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Bounds on Subsets of Coherent Configurations

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Dedicated to D. G. Higman

1. Introduction

Coherent configurations were introduced by D. G. Higman, initially in a 1970 paper [5] and then in a pair of papers [6; 7] that developed the theory. The definition was based on combinatorial properties of the orbitals of a group acting on a finite set, with the intention that the structure would be useful both in group theory and in combinatorics. Some later work (see e.g. [8; 9]) focused on combinatorial aspects, and this paper is in that spirit.

Coherent configurations are a generalization of association schemes, and much of the theory carries over or can be modified. One association scheme idea that has not been considered is Delsarte's theory of subsets. Schrijver [10] has found new bounds on codes by considering subsets of the Terwilliger algebra of the Hamming scheme. Essentially, the bounds use a subset of the coherent configuration whose fibers are the weight of the words. This motivated our investigation of subsets of general coherent configurations.

We begin by defining coherent configurations. We use the notation of [8]; see that paper for more details than are given here.

Let X be a finite set, and let $\{f_i\}_{i\in I}$ be a set of relations on X partitioning $X \times X$ so that:

(1) $f_i \cap \operatorname{diag}(X \times X) \neq \emptyset$ implies $f_i \subseteq \operatorname{diag}(X \times X)$;
(2) given $i \in I$, $(f_i)^t = f_{i^*}$ for some $i^* \in I$, where $(f_i)^t = \{(y,x) : (x,y) \in f_i\}$;
(3) given $(x,y) \in f_k$, $|\{z : (x,z) \in f_i, (z,y) \in f_j\}|$ is a constant p_{ij}^k depending only on i, j, and k.

Then $\mathcal{C} = (X,(f_i)_{i\in I})$ is said to be a *coherent configuration*.

If instead of (1) and (2) we have

(1′) $f_i = \operatorname{diag}(X \times X)$ for some $i \in I$ and
(2′) $(f_i)^t = f_i$ for all $i \in I$,

then $(X,(f_i)_{i\in I})$ is a (symmetric) association scheme.

In an association scheme, the identity is one of the relations defining the scheme, whereas in a coherent configuration the identity is a union of relations. Let $\Omega = \{\alpha \in I : f_\alpha \subseteq \operatorname{diag}(X \times X)\}$. The relations $\{f_\alpha\}_{\alpha\in\Omega}$ determine a partition of X into *fibers* X_α with $f_\alpha = \operatorname{diag}(X_\alpha \times X_\alpha)$. For each $i \in I$, we have $f_i \subseteq X_\alpha \times X_\beta$ for some $\alpha, \beta \in \Omega$.

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Let $n = |X|$ and $m_i = |f_i|$. Each relation f_i defines a $(0,1)$-matrix $A_i \in M_n(\mathbb{C})$. These matrices satisfy equations

$$A_i A_j = \sum_{k \in I} p_{ij}^k A_k,$$

and hence form a basis for an $|I|$-dimensional algebra $\mathcal{A}$ over $\mathbb{C}$, called the *adjacency algebra* of the configuration. The algebra $\mathcal{A}$ is closed both under usual matrix multiplication and under Hadamard (entrywise) multiplication.

Since $\mathcal{A}$ is semisimple, it follows that $\mathcal{A} = \sum_{s=1}^m B_s$, a direct sum of simple two-sided ideals; here $B_s \simeq M_{e_s}(\mathbb{C})$ for some positive integer e_s. Let $\Delta_1, \ldots, \Delta_m$ be the inequivalent absolutely irreducible representations of $\mathcal{A}$ with $\Delta_s \colon \mathcal{A} \to M_{e_s}(\mathbb{C})$. We choose the representations so that $\Delta_s(A^*) = (\Delta_s(A))^*$ for all $A \in \mathcal{A}$, where $*$ denotes (as usual) the conjugate transpose operator. Then $\Delta_s(A_{i^*}) = \Delta_s(A_i)^*$.

