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Points and Hyperplanes of the Universal Embedding Space of the Dual Polar Space $DW(5,q)$, q Odd

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Dedicated to the memory of Donald G. Higman

1. Introduction

A *partial linear rank-2 incidence geometry,* also called a *point-line geometry,* is a pair $\Gamma = (\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ whose elements are called *points* and a collection $\mathcal{L}$ of distinguished subsets of $\mathcal{P}$ whose elements are called *lines,* such that any two distinct points are contained in at most one line. The *point-collinearity graph* of Γ is the graph with vertex set $\mathcal{P}$ where two points are adjacent if they are collinear (i.e., lie on a common line). By a *subspace* of Γ we mean a subset S of $\mathcal{P}$ such that, if $l \in \mathcal{L}$ and $l \cap S$ contains at least two points, then $l \subset S$. A subspace S is *singular* if each pair of points in S is collinear—that is, if S is a clique in the collinearity graph of Γ. We say that $(\mathcal{P}, \mathcal{L})$ is a *Gamma space* (see [13]) if, for every $x \in \mathcal{P}$, $\{x\} \cup \Gamma(x)$ is a subspace. A subspace $S \neq \mathcal{P}$ is a *geometric hyperplane* if it meets every line.

Let e be a positive integer, p a prime, and V a 6-dimensional vector space over the finite field $\mathbb{F}_q$, $q = p^e$, equipped with a nondegenerate alternating form f. Then every vector $\bar{u} \in V$ is *isotropic,* that is, satisfies $f(\bar{u}, \bar{u}) = 0$. A subspace U of V is called *totally isotropic* (*with respect to* f) if $f(\bar{u}_1, \bar{u}_2) = 0$ for all $\bar{u}_1, \bar{u}_2 \in U$.

Associated with (V, f) is a polar space denoted by $W(5,q)$. Here, by a *polar space* we mean a point-line geometry (P, L) that satisfies the following properties:

1. (P, L) is a Gamma space and, for every point p and line l, p is collinear with some point of l (this means that p is collinear with one point or all points of l);
2. no point p is collinear with every other point; and
3. there is an integer n called the *rank* of (P, L) such that, if $S_0 \subset S_1 \subset \cdots \subset S_k$ is a properly ascending chain of singular subspaces, then $k \leq n$.

When $n = 2$, (P, L) is said to be a *generalized quadrangle*.

The points (resp. lines) of $W(5,q)$ are the 1-dimensional (resp. 2-dimensional) subspaces of V that are totally isotropic with respect to f and where incidence is containment. In $W(5,q)$, two points $\langle \bar{u}_1 \rangle_V$ and $\langle \bar{u}_2 \rangle_V$ are collinear if and only if $f(\bar{u}_1, \bar{u}_2) = 0$ (i.e., iff $\bar{u}_1$ and $\bar{u}_2$ are orthogonal).

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Also associated with the alternating form f of V is a *dual polar space* $DW(5,q)$. The points (resp. lines) of $DW(5,q)$ are the 3-spaces (resp. 2-spaces) of V that are totally isotropic with respect to f and where incidence is reverse containment. We denote the point-set and line-set of $DW(5,q)$ by $\mathcal{P}$ and $\mathcal{L}$, respectively. In the incidence system $(\mathcal{P},\mathcal{L})$, two "points" U_1 and U_2 are collinear if and only if $\dim(U_1 \cap U_2) = 2$. More generally, one can say that the distance $\mathrm{d}(U_1,U_2)$ (in the collinearity graph of $(\mathcal{P},\mathcal{L})$) between two points U_1 and U_2 of $DW(5,q)$ is equal to $3 - \dim(U_1 \cap U_2)$. The lines of the dual polar space $DW(5,q)$ are maximal singular subspaces, so this geometry is also a Gamma space.

Alternatively, the geometries (P,L) and $(\mathcal{P},\mathcal{L})$ can be defined as Lie incidence geometries (see [4]) by making use of a construction of Gamma spaces from a symmetrical orbital (orbit of the Symplectic group on the Cartesian products P^2 or $\mathcal{P}^2$; see [13]).

By Shult and Yanushka [21] or Cameron [1], the set of totally isotropic 3-spaces of V that contain a given 1-space of V is a convex subspace of diameter 2 of $DW(5,q)$. Such a convex subspace is called a *quad* of $DW(5,q)$. The points and lines contained in a quad define a generalized quadrangle that is isomorphic to the classical generalized quadrangle $Q(4,q)$ (Payne and Thas [16, Sec. 3.1]).

In this paper we are concerned with classifying all the geometric hyperplanes of $DW(5,q)$, q odd, that arise from an embedding (to be defined). In the Main Theorem we will show that there are always six isomorphism classes of such hyperplanes.

The notion of a geometric hyperplane was introduced by Veldkamp (see [23; 24]) in his characterization of polar geometries for the explicit purpose of proving that such a geometry is embeddable. Geometric hyperplanes have been studied in many other contexts as well: for example, they arise in the classification by Cohen and Shult of the affine polar spaces (see [3]) and in Cuypers' characterization of the graph on 2300 vertices with automorphism group Co_2, the second Conway group [8]. Removing a geometric hyperplane with certain properties from an incidence geometry often allows one to create interesting affine geometries, and this was the motivation of Pasini and Shpectorov [15] in studying uniform hyperplanes in dual polar spaces and of Cooperstein and Pasini [7] in proving that ovoidal hyperplanes do not exist in $DW(5,q)$.

The research carried out in this paper is part of a larger project of classifying all hyperplanes of finite dual polar spaces of small rank. A complete classification of all hyperplanes of the Hermitian dual polar space $DH(5,q^2)$ was obtained by De Bruyn and Pralle [11; 12]. All hyperplanes of the dual polar space $DQ^-(7,q)$ arising from an embedding were classified by De Bruyn [9]. The classification of all hyperplanes of the dual polar spaces $DQ(6,q)$ and $DQ(8,q)$ that arise from their spin embeddings was obtained by Cardinali, De Bruyn, and Pasini [2], De Bruyn [9], Shult [19], and Shult and Thas [20]. A complete list of all hyperplanes of $DW(5,q)$, q even, arising from an embedding was given by Pralle [17] (for $q = 2$) and by De Bruyn [10] (for arbitrary q even).

2. Technical Description of the Results

2.1. The Grassmann Embedding of $DW(5,q)$

We continue with the notation introduced in Section 1. Choose a basis $\mathcal{S} = \{\bar{v}_1, \bar{w}_1, \bar{v}_2, \bar{w}_2, \bar{v}_3, \bar{w}_3\}$ in V such that $f(\bar{v}_i, \bar{w}_i) = 1$ and $f(\bar{v}_i, \bar{v}_j) = f(\bar{w}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j) = 0$ for all $i, j \in \{1,2,3\}$ with $i \neq j$. Let $W := \bigwedge^3 V$ be the third exterior product of V that is a vector space of dimension $\binom{6}{3} = 20$ over $\mathbb{F}_q$. Define now a bilinear form $g(\cdot,\cdot)$ from $W \times W$ to $\mathbb{F}_q$ by setting $\alpha \wedge \beta$ equal to $g(\alpha,\beta)(\bar{v}_1 \wedge \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 \wedge \bar{v}_3 \wedge \bar{w}_3)$ for all $\alpha, \beta \in W$. Since $(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) \wedge (\bar{u}_4 \wedge \bar{u}_5 \wedge \bar{u}_6) = (-1)^9 (\bar{u}_4 \wedge \bar{u}_5 \wedge \bar{u}_6) \wedge (\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3)$ for all vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_6 \in V$, the form $g(\cdot,\cdot)$ is alternative. Obviously, it is also nondegenerate.

