual diameter* of W we mean $|\{i \mid 0 \le i \le D,\ E_iW \ne 0\}| - 1$. It turns out that the diameter of W is equal to the dual diameter of W [30, Cor. 3.3]. By [32, Lemma 3.4], $\dim E_i^*W \le 1$ for $0 \le i \le D$ if and only if $\dim E_iW \le 1$ for $0 \le i \le D$; in this case, W is called *thin*.

We finish this section with two lemmas.

LEMMA 4.1 [32, Lemmas 3.4, 3.9, 3.12]. *Let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then ρ, τ, d are nonnegative integers such that $\rho + d \le D$ and $\tau + d \le D$. Moreover, the following statements hold*:

(i) $E_i^*W \ne 0$ *if and only if* $\rho \le i \le \rho + d$ $(0 \le i \le D)$;
(ii) $W = \sum_{h=0}^{d} E_{\rho+h}^*W$ (*orthogonal direct sum*);
(iii) $E_iW \ne 0$ *if and only if* $\tau \le i \le \tau + d$ $(0 \le i \le D)$;
(iv) $W = \sum_{h=0}^{d} E_{\tau+h}W$ (*orthogonal direct sum*).

LEMMA 4.2 [26, Lemma 12.1]. *For $Y \in \mathrm{Mat}_X(\mathbb{C})$, the following are equivalent*:

(i) $Y \in T$;
(ii) $YW \subseteq W$ *for all irreducible T-modules W.*

5. The Split Decomposition

We shall make use of a certain decomposition of V called the *split decomposition*. The split decomposition was defined in [37] and discussed further in [26; 28]. In this section we recall some results on this topic.

DEFINITION 5.1 [37, Def. 5.1]. For $-1 \le i, j \le D$ we define

$$V_{i,j}^{\downarrow\downarrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV),$$
$$V_{i,j}^{\downarrow\uparrow} = (E_0^*V + \cdots + E_i^*V) \cap (E_DV + \cdots + E_{D-j}V).$$

In these two equations we interpret the right-hand side to be 0 if $i = -1$ or $j = -1$.

DEFINITION 5.2 [37, Def. 5.5]. With reference to Definition 5.1, for $(\mu, \nu) = (\downarrow, \downarrow)$ or $(\mu, \nu) = (\downarrow, \uparrow)$ we have $V_{i-1,j}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$ and $V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}$. Therefore

$$V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu} \subseteq V_{i,j}^{\mu\nu}.$$

With reference to this inclusion, we define $\tilde{V}_{i,j}^{\mu\nu}$ to be the orthogonal complement of the left-hand side in the right-hand side; that is,

$$\tilde{V}_{i,j}^{\mu\nu} = (V_{i-1,j}^{\mu\nu} + V_{i,j-1}^{\mu\nu})^{\perp} \cap V_{i,j}^{\mu\nu}.$$

The next lemma is a mild generalization of [37, Cor. 5.8].

LEMMA 5.3. *With reference to Definition 5.2, the following holds for* $(\mu, \nu) = (\downarrow, \downarrow)$ *and* $(\mu, \nu) = (\downarrow, \uparrow)$:

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{i,j}^{\mu\nu} \quad (\textit{direct sum}). \tag{6}$$

Proof. For $(\mu, \nu) = (\downarrow, \downarrow)$, this is just [37, Cor. 5.8]. For $(\mu, \nu) = (\downarrow, \uparrow)$, in the proof of [37, Cor. 5.8] replace the sequence $\{E_i\}_{i=0}^{D}$ by $\{E_{D-i}\}_{i=0}^{D}$. □

NOTE 5.4. Following [28, Def. 6.4], we call the sum (6) the (μ, ν)-*split decomposition of* V.

We now recall how split decompositions are related to irreducible T-modules. We begin with a definition.

DEFINITION 5.5 [37, Def. 4.1]. Let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. By the *displacement of* W *of the first kind* we mean the scalar $\rho + \tau + d - D$. By the *displacement of* W *of the second kind* we mean the scalar $\rho - \tau$. By the inequalities in Lemma 4.1, each kind of displacement is at least $-D$ and at most D.