Let z_s be the multiplicity of Δ_s in the standard module. One way of looking at the representations is that there exists a unitary matrix U such that, for all $A \in \mathcal{A}$,

$$U^* A U = \operatorname{diag}(\Delta_1(A), \ldots, \Delta_m(A)), \tag{1.1}$$

a block diagonal matrix with $\Delta_s(A)$ repeated z_s times.

The representations determine a basis $\mathcal{E}_{ij}^s$ of $\mathcal{A}$ such that $\mathcal{E}_{ij}^s \mathcal{E}_{kl}^t = \delta_{jk}\delta_{st}\mathcal{E}_{il}^s$ and $\Delta_t(\mathcal{E}_{ij}^s) = \delta_{st} E_{ij}^s$, where E_{ij}^s is the $e_s \times e_s$ matrix with (i, j)-entry 1 and all other entries 0. There are linear functionals a_{ij}^s such that, for $A \in \mathcal{A}$, we have $\Delta_s(A) = \sum_{i,j} a_{ij}^s(A) E_{ij}^s$. To simplify notation, choose a set L of symbols so that $\{a_\lambda\}_{\lambda \in L}$ are these linear functionals. If $a_\lambda = a_{ij}^s$, define $\bar{\lambda}$ to be the element of L such that $a_{\bar{\lambda}} = a_{ji}^s$. Let $\mathcal{E}_\lambda = \mathcal{E}_{ij}^s$ and $z_\lambda = z_s$. Note that $\mathcal{E}_{\bar{\lambda}} = \mathcal{E}_{ji}^s = (\mathcal{E}_{ij}^s)^* = \mathcal{E}_\lambda^*$ and $z_{\bar{\lambda}} = z_\lambda$.

We will require the following results from [8, Sec. 7]:

$$A_i = \sum_\lambda a_\lambda(A_i)\mathcal{E}_\lambda, \tag{1.2}$$

$$\mathcal{E}_\lambda = z_\lambda \sum_{i \in I} \frac{1}{m_i} \overline{a_\lambda(A_i)} A_i, \tag{1.3}$$

$$\sum_{i \in I} \frac{1}{m_i} a_\lambda(A_i)\overline{a_\mu(A_i)} = \frac{1}{z_\lambda}\delta_{\lambda\mu}. \tag{1.4}$$

2. Bounds on Subsets

Throughout this section, let $\mathcal{C} = (X, (f_i)_{i \in I})$ be a coherent configuration with parameters and matrices denoted as in Section 1. We wish to investigate subsets $Y \subseteq X$.

The adjacency algebra $\mathcal{A}$ acts on $V = \mathbb{C}^X$ via matrix multiplication. Let $\{\mathbf{e}_x\}_{x \in X}$ be the standard basis for V. For $Y \subseteq X$, let $\mathbf{y}$ be the characteristic vector $\sum_{x \in Y} \mathbf{e}_x$. Let $b_i = |\{Y \times Y \cap f_i\}|$. Then clearly $b_i = b_{i^*}$ and $b_i = \mathbf{y}^* A_i \mathbf{y} = \mathbf{y}^* A_{i^*}\mathbf{y}$.

Let

$$D(Y) = \sum_{i \in I} \frac{b_i}{m_i} A_i = \sum_{i \in I} \frac{\mathbf{y}^* A_{i^*} \mathbf{y}}{m_i} A_i.$$

This matrix is analogous to the outer distribution for a subset of an association scheme, and we will refer to it as the *distribution matrix* for the set Y.

Our main result (Theorem 2.3) is that $D(Y)$ is positive semidefinite. The proof is somewhat indirect. It begins by expressing the matrix in terms of the second basis $\{\mathcal{E}_\lambda\}$ and then considers its image under the irreducible representations.