For every point $x = \langle \bar{u}_1, \bar{u}_2, \bar{u}_3 \rangle_V$ of $DW(5,q)$, let $\varepsilon(x)$ denote the 1-space $\langle \bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3 \rangle_W$ of $W = \bigwedge^3 V$. This 1-space is independent from the generating set $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ of x. It is well known that the subspace M of W generated by all 1-spaces $\varepsilon(x)$, $x \in \mathcal{P}$, is 14-dimensional. One readily verifies that a basis of M is given by the set $\mathcal{S}_M := \{p_i \mid 1 \leq i \leq 14\}$, where

$$p_1 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3, \quad p_2 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3, \quad p_3 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{v}_3,$$

$$p_4 = \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3, \quad p_5 = \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_3, \quad p_6 = \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3,$$

$$p_7 = \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3, \quad p_8 = \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3,$$

$$p_9 = \bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 - \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{w}_3, \quad p_{10} = \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 - \bar{w}_1 \wedge \bar{v}_3 \wedge \bar{w}_3,$$

$$p_{11} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{v}_2 - \bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_3, \quad p_{12} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{w}_2 - \bar{w}_2 \wedge \bar{v}_3 \wedge \bar{w}_3,$$

$$p_{13} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{v}_3 - \bar{v}_2 \wedge \bar{w}_2 \wedge \bar{v}_3, \quad p_{14} = \bar{v}_1 \wedge \bar{w}_1 \wedge \bar{w}_3 - \bar{v}_2 \wedge \bar{w}_2 \wedge \bar{w}_3.$$

For all $i, j \in \{1, \ldots, 14\}$ we have $g(p_i, p_j) = 0$ except when $\{i, j\}$ is equal to $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$, $\{9,10\}$, $\{11,12\}$, or $\{13,14\}$. Hence, the form $g(\cdot,\cdot)$ defines a nondegenerate alternating form in the 14-space M. For every subspace U of M, let $U^{\perp_g} = \{m \in M \mid g(u,m) = 0 \text{ for all } u \in U\}$.

The map ε defines a full projective embedding of the dual polar space $DW(5,q)$ into the projective space $\mathrm{PG}(M) \cong \mathrm{PG}(13,q)$. This embedding is called the *Grassmann embedding* of $DW(5,q)$. If $q \neq 2$, then by results in [5] and [14] we know that the Grassmann embedding of $DW(5,q)$ is absolutely universal [18]. This implies that all full embeddings of $DW(5,q)$, $q \neq 2$, can be obtained from its Grassmann embedding by taking so-called quotients.

If π is a hyperplane of $\mathrm{PG}(M)$, then $\varepsilon^{-1}(\varepsilon(\mathcal{P}) \cap \pi)$ is a (geometric) hyperplane of $DW(5,q)$—namely, a proper subset of $\mathcal{P}$ intersecting each line of $DW(5,q)$ in either a unique point or the whole line. We will say that the hyperplane $\varepsilon^{-1}(\varepsilon(\mathcal{P}) \cap \pi)$ *arises from the embedding* ε.

2.2. The Automorphism Groups of $W(5,q)$ and $DW(5,q)$

Before proceeding to our main theorem, we describe the automorphism groups of $W(5,q)$ and $DW(5,q)$. Suppose θ is a permutation of the point-set of $W(5,q)$.

Then θ will be an automorphism of $W(5,q)$ if and only if it induces a permutation on the set of all ordered pairs of distinct collinear points of $W(5,q)$. Similarly, a permutation of $\mathcal{P}$ will be an automorphism of $DW(5,q)$ if and only if it induces a permutation of the set of all ordered pairs of distinct collinear points of $DW(5,q)$. It is not difficult to see that automorphism groups of $DW(5,q)$ and $W(5,q)$ are isomorphic.

That automorphisms of $W(5,q)$ induce automorphisms of $DW(5,q)$ is fairly straightforward. That automorphisms of $DW(5,q)$ induce automorphisms of $W(5,q)$ follows from two facts: (i) the quads of $DW(5,q)$ are characterized as the convex subspaces of diameter 2 and (ii) these are in one-to-one correspondence with the points of $W(5,q)$. We proceed to describe the group $\mathrm{Aut}(W(5,q)) \cong \mathrm{Aut}(DW(5,q))$.

Recall that $\mathcal{S} = \{\bar{v}_1, \bar{w}_1, \bar{v}_2, \bar{w}_2, \bar{v}_3, \bar{w}_3\}$ is a basis of V such that $f(\bar{v}_i, \bar{w}_i) = 1$ and $f(\bar{v}_i, \bar{v}_j) = f(\bar{w}_i, \bar{w}_j) = f(\bar{v}_i, \bar{w}_j) = 0$ for all $i, j \in \{1,2,3\}$ with $i \neq j$. A *similarity* of (V, f) is a linear transformation $\sigma \in \mathrm{GL}(V)$ such that $f(\sigma(\bar{u}_1), \sigma(\bar{u}_2)) = \lambda_\sigma \cdot f(\bar{u}_1, \bar{u}_2)$ for all $\bar{u}_1, \bar{u}_2 \in V$. Here λ_σ is a nonzero scalar that depends on σ but is independent of $\bar{u}_1$ and $\bar{u}_2$. We denote by $G_f \leq \mathrm{GL}(V)$ the group of all similarities. An *isometry* is a similarity σ with $\lambda_\sigma = 1$. We denote by S_f the group of all isometries; S_f is normal in G_f and is isomorphic to $\mathrm{Sp}(6, \mathbb{F}_q)$. Clearly, similarities induce automorphisms of $W(5,q)$. The kernel of the action of G_f on P is the center of G_f and consists of all the scalar transformations $\lambda \cdot I_V$, where λ is a nonzero scalar. Denote by PG_f the quotient $G_f/Z(G_f)$ and by PS_f the quotient of S_f by $S_f \cap Z(G_f)$. We note that $S_f \cap Z(G_f) = Z(S_f) = \langle -I_V \rangle$ and therefore $S_f Z(G_f)/Z(G_f) \cong S_f/Z(S_f)$. Thus, we may consider PS_f to be a subgroup of PG_f. The group PS_f is the simple group $P\,\mathrm{Sp}(6, \mathbb{F}_q)$. The index of PS_f in PG_f is 2 (see [22]). For $i = 1,2,3$, if σ^* is the linear transformation of V that fixes $\bar{v}_i$ and takes $\bar{w}_i$ to $d\bar{w}_i$ with d a given nonsquare in $\mathbb{F}_q$, then $\sigma^* \in G_f \setminus S_f$ and consequently $PG_f = PS_f\langle\sigma^*\rangle$. This describes the automorphisms of $W(5,q)$ that are induced by linear transformations of V. In addition, there are "field automorphisms".

For $\bar{u} \in V$, denote by $[\bar{u}]_{\mathcal{S}}$ the coordinate vector of $\bar{u}$ with respect to the basis $\mathcal{S}$ of V. For every $\gamma \in \mathrm{Aut}(\mathbb{F}_q)$, define a map $T_\gamma \colon V \to V$ by $[T_\gamma(\bar{v})]_{\mathcal{S}} = \gamma([\bar{v}]_{\mathcal{S}})$. Then T_γ induces a permutation of the point-set of $W(5,q)$ that preserves orthogonality and therefore induces an automorphism of $W(5,q)$. If $A = \{T_\gamma \mid \gamma \in \mathrm{Aut}(\mathbb{F}_q)\}$, then $\mathrm{Aut}(W(5,q)) = PG_f A = PS_f\langle\sigma^*\rangle A$.

2.3. The Main Results

Every element θ of G_f gives rise to a unique element $\theta' \in \mathrm{GL}\big(\bigwedge^3 V\big)$ such that $\theta'(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = \theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3)$ for all $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$. Obviously, θ' fixes M and hence gives rise to an element $\hat{\theta} \in \mathrm{GL}(M)$. For all $\alpha, \beta \in W = \bigwedge^3 V$,

$$g(\theta'(\alpha), \theta'(\beta)) = \det(\theta) \cdot g(\alpha, \beta). \tag{2.1}$$

Hence $\hat{\theta}$ is a similarity of (M, g). Now define $\widehat{G_f} := \{\hat{\theta} \mid \theta \in G_f\}$ and $\widehat{S_f} := \{\hat{\theta} \mid \theta \in S_f\}$. By (2.1),

$$(\phi(U))^{\perp_g} = \phi(U^{\perp_g}) \tag{2.2}$$

for every $\phi \in \widehat{G_f}$ and every subspace U of M.