LEMMA 5.6 [37, Thm. 6.2]. *For* $-D \le \delta \le D$, *the following coincide*:

(i) *the subspace of* V *spanned by the irreducible* T*-modules for which* δ *is the displacement of the first kind*; *and*

(ii) $\sum \tilde{V}_{ij}^{\downarrow\downarrow}$, *where the sum is over all ordered pairs* i, j $(0 \le i, j \le D)$ *such that* $i + j = \delta + D$.

LEMMA 5.7. *For* $-D \le \delta \le D$, *the following coincide*:

(i) *the subspace of* V *spanned by the irreducible* T*-modules for which* δ *is the displacement of the second kind*; *and*

(ii) $\sum \tilde{V}_{ij}^{\downarrow\uparrow}$, *where the sum is over all ordered pairs* i, j $(0 \le i, j \le D)$ *such that* $i + j = \delta + D$.

Proof. In the proof of [37, Thm. 6.2], replace the sequence $\{E_i\}_{i=0}^{D}$ with the sequence $\{E_{D-i}\}_{i=0}^{D}$. □

6. A Homomorphism $\vartheta : \boxtimes_q \to T$

We now impose an assumption on Γ.

ASSUMPTION 6.1. We fix complex scalars $q, \eta, \eta^*, u, u^*, v, v^*$ with q, u, u^*, v, v^* nonzero and $q^{2i} \neq 1$ for $1 \le i \le D$. We assume that Γ has a Q-polynomial structure with eigenvalue sequence

$$\theta_i = \eta + uq^{2i-D} + vq^{D-2i} \quad (0 \le i \le D)$$

and dual eigenvalue sequence

$$\theta_i^* = \eta^* + u^* q^{2i-D} + v^* q^{D-2i} \quad (0 \leq i \leq D).$$

Moreover, we assume that each irreducible T-module is thin.

REMARK 6.2. In the notation of Bannai and Ito [1, p. 263], the Q-polynomial structure from Assumption 6.1 is type I with $s \neq 0$ and $s^* \neq 0$. We caution the reader that the scalar denoted q in [1, p. 263] is the same as our scalar q^2.

EXAMPLE 6.3 [3]. The ordinary cycles are the only known distance-regular graphs that satisfy Assumption 6.1.

Under Assumption 6.1 we will display a $\mathbb{C}$-algebra homomorphism $\vartheta : \boxtimes_q \to T$. To describe this homomorphism we define two matrices in $\mathrm{Mat}_X(\mathbb{C})$, called Φ and Ψ.

DEFINITION 6.4. With reference to Lemma 5.3 and Assumption 6.1, let Φ (resp. Ψ) denote the unique matrix in $\mathrm{Mat}_X(\mathbb{C})$ that acts on $\tilde{V}_{ij}^{\downarrow\downarrow}$ (resp. $\tilde{V}_{ij}^{\downarrow\uparrow}$) as $q^{i+j-D}I$ for $0 \leq i, j \leq D$. Observe that each of Φ, Ψ is invertible.

LEMMA 6.5. *Under Assumption 6.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then Φ and Ψ act on W as $q^{\rho+\tau+d-D}I$ and $q^{\rho-\tau}I$, respectively.*

Proof. Concerning Φ, abbreviate $\delta = \rho + \tau + d - D$ and recall that this is the displacement of W of the first kind. We show that Φ acts on W as $q^\delta I$. Let V_δ denote the common subspace from parts (i) and (ii) of Lemma 5.6. By Lemma 5.6(i) we have $W \subseteq V_\delta$. In Lemma 5.6(ii), V_δ is expressed as a sum. The matrix Φ acts on each term of this sum as $q^\delta I$ by Definition 6.4, so Φ acts on V_δ as $q^\delta I$. By these comments, Φ acts on W as $q^\delta I$ and this proves our assertion concerning Φ. Our assertion concerning Ψ is similarly proved using the displacement of the second kind and Lemma 5.7. □

LEMMA 6.6. *Under Assumption 6.1, the matrices Φ and Ψ are central elements of T.*

Proof. The matrices Φ and Ψ are contained in T by Lemma 4.2 and Lemma 6.5. These matrices are central in T because, by Lemma 6.5, they act as a scalar multiple of the identity on every irreducible T-module. □

The following theorem is our main result.