LEMMA 2.1.

$$D(Y) = \sum_\lambda \frac{\mathbf{y}^*\mathcal{E}_{\bar\lambda}\mathbf{y}}{z_\lambda}\mathcal{E}_\lambda.$$

Proof.

$$\begin{aligned}
D(Y) &= \sum_{i\in I} \frac{\mathbf{y}^* A_{i^*}\mathbf{y}}{m_i} A_i \\
&= \sum_i \frac{1}{m_i}\mathbf{y}^*\Bigl(\sum_\lambda a_\lambda(A_{i^*})\mathcal{E}_\lambda\Bigr)\mathbf{y}\Bigl(\sum_\mu a_\mu(A_i)\mathcal{E}_\mu\Bigr) && \text{by (1.2)} \\
&= \sum_i \sum_{\lambda,\mu} \frac{1}{m_i}(\mathbf{y}^*\mathcal{E}_\lambda\mathbf{y})\overline{a_{\bar\lambda}(A_i)}a_\mu(A_i)\mathcal{E}_\mu \\
&= \sum_{\lambda,\mu} \mathbf{y}^*\mathcal{E}_\lambda\mathbf{y}\Bigl(\sum_i \frac{1}{m_i}\overline{a_{\bar\lambda}(A_i)}a_\mu(A_i)\Bigr)\mathcal{E}_\mu \\
&= \sum_{\lambda,\mu} \mathbf{y}^*\mathcal{E}\mathbf{y}\Bigl(\delta_{\bar\lambda\mu}\frac{1}{z_\mu}\Bigr)\mathcal{E}_\mu && \text{by (1.4)} \\
&= \sum_\lambda \frac{1}{z_{\bar\lambda}}\mathbf{y}^*\mathcal{E}_\lambda\mathbf{y}\mathcal{E}_{\bar\lambda} = \sum_\lambda \frac{1}{z_\lambda}\mathbf{y}^*\mathcal{E}_{\bar\lambda}\mathbf{y}\mathcal{E}_\lambda.
\end{aligned}$$

□

THEOREM 2.2. *The matrix $\Delta_s(D(Y))$ is positive semidefinite.*

Proof. Let $M_s = \Delta_s(D(Y))$.

By Lemma 2.1,

$$\begin{aligned}
M_s &= \Delta_s\Bigl(\sum_\lambda \frac{\mathbf{y}^*\mathcal{E}_{\bar\lambda}\mathbf{y}}{z_\lambda}\mathcal{E}_\lambda\Bigr) \\
&= \sum_\lambda \frac{1}{z_\lambda}\mathbf{y}^*\mathcal{E}_{\bar\lambda}\mathbf{y}\Delta_s(\mathcal{E}_\lambda).
\end{aligned}$$

Observe that $\Delta_s(\mathcal{E}_\lambda)$ is nonzero only when $\mathcal{E}_\lambda = \mathcal{E}^s_{ij}$, in which case $\mathcal{E}_{\bar\lambda} = \mathcal{E}^s_{ji}$ and $z_\lambda = z_s$. Therefore,

$$M_s = \frac{1}{z_s}\sum_{i,j}\mathbf{y}^*\mathcal{E}^s_{ji}\mathbf{y}E^s_{ij}.$$

Now

$$M_s^* = \frac{1}{z_s}\sum_{i,j}(\mathbf{y}^*\mathcal{E}^s_{ji}\mathbf{y})^*(E^s_{ij})^* = \frac{1}{z_s}\sum_{i,j}\mathbf{y}^*\mathcal{E}^s_{ij}\mathbf{y}E^s_{ji} = M_s,$$

so M_s is hermitian.

Let $\mathbf{x} = (x_i)$ be any vector in $\mathbb{C}^{e_s}$. We will show that $\mathbf{x}^*(z_s M_s)\mathbf{x} \geq 0$.