Now suppose as before that $\gamma \in \mathrm{Aut}(\mathbb{F}_q)$, and let $\mathcal{B}$ be the basis $\{\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3, \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3\} \cup \{\bar{v}_i \wedge \bar{v}_j \wedge \bar{w}_k, \bar{v}_k \wedge \bar{w}_i \wedge \bar{w}_j \mid 1 \leq i,j,k \leq 3,\ i < j\}$ of W. Let T'_γ be the $\mathbb{F}_p$-linear map of W defined by $[T'_\gamma(\alpha)]_{\mathcal{B}} = \gamma([\alpha]_{\mathcal{B}})$. We have $T'_\gamma(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = T_\gamma(\bar{u}_1) \wedge T_\gamma(\bar{u}_2) \wedge T_\gamma(\bar{u}_3)$ for all $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$. For all $\alpha, \beta \in W$, we have

$$g(T'_\gamma(\alpha), T'_\gamma(\beta)) = \gamma(g(\alpha,\beta)). \tag{2.3}$$

Note that T'_γ fixes each of the vectors of the basis $\mathcal{S}_M$ of M and hence induces an $\mathbb{F}_p$-linear map $\widehat{T_\gamma}\colon M \to M$. By (2.3),

$$(\widehat{T_\gamma}(U))^{\perp_g} = \widehat{T_\gamma}(U^{\perp_g}) \tag{2.4}$$

for every subspace U of M.

Let $\overline{G_f}$ (resp. $\widehat{\overline{G_f}}$) denote the group of $\mathbb{F}_p$-linear maps of V (resp. W) generated by G_f and T_γ, $\gamma \in \mathrm{Aut}(\mathbb{F}_q)$ (resp. $\widehat{G_f}$ and $\widehat{T_\gamma}$, $\gamma \in \mathrm{Aut}(\mathbb{F}_q)$). By our previous discussion, for every $\theta \in \overline{G_f}$ there exists a unique $\mathbb{F}_p$-linear map $\theta'\colon W \to W$ such that $\theta'(\bar{u}_1 \wedge \bar{u}_2 \wedge \bar{u}_3) = \theta(\bar{u}_1) \wedge \theta(\bar{u}_2) \wedge \theta(\bar{u}_3)$ for all $\bar{u}_1, \bar{u}_2, \bar{u}_3 \in V$. The map θ' stabilizes M and hence induces an $\mathbb{F}_p$-linear map $\hat{\theta}\colon M \to M$. Obviously, $\hat{\theta} \in \widehat{\overline{G_f}}$. Moreover, the map $\theta \mapsto \hat{\theta}$ is an isomorphism between the groups $\overline{G_f}$ and $\widehat{\overline{G_f}}$.

It is the main purpose of this paper to determine the orbits of the group $\mathrm{Aut}(DW(5,q))$ on the hyperplanes of $DW(5,q)$, q odd, that arise from its Grassmann embedding. Because the Grassmann embedding of $DW(5,q)$, q odd, is absolutely universal, it follows that the hyperplanes of $DW(5,q)$, q odd, arising from the Grassmann embedding are all the hyperplanes of that dual polar space that arise from an embedding.

Determining the orbits of $\mathrm{Aut}(DW(5,q))$ on the hyperplanes of $DW(5,q)$ is equivalent to the enumeration of all $\widehat{\overline{G_f}}$-orbits on the hyperplanes of M. By equations (2.2) and (2.4), this is equivalent to enumerating the orbits of $\widehat{\overline{G_f}}$ on the 1-spaces of M—that is, the points of $\mathrm{PG}(M)$. We will achieve our objective by first enumerating the orbits of $\widehat{S_f}$ on the 1-spaces of M and then determining when these $\widehat{S_f}$-orbits fuse when the group is extended to all of $\mathrm{Aut}(DW(5,q))$.

Before stating the Main Theorem, we need to define some extra vectors in M. Unless indicated otherwise, in the sequel we will always assume that q is an odd prime power. Let $d \in \mathbb{F}_q$ be such that d is a nonsquare (if -1 is a nonsquare then we take d equal to -1). Define the following additional vectors of M:

$$p_{15} = p_1 + p_4, \quad p_{16} = p_1 + dp_4, \quad p_{17} = p_1 + p_4 + p_6, \quad p_{18} = p_1 + p_8,$$
$$p_{19} = p_1 + dp_8, \quad p_{20} = dp_1 + p_4 + p_6 + p_7, \quad p_{21} = dp_2 + dp_3 + dp_5 + p_8.$$

Also, set $P_i = \langle p_i \rangle_W$ and $H_i = \varepsilon^{-1}(P_i^{\perp_g} \cap \varepsilon(\mathcal{P}))$ for every $i \in \{1, \ldots, 21\}$. We can now state our main theorem.

Main Theorem. *Let q be an odd prime power. Then the group* $\mathrm{Aut}(DW(5,q))$ *has six orbits on the geometric hyperplanes of $DW(5,q)$ that arise from an embedding with representatives H_1, H_{15}, H_{16}, H_{17}, H_{18}, and H_{20}. The sizes of the orbits are given in Table 1.*

Table 1 The orbits of $\mathrm{Aut}(DW(5,q))$, q odd, on the geometric hyperplanes of $DW(5,q)$

Type	Representative	Orbit size
I	H_1	$(q^3+1)(q^2+1)(q+1)$
II	H_{15}	$\frac{(q^6-1)q^2(q^2+1)}{2(q-1)}$
III	H_{16}	$\frac{(q^6-1)q^2(q^2-1)}{2(q-1)}$
IV	H_{17}	$q^3(q^6-1)(q^2+1)(q+1)$
V	H_{18}	$\frac{q^6(q^4-1)(q^3+1)}{2}$
VI	H_{20}	$\frac{q^6(q^4-1)(q^3-1)}{2}$

The Main Theorem is a consequence of the following two results, which we will prove in Sections 3 and 4 (respectively).

Point Enumeration Theorem. (i) *If* -1 *is a nonsquare in* $\mathbb{F}_q$, *q odd, then the group* $\widehat{S_f}$ *has six orbits on the point-set of* $\mathrm{PG}(M)$ *with representatives* P_1, P_{15}, P_{16}, P_{17}, P_{18}, *and* P_{20}. *The orbit sizes and the stabilizers of each representative are given in Table 2.*

(ii) *If* -1 *is a square in* $\mathbb{F}_q$, *q odd, then the group* $\widehat{S_f}$ *has eight orbits on the point-set of* $\mathrm{PG}(M)$ *with representatives* P_1, P_{15}, P_{16}, P_{17}, P_{18}, P_{19}, P_{20}, *and* P_{21}. *The orbit sizes and stabilizers are given in Table 3.*

To prove the Point Enumeration Theorem, we will show in both cases that the conjectured representatives given in the tables are all in different orbits; we then compute their stabilizers and hence their orbit sizes. Since in both cases the sum of the orbit sizes is $\frac{q^{14}-1}{q-1}$, it will follow that we have enumerated all the $\widehat{S_f}$-orbits on the points of M.

Fusion Theorem. (i) *Assume that* -1 *is a nonsquare in* $\mathbb{F}_q$ *with q odd. Then the automorphisms of* $DW(5,q)$ *induced by* σ^* *and* T_γ, $\gamma \in \mathrm{Aut}(\mathbb{F}_q)$, *fix each of the* $\widehat{S_f}$*-orbits of the hyperplanes* H_1, H_{15}, H_{16}, H_{17}, H_{18}, *and* H_{20}.

(ii) *Assume that* -1 *is a square in* $\mathbb{F}_q$ *with q odd. Then the automorphisms of* $DW(5,q)$ *induced by* σ^* *and* T_γ, $\gamma \in \mathrm{Aut}(\mathbb{F}_q)$, *fix each of the* $\widehat{S_f}$*-orbits of the hyperplanes* H_1, H_{15}, H_{16}, *and* H_{17}. *On the other hand, the* $\widehat{S_f}$*-orbits of* H_{18} *and* H_{19} *become a single orbit, as do the* $\widehat{S_f}$*-orbits of* H_{20} *and* H_{21}.

The Main Theorem classifies all hyperplanes of $DW(5,q)$, q odd, arising from an embedding. As previously mentioned, all hyperplanes of $DW(5,q)$, q even, arising from an embedding were classified in [17] (for $q = 2$, with the aid of a computer) and [10] (for arbitrary $q = 2^m$, without the use of a computer).

