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On Problems Concerning the Bruhat Decomposition and Structure Constants of Hecke Algebras of Finite Chevalley Groups

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This paper is dedicated to the memory of Donald G. Higman

1. Introduction

Let G be a finite Chevalley group over a finite field $k = F_q$ of characteristic p (as in [15] or [3]). Let B be a Borel subgroup of G with $U = O_p(B)$ (the unipotent radical of B), and let T be a maximal torus such that $B = UT$. Let $W = N_G(T)/T$ be the Weyl group of G. Then W is a finite Coxeter group with distinguished generators $S = \{s_1, \dots, s_n\}$.

Let Φ be the root system associated with W, with $\{\alpha_1, \dots, \alpha_n\}$ the set of simple roots corresponding to the generators $s_i \in S$, and let Φ_+ (resp. Φ_-) be the set of positive (resp. negative) roots associated with them.

By the Bruhat decomposition, the (U, U)-double cosets are parameterized by the elements of $N = N_G(T)$ and the (B, B)-double cosets are parameterized by the elements of W. The main result is a description of the set

$$B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$$

of representatives of the left B-cosets in the intersection

$$BwB \cap yBx^{-1}B,$$

for elements $\dot{w}, \dot{x}, \dot{y}$ in N corresponding to elements w, x, y in W, by an algorithm based on a reduced expression of w in terms of the generators $s_1, \dots, s_n$ of W. Its cardinality was shown by Iwahori [12] to be the structure constant

$$[e_w e_x : e_y]$$

for standard basis elements e_w, e_x, e_y of the Iwahori Hecke algebra $\mathcal{H}(G, B)$. A formula for the structure constants $[e_w e_x : e_y]$ was proved by Kawanaka [13] and is stated as follows:

$$[e_w e_x : e_y] = \sum_{\tau} q^{a(\tau)}(q-1)^{b(\tau)},$$

where the sum is taken over a set of subexpressions τ of a fixed reduced expression of w. The subexpressions are called *K-sequences for* (w, x, y) in what follows and were first defined in Kawanaka's paper [13]. The nonnegative integers $a(\tau)$ and

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$b(\tau)$ are defined by properties of the subexpressions. The description of the sets $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ can be viewed as a geometric interpretation of Kawanaka's formula. (We use the notation U_w for $U_{\dot{w}} = U \cap \dot{w}B_-\dot{w}^{-1}$ for a representative $\dot{w} \in N$ of $w \in W$ and B_- for the Borel subgroup opposite to B; then $B\dot{w}B = U_w\dot{w}B$ with uniqueness of expression.)

In more detail, a nonempty set $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ can be identified with the set of triples (u, b, v), with $u \in U_w$, $b \in B$, and $v \in U_{x^{-1}}$, that satisfy the *structure equation*

$$u\dot{w}b = \dot{y}v\dot{x}^{-1}.$$

For such a triple (u, b, v) the elements b and v are uniquely determined by $u \in U_w$, so it is natural to focus attention on the sets

$$U(w, x, y) = \{u \in U_w : u\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1} \neq \emptyset\}.$$

The first part of the algorithm in Section 2 shows that each set $U(w, x, y)$ can be expressed as the disjoint union of nonempty subsets U_τ parameterized by K-sequences for (w, x, y) and constructed from the root subgroups of G (see [4] for a previous version of this result.) The second part of the algorithm gives all solutions of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ with $u \in U_\tau$.

The sets $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ are of interest not only in the case of finite Chevalley groups but also for Chevalley groups $\mathbf{G}$ over the algebraically closed field $\bar{k}$. These are connected semisimple algebraic groups over $\bar{k}$ [15, Sec. 5] with a Frobenius endomorphism F such that the group $\mathbf{G}^F$ of fixed points under the action of F is a finite Chevalley group G as considered earlier. The algorithms in Section 2 also apply to this situation. We may assume that the representatives $\dot{w}, \dot{x}, \dots$ belong to $\mathbf{G}^F$. Then the set of solutions (u, b, v) of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ corresponding to $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ and the sets corresponding to $U(w, x, y)$ and U_τ can be viewed as F-stable locally closed subvarieties of $\mathbf{G}$.

The proof of the algorithm is an improved and extended version of the method used in [4] to calculate $U(w, x, y)$.

A *Gelfand–Graev representation* γ of G is an induced representation ψ^G for a linear representation ψ of U in general position; that is, $\psi \,|\, U_{\alpha_i} \neq 1$ for each simple root subgroup U_{α_i}, $1 \leq i \leq n$. Let

$$e = |U|^{-1} \sum_{u \in U} \psi(u^{-1})u$$

be the primitive idempotent affording ψ in the group algebra $\mathbb{C}U$ of U over the field of complex numbers. Then ψ^G is afforded by the left $\mathbb{C}G$-module $\mathbb{C}Ge$. The *Hecke algebra* of γ is the subalgebra $\mathcal{H} = e\mathbb{C}Ge$ of $\mathbb{C}G$ and is isomorphic to $(\mathrm{End}_{\mathbb{C}}\, \mathbb{C}Ge)^\circ$. It is known that $\mathcal{H}$ is a commutative algebra, so that a Gelfand–Graev representation γ is multiplicity-free.

A basis for $\mathcal{H}$ is given by the nonzero elements of the form ene, $n \in N$. The standard basis elements are the nonzero elements of the form $c_n = \mathrm{ind}(n)ene$, where $\mathrm{ind}(n) = |U : nUn^{-1} \cap U|$; they are parameterized by the set $\mathcal{N}$ consisting of those elements $n \in N$ such that $ene \neq 0$. The structure constants $[c_\ell c_m : c_n]$ in the formulas

$$c_\ell c_m = \sum_n [c_\ell c_m : c_n] c_n, \quad \ell, m, n \in \mathcal{N},$$

are algebraic integers.

We mention two methods for computing the structure constants $[c_\ell c_m : c_n]$, $\ell, m, n \in \mathcal{N}$. The first was used by Chang [2] to compute some of the structure constants for the Hecke algebra of the Gelfand–Graev representation of $\mathrm{GL}_3(k)$. One computes $\ell U m \cap U n U$ for $\ell, m \in \mathcal{N}$ and all $n \in N$. This gives $\ell e m$ as a linear combination of certain elements unu' with $u, u' \in U$ and $n \in N$. Then multiply on the left and right by e and use the facts that $ue = \psi(u)e = eu$ for all elements $u \in U$ and $ene = 0$ for $n \notin \mathcal{N}$. The result is a formula for $e\ell e \cdot eme$ as a linear combination of the elements ene, $n \in \mathcal{N}$.

The second method uses [7, Prop. 11.30] to obtain

$$[c_\ell c_m : c_n] = |U| \sum_{y \in U\ell U \cap nUm^{-1}U} a_\ell(y) a_m(y^{-1}n)$$

(or 0 if the intersection $U\ell U \cap nUm^{-1}U = \emptyset$). In this formula,

$$a_\ell(y) = |U|^{-1}\psi(u_1^{-1})\psi(u_2^{-1}) \quad \text{and} \quad y = u_1 \ell u_2 \in U\ell U \quad \text{for } u_i \in U,$$

and $a_n(y) = 0$ if $y \notin UnU$.

Let $U_n = U \cap nB_- n^{-1}$ for $n \in N$. The set $U\ell U \cap nU_{m^{-1}}m^{-1}$ is a set of representatives for the left U-cosets in the intersection of one (U, U)-double coset with the translate of another—namely, $U\ell U \cap nUm^{-1}U$. From the preceding formula, the structure constants are determined if the sets $U\ell U \cap nU_{m^{-1}}m^{-1}$ are known.