Consider the matrix $T = \sum_{i=1}^{e_s} \overline{x_i}\mathcal{E}_{i1}^s$. Then TT^* is positive semidefinite, and

$$TT^* = \left(\sum_{i=1}^{e_s} \overline{x_i}\mathcal{E}_{i1}^s\right)\left(\sum_{j=1}^{e_s} \overline{x_j}\mathcal{E}_{j1}^s\right)^* = \sum_{i,j} \overline{x_i}\mathcal{E}_{i1}^s x_j \mathcal{E}_{1j}^s = \sum_{i,j} \overline{x_i} x_j \mathcal{E}_{ij}^s.$$

Now

$$\begin{aligned}\mathbf{x}^*(z_s M_s)\mathbf{x} &= \mathbf{x}^*\left(\sum_{i,j} \mathbf{y}^*\mathcal{E}_{ij}^s\mathbf{y}E_{ij}^s\right)\mathbf{x} \\ &= \sum_{i,j} \mathbf{y}^*\mathcal{E}_{ij}^s\mathbf{y}\mathbf{x}^*E_{ij}^s\mathbf{x} \\ &= \sum_{i,j} \overline{x_i}x_j\mathbf{y}^*\mathcal{E}_{ij}^s\mathbf{y} \\ &= \mathbf{y}^*TT^*\mathbf{y} \geq 0.\end{aligned}$$

Therefore, M_s is positive semidefinite. □

THEOREM 2.3. *The distribution matrix $D(Y)$ is positive semidefinite.*

Proof. This follows directly from Theorem 2.2 and (1.1). □

If $\mathcal{C}$ is an association scheme, then $nD(Y)$ is the distribution matrix for Y (in the sense of [3, Sec. 12.6]). The positive semidefiniteness of $nD(Y)$ implies Delsarte's inequality $\mathbf{a}Q \geq 0$, where Q is the second eigenmatrix and $\mathbf{a}$ is the inner distribution of Y.

3. Connection to Groups and Association Schemes

Let $(X, (f_i)_{i\in I})$ be an association scheme with automorphism group G that is transitive on X and on each relation. For any subset Y of X, let $\mathbf{y}$ be the characteristic vector of Y and $\mathbf{y}^g$ the characteristic vector of Y^g. Then $\sum_{g\in G} \mathbf{y}^g(\mathbf{y}^g)^*$ is a positive semidefinite matrix that is in $\mathcal{A}$. This implies Delsarte's inequality for such schemes (and other methods prove it in general). Schrijver's innovation in [10] was to sum over a subgroup H instead and to show that the resulting matrix is in the Terwilliger algebra of the Hamming scheme. We will show that Theorem 2.3 is an extension of this idea to general coherent configurations.

In order to use this idea, we need to have a group that acts on $\mathcal{C}$. The simplest way to do this is to have a configuration that is defined by the group. Take a group G that acts faithfully on a finite set X, and let $\{f_i\}_{i\in I}$ be the orbitals of G on X. Then $(X, (f_i)_{i\in I})$ is a coherent configuration, and such configurations are the examples motivating Higman's definition.

Let $Y \subset X$ with $\mathbf{y}$ and $\mathbf{y}^g$ as before. In this situation, the relevant matrix is $\sum_{g\in G} \mathbf{y}^g(\mathbf{y}^g)^*$, which is clearly positive semidefinite. The (u, w)-entry is the number of $g \in G$ with $(u^{g^{-1}}, w^{g^{-1}}) \in Y \times Y$, and this is constant on the orbitals; hence this matrix is in $\mathcal{A}$. The question is what to do when we have no group. As the next theorem shows, the matrix to use is $D(Y)$, and in this sense Theorem 2.3 is a generalization.

THEOREM 3.1.
$$\sum_{g\in G} \mathbf{y}^g(\mathbf{y}^g)^* = |G| \sum_{i\in I} \frac{b_i}{m_i} A_i.$$

Proof. Let $M = \sum_{g\in G} \mathbf{y}^g(\mathbf{y}^g)^*$. Since $M \in \mathcal{A}$, we can write $M = \sum_{i\in I} c_i A_i$ for some scalars c_i. Then $M \circ A_i = c_i A_i$.