Table 2 The $\widehat{S_f}$-orbits on the points of $\mathrm{PG}(M)$ when -1 is a nonsquare in $\mathbb{F}_q$

Type	Representative	Orbit size	Stabilizer
I	P_1	$(q^3+1)(q^2+1)(q+1)$	$q^6.\mathrm{GL}(3,q)$
II	P_{15}	$\frac{(q^6-1)q^2(q^2+1)}{2(q-1)}$	$q^5.\mathrm{SL}(2,q) \times \mathrm{SL}(2,q) \times \mathbb{Z}_{q-1}.2$
III	P_{16}	$\frac{(q^6-1)q^2(q^2-1)}{2(q-1)}$	$q^5.\mathrm{SL}(2,q^2) \times \mathbb{Z}_{q-1}.2$
IV	P_{17}	$q^3(q^6-1)(q^2+1)(q+1)$	$q^5.\mathrm{SL}(2,q)\ \ \mathbb{Z}_{q-1}.2$
V	P_{18}	$\frac{q^6(q^4-1)(q^3+1)}{2}$	$\mathbb{Z}_2 \times \mathrm{SL}(3,q)$
VI	P_{20}	$\frac{q^6(q^4-1)(q^3-1)}{2}$	$\mathbb{Z}_2 \times \mathrm{SU}(3,q)$

Table 3 The $\widehat{S_f}$-orbits on the points of $\mathrm{PG}(M)$ when -1 is a square in $\mathbb{F}_q$

Type	Representative	Orbit size	Stabilizer
I	P_1	$(q^3+1)(q^2+1)(q+1)$	$q^6.\mathrm{GL}(3,q)$
II	P_{15}	$\frac{(q^6-1)q^2(q^2+1)}{2(q-1)}$	$q^5.\mathrm{SL}(2,q) \times \mathrm{SL}(2,q) \times \mathbb{Z}_{q-1}.2$
III	P_{16}	$\frac{(q^6-1)q^2(q^2-1)}{2(q-1)}$	$q^5.\mathrm{SL}(2,q^2) \times \mathbb{Z}_{q-1}.2$
IV	P_{17}	$q^3(q^6-1)(q^2+1)(q+1)$	$q^5.\mathrm{SL}(2,q)\ \ \mathbb{Z}_{q-1}.2$
Va	P_{18}	$\frac{q^6(q^4-1)(q^3+1)}{4}$	$\mathbb{Z}_2 \times \mathrm{SL}(3,q).2$
Vb	P_{19}	$\frac{q^6(q^4-1)(q^3+1)}{4}$	$\mathbb{Z}_2 \times \mathrm{SL}(3,q).2$
VIa	P_{20}	$\frac{q^6(q^4-1)(q^3-1)}{4}$	$\mathbb{Z}_2 \times \mathrm{SU}(3,q).2$
VIb	P_{21}	$\frac{q^6(q^4-1)(q^3-1)}{4}$	$\mathbb{Z}_2 \times \mathrm{SU}(3,q).2$

Several combinatorial properties of the hyperplanes of $DW(5,q)$, q odd, that arise from an embedding were already obtained by the authors in [6]. For each hyperplane H of $DW(5,q)$, q odd, they determined (using purely combinatorial and geometrical techniques) the total number of quads Q for which $Q \cap H$ is a certain configuration of points in Q and the total number of points x for which $\Delta(x) \cap H$ is a certain configuration of points in $\Delta(x)$. Here, $\Delta(x)$ denotes the set of points equal to or collinear with x. On the basis of these combinatorial properties, the authors were able to divide the set of hyperplanes of $DW(5,q)$, q odd, into six classes: Type I hyperplanes, Type II hyperplanes, $\ldots$, Type VI hyperplanes. This terminology is consistent with that used in our paper. By the Main Theorem, each of the six classes defined in [6] is actually an isomorphism class.

3. Proof of the Point Enumeration Theorem

3.1. Notation and a Few Lemmas

We will continue with the notation introduced in Sections 1 and 2.

Let $\delta \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ such that $\delta^2 = d$. We may suppose that (i) $\bar{w}_i = \delta \bar{v}_i$ for every $i \in \{1, 2, 3\}$ and (ii) V is a 3-dimensional vector space over $\mathbb{F}_{q^2}$ with basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ and a 6-dimensional vector space over $\mathbb{F}_q$ with basis $\mathcal{S} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3\}$. Recall that $\bigwedge^3 V$ must be regarded as the third exterior power of V as a vector space over the field $\mathbb{F}_q$.

LEMMA 3.1. *If τ is an $\mathbb{F}_{q^2}$-linear transformation of V with* $\det(\tau) = 1$, *then $\hat{\tau}$ centralizes the vectors p_{20} and p_{21}.*

Proof. Let E_{ij} denote the 3×3 matrix with a 1 in the (i, j)th entry and 0s elsewhere, and set $\chi_{ij} = \{I_3 + \alpha E_{ij} \mid \alpha \in \mathbb{F}_{q^2}\}$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. Also, set $w_1 = E_{12} - E_{21} + E_{33}$ and $w_2 = E_{11} + E_{23} - E_{32}$. Then the group $\mathrm{SL}(3, q^2)$ is generated by χ_{13}, w_1, and w_2. Hence it suffices to prove that the induced action of each of these centralizes p_{20} and p_{21}.

Let $\alpha = a + b\delta$ where $a, b \in \mathbb{F}_q$, and suppose that τ is the $\mathbb{F}_{q^2}$-linear transformation of V whose associated matrix with respect to the basis $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is equal to $I_3 + \alpha E_{13}$. Then the matrix of τ with respect to $\mathcal{S}$ is $\left(\begin{smallmatrix} A & dB \\ B & A \end{smallmatrix}\right)$, where $A = I_3 + aE_{13}$ and $B = bE_{13}$. It is now quite straightforward to compute the induced action of τ on p_{20} and p_{21}: $\hat{\tau}(p_{20})$ is equal to

$$\begin{aligned}
&\hat{\tau}(d\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3) \\
&\quad = d[\bar{v}_1 \wedge \bar{v}_2 \wedge (a\bar{v}_1 + \bar{v}_3 + b\bar{w}_1)] + \bar{v}_1 \wedge \bar{w}_2 \wedge ((db)\bar{v}_1 + a\bar{w}_1 + \bar{w}_3) \\
&\qquad + \bar{w}_1 \wedge \bar{v}_2 \wedge ((db)\bar{v}_1 + a\bar{w}_1 + \bar{w}_3) + \bar{w}_1 \wedge \bar{w}_2 \wedge (a\bar{v}_1 + \bar{v}_3 + b\bar{w}_1) \\
&\quad = d\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + (db)\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_1 + \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + a\bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_1 \\
&\qquad + \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + (db)\bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_1 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3 + a\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_1 \\
&\quad = p_{20},
\end{aligned}$$

since $\bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_1 = -\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_1$ and $\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_1 = -\bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_1$.

Similarly, $\hat{\tau}(p_{21})$ is equal to

$$\begin{aligned}
&\hat{\tau}(d\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + d\bar{v}_1 \wedge \bar{w}_2 \wedge \bar{v}_3 + d\bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3) \\
&\quad = d[\bar{v}_1 \wedge \bar{v}_2 \wedge ((db)\bar{v}_1 + a\bar{w}_1 + \bar{w}_3)] + d[\bar{v}_1 \wedge \bar{w}_2 \wedge (a\bar{v}_1 + \bar{v}_3 + b\bar{w}_1)] \\
&\qquad + d[\bar{w}_1 \wedge \bar{v}_2 \wedge (a\bar{v}_1 + \bar{v}_3 + b\bar{w}_1)] + \bar{w}_1 \wedge \bar{w}_2 \wedge ((db)\bar{v}_1 + a\bar{w}_1 + \bar{w}_3) \\
&\quad = (da)\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_1 + d\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + (db)\bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_1 + d\bar{v}_1 \wedge \bar{w}_2 \wedge \bar{v}_3 \\
&\qquad + (da)\bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_1 + d\bar{w}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + (db)\bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_1 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 \\
&\quad = p_{21}.
\end{aligned}$$