The sets $U\ell U \cap nU_{m^{-1}}m^{-1}$ and the structure constants $[c_\ell c_m : c_n]$ are calculated in Section 3 (Corollary 3.2) using the result obtained in Section 2 concerning the sets

$$B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}.$$

In Section 4 we explain connections between the description of $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ and Deodhar's results [10] on the sets $BwB \cap B_- xB$ for elements x and w in W such that $x \le w$ with respect to the Bruhat order on the Coxeter group W. A set of representatives of the left B-cosets in $BwB \cap B_- xB$ is given by the set $B\dot{w}B \cap \dot{w}_0 U_{w_0 x}(w_0 x)^{\cdot}$ whose cardinality is the structure constant $[e_w e_{(w_0 x)^{-1}} : e_{w_0}]$ in $\mathcal{H}(G, B)$ and is given by Kawanaka's theorem as $\sum_\tau q^{a(\tau)}(q-1)^{b(\tau)}$, where w_0 is the element of maximal length in W and the sum is taken over K-sequences for $(w, (w_0 x)^{-1}, w_0)$. In [10], Deodhar proved that the cardinality of the set $B\dot{w}B \cap \dot{w}_0 U_{w_0 x}(w_0 x)^{\cdot}$ is given by the sum $\sum_\sigma q^{m(\sigma)}(q-1)^{n(\sigma)}$ over another set of subexpressions σ of a reduced expression of w and that this cardinality is equal to the Kazhdan–Lusztig polynomial $R_{x,w}(q)$. The main point of Section 4 is to show that the two kinds of subexpressions of w coincide and that the integers $a(\tau)$ and $b(\tau)$ coincide with the integers $m(\sigma)$ and $n(\sigma)$ associated with the subexpression $\sigma = \tau$. The result is a formula for the Kazhdan–Lusztig polynomial $R_{x,w}(q)$ as a sum over K-sequences for $(w, (w_0 x)^{-1}, w_0)$.

The values of the irreducible representations of the commutative semisimple algebra $\mathcal{H}$ on standard basis elements are obtained as eigenvalues of matrices giving the regular representation of $\mathcal{H}$ and whose entries are the structure constants

$[c_\ell c_m : c_n]$. This idea was first applied by Frobenius ([11]; see also [6, Chap. II]) to define the characters of a finite group in terms of the eigenvalues of matrices affording the regular representation of the center of the group algebra. The entries of these matrices are structure constants of the center of the group algebra with respect to the basis consisting of class sums.

The following result recalls well-known facts about the irreducible representations of the commutative semisimple algebra $\mathcal{H}$ referred to in the preceding paragraph.

PROPOSITION 1.1. *Let $\{c_n\}$ be the standard basis elements of $\mathcal{H}$, and let A_ℓ be the matrix of left multiplication by c_ℓ with respect to the standard basis $\{c_n, n \in \mathcal{N}\}$ so that $A_\ell = ([c_\ell c_m : c_n])$. The primitive idempotents $\varepsilon = \varepsilon_f$ in $\mathcal{H}$ form a basis of $\mathcal{H}$ such that each idempotent ε corresponds to a unique irreducible representation f of $\mathcal{H}$ and to an irreducible component of the Gelfand–Graev representation γ. Moreover, each idempotent ε_f is an eigenvector for each matrix A_ℓ such that the corresponding eigenvalues are the values $f(c_\ell)$ of the irreducible representation f at the basis element c_ℓ.*

The values of the irreducible representations $f(c_\ell)$ and the primitive idempotents ε_f were calculated in [5] (see also [8] and Section 3) using the virtual repesentations $R_{\mathbf{T},\theta}$ defined by Deligne and Lusztig [9]. It would be interesting to investigate whether these formulas can be related to the eigenvalues of the matrices A_ℓ. A problem involving the structure constants and the irreducible representations of $\mathcal{H}$, with an example for a Gelfand–Graev representation of $\mathrm{SL}_2(k)$, is discussed at the end of Section 3.

2. Calculation of $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$

Let G denote a finite Chevalley group over k as in Section 1.

For each root α, there is a homomorphism (see [15, p. 46]) $\varphi = \varphi_\alpha : \mathrm{SL}_2(k) \to G$ such that φ takes

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \to x_\alpha(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \to x_{-\alpha}(t),$$

$$\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \to w_\alpha(t) \in N, \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to h_\alpha(t) \in T$$

for all $t \in k$. The elements $w_\alpha(t)$ and $h_\alpha(t)$ are given by

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \quad \text{and} \quad h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$$

[15, p. 30]. By [15, Lemma 83, p. 242], if $w = s_k \cdots s_1$ is a reduced expression of an element $w \in W$ then $\dot{w} = \dot{s}_k \cdots \dot{s}_1$, with $\dot{s}_i = w_{\alpha_i}(t_i)$ for some fixed choice of $t_i \in k - \{0\}$, is a representative in N of w that is independent of the choice of the reduced expression chosen. In what follows we assume that representatives $\dot{x}$ of all elements $x \in W$ have been chosen in this way for a fixed choice of representatives $\dot{s}_i$ of the generators $s_i \in S$.

We also need a consequence of the Bruhat decomposition for SL_2 and an application of the homomorphism $\varphi_{\alpha_i} : \mathrm{SL}_2 \to G$ described previously. Hereafter we shall use the notation X^* for the set of elements $x \neq 1$ in a group X.

LEMMA 2.1. *Let $s_i \in S$ correspond to the simple root α_i and assume, as usual, that $\dot{s}_i = w_{\alpha_i}(t_i)$ for $t_i \in k$. Then there exist bijections $f_i, g_i : U_{\alpha_i}^* \to U_{\alpha_i}^*$ such that, for each $u \in U_{\alpha_i}^*$, there exist $t_i(u) \in T$ such that*

$$\dot{s}_i^{-1} u \dot{s}_i = f_i(u) \dot{s}_i t_i(u) g_i(u).$$

As in [10], a *subexpression* τ of a fixed reduced expression $w = s_k \cdots s_1$ is a sequence $\tau = (\tau_k, \ldots, \tau_1, \tau_0)$ of elements of W such that $\tau_i \tau_{i-1}^{-1} \in \{1, s_i\}$ for $i = 1, \ldots, k$ and $\tau_0 = 1$. Then the set of terminal elements τ_k of subexpressions of $w = s_k \cdots s_1$ coincides with the set of elements $x \in W$ such that $x \leq w$ in the Chevalley–Bruhat order (see [10]). A subexpression $\tau = (\tau_k, \ldots, \tau_1, \tau_0)$ is called a *K-sequence relative to the triple* $w = s_k \cdots s_1, x, y$ of elements of W if it satisfies the conditions (2.10)(a–c) of [13] (see also [1, (3.19)], where the conditions are stated in a different way). It is understood that a K-sequence for the triple (w, x, y) is always given with reference to a fixed reduced expression $w = s_k \cdots s_1$. Let $J_\tau = \{ j : \tau_j \tau_{j-1}^{-1} = s_j \} \cup \{0\}$. Then the defining conditions for a K-sequence state that $\tau_k x = y$ and

$$\ell(s_p \tau_j x) < \ell(\tau_j x)$$

for each $j \in J_\tau$ and p in the interval between j and the next element in J_τ (or simply all $p > j$ if j is the maximal element of J_τ). For each K-sequence τ, set

$$J_\tau^- = \{ j \in J_\tau : \ell(s_j \tau_{j'} x) < \ell(\tau_{j'} x) \},$$

where $j' \in J_\tau$ is the predecessor of j, and define a pair of nonnegative integers by

$$a(\tau) = |J_\tau^-|, \qquad b(\tau) = k - |J_\tau| + 1 = \mathrm{card}\{ j > 0 : \tau_j \tau_{j-1}^{-1} = 1 \}.$$

We can now state a more precise version of Kawanaka's theorem [13, Lemma 2.14b]. For a finite Chevalley group G over $k = F_q$, let e_w, e_x, e_y be standard basis elements of the Iwahori Hecke algebra $\mathcal{H}(G, B)$. Then

$$[e_w e_x : e_y] = |B \dot{w} B \cap \dot{y} U_{x^{-1}} \dot{x}^{-1}| = \sum_\tau q^{a(\tau)} (q-1)^{b(\tau)},$$

where the sum is taken over all K-sequences τ for w, x, y and where $a(\tau)$ and $b(\tau)$ are the nonnegative integers defined previously.

Fix w, x, y in W, and let $w = s_k \cdots s_1$, $s_i \in S$, be a reduced expression. Let $\dot{x}, \dot{y}, \dot{w}$ be chosen as in the preceding paragraph. We shall decompose $B \dot{w} B \cap \dot{y} U_{x^{-1}} \dot{x}^{-1}$, assuming it is nonempty, as a union of nonempty subsets parameterized by the K-sequences for (w, x, y) corresponding to the decomposition of $U(w, x, y)$ obtained in [4].

We first require some preliminary results, which provide an inductive construction of K-sequences and a matching inductive construction of the sets $U(w, x, y) = \{ u \in U_w : u \dot{w} B \cap \dot{y} U_{x^{-1}} \dot{x}^{-1} \neq \emptyset \}$. The sets $U(w, x, y)$ are invariant under conjugation by elements of T. The sets $U(w, x, y)$ are also independent of the choice of representatives $\dot{w}, \dot{x}, \dot{y}$ of w, x, y, but the sets $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ are not. Nevertheless, the sets $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ are obtained by the algorithm given in this section.