THEOREM 6.7. *Under Assumption 6.1, there exists a $\mathbb{C}$-algebra homomorphism $\vartheta : \boxtimes_q \to T$ such that*

$$A = \eta I + u\Phi\Psi^{-1}\vartheta(x_{01}) + v\Psi\Phi^{-1}\vartheta(x_{12}) \quad \textit{and} \tag{7}$$

$$A^* = \eta^* I + u^*\Phi\Psi\vartheta(x_{23}) + v^*\Psi^{-1}\Phi^{-1}\vartheta(x_{30}). \tag{8}$$

The proof is an easy consequence of the following two lemmas.

LEMMA 6.8. *Under Assumption 6.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then there exists a $\boxtimes_q$-module structure on W such that the adjacency matrix A acts as $\eta I + uq^{2\tau+d-D}x_{01} + vq^{D-d-2\tau}x_{12}$ and the dual adjacency matrix A^* acts as $\eta^* I + u^* q^{2\rho+d-D}x_{23} + v^* q^{D-d-2\rho}x_{30}$. This $\boxtimes_q$-module structure is irreducible.*

Proof. By [22, Ex. 1.4] and since the T-module W is thin, the pair A, A^* acts on W as a Leonard pair in the sense of [35, Def. 1.1]. In the notation of [35, Def. 5.1], this Leonard pair has an eigenvalue sequence $\{\theta_{\tau+i}\}_{i=0}^d$ and a dual eigenvalue sequence $\{\theta^*_{\rho+i}\}_{i=0}^d$ in view of Lemma 4.1. To motivate what follows we note that

$$\theta_{\tau+i} = \eta + uq^{2\tau+d-D}q^{2i-d} + vq^{D-d-2\tau}q^{d-2i} \quad \text{and}$$
$$\theta^*_{\rho+i} = \eta^* + u^*q^{2\rho+d-D}q^{2i-d} + v^*q^{D-d-2\rho}q^{d-2i}$$

for $0 \le i \le d$. In both of these equations, the coefficients of q^{2i-d} and q^{d-2i} are nonzero; hence the action of A, A^* on W is a Leonard pair of q-Racah type in the sense of [36, Ex. 5.3]. Referring to this Leonard pair, let $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denote the first (resp. second) split sequence [35, Sec. 7] associated with the eigenvalue sequence $\{\theta_{\tau+i}\}_{i=0}^d$ and the dual eigenvalue sequence $\{\theta^*_{\rho+i}\}_{i=0}^d$. By [35, Sec. 7], each of φ_i, ϕ_i is nonzero for $1 \le i \le d$. By [36, Ex. 5.3], there exists a nonzero $r \in \mathbb{C}$ such that, for $1 \le i \le d$:

$$\begin{aligned}\varphi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\\ &\quad \times (q^{d-i} - r^{-1}q^{i-1})(uu^*rq^{2\tau+2\rho+d+i-2D} - vv^*q^{2D-2d-2\tau-2\rho+1-i});\\ \phi_i &= (q^i - q^{-i})(q^{d-i+1} - q^{i-d-1})\\ &\quad \times (urq^{2\tau+d-D+1-i} - vq^{D-2d-2\tau+i})(u^*q^{2\rho+d-D+i-1} - v^*r^{-1}q^{D-2\rho-i}).\end{aligned}$$

Observe that r is not among $q^{d-1}, q^{d-3}, \dots, q^{1-d}$ because each of $\varphi_1, \varphi_2, \dots, \varphi_d$ is nonzero. By [35, Sec. 7] there exists a basis $\{v_i\}_{i=0}^d$ of W such that

$$Av_i = \theta_{\tau+d-i}v_i + v_{i+1} \quad (0 \le i \le d-1), \qquad Av_d = \theta_\tau v_d,$$
$$A^*v_i = \theta^*_{\rho+i}v_i + \phi_i v_{i-1} \quad (1 \le i \le d), \qquad A^*v_0 = \theta^*_\rho v_0.$$