The matrix of w_1 with respect to $\mathcal{S}$ is $\left(\begin{smallmatrix} A & O \\ O & A \end{smallmatrix}\right)$, where $A = E_{12} - E_{21} + E_{33}$ and O is the 3×3 matrix with all entries equal to 0. Now $\widehat{w_1}(p_{20})$ is equal to

$$\begin{aligned}
&\widehat{w_1}(d\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{v}_3 + \bar{v}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{v}_2 \wedge \bar{w}_3 + \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{v}_3) \\
&\quad = d(-\bar{v}_2) \wedge \bar{v}_1 \wedge \bar{v}_3 + (-\bar{v}_2) \wedge \bar{w}_1 \wedge \bar{w}_3 \\
&\qquad + (-\bar{w}_2) \wedge \bar{v}_1 \wedge \bar{w}_3 + (-\bar{w}_2) \wedge \bar{w}_1 \wedge \bar{v}_3 \\
&\quad = p_{20},
\end{aligned}$$

since $(-\bar{v}_2) \wedge \bar{v}_1 = \bar{v}_1 \wedge \bar{v}_2$, $(-\bar{w}_2) \wedge \bar{w}_1 = \bar{w}_1 \wedge \bar{w}_2$, $(-\bar{v}_2) \wedge \bar{w}_1 = \bar{w}_1 \wedge \bar{v}_2$, and $(-\bar{w}_2) \wedge \bar{v}_1 = \bar{v}_1 \wedge \bar{w}_2$.

In an entirely similar way, one shows that $\widehat{w_1}(p_{21}) = p_{21}$, $\widehat{w_2}(p_{20}) = p_{20}$, and $\widehat{w_2}(p_{21}) = p_{21}$. □

Recall that every point $x \in \mathcal{P}$ gives rise to a 1-space $\varepsilon(x)$ of M, that is, a point $\varepsilon(x)$ of $\mathrm{PG}(M)$. For a line $l \in \mathcal{L}$, we define $\varepsilon(l) := \{\varepsilon(x) \mid x \in l\}$. We denote by $\tilde{l}$ the 2-space of M generated by the 1-spaces $\varepsilon(x)$, $x \in l$. We put $\tilde{\mathcal{P}} = \hat{\mathcal{P}} = \{\varepsilon(x) \mid x \in \mathcal{P}\}$, $\hat{\mathcal{L}} = \{\varepsilon(l) \mid l \in \mathcal{L}\}$, and $\tilde{\mathcal{L}} = \{\tilde{l} \mid l \in \mathcal{L}\}$.

Let X be a point of $\mathrm{PG}(V)$. By abuse of notation, we will also write $X \in \mathrm{PG}(V)$. The set $Q(X) = \{x \in \mathcal{P} \mid X \subset x \subset X^{\perp}\}$ is a convex subspace of $DW(5,q)$ that defines a generalized quadrangle isomorphic to $Q(4,q)$. We set $\mathcal{Q} := \{Q(X) \mid X \in \mathrm{PG}(V)\}$ and refer to the elements of $\mathcal{Q}$ as *quads* of $DW(5,q)$. For $Q \in \mathcal{Q}$, we will denote by $\hat{Q}$ the collection $\{\varepsilon(x) \mid x \in Q\}$ and by $\tilde{Q}$ the subspace of M spanned by the elements of $\hat{Q}$. We refer to both $\hat{Q}$ and $\tilde{Q}$ as the *quads* of M. Set $\tilde{\mathcal{Q}} = \{\tilde{Q} \mid Q \in \mathcal{Q}\}$.

For every point u of $DW(5,q)$, $\Delta(u)$ denotes the set of points of $DW(5,q)$ that are collinear or equal to u. If $P = \varepsilon(u) \in \tilde{\mathcal{P}}$, then we define $\Delta(P) := \varepsilon(\Delta(u))$ and $M(P)$ is the subspace of M spanned by the elements of $\Delta(P)$. We call $M(P)$ the *hemisphere* of P.

LEMMA 3.2. *Let $X, Y \in \mathrm{PG}(V)$. Then the following statements hold*:

(i) *if $X \perp_f Y$, then $\tilde{Q}(X) \cap \tilde{Q}(Y) \in \tilde{\mathcal{L}}$*;

(ii) *if X and Y are not orthogonal, then $\tilde{Q}(X) \cap \tilde{Q}(Y) = 0$.*

Proof. (i) The group $\overline{G_f}$ is transitive on pairs $\{X, Y\}$ of 1-spaces of V such that $X \perp_f Y$. Therefore we can take $X = \langle \bar{v}_1 \rangle$ and $Y = \langle \bar{v}_2 \rangle$. Then

$$\tilde{Q}(X) = \langle p_1, p_2, p_3, p_4, p_9 \rangle, \qquad \tilde{Q}(Y) = \langle p_1, p_2, p_5, p_6, p_{11} \rangle,$$

and $\tilde{Q}(X) \cap \tilde{Q}(Y) = \langle p_1, p_2 \rangle \in \tilde{\mathcal{L}}$.

(ii) The group $\overline{G_f}$ is also transitive on pairs $\{X, Y\}$ of 1-spaces of V such that X and Y are nonorthogonal with respect to f. We can take $X = \langle \bar{v}_1 \rangle$ and $Y = \langle \bar{w}_1 \rangle$. Now $\tilde{Q}(X) = \langle p_1, p_2, p_3, p_4, p_9 \rangle$ and $\tilde{Q}(Y) = \langle p_5, p_6, p_7, p_8, p_{10} \rangle$. Therefore, $\tilde{Q}(X) \cap \tilde{Q}(Y) = 0$ as claimed. □

This result implies the next corollary, which is fundamental.

COROLLARY 3.3. *Let $\tilde{Q} \in \tilde{\mathcal{Q}}$ and $P \in \mathrm{PG}(\tilde{Q}) \setminus \tilde{\mathcal{P}}$. Then $\tilde{Q}$ is the unique quad of M that contains P.*

LEMMA 3.4. *Let $P \in \tilde{\mathcal{P}}$ and $Q \in \tilde{\mathcal{Q}}$ be such that $P \notin Q$. Let R denote the unique point of $Q \cap \tilde{\mathcal{P}}$ at distance* 1 *from P. Then $M(P) \cap Q = R$.*

Proof. Since $\widehat{\widehat{G_f}}$ is transitive on the pairs (P, Q) with $P \in \tilde{\mathcal{P}}$, $Q \in \tilde{\mathcal{Q}}$, and $P \notin Q$, we may without loss of generality suppose that $Q = \tilde{Q}(\langle \bar{v}_1 \rangle)$ and $P = \langle p_8 \rangle$. Then $R = \langle p_4 \rangle$. Now $Q = \langle p_1, p_2, p_3, p_4, p_9 \rangle$ and $M(P) = \langle p_4, p_6, p_7, p_8, p_{10}, p_{12}, p_{14} \rangle$; hence $M(P) \cap Q = \langle p_4 \rangle = R$. □

COROLLARY 3.5. *Let $P \in \tilde{\mathcal{P}}$ and $R \in \mathrm{PG}(M(P)) \setminus \tilde{\mathcal{P}}$. If R is contained in a quad, then this quad necessarily contains P.*

LEMMA 3.6. *Let $\tilde{Q} \in \tilde{\mathcal{Q}}$ and $R \in \mathrm{PG}(\tilde{Q}) \setminus \tilde{P}$. Then there exists a $P \in \tilde{\mathcal{P}}$ such that $R \in \mathrm{PG}(M(P))$.*

Proof. Let $\tilde{L}$ be contained in $\tilde{Q}$ where $L \in \mathcal{L}$. Then $\tilde{Q} = \bigcup_{P \in \tilde{L}} \langle \tilde{Q} \cap \Delta(P) \rangle \subset \bigcup_{P \in \tilde{Q}} M(P)$. □

Our next lemma shows that, if a point is contained in two distinct hemispheres, then in fact it is contained in a quad.