LEMMA 2.2. *Let $w = s_k \cdots s_1$ be a fixed reduced expression of w for $k \geq 1$, and let $\tau' = (\tau_{k-1}, \dots, \tau_1, \tau_0)$ be a K-sequence for $s_k w, x', y'$ with $x', y' \in W$. (In case $\ell(w) = 1$ it is understood that $\tau' = \tau_0$ is a K-sequence for $1, x', x'$.) The K-sequences $\tau = (\tau_k, \tau_{k-1}, \dots, \tau_1, \tau_0) = (\tau_k, \tau')$ for w, x, y (obtained from τ' by adding τ_k), and the integers $a(\tau)$ and $b(\tau)$ associated with them, are as follows.*

(i) *$\tau_k\tau_{k-1}^{-1} = s_k$, $x = x'$, $y = s_k y'$, and $\ell(s_k y') > \ell(y')$. In this case $a(\tau) = a(\tau')$ and $b(\tau) = b(\tau')$.*

(ii) *$\tau_k\tau_{k-1}^{-1} = s_k$, $x = x'$, $y = s_k y'$, and $\ell(s_k y') < \ell(y')$. In this case $a(\tau) = a(\tau') + 1$ and $b(\tau) = b(\tau')$.*

(iii) *$\tau_k\tau_{k-1}^{-1} = 1$, $x = x'$, $y = y'$, and $\ell(s_k y') < \ell(y')$. In this case $a(\tau) = a(\tau')$ and $b(\tau) = b(\tau') + 1$.*

The proof is included in Kawanaka's proof of Lemma 2.14b in [13].

In the discussion to follow, we use the fact that $U_w = U_{\alpha_k}\dot{s}_k U_{s_k w}\dot{s}_k^{-1}$ with uniqueness, so that each element $u \in U_w$ has the form $u = u_0\dot{s}_k u_1 \dot{s}_k^{-1}$ for uniquely determined elements $u_0 \in U_{\alpha_k}$ and $u_1 \in U_{s_k w}$. We also require that for each element $w \in W$ there is an isomorphism of subgroups $U = U_w U_{ww_0} \cong U_w \times U_{ww_0}$, where w_0 is the unique element of maximal length in W.

LEMMA 2.3. *Let $w = s_k \cdots s_1$ be a fixed reduced expression of w for $k \geq 1$, and let $x, y \in W$. Then $U(w, x, y)$ is either empty or is obtained from the nonempty sets $U(s_k w, x', y')$ for elements $x', y' \in W$ as follows.*

(i) *Let $\ell(s_k y) < \ell(y)$, $x' = x$, and $y' = s_k y$. Then*
$$U(w, x, y) = \dot{s}_k U(s_k w, x, s_k y)\dot{s}_k^{-1}.$$

(ii) *Let $\ell(s_k y) > \ell(y)$, $x' = x$, and $y' = s_k y$. Then there is a bijection of sets*
$$U(w, x, y) = U_{\alpha_k}\dot{s}_k U(s_k w, x, s_k y)\dot{s}_k^{-1} \cong U_{\alpha_k} \times U(s_k w, x, s_k y).$$

(iii) *Let $\ell(s_k y) < \ell(y)$, $x' = x$, and $y' = y$. Then $U(w, x, y)$ consists of the elements $u = u_0\dot{s}_k u_1\dot{s}_k^{-1} \in U_w$, with $u_0 \in U^*_{\alpha_k}$ and $u_1 \in U_{s_k w}$, such that $\pi(g_k(u_0)u_1) \in U(s_k w, x, y)$, $g_k(u_0)$ is as in Lemma 2.1, and π is the projection $\pi : U \to U_{s_k w}$ accompanying the decomposition $U = U_{s_k w} \times U_{s_k w w_0}$. The map $u = u_0\dot{s}_k u_1\dot{s}_k^{-1} \to \pi(g_k(u_0)u_1)$ from $U(w, x, y) \to U(s_k w, x, y)$ is surjective. There is a bijection of sets,*
$$U(w, x, y) \cong U^*_{\alpha_k} \times U(s_k w, x, y).$$

For the proofs of (i) and (ii), we refer to [4, pp. 40, 42]. We begin the proof of (iii) by observing that, for all $u_0 \in U^*_{\alpha_k}$ and $v \in U_{s_k w}$, we have $\pi(u_0\pi(u_0^{-1}v)) = v$. This follows from the decomposition $U = U_{s_k w} \times U_{s_k w w_0}$ and is the beginning of the proof that the map $u = u_0\dot{s}_k u_1\dot{s}_k^{-1} \to \pi(g_k(u_0)u_1)$ from $U(w, x, y) \to U(s_k w, x, y)$ is surjective.

Now let $u' \in U(s_k w, x, y)$ and let
$$u'\dot{s}_k^{-1}\dot{w}b' = \dot{y}v'\dot{x}^{-1}$$

with $b' \in B$ and $v' \in U_{x^{-1}}$. By the previous remark and the fact that g_k is a bijection, there exist $u_0 \in U^*_{\alpha_k}$, $u_1 \in U_{s_k w}$, and $u = u_0 \dot{s}_k u_1 \dot{s}_k^{-1} \in U_w$ such that $\pi(g_k(u_0)u_1) = u'$. We aim to prove that $u \in U(w, x, y)$. For later use we shall not only prove that $u \in U(w, x, y)$ but also will obtain all solutions of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ from the solutions (u', b', v'), assumed known, of the structure equation $u'\dot{s}_k^{-1}\dot{w}b' = \dot{y}v'\dot{x}^{-1}$.

We have $g_k(u_0)u_1 = u'u''$ with $u'' \in U_{s_k w w_0}$ determined by the factorization $U = U_{s_k w} U_{s_k w w_0}$. Then the structure equation $u'\dot{s}_k^{-1}\dot{w}b' = \dot{y}v'\dot{x}^{-1}$ becomes $g_k(u_0)u_1\dot{s}_k^{-1}\dot{w}b'' = \dot{y}v'\dot{x}^{-1}$ with $b'' = (\dot{s}_k^{-1}\dot{w})^{-1}(u'')^{-1}\dot{s}_k^{-1}\dot{w}b' \in B$. Put $z = t_k(u_0)^{-1}\dot{s}_k^{-1} f_k(u_0)^{-1}\dot{s}_k^{-1}$; then, solving for $g_k(u_0)$, by Lemma 2.1 the structure equation becomes

$$u_0 \dot{s}_k u_1 \dot{s}_k^{-1}\dot{w}b'' = \dot{y}\dot{y}^{-1}z^{-1}\dot{y}v'\dot{x}^{-1},$$

with $\dot{y}^{-1}z^{-1}\dot{y} = \tilde{u}t$ for uniquely determined elements $\tilde{u} \in U$ and $t \in T$ because $\ell(s_k y) < \ell(y)$ and so $\dot{y}^{-1}(z^{-1})\dot{y} \in B$. We then have $tv'\dot{x}^{-1} = tv't^{-1}\dot{x}^{-1}\dot{x}t\dot{x}^{-1} \in U_{x^{-1}}\dot{x}^{-1}t'$ for $t' \in T$. If $\tilde{u} \in U_{x^{-1}}$ then the required solutions of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ with $u \in U(w, x, y)$ are $(u = u_0\dot{s}_k u_1 \dot{s}_k^{-1}, b''t'^{-1}, \tilde{u}tv't^{-1})$. If $\tilde{u} \notin U_{x^{-1}}$ then the situation is handled as follows. One has $\tilde{u}tv't^{-1} = v_1 v_1'$ with $v_1 \in U_{x^{-1}}$ and $v_1' \in U_{x^{-1}w_0}$. Then v_1' can be conjugated past $\dot{x}^{-1}$ and absorbed in the contribution to the solution of the equation in B, since $\dot{x}v_1'\dot{x}^{-1} \in B$. This completes, among other things, the proof that $u \in U(w, x, y)$.