Let τ be the linear function such that $\tau(A)$ is equal to the sum of the entries of the matrix A. Then
$$\tau(c_i A_i) = c_i \tau(A_i) = c_i m_i = \tau(M \circ A_i) = \sum_{g\in G} \tau(\mathbf{y}^g(\mathbf{y}^g)^* \circ A_i).$$

Since $\mathbf{y}^g(\mathbf{y}^g)^*$ is a symmetric matrix, $\tau(\mathbf{y}^g(\mathbf{y}^g)^* \circ A_i) = \mathrm{tr}(\mathbf{y}^g(\mathbf{y}^g)^* A_i)$, where "tr" denotes the trace function. Let P_g be the permutation matrix corresponding to the action of $g \in G$. Then
$$\begin{aligned}\tau(\mathbf{y}^g(\mathbf{y}^g)^* \circ A_i) &= \mathrm{tr}((P_g\mathbf{y})(P_g\mathbf{y})^* A_i)\\ &= \mathrm{tr}((P_g\mathbf{y}\mathbf{y}^*P_g^*)P_g A_i P_g^*) \quad \text{since } g \text{ fixes } f_i\\ &= \mathrm{tr}(\mathbf{y}\mathbf{y}^*A_i) = \mathrm{tr}(\mathbf{y}^*A_i\mathbf{y})\\ &= b_i.\end{aligned}$$

Therefore, $c_i = (b_i/m_i)|G|$. □

4. Application to Partial Geometries

A *partial geometry* with parameters s, t, α, which we denote as pG(s, t, α), is an incidence structure $(\mathcal{P}, \mathcal{B})$ of points and lines such that the following conditions hold:

(1) every point lies on $t + 1$ lines;
(2) every line contains $s + 1$ points;
(3) two points lie on at most one line; and
(4) if a point x is not incident with a line B, then there are exactly α points on B collinear with x.

Partial geometries have been extensively studied (see e.g. [1]); we use the standard results about these structures. In particular, note that collinearity of points defines a strongly regular graph on $\mathcal{P}$ called the *point graph*. Similarly, defining lines to be adjacent if they intersect gives a strongly regular graph, the *line graph*.

Given a pG(s, t, α), let $X = \mathcal{P} \cup \mathcal{B}$ and define relations on X as follows:

$$f_1 = \mathrm{diag}(\mathcal{P} \times \mathcal{P});$$
$$f_2 = \{(x, y) : x \text{ and } y \text{ are collinear points}\};$$
$$f_3 = \{(x, y) : x \text{ and } y \text{ are not collinear}\};$$
$$f_4 = \mathrm{diag}(\mathcal{B} \times \mathcal{B});$$

Table 1 Irreducible representations

	Δ_1	Δ_2	Δ_3	Δ_4
A_1	E_{11}	E_{11}	1	0
A_2	$s(t+1)E_{11}$	$(s-\alpha)E_{11}$	$-t-1$	0
A_3	$\frac{st(s+1-\alpha)}{\alpha}E_{11}$	$(-s-1+\alpha)E_{11}$	t	0
A_4	E_{22}	E_{22}	0	1
A_5	$t(s+1)E_{22}$	$(t-\alpha)E_{22}$	0	$-s-1$
A_6	$\frac{st(t+1-\alpha)}{\alpha}E_{22}$	$(-t-1+\alpha)E_{22}$	0	s
A_7	$\sqrt{(s+1)(t+1)}E_{12}$	$\sqrt{s+t+1-\alpha}E_{12}$	0	0
A_8	$\frac{st\sqrt{(s+1)(t+1)}}{\alpha}E_{12}$	$-\sqrt{s+t+1-\alpha}E_{12}$	0	0
A_9	$\sqrt{(s+1)(t+1)}E_{21}$	$\sqrt{s+t+1-\alpha}E_{21}$	0	0
A_{10}	$\frac{st\sqrt{(s+1)(t+1)}}{\alpha}E_{21}$	$-\sqrt{s+t+1-\alpha}E_{21}$	0	0

$$f_5 = \{(B, C) : B \text{ and } C \text{ are lines that intersect}\};$$
$$f_6 = \{(B, C) : B \text{ and } C \text{ are disjoint lines}\};$$
$$f_7 = f_9^t = \{(x, B) : \text{point } x \text{ lies on line } B\};$$
$$f_8 = f_{10}^t = \{(x, B) : x \text{ does not lie on } B\}.$$

Then $(X, (f_i)_{i=1}^{10})$ is a coherent configuration with two fibers $\mathcal{P}$ and $\mathcal{B}$, and it is included in the configurations studied by Higman in [9]. In fact, partial geometries are the strongly regular designs with $a_1 = a_2 = 1$ and $b_1 = b_2 = 0$ (in the terminology of that paper).