LEMMA 3.7. *Let P and P' be distinct points of $\tilde{\mathcal{P}}$, and let $X \in \mathrm{PG}(M(P) \cap M(P'))$. Then there is a quad $\tilde{Q}$ containing P such that $X \subset \tilde{Q}$.*

Proof. For every $t \in \{1, 2, 3\}$, $\widehat{\widehat{G_f}}$ is transitive on the pairs (P, P') of points of $\tilde{\mathcal{P}}$ with $d(P, P') = t$. Therefore we can take (P, P') to be one of (P_1, P_2), (P_1, P_4), or (P_1, P_8). For every $i \in \{1, 2, 4, 8\}$, set $M_i = M(P_i)$. Then

$$M_1 = \langle p_1, p_2, p_3, p_5, p_9, p_{11}, p_{13} \rangle, \quad M_2 = \langle p_1, p_2, p_4, p_6, p_9, p_{11}, p_{14} \rangle,$$
$$M_4 = \langle p_2, p_3, p_4, p_8, p_9, p_{12}, p_{14} \rangle, \quad M_8 = \langle p_4, p_6, p_7, p_8, p_{10}, p_{12}, p_{14} \rangle.$$

Now $M_1 \cap M_2 = \langle p_1, p_2, p_9, p_{11} \rangle$. This space is covered by $\bigcup \tilde{Q}(\langle \bar{v} \rangle)$ where $\bar{v} \in \langle \bar{v}_1, \bar{v}_2 \rangle$. Also $M_1 \cap M_4 = \langle p_2, p_3, p_9 \rangle$ and this is contained in $\tilde{Q}(\langle \bar{v}_1 \rangle)$. Finally, $M_1 \cap M_8 = 0$. □

This lemma has an important corollary as follows.

COROLLARY 3.8. *Assume $X \in \mathrm{PG}(M(P))$ for $P \in \tilde{\mathcal{P}}$ and assume X is not contained in a quad that contains P. Then P is the unique point of $\tilde{\mathcal{P}}$ for which $X \in \mathrm{PG}(M(P))$.*

3.2. Points Contained in At Least One Hemisphere

We now show that the points P_1, P_{15}, P_{16}, and P_{17} are in distinct orbits of $\widehat{S_f}$, with orbit sizes and stabilizers as shown in Tables 2 and 3. We also show that the union of these orbits comprises all points of $\mathrm{PG}(M)$ that are contained in at least one hemisphere.

The orbit of P_1 is just $\tilde{\mathcal{P}}$. There are $(q^3 + 1)(q^2 + 1)(q + 1)$ such points, and the stabilizer $S_{P_1} := (\widehat{S_f})_{P_1}$ of P_1 is isomorphic to the subgroup of S_f that fixes a maximal totally isotropic subspace of V. The group S_{P_1} has a normal elementary abelian subgroup $E(P_1)$ of order q^6. This subgroup has a complement $L(P_1) \cong \mathrm{GL}(3, q)$, which justifies the entries of line I of Table 3 and Table 4.

For a point X of $\mathrm{PG}(V)$, the stabilizer in $\widehat{S_f}$ of $\tilde{Q}(X)$ is isomorphic to $S_X := (S_f)_X$. The group S_X has a normal subgroup $E(X)$ of order q^5, which is a special group. This subgroup has a complement $L(X)$ that is isomorphic to $L(X)' \times Z(X)$, where $L(X)' \cong \mathrm{Sp}(4, q)$ is the commutator subgroup of $L(X)$ and $Z(X) \cong \mathbb{Z}_{q-1}$. Note that $L(X)'/Z(L(X)') \cong \Omega(5, q)$. In fact, the group $L(X)$ preserves a quadratic form on $\tilde{Q}(X)$, which we now describe.

Let $X = \langle \bar{v}_1 \rangle$, and set $V(X) = \langle \bar{v}_2, \bar{w}_2, \bar{v}_3, \bar{w}_3 \rangle$. Observe that $X \wedge \bigwedge^2(X^\perp) = X \wedge \bigwedge^2(V(X))$ has dimension 6. We denote this space by $D(X)$. Any vector $\bar{v}$ is in $D(X)$ and can be written as $\bar{v}_1 \wedge \alpha$ for $\alpha \in \bigwedge^2(V(X))$. Also, for $\alpha, \beta \in \bigwedge^2(V(X))$, we have that $\alpha \wedge \beta$ is a multiple of $\bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3$. Thus, define $b\colon \bigwedge^2(V(X)) \times \bigwedge^2(V(X)) \to \mathbb{F}_q$ by $\alpha \wedge \beta = b(\alpha, \beta)(\bar{v}_2 \wedge \bar{v}_3 \wedge \bar{w}_2 \wedge \bar{w}_3)$. This defines a nondegenerate symmetric bilinear form of Witt index 3. Now define $\hat{b}\colon D(X) \times D(X) \to \mathbb{F}_q$ by $\hat{b}(\bar{v}_1 \wedge \alpha, \bar{v}_1 \wedge \beta) = b(\alpha, \beta)$; this also is a nondegenerate symmetric bilinear form of Witt index 3. The space $\tilde{Q}(X)$ is the subspace of $D(X)$ that is orthogonal to $\bar{v}_1 \wedge \bar{v}_2 \wedge \bar{w}_2 + \bar{v}_1 \wedge \bar{v}_3 \wedge \bar{w}_3$ with respect to $\hat{b}$. The group $L(X)$ has three orbits on the projective points of $\tilde{Q}(X)$: the singular points of the quadratic form $\hat{b}$, which are the points of $\hat{Q}(X)$; and the two classes of nonsingular points with respect to $\hat{b}$. Note that $\hat{b}(p_9, p_9) = \hat{b}(p_{15}, p_{15}) = 2$. Also, $p_9^{\perp_{\hat{b}}} = \langle p_1, p_2, p_3, p_4 \rangle$, which is a nondegenerate hyperbolic subspace of $(\tilde{Q}(X), \hat{b})$. On the other hand, $\hat{b}(p_{16}, p_{16}) = 2d$ and, since $\hat{b}(p_{15}, p_{15}) \cdot \hat{b}(p_{16}, p_{16}) = 4d$ is a nonsquare, it follows that P_{15} and P_{16} are in different classes of nonsingular points of $(\tilde{Q}(X), \hat{b})$ and hence are representatives of the two classes. Since there are $\frac{q^6-1}{q-1}$ quads $Q(X)$ for $X \in \mathrm{PG}(V)$ and since, for each X, there are $\frac{q^2(q^2+1)}{2}$ points in the class of P_{15} contained in $\tilde{Q}(X)$ and $\frac{q^2(q^2-1)}{2}$ points in the class of P_{16} contained in $\tilde{Q}(X)$, the entries of lines II and III of Table 3 and Table 4 are justified.

We now make use of Corollary 3.3 and simple counting to show that, for $P \in \tilde{\mathcal{P}}$, there are points in $M(P)$ that are not from classes I, II, or III.

LEMMA 3.9. *The following statements hold for a point $P \in \tilde{\mathcal{P}}$:*

(i) *the number of points of type I in* $\mathrm{PG}(M(P))$ *is* $1 + q(q^2 + q + 1)$;
(ii) *the number of points of type II in* $\mathrm{PG}(M(P))$ *is* $\frac{q^2(q^2+q+1)(q+1)}{2}$;
(iii) *the number of points of type III in* $\mathrm{PG}(M(P))$ *is* $\frac{q^2(q^2+q+1)(q-1)}{2}$;
(iv) *there are* $q^3(q^3 - 1)$ *points in* $\mathrm{PG}(M(P))$ *that do not belong to a quad.*

Proof. (i): The points of type I in $M(P)$ are precisely $\Delta(P)$. There are $q^2 + q + 1$ lines on P, each with q points of $\Delta(P)$ apart from P.

(ii) and (iii): The point P belongs to $q^2 + q + 1$ quads. For a quad $\tilde{Q}$ containing P, we know that $M(P) \cap \tilde{Q}$ is the hyperplane of $\tilde{Q}$ spanned by $\Delta(P) \cap \hat{Q}$. A simple count yields that $M(P) \cap \tilde{Q}$ contains $\frac{q^2(q+1)}{2}$ points of type II and $\frac{q^2(q-1)}{2}$ points of type III. The second and third parts follow from this.