Conversely, one must prove that, if $u = u_0\dot{s}_k u_1 \dot{s}_k^{-1} \in U(w, x, y)$ as before, then $\pi(g_k(u_0)u_1) \in U(s_k w, x, y)$. We have

$$u_0\dot{s}_k u_1 \dot{s}_k^{-1}\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1} \neq \emptyset.$$

Multiply the left-hand side by $\dot{s}_k\dot{s}_k^{-1}$ and apply Lemma 2.1 to $\dot{s}_k^{-1}u_0\dot{s}_k$. Using the assumption that $\ell(s_k y) < \ell(y)$, it follows by reasoning similar to that used previously that

$$g_k(u_0)u_1\dot{s}_k^{-1}\dot{w}B \cap \dot{y}B\dot{x}^{-1} \neq \emptyset;$$

further use of the argument in the preceding paragraph yields $\pi(g_k(u_0)u_1) \in U(s_k w, x, y)$, completing the proof.

From the first statement in part (iii) it follows that there is a bijection

$$U(w, x, y) \cong Y = \{(u_0, u_1) \in U^*_{\alpha_k} \times U_{s_k w} : \pi(u_0 u_1) \in U(s_k w, x, y)\}$$

given by $u_0\dot{s}_k u_1\dot{s}_k^{-1} \to (g_k(u_0), u_1)$ for $u_0\dot{s}_k u_1\dot{s}_k^{-1} \in U(w, x, y)$. One then shows that the map

$$\xi : Y \to U^*_{\alpha_k} \times U(s_k w, x, y)$$

given by $\xi(u_0, u_1) = (u_0, \pi(u_0 u_1))$ is surjective, as follows. Let $u_0 \in U^*_{\alpha_k}$ and $v \in U(s_k w, x, y)$; then

$$\xi(u_0, \pi(u_0^{-1}v)) = (u_0, \pi(u_0\pi(u_0^{-1}v))) = (u_0, v).$$

Finally, define the map

$$\eta : U^*_{\alpha_k} \times U(s_k w, x, y) \to Y$$

by setting $\eta(u_0, v) \to (u_0, \pi(u_0^{-1}v))$; then it is easily verified, using the factorization $U = U_w U_{w w_0} = U_{w w_0} U_w$ and the projection π, that ξ and η are inverses of each other. This completes the proof of the last statement in Lemma 2.3—namely, that there is a bijection

$$U(w,x,y) \cong U^*_{\alpha_k} \times U(s_k w,x,y).$$

Let w,x,y be arbitrary elements of W such that $U(w,x,y) \neq \emptyset$. We shall describe, by induction on $\ell(w)$ for a fixed reduced expression $w = s_k \cdots s_1$ or $s_0 = 1$, a family of nonempty T-invariant subsets U_τ of $U(w,x,y)$ corresponding to K-sequences τ for (w,x,y) such that $U(w,x,y)$ is the disjoint union

$$U(w,x,y) = \bigcup_\tau U_\tau.$$

First assume $w = s_0 = 1$. Then there is exactly one K-sequence $\tau = \tau_0$ for $(1,x,y)$ in case $x = y$ and none if $x \neq y$. If $x = y$ then $U(1,x,x) = \{1\}$ because $B \cap \dot{x}U_{x^{-1}}\dot{x}^{-1} = 1$, since $\dot{x}U_{x^{-1}}\dot{x}^{-1} \subseteq U_-$ and $B \cap U_- = 1$. So we put $U_\tau = 1$ in this case.

Next let $w \neq 1$ and let $\tau = (\tau_k, \tau')$ be a K-sequence for (w,x,y) with τ' a K-sequence for $(s_k w, x', y')$, as in one of the three cases in Lemma 2.2. Assume as an induction hypothesis that $U_{\tau'}$ is the nonempty subset of $U(s_k w, x', y')$ corresponding to τ', and define $U_\tau \subseteq U(w,x,y)$ according to the three cases in Lemma 2.2 as follows.

In case (i) we have $\ell(s_k y) < \ell(y)$, $x = x'$, and $y = s_k y'$. In this case, put

$$U_\tau = \dot{s}_k U_{\tau'} \dot{s}_k^{-1},$$

a nonempty subset of $U(w,x,y)$ by Lemma 2.3(i).

In case (ii) we have $\ell(s_k y) > \ell(y)$, $x' = x$, and $y' = s_k y$. Then put

$$U_\tau = U_{\alpha_k} \dot{s}_k U_{\tau'} \dot{s}_k^{-1},$$

and check that U_τ is a nonempty subset of $U(w,x,y)$ (by part (ii) of Lemma 2.3) and that there is a bijection of sets $U_\tau \cong U_{\alpha_k} \times U_{\tau'}$. In case (iii) we have $\ell(s_k y) < \ell(y)$, $x' = x$, and $y' = y$, so we may assume that $U_{\tau'}$ is a nonempty subset of $U(s_k w, x, y)$. Let U_τ be the set of elements $u_0 \dot{s}_k u_1 \dot{s}_k^{-1} \in U_w$, with $u_0 \in U^*_{\alpha_k}$ and $u_1 \in U_{s_k w}$, such that $\pi(g_k(u_0)u_1) \in U_{\tau'}$. By the first statement in Lemma 2.3(iii), U_τ is a nonempty subset of $U(w,x,y)$. Moreover, by the second statement in Lemma 2.3(iii), one has the bijection of sets

$$U_\tau \cong U^*_{\alpha_k} \times U_{\tau'}.$$

At this point it follows—by the preceding discussion, Lemma 2.3, and the appropriate induction hypothesis for the sets $U(s_k w, x', y')$—that $U(w,x,y)$ is indeed the disjoint union of the sets U_τ corresponding to K-sequences for (w,x,y).

The elements $u \in U(w,x,y)$ are the first entries in solutions u,b,v of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$. Given a solution u,b,v of the structure equation, by Lemma 2.4 the other elements b,v and also the sets $B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ are uniquely determined if u is known.

LEMMA 2.4. *Let (u,b,v) be a solution of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ for $\dot{w},\dot{x},\dot{y}$. Then b and v are uniquely determined by u.*

Let $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ and $u\dot{w}b' = \dot{y}v'\dot{x}^{-1}$ for $u \in U_w$ with $b, b' \in B$ and $v, v' \in U_{x^{-1}}$. Then

$$(u\dot{w}b')^{-1}u\dot{w}b = (\dot{y}v'\dot{x}^{-1})^{-1}\dot{y}v\dot{x}^{-1}$$

and

$$b'^{-1}b = \dot{x}v'^{-1}v\dot{x}^{-1} \in B \cap U_- = 1.$$

Then $b = b'$ and $v = v'$, completing the proof of the lemma.

COROLLARY 2.1. *With notation as before, for all $w, x, y \in W$ one has that*

$$|U(w, x, y)| = |B\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}|$$

and is given by Kawanaka's formula $\sum_\tau q^{a(\tau)}(q-1)^{b(\tau)}$ in terms of K-sequences for w, x, and y.

The next step is the construction of all solutions (u, b, v) of the structure equation for $(\dot{w}, \dot{x}, \dot{y})$ using induction on $\ell(w)$, starting from a fixed reduced expression $w = s_k \cdots s_1$ of w. We have shown that each element $u \in U(w, x, y)$ belongs to U_τ for a unique K-sequence τ. For such elements u the solutions (u, b, v) of the structure equation are obtained by the algorithm to follow. In case $w = 1$ there is just one K-sequence $\tau = \tau_0$ for $(1, x, y)$ with $x = y$ and $U_\tau = \{1\}$ and none if $x \neq y$, so $U(1, x, x) = U_\tau = \{1\}$.