For the rest of the section, assume that $(\mathcal{P}, \mathcal{B})$ is a pG(s, t, α) with coherent configuration $\mathcal{C} = (X, (f_i)_{i=1}^{10})$ defined as before. With this configuration, we have:

$$m_1 = |\mathcal{P}| = (s+1)(st+\alpha)/\alpha;$$
$$m_2 = s(s+1)(t+1)(st+\alpha)/\alpha;$$
$$m_3 = st(s+1)(st+\alpha)(s+1-\alpha)/\alpha^2;$$
$$m_4 = |\mathcal{B}| = (t+1)(st+\alpha)/\alpha;$$
$$m_5 = t(s+1)(t+1)(st+\alpha)/\alpha;$$
$$m_6 = st(t+1)(st+\alpha)(t+1-\alpha)/\alpha^2;$$
$$m_7 = m_9 = (s+1)(t+1)(st+\alpha))/\alpha;$$
$$m_8 = m_{10} = st(s+1)(t+1)(st+\alpha)/\alpha^2.$$

Table 1 gives the irreducible representations for $\mathcal{C}$.

Suppose $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{B}' \subseteq \mathcal{B}$ are subsets such that no point of $\mathcal{P}'$ lies on any line of $\mathcal{B}'$. Let $Y = \mathcal{P}' \cup \mathcal{B}'$ and $b_i = |Y \times Y \cap f_i|$. Let $p = |\mathcal{P}'|$ and $q = |\mathcal{B}'|$. Then $b_1 = p$, $b_4 = q$, $b_7 = b_9 = 0$, and $b_8 = b_{10} = pq$.

THEOREM 4.1. *In the situation just described,*

$$(st(t+1)p + t(s-\alpha)b_2 - (t+1)\alpha b_3)(st(s+1)q + s(t-\alpha)b_5 - (s+1)\alpha b_6) \geq (s+t+1-\alpha)\alpha^2 p^2 q^2.$$

Proof. By Theorem 2.2, the matrix

$$\Delta_2(D(Y)) = \begin{bmatrix} \dfrac{1}{|\mathcal{P}|}\left(p + \dfrac{(s-\alpha)b_2}{s(t+1)} - \dfrac{\alpha b_3}{st}\right) & \dfrac{-\sqrt{s+t+1-\alpha}}{|\mathcal{B}|(t+1)}\left(\dfrac{\alpha pq}{st}\right) \\ \dfrac{-\sqrt{s+t+1-\alpha}}{|\mathcal{P}|(s+1)}\left(\dfrac{\alpha pq}{st}\right) & \dfrac{1}{|\mathcal{B}|}\left(q + \dfrac{(t-\alpha)b_5}{t(s+1)} - \dfrac{\alpha b_6}{st}\right) \end{bmatrix}$$

is positive semidefinite and so has determinant ≥ 0. This implies that

$$\left(p + \frac{(s-\alpha)b_2}{s(t+1)} - \frac{\alpha b_3}{st}\right)\left(q + \frac{(t-\alpha)b_5}{t(s+1)} - \frac{\alpha b_6}{st}\right) \geq \frac{(s+t+1-\alpha)\alpha^2 p^2 q^2}{s^2 t^2 (s+1)(t+1)}, \quad (4.1)$$

and the result follows. □

Haemers [4, Chap. 3] used interlacing to prove an inequality for sets of nonincident points and lines in a partial geometry. His inequality involves only p and q and so doesn't include the same kind of information about the structure of the sets.