(iv): The number of points that have been accounted for is

$$1 + q + q^2 + q^3 + (q^2 + q + 1)\left[\frac{q^2(q+1)}{2} + \frac{q^2(q-1)}{2}\right] = 1 + q + q^2 + 2q^3 + q^4 + q^5.$$

Since $|\mathrm{PG}(M(P))| = \frac{q^7-1}{q-1}$, there are $q^6 - q^3 = q^3(q^3 - 1)$ remaining points. □

LEMMA 3.10. *The stabilizer S_P of a point $P \in \tilde{\mathcal{P}}$ is transitive on the points of* $\mathrm{PG}(M(P))$ *that do not belong to quads.*

Proof. Since $\widehat{S_f}$ is transitive on $\tilde{\mathcal{P}}$, we can take $P = P_2$ and $M(P) = M_2$. Recall that $S_P = E(P) \cdot L(P)$, where $E(P)$ is elementary abelian of order q^6 and $L(P) \cong \mathrm{GL}(3, q)$. The subgroup $E(P)$ fixes every projective line of the form $P + P'$ with $P' \in \Delta(P) \setminus \{P\}$ and, for such a line, is transitive on $\mathrm{PG}(P+P') \setminus \{P\}$. This implies that $E(P)$ acts trivially on the 6-dimensional quotient space $M(P)/P$. The action of the complement, $L(P)$, on $M(P)/P$ is equivalent to the action of $\mathrm{GL}(3, q)$ on the space $\mathrm{Sym}(3, q)$ of 3×3 symmetric matrices for which the action is given by $g \circ m = g^{\mathrm{T}} m g$ (where g^{T} denotes the transpose of the matrix g). Under this action, every matrix is equivalent to a diagonal matrix and there are six orbits on nonzero vectors, two each for rank 1, 2, and 3. Representatives for the orbits on vectors are as follows:

$$(1)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\quad (2)\ \begin{pmatrix} d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\quad (3)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$(4)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{pmatrix};\quad (5)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\quad (6)\ \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}.$$

Note that the vectors in (1) and (2) give rise to the same point of $\mathrm{PG}(\mathrm{Sym}(3, q))$, as do the vectors in (5) and (6); however, the vectors in (3) and (4) do not.

Consequently, $L(P)$ has four orbits on the points of $M(P)/P$. However, for any 2-space U of $M(P)$ containing P, the group $E(P)$ is transitive on $\mathrm{PG}(U) \setminus \{P\}$ and therefore S_P has four orbits on the points of $\mathrm{PG}(M(P)) \setminus \{P\}$. The point P_1 is a representative of one orbit, and the points P_{15} and P_{16} are the representatives of two other orbits. Thus, there is one other orbit consisting of all those points of $\mathrm{PG}(M(P))$ that do not belong to quads. □

The point P_{17} is a point of $M(P_2)$ that does not belong to a quad. In view of Corollary 3.8 and Lemmas 3.9 and 3.10, it now follows that the orbit of P_{17} has $|\tilde{\mathcal{P}}| \times (q^6 - q^3) = q^3(q^6 - 1)(q^2 + 1)(q + 1)$.

3.3. Points Not Belonging to a Hemisphere

We now turn our attention to points that do not belong to $M(P)$ for any point $P \in \tilde{\mathcal{P}}$.

Since the group $\widehat{S_f}$ is transitive on $\tilde{\mathcal{P}}$ and since, for a point $P \in \tilde{\mathcal{P}}$, the normal abelian group $E(P)$ acts regularly on the points P' with $d(P, P') = 3$, it follows that $\widehat{S_f}$ is transitive on ordered pairs (P, P') of points from $\tilde{\mathcal{P}}$ at distance 3. One such pair is (P_1, P_8). By [6, Cor. 5.3], an element of $\widehat{S_f}$ that stabilizes a given point of $\langle P_1, P_8 \rangle \setminus \{P_1, P_8\}$ must either stabilize the ordered pair (P_1, P_8) or interchange P_1 and P_8.

The stabilizer $S_{(P_1, P_8)}$ of the ordered pair (P_1, P_8) is isomorphic to $\mathrm{GL}(3, q)$. The normal subgroup $\mathrm{SL}(3, q)$ acts trivially on both the points P_1 and P_8, whereas an element of $Z(S_{(P_1,P_8)})$ will multiply p_8 by a scalar a and multiply p_1 by $1/a$. Such an element takes the point $\langle p_1+p_8\rangle$ to $\langle p_1+a^2p_8\rangle$.

There is also a group element that interchanges the points P_1 and P_8 and, specifically, takes p_1 to p_8 and p_8 to $-p_1$. This transformation takes the point $\langle p_1+p_8\rangle$ to $\langle p_1-p_8\rangle$. If -1 is a nonsquare in $\mathbb{F}_q$ then all the points of $\langle P_1,P_8\rangle\setminus\{P_1,P_8\}$ are in the same orbit. On the other hand, if -1 is a square in $\mathbb{F}_q$ then $\langle p_1+p_8\rangle$ and $\langle p_1+dp_8\rangle$ are in different orbits. We obtain in the former case a single orbit with representative P_{18} and orbit size $\frac{q^6(q^4-1)(q^3+1)}{2}$; in the latter case we have two orbits, with representatives P_{18} and P_{19}, each with orbit size $\frac{q^6(q^4-1)(q^3+1)}{4}$.

We next show that the group S_f contains a subgroup $G\cong \mathrm{GU}(3,q^2)$. Recall that δ is an element of $\mathbb{F}_{q^2}$ such that $\delta^2=d$ and $\bar{w}_i=\delta\bar{v}_i$ for every $i\in\{1,2,3\}$. For any $\alpha\in\mathbb{F}_{q^2}$, put $\bar{\alpha}:=\alpha^q$. Note that for $\alpha=a+b\delta$ we have $\bar{\alpha}=a-b\delta$.

Now define a map $h\colon V\times V\to\mathbb{F}_{q^2}$ as follows ($\alpha_i,\beta_i\in\mathbb{F}_{q^2}$):

$$h\left(\sum_{i=1}^{3}\alpha_i\bar{v}_i,\ \sum_{i=1}^{3}\beta_i\bar{v}_i\right)=\frac{1}{2\bar{\delta}}\sum_{i=1}^{3}\alpha_i\bar{\beta}_i.$$

Since $\mathrm{tr}(\delta)=0$, this defines a skew Hermitian form on V. It then follows that the map $f'\colon V\times V\to\mathbb{F}_q$ given by $f'(\bar{v},\bar{w})=\mathrm{tr}(h(\bar{v},\bar{w}))$ is an alternating form. We claim that $f'=f$. We compute $f'(\bar{v}_i,\bar{v}_j)$, $f'(\bar{w}_i,\bar{w}_j)$, and $f'(\bar{v}_i,\bar{w}_j)$ for $i\neq j$ and $f'(\bar{v}_i,\bar{w}_i)$ for $i=1,2,3$.

By definition, $h(\bar{v}_i,\bar{v}_j)=h(\bar{w}_i,\bar{w}_j)=h(\bar{v}_i,\bar{w}_j)=0$ for $i\neq j$; consequently, we need only compute $f'(\bar{v}_i,\bar{w}_i)$. By definition this is $\mathrm{tr}(h(\bar{v}_i,\delta\bar{v}_i))=\mathrm{tr}(\bar{\delta}/2\bar{\delta})=\mathrm{tr}(1/2)=1$, so our claim holds.

It now follows that if σ is an isometry of (V,h)—that is, a unitary transformation—then σ is an isometry of the symplectic space (V,f). Therefore, if $G=\{\sigma\in\mathrm{GL}_{\mathbb{F}_{q^2}}(V)\mid h(\sigma(\bar{u}_1),\sigma(\bar{u}_2))=h(\bar{u}_1,\bar{u}_2)\ \forall\bar{u}_1,\bar{u}_2\in V\}$, then $G\cong\mathrm{GU}(3,q^2)$ and $G<S_f$. Let G' be the derived subgroup of G; then G' is isomorphic to $\mathrm{SU}(3,q^2)$. By Lemma 3.1, it follows that $\widehat{G'}$ centralizes $\langle p_{20},p_{21}\rangle$.

We next determine the stabilizer of the point P_{20}. We will first show in a series of lemmas that if $\bar{v},\bar{w}\in\langle p_{20},p_{21}\rangle$ and if $\theta\in S_f$ satisfies $\theta(\bar{v})=\bar{w}$, then $\theta(\langle p_{20},p_{21}\rangle)=\langle p_{20},p_{21}\rangle$.