Now let $\ell(w) = 1$, so $\dot{w} = \dot{s}_1$, and let $\tau = (\tau_1, \tau_0)$ be a K-sequence for (s_1, x, y) corresponding to one of the three cases in Lemma 2.2, starting from the K-sequence τ_0 for $(1, x', y')$ with $x' = y'$. For case (i), $\tau_1 = s_1, x = x', y = s_1y'$, and $\ell(s_1y') > \ell(y')$. Then, by the previous discussion, $U_\tau = \dot{s}_1U_{\tau_0}\dot{s}_1^{-1} = \{1\}$ and the unique solution of the structure equation $u\dot{s}_1b = \dot{s}_1\dot{y}'v(\dot{y}')^{-1}$ with $u = 1$ is $(1, 1, 1)$. For case (ii), $\tau_1 = s_1$, $x = x'$, $y = s_1y'$, and $\ell(s_1y') < \ell(y')$. In this case, $U_\tau = U_{\alpha_1}$ and the solutions of the structure equation $u\dot{s}_1^{-1}b = \dot{y}v\dot{x}^{-1}$ for $u \in U_{\alpha_1}$ are

$$(u, 1, \dot{y}^{-1}u\dot{y}),$$

noting that $\dot{y}^{-1}u\dot{y} \in U_{x^{-1}}$. For case (iii), $\tau_1 = \tau_0$, $x = x'$, $y = y' = x'$, and $\ell(s_1y) < \ell(y)$. In this case,

$$U_w = U_{s_1} = U_{\alpha_1} \quad \text{and} \quad U_\tau = \{u_0 \in U^*_{\alpha_1} : \pi(g_1(u_0)) \in U_{\tau_0} = 1\} = U^*_{\alpha_1}.$$

First assume $\ell(x) = 1$. Then the assumptions imply that $x = s_1$, $\dot{x} = \dot{y} = \dot{s}_1$, and there is a unique solution of the structure equation $u_0\dot{s}_1b = \dot{s}_1v\dot{s}_1^{-1}$ using the homomorphism $\varphi_{\alpha_1}\colon \mathrm{SL}_2(k) \to G$ and the Bruhat decomposition in $\mathrm{SL}_2(k)$. Now let $\ell(x) > 1$; then $\ell(s_1x) < \ell(x)$ implies $\dot{x} = \dot{s}_1\dot{x}_1$ with $\ell(s_1x_1) > \ell(x_1)$. For each $v \in U^*_{\alpha_1}$ one has $\dot{x}_1^{-1}v\dot{x}_1 \in U$ because $\ell(s_1x_1) > \ell(x_1)$. Moreover,

$$\dot{s}_1\dot{x}_1(\dot{x}_1^{-1}v\dot{x}_1)\dot{x}_1^{-1}\dot{s}_1^{-1} = \dot{s}_1v\dot{s}_1^{-1} \in U_-$$

and so $\dot{x}_1^{-1}v\dot{x}_1 \in U_{x^{-1}}$. The unique solution of the structure equation $u_0\dot{s}_1b = \dot{s}_1v\dot{s}_1^{-1}$ with $u_0 \in U^*_{\alpha_1}$ from the case $\ell(x) = 1$ now yields the unique solution $(u, b, \dot{x}_1^{-1}v\dot{x}_1)$ of the structure equation for (s_1, x, y); that is,

$$u_0\dot{s}_1b = \dot{y}\dot{x}_1^{-1}v\dot{x}_1\dot{x}^{-1}$$

since $\dot{y}\dot{x}_1^{-1} = \dot{s}_1$ and $\dot{x}_1\dot{x}^{-1} = \dot{s}_1^{-1}$. Note that in case (iii) there is no solution to the structure equation $u\dot{s}_1 b = \dot{y}v\dot{x}^{-1}$ in case $u = 1$ and $x = y$, since this would contradict the fact that $B\dot{y}^{-1}\dot{s}_1 B \neq B\dot{x}^{-1}B$ by the uniqueness part of the Bruhat decomposition. This completes the discussion of the solutions of the structure equation for the case $\ell(w) = 1$.

We shall now prove the inductive step. Assume $\ell(w) > 1$, and let τ be a K-sequence for w, x, y such that $\tau = (\tau_k, \tau')$ with τ' a K-sequence for $(s_k w, x', y')$ corresponding to the three cases of Lemmas 2.2 and 2.3. We further assume that U_τ is a nonempty subset of $U(w, x, y)$ obtained from $U_{\tau'} \subseteq U(s_k w, x', y')$, as in the discussion following Lemma 2.3. We consider the cases (i)–(iii) separately, and in each case we obtain the solutions of the structure equation for each element of U_τ starting from solutions of the structure equation for $(s_k w, x', y')$.

(i) $\tau_k\tau_{k-1}^{-1} = s_k$, $x = x'$, $y = s_k y'$, and $\ell(s_k y') > \ell(y')$. Then $U_\tau = \dot{s}_k U_{\tau'}\dot{s}_k^{-1}$. Consider as an induction hypothesis that the solutions of the structure equation $u'\dot{s}_k^{-1}\dot{w}b' = \dot{y}'v'\dot{x}'^{-1}$ with $u' \in U_{\tau'}$, $b' \in B$, and $v' \in U_{x'^{-1}}$ are known. For each such solution (u', b', v'), the equation

$$\dot{s}_k u'\dot{s}_k^{-1}\dot{w}b' = \dot{s}_k\dot{y}'v'(\dot{x}')^{-1} = \dot{y}v'\dot{x}^{-1}$$

clearly gives all solutions (u, b, v) of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ for $u \in U_\tau$.

(ii) $\tau_k\tau_{k-1}^{-1} = s_k$, $x = x'$, $y = s_k y'$, and $\ell(s_k y') < \ell(y')$. Then $U_\tau = U_{\alpha_k}\dot{s}_k U_{\tau'}\dot{s}_k^{-1}$, and the induction hypothesis in this case gives the solutions (u', b', v') of the structure equations $u'\dot{s}_k\dot{w}b' = \dot{y}'v'\dot{x}'^{-1}$ for each $u' \in U_{\tau'}$. Then $\dot{y} = \dot{s}_k^{-1}\dot{y}'$ and the equations become $\dot{s}_k^{-1}u'\dot{s}_k\dot{w}b' = \dot{y}v'\dot{x}'^{-1}$. Multiply each such equation by $u_0 \in U_{\alpha_k}$; then

$$u_0\dot{s}_k^{-1}u'\dot{s}_k\dot{w}b' = \dot{y}\dot{y}^{-1}u_0\dot{y}v'\dot{x}^{-1},$$

noting that $\dot{y}^{-1}u_0\dot{y} \in U$ because $\ell(s_k y) > \ell(y)$. If $\dot{y}^{-1}U_{\alpha_k}\dot{y} \subseteq U_{x^{-1}}$ then the elements

$$(u_0\dot{s}_k^{-1}u'\dot{s}_k, b', \dot{y}^{-1}u_0\dot{y}v')$$

are a complete set of solutions of the structure equations for (w, x, y) with $u_0\dot{s}_k^{-1}u'\dot{s}_k \in U_\tau$. If, on the other hand, an element $\dot{y}^{-1}u_0\dot{y} \notin U_{x^{-1}}$, then $\dot{y}^{-1}u_0\dot{y}v' = v_1v_1'$ with $v_1 \in U_{x^{-1}}$ and $v_1' \in U_{x^{-1}w_0}$; the solution of the structure equation for (w, x, y) in this case is

$$(u_0\dot{s}_k^{-1}u'\dot{s}_k, b'(\dot{x}v_1'\dot{x}^{-1})^{-1}, v_1)$$

since $\dot{x}v_1'\dot{x}^{-1} \in U$.

(iii) $\tau_k\tau_{k-1}^{-1} = 1$, $x = x'$, $y = y'$, and $\ell(s_k y) < \ell(y)$. We first recall the connection between U_τ and $U_{\tau'}$. By the discussion following Lemma 2.3, we have $U_\tau \cong U_{\alpha_k}^* \times U_{\tau'}$ and

$$U_\tau = \{u_0\dot{s}_k u_1\dot{s}_k^{-1} \in U_{\alpha_k}^*\dot{s}_k U_{s_k w}\dot{s}_k^{-1} : \pi(g_k(u_0)u_1) \in U_{\tau'}\},$$

where $\dot{s}_k^{-1}u_0\dot{s}_k = f_k(u_0)\dot{s}_k t_k(u_0)g_k(u_0)$ as in Lemma 2.1 and where $\pi : U \to U_{s_k w}$ is the projection associated with the decomposition $U = U_{s_k w}U_{s_k w w_0}$. Let

$u' \in U_{\tau'} \subseteq U(s_k w, x, y)$ and let $u_0 \dot{s}_k u_1 \dot{s}_k^{-1} \in U_\tau$ correspond to u' as described previously. We then obtain, by the argument given in the proof of the first part of Lemma 2.3(iii), all solutions of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ from the solutions (u', b', v'), assumed known, of the structure equation $u'\dot{s}_k^{-1}\dot{w}b' = \dot{y}v'\dot{x}^{-1}$.

This completes the proof of the following theorem.

THEOREM 2.1. *Let $\dot{w}, \dot{x}, \dot{y}$ be representatives, chosen as before, of elements w, x, y of the Weyl group W of the Chevalley group G, and assume that $U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ is nonempty. We may identify $U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ with the set of solutions (u, b, v) of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ with $u \in U_w$, $b \in B$, and $v \in U_{x^{-1}}$; furthermore, we shall identify $(U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1})_\tau$ with the set of solutions of the equation such that $u \in U_\tau$.*

(i) *The set*
$$U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$$
is the disjoint union of subsets $(U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1})_\tau$ indexed by K-sequences τ for (w, x, y).