We consider further the special case where $\mathcal{P}'$ and $\mathcal{B}'$ are co-cliques in the point and line graphs.

COROLLARY 4.2. *Suppose that $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{B}' \subseteq \mathcal{B}$ are such that no two points of $\mathcal{P}'$ are collinear, no two lines of $\mathcal{B}'$ intersect, and no point of $\mathcal{P}'$ lies on any line of $\mathcal{B}'$. Let $p = |\mathcal{P}'|$ and $q = |\mathcal{B}'|$. Then*

$$((s+1)(t+1) - \alpha p)((s+1)(t+1) - \alpha q) \geq (s+1)(t+1)(s+t+1-\alpha). \quad (4.2)$$

Proof. As before, let $Y = \mathcal{P}' \cup \mathcal{B}'$. The conditions on $\mathcal{P}'$ and $\mathcal{B}'$ imply that $b_2 = 0$, $b_3 = p^2 - p$, $b_5 = 0$, and $b_6 = q^2 - q$.

Since $\Delta_2(D(Y))$ is positive semidefinite, the diagonal entries are nonnegative. This implies

$$p\left(\frac{st + \alpha - \alpha p}{\alpha}\right) \geq 0 \quad \text{and} \quad q\left(\frac{st + \alpha - \alpha q}{\alpha}\right) \geq 0;$$

hence

$$p \leq \frac{st + \alpha}{\alpha} \quad \text{and} \quad q \leq \frac{st + \alpha}{\alpha}. \quad (4.3)$$

(Inequalities (4.3) are standard results that are easily shown combinatorially from the assumptions on $\mathcal{P}'$ and $\mathcal{B}'$.) If $p = 0$ or $q = 0$, then (4.2) reduces to one of the two inequalities in (4.3).

Now assume that $p \neq 0$ and $q \neq 0$. By (4.1),

$$pq\left(\frac{st+\alpha-\alpha p}{st}\right)\left(\frac{st+\alpha-\alpha q}{st}\right) \geq \frac{(s+t+1-\alpha)\alpha^2 p^2 q^2}{s^2t^2(s+1)(t+1)}$$

and so

$$\alpha^2 pq - (s+1)(t+1)\alpha(p+q) + (s+1)^2(t+1)^2 \\ \geq (s+1)^2(t+1)^2 - (s+1)(t+1)(st+\alpha).$$

Inequality (4.2) follows from this. □

The bound is tight, as the following example shows.

There is a unique pG$(2,2,1)$ (up to isomorphism) that has fifteen points and fifteen lines. Let $\mathcal{P} = \{1, 2, \ldots, 15\}$, and take as lines the 3-subsets

$$\{1,2,3\},\ \{3,4,5\},\ \{5,6,7\},\ \{7,8,9\},\ \{9,10,1\},$$
$$\{1,6,13\},\ \{2,7,14\},\ \{3,8,15\},\ \{4,9,11\},\ \{5,10,12\},$$
$$\{2,11,12\},\ \{4,13,14\},\ \{6,11,15\},\ \{8,12,13\},\ \{10,14,15\}.$$

This partial geometry is a generalized quadrangle, and by Corollary 4.2 we have $(9-p)(9-q) \geq 36$. The maximum value of p satisfying this is $p = 5$ (which also follows from (4.3)). If $p = 5$, the only possibility is that $q = 0$ and then (4.2) holds with equality. Such a set $\mathcal{P}'$ is an ovoid in the generalized quadrangle. Ovoids exist; for example, let $\mathcal{P}' = \{1, 4, 7, 12, 15\}$. Similarly, $q \leq 5$. If $q = 5$, then $p = 0$ and the set $\mathcal{B}'$ is a spread; such sets exist.

The other values giving equality in (4.2) are $p = 3$ and $q = 3$. Letting $\mathcal{P}' = \{8, 10, 11\}$ and $\mathcal{B}' = \{\{1, 6, 13\}, \{2, 7, 14\}, \{3, 4, 5\}\}$ gives an example.

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