For convenience we adjust this basis slightly. For $1 \le i \le d$ define

$$\gamma_i = (q^i - q^{-i})(urq^{2\tau+d-D+1-i} - vq^{D-2d-2\tau+i}).$$

In this equation the right-hand side is nonzero because it is a factor of ϕ_i, so $\gamma_i \ne 0$. Define $u_i = (\gamma_1\gamma_2\cdots\gamma_i)^{-1}v_i$ for $0 \le i \le d$ and note that $\{u_i\}_{i=0}^d$ is a basis for W. By the construction, we have

$$Au_i = \theta_{\tau+d-i}u_i + \gamma_{i+1}u_{i+1} \quad (0 \le i \le d-1), \qquad Au_d = \theta_\tau u_d,$$
$$A^*u_i = \theta^*_{\rho+i}u_i + \phi_i\gamma_i^{-1}u_{i-1} \quad (1 \le i \le d), \qquad A^*u_0 = \theta^*_\rho u_0.$$

We let each standard generator of $\boxtimes_q$ act linearly on W; to define this action, we specify what it does to the basis $\{u_i\}_{i=0}^d$. Here are the details:

$$x_{01}.u_i = q^{d-2i}u_i + (q^d - q^{d-2i-2})q^{1-d}ru_{i+1} \quad (0 \le i \le d-1), \qquad x_{01}.u_d = q^{-d}u_d;$$
$$x_{12}.u_i = q^{2i-d}u_i + (q^{-d} - q^{2i+2-d})u_{i+1} \quad (0 \le i \le d-1), \qquad x_{12}.u_d = q^d u_d;$$
$$x_{23}.u_i = q^{2i-d}u_i + (q^d - q^{2i-2-d})u_{i-1} \quad (1 \le i \le d), \qquad x_{23}.u_0 = q^{-d}u_0;$$
$$x_{30}.u_i = q^{d-2i}u_i + (q^{-d} - q^{d-2i+2})q^{d-1}r^{-1}u_{i-1} \quad (1 \le i \le d), \qquad x_{30}.u_0 = q^d u_0;$$
$$x_{13}.u_i = q^{2i-d}u_i \quad (0 \le i \le d);$$
$$x_{31}.u_i = q^{d-2i}u_i \quad (0 \le i \le d);$$
$$\begin{aligned} x_{02}.u_i = {} & (1-rq^{-d-1})\frac{(1-q^{2d-2i+2})(1-q^{2d-2i+4})\cdots(1-q^{2d})q^{d-2i}}{(1-rq^{d-1-2i})(1-rq^{d+1-2i})\cdots(1-rq^{d-1})}u_0 \\ & + (1-rq^{d+1})(1-rq^{-d-1}) \\ & \times \sum_{h=1}^{i} \frac{(1-q^{2d-2i+2})(1-q^{2d-2i+4})\cdots(1-q^{2d-2h})q^{d-2i}}{(1-rq^{d-1-2i})(1-rq^{d+1-2i})\cdots(1-rq^{d+1-2h})}u_h \\ & + \frac{(q^{2i+2}-1)r}{q^{2i+1}(1-rq^{d-1-2i})}u_{i+1} \quad (0 \le i \le d-1); \end{aligned}$$
$$\begin{aligned} x_{02}.u_d = {} & \frac{(1-q^2)(1-q^4)\cdots(1-q^{2d})q^{-d}}{(1-rq^{1-d})(1-rq^{3-d})\cdots(1-rq^{d-1})}u_0 \\ & + (1-rq^{d+1})\sum_{h=1}^{d} \frac{(1-q^2)(1-q^4)\cdots(1-q^{2d-2h})q^{-d}}{(1-rq^{1-d})(1-rq^{3-d})\cdots(1-rq^{d+1-2h})}u_h; \end{aligned}$$
$$\begin{aligned} x_{20}.u_0 = {} & (1-rq^{d+1})\sum_{h=0}^{d-1} \frac{(1-q^2)(1-q^4)\cdots(1-q^{2h})r^h q^{h-dh-d}}{(1-rq^{1-d})(1-rq^{3-d})\cdots(1-rq^{2h-d+1})}u_h \\ & + \frac{(1-q^2)(1-q^4)\cdots(1-q^{2d})r^d q^{-d^2}}{(1-rq^{1-d})(1-rq^{3-d})\cdots(1-rq^{d-1})}u_d; \end{aligned}$$
$$\begin{aligned} x_{20}.u_i = {} & \frac{q^d - q^{2i-2-d}}{1-rq^{2i-d-1}}u_{i-1} + (1-rq^{d+1})(1-rq^{-d-1}) \\ & \times \sum_{h=i}^{d-1} \frac{(1-q^{2i+2})(1-q^{2i+4})\cdots(1-q^{2h})r^{h-i}q^{(d+1)i-(d-1)h-d}}{(1-rq^{2i-d-1})(1-rq^{2i-d+1})\cdots(1-rq^{2h-d+1})}u_h \\ & + (1-rq^{-d-1}) \\ & \times \frac{(1-q^{2i+2})(1-q^{2i+4})\cdots(1-q^{2d})r^{d-i}q^{di+i-d^2}}{(1-rq^{2i-d-1})(1-rq^{2i-d+1})\cdots(1-rq^{d-1})}u_d \quad (1 \le i \le d). \end{aligned}$$