(ii) *Each subset $(U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1})_\tau$ is described by the set of solutions (u, b, v) of the structure equation*
$$u\dot{w}b = \dot{y}v\dot{x}^{-1},$$
with $u \in U_\tau$, $b \in B$, and $v \in U_{x^{-1}}$ corresponding to a given K-sequence τ, and is obtained by the algorithm given in this section based on a fixed reduced expression for the element $w \in W$ in terms of the distinguished generators $s_1, \dots, s_n$ of W.

(iii) *Each subset U_τ, and each set of solutions (u, b, v) of the structure equation corresponding to $(U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1})_\tau$, is in bijective correspondence with the product of root subgroups*
$$\prod_{\alpha \in \Phi_\tau} U_\alpha \times \prod_{\beta \in \Phi_\tau^*} U_\beta^*$$
for subsets Φ_τ and Φ_τ^* of the positive root system Φ_+ of cardinalities $a(\tau)$ and $b(\tau)$, respectively.

3. The Sets $U\ell U \cap nU_{m^{-1}}m$ for $\ell, m, n \in \mathcal{N}$ and the Structure Constants $[c_\ell c_m : c_n]$

The formula in the Introduction (from [7, Prop. 11.30]) for the structure constants $[c_\ell c_m : c_n]$ involves the sets $U\ell U \cap nU_{m^{-1}}m^{-1}$ for $\ell, m, n \in \mathcal{N}$. We shall obtain a description of these constants in terms of the sets $U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$ calculated in Section 2.

Let ℓ, m, n correspond to w, x, y in W, and let $\dot{w}, \dot{x}, \dot{y}$ be representatives in N of w, x, y chosen as in Section 2. Then $\ell \in \dot{w}T$, $m \in \dot{x}T$, and $n \in \dot{y}T$, and we can write $\ell = \dot{w}t_w = t_w'\dot{w}$, $m = \dot{x}t_x = t_x'\dot{x}$, and $n = \dot{y}t_y = t_y'\dot{y}$ for uniquely determined elements $t_w, t_w', \dots$ in T. Note that $t_w' = \dot{w}t_w\dot{w}^{-1}, \dots$; this means that the elements

t'_w, t'_x, t'_y are given by the algorithms for multiplication in a Chevalley group if the elements t_w, t_x, and t_y are known.

PROPOSITION 3.1. *Let (ℓ, m, n) and $(\dot{w}, \dot{x}, \dot{y})$ be as before with $\ell = \dot{w}t_w = t'_w\dot{w}, \dots$ for elements $t_w, t'_w, \dots$ in T. Then*

$$U\ell B \cap nU_{m^{-1}}m^{-1} = t'_y(U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1})(t'_x)^{-1}.$$

Moreover, if (u, b, v) is a solution of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$, then

$$u' = t'_y u(t'_y)^{-1}, \quad b' = (t_w)^{-1}\dot{w}^{-1}t'_y\dot{w}b(t'_x)^{-1}, \quad v' = v$$

is a solution of the equation $u'\ell b' = nv'm^{-1}$ with $u' \in U_w$, $b' \in B$, and $v' \in U_{x^{-1}}$, and b' and v' are uniquely determined by u'.

The first result follows from the formulas relating $\dot{x}, \dot{y}$ to m, n and the fact that U_w is stable under conjugation by elements of T, so that

$$t'_y U_w\dot{w}B(t'_x)^{-1} = U_w\ell B.$$

For the second statement, from

$$u\dot{w}b = \dot{y}v\dot{x}^{-1} \in U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1}$$

we obtain, from the first part of the proof,

$$t'_y u\dot{w}b(t'_x)^{-1} = t'_y\dot{y}v\dot{x}^{-1}(t'_x)^{-1} \in U\ell B \cap nU_{m^{-1}}m^{-1}$$

and

$$t'_y u(t'_y)^{-1}\dot{w}t_w(t_w)^{-1}\dot{w}^{-1}t'_y\dot{w}b(t'_x)^{-1} = nvm^{-1}.$$

The last statement follows from Lemma 2.4.

COROLLARY 3.1. *A solution (u, b, v) of the structure equation $u\dot{w}b = \dot{y}v\dot{x}^{-1}$ corresponds to an element $u'\ell b' = nv'm^{-1} \in U\ell U \cap nU_{m^{-1}}m^{-1}$ if and only if $b' \in U$ or (what amounts to the same thing) iff $tt_w^{-1}\dot{w}^{-1}t'_y\dot{w}(t'_x)^{-1} = 1$, where $b = t\tilde{u}$ for $t \in T$ and $\tilde{u} \in U$. When t satisfies these conditions, $b' = t'_x\tilde{u}(t'_x)^{-1} \in U$.*

COROLLARY 3.2. *Let $\ell, m, n \in \mathcal{N}$ correspond to $w, x, y \in W$ as before.*

(i) *The set $U\ell U \cap nU_{m^{-1}}m^{-1}$ is the disjoint union of subsets $(U\ell B \cap nU_{m^{-1}}m^{-1})_\tau$ that corresponds to those K-sequences τ for w, x, and y such that the set $t'_y(U_w\dot{w}B \cap \dot{y}U_{x^{-1}}\dot{x}^{-1})_\tau(t'_x)^{-1}$ is nonempty and consists of elements $u'\ell b' = nv'm^{-1}$ with $b' \in U$ as in Corollary 3.1.*
(ii) *The structure constants of standard basis elements c_ℓ, c_m, c_n of the Hecke algebra $\mathcal{H}$ of a Gelfand–Graev representation γ of G are given by the formula*

$$[c_\ell c_m : c_n] = \sum_\tau \sum_{u\ell u_1 = nvm^{-1} \in (U\ell U \cap nU_{m^{-1}}m^{-1})_\tau} \psi((uu_1)^{-1}v).$$

As an example we calculate some structure constants of the Hecke algebra of a Gelfand–Graev representation of $\mathrm{SL}_2(k)$ (see [5, Sec. 5]). We start with the solutions (u, b, v) of the structure equation $u\dot{s}b = \dot{s}v\dot{s}^{-1}$ with $\dot{s} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Let

$$u = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} c & cd \\ 0 & c^{-1} \end{pmatrix}, \quad v = \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}.$$

Then the structure equation has a unique solution for each $u \neq 1$ in U given by $c = -a^{-1}$, $d = a$, and $e = -a^{-1}$.

Let $\lambda, \mu, \nu \in k^*$ and put

$$\ell = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}, \quad m = \begin{pmatrix} 0 & \mu \\ -\mu^{-1} & 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 & \nu \\ -\nu^{-1} & 0 \end{pmatrix}.$$

Then $\ell, m, n \in \mathcal{N}$, and we can now compute $[c_\ell c_n : c_m]$. The elements ℓ, m, n are related to $\dot{s}$ as in the discussion before Proposition 3.1. There is just one K-sequence for (s, s, s). The solutions of the structure equation $u'\ell u_1' = nv'm^{-1}$ are, by Proposition 3.1 and Corollary 3.1,

$$u' = \begin{pmatrix} 1 & \nu^2 a \\ 0 & 1 \end{pmatrix}, \quad u_1' = \begin{pmatrix} 1 & \mu^2 a \\ 0 & 1 \end{pmatrix}, \quad v' = \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix},$$

with a as before. Then a is determined by the condition (from Corollary 3.1) that $b' \in U$ and is given by $a = -\lambda\mu^{-1}\nu^{-1}$. Upon substituting the computed value of a in the matrices u', u_1', v' and applying Corollary 3.2, one has

$$[c_\ell c_m : c_n] = \chi(\lambda\mu\nu^{-1} + \lambda\mu^{-1}\nu + \lambda^{-1}\mu\nu),$$

where the linear character ψ defining the Gelfand–Graev representation of $\mathrm{SL}_2(k)$ is given by $\psi(u) = \chi(a)$ with $u = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and χ a nontrivial additive character of the additive group of k.

To conclude this section, we recall the main result of [5] and state a problem concerning it, illustrated by an example involving $\mathrm{SL}_2(k)$. Let $\mathbf{G} = G(\bar{k})$ be a Chevalley group over the algebraic closure $\bar{k}$ of k with a Frobenius endomorphism F such that the group of fixed elements $\mathbf{G}^F$ is the Chevalley group G over k considered previously. Let $\mathbf{B} = \mathbf{U}\mathbf{T}_0$ be the F-stable Borel subgroup of $\mathbf{G}$ containing the split F-stable maximal torus $\mathbf{T}_0$ with $\mathbf{U}$ the unipotent radical of $\mathbf{B}$ such that $B = \mathbf{B}^F$, $U = \mathbf{U}^F$, and $T_0 = \mathbf{T}_0^F$ are the subgroups of G appearing in Proposition 3.1. Let $\gamma = \psi^G$ be a Gelfand–Graev representation of $G = \mathbf{G}^F$ as in Section 1, with $\mathcal{H}$ the Hecke algebra of γ and $c_\ell, c_m, \ldots$ the standard basis elements of $\mathcal{H}$. We can now state a formula (cf. [5, Thm. 4.2]) for the values of the irreducible representations of $\mathcal{H}$ on standard basis elements of $\mathcal{H}$. The proof is based on the character formula for the virtual representations $R^{\mathbf{G}}_{\mathbf{T},\theta}$ of Deligne and Lusztig [9] in a suitable algebraically closed field K of characteristic 0.

THEOREM 3.1. *The irreducible representations of $\mathcal{H}$ in the field K are parameterized by the geometric conjugacy classes of pairs $(\mathbf{T}, \theta)$ consisting of an F-stable maximal torus $\mathbf{T}$ in $\mathbf{G}$ and an irreducible representation θ of the finite torus $T = \mathbf{T}^F$. Each irreducible representation $f_{\mathbf{T},\theta}$ of $\mathcal{H}$ can be factored,*

$$f_{\mathbf{T},\theta} = \tilde{\theta} \circ f_{\mathbf{T}},$$

with $f_{\mathbf{T}}$ a homomorphism of algebras from $\mathcal{H}$ to KT, independent of θ, and $\tilde{\theta}$ is an extension of θ to an irreducible representation of the group algebra KT. Let

$f_{\mathbf{T}}(c_\ell) = \sum_{t\in T} f_{\mathbf{T}}(c_\ell)(t)t$. Then $f_{\mathbf{T}}(c_\ell)(t)$ is given by the following formula involving the Green function $Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}$ and the linear character ψ of U such that $\gamma = \psi^G$:

$$f_{\mathbf{T}}(c_\ell)(t) = \operatorname{ind}\ell\langle Q_{\mathbf{T}}^{\mathbf{G}}, \gamma\rangle^{-1}|U|^{-1}|C_{\mathbf{G}}(t)^{\circ F}|^{-1} \times \sum_{g\in G, u\in U, (gu\ell g^{-1})_{ss}=t} \psi(u^{-1}) Q_{\mathbf{T}}^{C_{\mathbf{G}}(t)^\circ}((gu\ell g^{-1})_{uni}),$$

where $g = g_{ss}g_{uni}$ is the Jordan decomposition of an element $g \in G$ and $\langle\lambda, \mu\rangle$ is the scalar product of K-valued class functions on G.

The next result simply expresses the fact that $f_{\mathbf{T}}$ is a homomorphism of algebras from $\mathcal{H}$ to KT.

COROLLARY 3.3 (Torus homomorphism identity). *The elements $f_{\mathbf{T}}(c_\ell)(t)$ satisfy the identity*

$$\sum_{s\in T} f_{\mathbf{T}}(c_\ell)(ts) f_{\mathbf{T}}(c_m)(s^{-1}) = \sum_n [c_\ell c_m : c_n] f_{\mathbf{T}}(c_n)(t)$$

for all $t \in T$ and standard basis elements c_ℓ, c_m, c_n of $\mathcal{H}$.

As an illustration, we work out this torus homomorphism identity for the Coxeter torus $\mathbf{T}$ in the finite Chevalley group $G = \mathrm{SL}_2(k) = \mathbf{G}^F$ with $\mathbf{G} = \mathrm{SL}_2(\bar{k})$ and $k = F_q$. As in [5, Sec. 5], we let ψ^G be a Gelfand–Graev representation of G with

$$\psi(u) = \chi(a) \quad \text{and} \quad u = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

and let χ be a nontrivial character of the additive group of k. For the Coxeter torus $\mathbf{T}$ of $\mathbf{G}$ one has

$$T = \mathbf{T}^F \cong C,$$

where $C = \{\xi \in F_{q^2} : \xi^{q+1} = 1\}$, so $|C| = q + 1$. Let $c_\ell, c_m, \dots$ be the standard basis elements of the Hecke algebra $\mathcal{H}$, as in the example considered earlier in this section. By [5, Thm. 5.2],

$$f_{\mathbf{T}}(c_\ell)(\xi) = -\chi(\lambda(\xi + \xi^{-1})), \quad \xi \in C,$$

where we have identified C with the finite torus $T = \mathbf{T}^F$.

The torus homomorphism identity for the Coxeter torus in $\mathrm{SL}_2(k)$ then becomes

$$\sum_{\xi\in C} \chi(\xi(\lambda\gamma + \mu) + \xi^q(\lambda\gamma^q + \mu)) = -\sum_{\nu\in k^*} \chi(\nu + (\lambda\gamma + \mu)(\lambda\gamma^q + \mu)\nu^{-1}) + q\delta_{\lambda\gamma+\mu,0}$$

for all $\gamma \in C$. We have used the formula for the structure constants $[c_\ell c_m : c_n]$ from earlier in the section to work out the right-hand side of this identity.

The preceding identity is a consequence of Theorem 3.1. It relates an exponential sum over the quadratic extension field F_{q^2} of k to an exponential sum over the field $k = F_q$ and can also be proved directly, either by an analysis of quadratic equations over k (as in [2, Lemma 1.2]) or as an application of the Davenport–Hasse theorem for Gauss sums. The problem mentioned before is to work out the coefficient functions $f_{\mathbf{T}}(c_\ell)(t)$ as exponential sums using Theorem 3.1, in the general case, and then to prove the torus homomorphism identities directly using the formula for the structure constants $[c_\ell c_m : c_n]$ from Corollary 3.2. A solution to the problem would give an alternative approach to the computation of the values of the irreducible representations $f_{\mathbf{T},\theta}$ of $\mathcal{H}$ (or the spherical functions on G associated with a Gelfand–Graev representation) and might be of independent interest. The problem was solved for all three classes of maximal tori in the groups $\mathrm{GL}_3(k)$ by Chang [2].

4. On $BwB \cap B_- xB$ and the Kazhdan–Lusztig Polynomials $R_{x,w}$

In [10, Thm. 1.3], Deodhar proved that the polynomials $R_{x,w}$ of Kazhdan and Lusztig [14] are given by a formula

$$R_{x,w}(q) = \sum_{\sigma \in \mathcal{D}, \pi(\sigma) = x} q^{m(\sigma)}(q-1)^{n(\sigma)},$$

where the sum is taken over the set $\mathcal{D}$ of distinguished subexpressions σ of a reduced expression $w = t_1 \cdots t_k$ of $w \in W$ and where the t_i belong to the set of distinguished generators S of W. Deodhar also stated that, if G is a finite Chevalley group over the field k of q elements, then $R_{x,w} = |(BwB \cap B_- xB)/B|$. In this section we shall prove that the distinguished subexpressions coincide with the set of K-sequences for $(w, (w_0 x)^{-1}, w_0)$, and we obtain a different formula for the polynomials $R_{x,w}$ by using K-sequences instead of distinguished subexpressions.

In what follows we shall use well-known properties of the Bruhat order $x \leq y$ in W; we continue to use the notation $\dot{w}$ for representatives in N of elements $w \in W$.

THEOREM 4.1. *Let G be a finite Chevalley group over the field k of q elements, and let x and w be elements of the Weyl group W of G. Let B_- be the opposite Borel subgroup to B; thus $B_- = w_0 B w_0$, where w_0 is the element of maximal length in W.*

(i) *The set $BwB \cap B_- xB$ is nonempty if and only if $x \leq w$. In that case*

$$\left| \frac{BwB \cap B_- xB)}{B} \right| = |U_w \dot{w} B \cap \dot{w}_0 U_{w_0 x} (w_0 x)^{\cdot}| = \sum_\tau q^{a(\tau)}(q-1)^{b(\tau)},$$

where the sum is taken over all K-sequences τ for $(w, (w_0 x)^{-1}, w_0)$.