In the preceding formulas, the denominators are nonzero because r is not among $q^{d-1}, q^{d-3}, \ldots, q^{1-d}$. One may check (or see [27]) that the actions just described satisfy the defining relations for $\boxtimes_q$ from Definition 2.1, so these actions induce a $\boxtimes_q$-module structure on W. Comparing the action of A (resp. A^*) on $\{u_i\}_{i=0}^d$ with the actions of x_{01}, x_{12} (resp. x_{23}, x_{30}) on $\{u_i\}_{i=0}^d$, we find that

$$A = \eta I + uq^{2\tau+d-D}x_{01} + vq^{D-d-2\tau}x_{12} \quad \text{and}$$
$$A^* = \eta^* I + u^* q^{2\rho+d-D}x_{23} + v^* q^{D-d-2\rho}x_{30}$$

on W. By these equations and since the T-module W is irreducible, we find that the $\boxtimes_q$-module W is irreducible. The result follows. □

LEMMA 6.9. *Under Assumption 6.1, let W denote an irreducible T-module and consider the $\boxtimes_q$-action on W from Lemma 6.8. Then the following equations hold on W:*

$$A = \eta I + u\Phi\Psi^{-1}x_{01} + v\Psi\Phi^{-1}x_{12};$$

$$A^* = \eta^* I + u^*\Phi\Psi x_{23} + v^*\Psi^{-1}\Phi^{-1}x_{30}.$$

Proof. Combine Lemma 6.5 and Lemma 6.8. □

Proof of Theorem 6.7. We start with a comment. Let W and W' denote irreducible T-modules, and consider the $\boxtimes_q$-module structure on W and W' from Lemma 6.8. From the construction we may assume that if the T-modules W and W' are isomorphic then the $\boxtimes_q$-modules W and W' are isomorphic. With that comment out of the way, we proceed to the main argument. The standard module V decomposes into a direct sum of irreducible T-modules. By Lemma 6.8, each irreducible T-module in this decomposition supports a $\boxtimes_q$-module structure. Combining these $\boxtimes_q$-modules yields a $\boxtimes_q$-module structure on V. This module structure induces a $\mathbb{C}$-algebra homomorphism $\vartheta : \boxtimes_q \to \mathrm{Mat}_X(\mathbb{C})$. The map ϑ satisfies (7) and (8) in view of Lemma 6.9. To finish the proof it suffices to show that $\vartheta(\boxtimes_q) \subseteq T$. Toward this end we pick $\zeta \in \boxtimes_q$ and show $\vartheta(\zeta) \in T$. Since T is semisimple (and by our preliminary comment) there exists a $B \in T$ that acts on each irreducible T-module in the preceding decomposition as $\vartheta(\zeta)$. The T-modules in this decomposition span V, so $\vartheta(\zeta)$ coincides with B on V; hence $\vartheta(\zeta) = B$ and, in particular, $\vartheta(\zeta) \in T$ as desired. We have now shown that $\vartheta(\boxtimes_q) \subseteq T$, and the result follows. □

REMARK 6.10. In Theorem 6.7 we obtained a $\mathbb{C}$-algebra homomorphism ϑ: $\boxtimes_q \to T$. In Theorem 3.3 we displayed four $\mathbb{C}$-algebra homomorphisms from $U_q(\widehat{\mathfrak{sl}}_2)$ into $\boxtimes_q$. Composing these homomorphisms with ϑ yields four $\mathbb{C}$-algebra homomorphisms from $U_q(\widehat{\mathfrak{sl}}_2)$ into T.

We conjecture that the conclusion of Theorem 6.7 still holds if we weaken Assumption 6.1 by no longer requiring that each irreducible T-module be thin.

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