(ii) *The K-sequences τ for $(w, (w_0 x)^{-1}, w_0)$ computed from a reduced expression $w = s_k \cdots s_1$ coincide with the distinguished subexpressions σ of the reduced expression $w = t_1 \cdots t_k$, where $t_1 = s_k$, $t_2 = s_{k-1}, \ldots$ such that $\pi(\sigma) = x$. Moreover, for corresponding subexpressions τ and σ, the nonnegative integers $a(\tau)$ and $b(\tau)$ associated with τ coincide with the integers $m(\sigma)$ and $n(\sigma)$ associated with σ.*

(iii) *The Kazhdan–Lusztig polynomial*

$$R_{x,w} = \left| \frac{BwB \cap B_- xB}{B} \right| \quad \textit{with } x \leq w$$

and is equal to the formula stated in part (i) *involving the sum over K-sequences for* $(w, (w_0 x)^{-1}, w_0)$.

We first examine part (i) of the theorem. The set $BwB \cap B_- xB$ is nonempty if and only if there exists at least one K-sequence τ for $(w, (w_0 x)^{-1}, w_0)$. For such a K-sequence $\tau = (\tau_k, \dots, \tau_1, \tau_0)$ we have $\tau_k (w_0 x)^{-1} = w_0$, so $\tau_k = x$. Since the terminal elements τ_k of K-sequences satisfy the condition $\tau_k \leq w$, we have $x \leq w$. Later we will use Deodhar's results in [10] to prove that if $x \leq w$ then the intersection is nonempty. Assuming the intersection is nonempty, the rest of part (i) follows from Kawanaka's results (Lemma 2.14(b) and Theorem 2.6(b) in [13]) stated in Corollary 2.1.

Let $w = s_k \cdots s_1$ be a reduced expression, and let $\tau = (\tau_k, \dots, \tau_1, \tau_0)$ be a K-sequence for $(w, (w_0 x)^{-1}, w_0)$. We shall prove that τ is then a distinguished subexpression $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_k)$ of the reduced expression $t_1 \cdots t_k = w = s_k \cdots s_1$ such that $\pi(\sigma) = x$. To be a subexpression means (see [10, p. 502]) that $\sigma_0 = 1$ and $\sigma_{j-1}^{-1} \sigma_j \in \{1, t_j\}$ for $1 \leq j \leq k$ and also that the map π from the set of subexpressions σ to the elements x such that $x \leq w$ is defined by $\pi(\sigma) = \sigma_k$. To be a *distinguished* subexpression [10, Def. 2.3] means that the following inequalities hold in the Bruhat order: $\sigma_j \leq \sigma_{j-1} t_j$ for $1 \leq j \leq k$.

The assumption that τ is a K-sequence for $(w, (w_0 x)^{-1}, w_0)$ implies that $\tau_k = x \leq w$, as we have seen in the discussion of part (i). The definition of K-sequence requires further that, if $j \in J_\tau = \{0\} \cup \{j : \tau_j = s_j \tau_{j-1}\}$, then $\ell(s_p \tau_j (w_0 x)^{-1}) < \ell(\tau_j (w_0 x)^{-1})$ for p in the interval between j and the next element of J_τ (taken in increasing order), or simply that all $p > j$ if j is the maximal element of J_τ. In order to prove that τ is a distinguished subexpression, it is enough to consider the case $\tau_{j+1} = \tau_j$ (or $\sigma_{k-j} = \sigma_{k-j-1}$ for corresponding subexpressions of $w = t_1 \cdots t_k$). Then the preceding condition implies, by the length-reversing properties of multiplication by w_0, that $\ell(s_{j+1} \tau_j x^{-1}) > \ell(\tau_j x^{-1})$. Now $x = \tau_k$ and $\tau_j x^{-1} = \tau_j \tau_k^{-1}$. In order to work with this expression we write the given subexpression as $\tau_k = \bar{s}_k \cdots \bar{s}_1$ with $\bar{s}_i = s_i$ or 1 according to the definition of τ. The condition $\ell(s_{j+1} \tau_j x^{-1}) > \ell(\tau_j x^{-1})$ becomes, upon setting $x = \tau_k$ and using our convention, $\ell(s_{j+1} \bar{s}_{j+1} \cdots \bar{s}_k) > \ell(\bar{s}_{j+1} \cdots \bar{s}_k)$. Since the length function is invariant under taking inverses, the preceding formula implies that $\ell(\bar{s}_k \cdots \bar{s}_{j+1} s_{j+1}) > \ell(\bar{s}_k \cdots \bar{s}_{j+1})$ or, in terms of the Bruhat order, $\bar{s}_k \cdots \bar{s}_{j+1} s_{j+1} \geq \bar{s}_k \cdots \bar{s}_{j+1}$. The condition $\tau_{j+1} = \tau_j$ implies that $\bar{s}_k \cdots \bar{s}_{j+1} = \bar{s}_k \cdots \bar{s}_{j+2}$. Combining these pieces of information and substituting $s_k = t_1$, $s_{k-1} = t_2, \dots$, we obtain $\sigma_{k-j} t_{k-j} \geq \sigma_{k-j}$ whenever $\sigma_{k-j} = \sigma_{k-j-1}$, and this means that the subexpression σ corresponding to τ is a distinguished subexpression.

Let τ be a K-sequence for $(w, (w_0 x)^{-1}, w_0)$ identified with the distinguished subexpression σ as in the preceding paragraph. We shall prove that the nonnegative integers $a(\tau)$ and $b(\tau)$ associated with τ coincide with the integers $m(\sigma)$ and $n(\sigma)$

associated with σ as in [10, p. 502]. We clearly have $b(\tau) = n(\sigma)$ because both are equal to the cardinality of the set $\{j : \tau_{j-1} = \tau_j\}$ for $1 \leq j \leq k$.

Next consider $a(\tau)$. We have $a(\tau) = |J_\tau^-|$, where

$$J_\tau^- = \{j \in J_\tau : \ell(s_j \tau_{j'} (w_0 x)^{-1}) < \ell(\tau_{j'} (w_0 x)^{-1})\}$$

for j' the predecessor of j in J_τ. By the reasoning used earlier, $x = \tau_k$ and $J_\tau^- = \{j \in J_\tau : \ell(s_j \tau_{j'} \tau_k^{-1}) > \ell(\tau_{j'} \tau_k^{-1})\}$. Using the convention introduced previously, we have $\tau_{j'} \tau_k^{-1} = \bar{s}_{j'+1} \cdots \bar{s}_k$. Since j is the next element in J_τ to j', it follows that $\tau_{j'} = \tau_{j'+1} = \cdots = \tau_{j-1}$ and $\tau_{j'} \tau_k^{-1} = \tau_{j-1} \tau_k^{-1} = \bar{s}_j \cdots \bar{s}_k$. The condition that $j \in J_\tau^-$ is $\ell(s_j \bar{s}_j \cdots \bar{s}_k) > \ell(\bar{s}_j \cdots \bar{s}_k)$. But $s_j \bar{s}_j \cdots \bar{s}_k = \bar{s}_{j+1} \cdots \bar{s}_k$ and so the condition becomes, after taking inverses, $\ell(\bar{s}_k \cdots \bar{s}_{j+1}) > \ell(\bar{s}_k \cdots \bar{s}_{j+1} s_j)$. The preceding steps can be reversed, and it follows that

$$a(\tau) = |\{j : \sigma_{j-1} > \sigma_j\}| = m(\sigma)$$

by the definition of $m(\sigma)$ in [10, p. 502].

Now assume that the intersection $BwB \cap w_0 B w_0 x$ is nonempty for some element $x \leq w$. Then we have, by part (i) of the proposition and [10, Cor. 1.2],

$$\begin{aligned} \frac{|BwB \cap w_0 B w_0 x B|}{|B|} &= \sum_\tau q^{a(\tau)} (q-1)^{b(\tau)} \leq \sum_\sigma q^{m(\sigma)} (q-1)^{n(\sigma)} \\ &= \frac{|BwB \cap w_0 B w_0 x B|}{|B|}, \end{aligned}$$

where the first sum is taken over all K-sequences for $(w, (w_0 x)^{-1}, w_0)$ and the second over all distinguished subexpressions σ with $\pi(\sigma) = x$. It follows that $\leq$ can be replaced by $=$, and we conclude that the two sets of subexpressions τ and σ coincide.

Finally, let $x \leq w$. Then by [10, Sec. 5] there exists a distinguished subexpression σ such that $\pi(\sigma) = x$, and it follows that the intersection $BwB \cap w_0 B w_0 x B$ is nonempty (see the proof of Corollary 1.2 in [10, p. 505]). Then part (iii) of the proposition follows from [10, Thm. 1.3] and the preceding paragraph. At this point we have also proved part (ii) of the theorem.

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