Let V' denote the 6-dimensional vector space over $\mathbb{F}_{q^2}$ with basis $\mathcal{S}$. For a vector $\bar{x}=a_1\bar{v}_1+a_2\bar{v}_2+a_3\bar{v}_3+b_1\bar{w}_1+b_2\bar{w}_2+b_3\bar{w}_3\in V'$ we define $\bar{x}^q=a_1^q\bar{v}_1+a_2^q\bar{v}_2+a_3^q\bar{v}_3+b_1^q\bar{w}_1+b_2^q\bar{w}_2+b_3^q\bar{w}_3$. For $\theta\in\mathrm{GL}(V)$ we denote by $\bar{\theta}$ the element induced by θ in $\mathrm{GL}(V')$ and by $\overline{\theta'}$ the corresponding element of $\mathrm{GL}\left(\bigwedge^3 V'\right)$.

LEMMA 3.11. *Let $\{\bar{e}_1,\bar{e}_2,\ldots,\bar{e}_6\}$ and $\{\bar{e}'_1,\bar{e}'_2,\ldots,\bar{e}'_6\}$ be two bases of V' such that $\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\bar{e}_4\wedge\bar{e}_5\wedge\bar{e}_6=\bar{e}'_1\wedge\bar{e}'_2\wedge\bar{e}'_3+\bar{e}'_4\wedge\bar{e}'_5\wedge\bar{e}'_6$. Then $\{\langle\bar{e}_1,\bar{e}_2,\bar{e}_3\rangle,\langle\bar{e}_4,\bar{e}_5,\bar{e}_6\rangle\}=\{\langle\bar{e}'_1,\bar{e}'_2,\bar{e}'_3\rangle,\langle\bar{e}'_4,\bar{e}'_5,\bar{e}'_6\rangle\}$.*

Proof. Put $\alpha:=\bar{e}_1\wedge\bar{e}_2\wedge\bar{e}_3+\bar{e}_4\wedge\bar{e}_5\wedge\bar{e}_6=\bar{e}'_1\wedge\bar{e}'_2\wedge\bar{e}'_3+\bar{e}'_4\wedge\bar{e}'_5\wedge\bar{e}'_6$. For every vector $\bar{x}$ of V', let $A_{\bar{x}}$ denote the subspace of V' consisting of all vectors $\bar{y}$ satisfying $\alpha\wedge\bar{x}\wedge\bar{y}=0$. Let B be the subset of V' consisting of all vectors $\bar{x}$ of V' such that $\dim(A_{\bar{x}})\geq 4$. We will now prove that $B=\langle\bar{e}_1,\bar{e}_2,\bar{e}_3\rangle\cup\langle\bar{e}_4,\bar{e}_5,\bar{e}_6\rangle$.

In a completely similar way, one can also prove that $B = \langle \bar{e}_1', \bar{e}_2', \bar{e}_3' \rangle \cup \langle \bar{e}_4', \bar{e}_5', \bar{e}_6' \rangle$, which then implies that $\{\langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle, \langle \bar{e}_4, \bar{e}_5, \bar{e}_6 \rangle\} = \{\langle \bar{e}_1', \bar{e}_2', \bar{e}_3' \rangle, \langle \bar{e}_4', \bar{e}_5', \bar{e}_6' \rangle\}$.

Put $\bar{x} = \delta_1 \bar{e}_1 + \delta_2 \bar{e}_2 + \cdots + \delta_6 \bar{e}_6$ and $\bar{y} = a_1 \bar{e}_1 + a_2 \bar{e}_2 + \cdots + a_6 \bar{e}_6$. Then the fact that $\alpha \wedge \bar{x} \wedge \bar{y} = 0$ implies that

$$\begin{bmatrix} -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_3 & 0 & \delta_1 & 0 & 0 & 0 \\ 0 & -\delta_3 & \delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_5 & \delta_4 & 0 \\ 0 & 0 & 0 & -\delta_6 & 0 & \delta_4 \\ 0 & 0 & 0 & 0 & -\delta_6 & \delta_5 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So, $\dim(V_{\bar{x}}) \geq 4$ if and only if the rank of

$$\begin{bmatrix} -\delta_2 & \delta_1 & 0 & 0 & 0 & 0 \\ -\delta_3 & 0 & \delta_1 & 0 & 0 & 0 \\ 0 & -\delta_3 & \delta_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta_5 & \delta_4 & 0 \\ 0 & 0 & 0 & -\delta_6 & 0 & \delta_4 \\ 0 & 0 & 0 & 0 & -\delta_6 & \delta_5 \end{bmatrix}$$

is at most 2. This happens precisely when $(\delta_1, \delta_2, \delta_3) = (0, 0, 0)$ or $(\delta_4, \delta_5, \delta_6) = (0, 0, 0)$, that is, when $\bar{x} \in \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle \cup \langle \bar{e}_4, \bar{e}_5, \bar{e}_6 \rangle$. □

The proof of the following lemma is straightforward.

LEMMA 3.12. *For all $a, b \in \mathbb{F}_q$, $(a + b\delta) \cdot (\bar{w}_1 + \delta \bar{v}_1) \wedge (\bar{w}_2 + \delta \bar{v}_2) \wedge (\bar{w}_3 + \delta \bar{v}_3) + (a - b\delta) \cdot (\bar{w}_1 - \delta \bar{v}_1) \wedge (\bar{w}_2 - \delta \bar{v}_2) \wedge (\bar{w}_3 - \delta \bar{v}_3) = 2a \cdot p_{21} + 2bd \cdot p_{20}$.*

COROLLARY 3.13. *The vectors of the 2-space $\langle p_{20}, p_{21} \rangle$ of $\bigwedge^3 V$ are precisely the vectors of the form $(a + b\delta) \cdot (\bar{w}_1 + \delta \bar{v}_1) \wedge (\bar{w}_2 + \delta \bar{v}_2) \wedge (\bar{w}_3 + \delta \bar{v}_3) + (a - b\delta) \cdot (\bar{w}_1 - \delta \bar{v}_1) \wedge (\bar{w}_2 - \delta \bar{v}_2) \wedge (\bar{w}_3 - \delta \bar{v}_3)$, where $a, b \in \mathbb{F}_q$.*

By Lemma 3.11 and Corollary 3.13, we have the following result.

COROLLARY 3.14. *If $\theta \in S_f$ such that $\hat{\theta}$ maps a nonzero vector of $\langle p_{20}, p_{21} \rangle$ to a nonzero vector of $\langle p_{20}, p_{21} \rangle$, then $\hat{\theta}$ stabilizes $\langle p_{20}, p_{21} \rangle$. Moreover, one of the following statements holds.*

(i) *$\overline{\theta'}$ stabilizes the 1-spaces $\langle (\bar{w}_1 + \delta \bar{v}_1) \wedge (\bar{w}_2 + \delta \bar{v}_2) \wedge (\bar{w}_3 + \delta \bar{v}_3) \rangle$ and $\langle (\bar{w}_1 - \delta \bar{v}_1) \wedge (\bar{w}_2 - \delta \bar{v}_2) \wedge (\bar{w}_3 - \delta \bar{v}_3) \rangle$ of $\bigwedge^3 V'$.*

(ii) *$\overline{\theta'}$ interchanges the 1-spaces $\langle (\bar{w}_1 + \delta \bar{v}_1) \wedge (\bar{w}_2 + \delta \bar{v}_2) \wedge (\bar{w}_3 + \delta \bar{v}_3) \rangle$ and $\langle (\bar{w}_1 - \delta \bar{v}_1) \wedge (\bar{w}_2 - \delta \bar{v}_2) \wedge (\bar{w}_3 - \delta \bar{v}_3) \rangle$ of $\bigwedge^3 V'$.*

Let W_f denote the subgroup of S_f consisting of all $\theta \in S_f$ for which $\hat{\theta}$ stabilizes $\langle p_{20}, p_{21} \rangle$. Let U_f denote the normal subgroup of W_f consisting of all $\theta \in W_f$

for which case (i) of Corollary 3.14 occurs. Put $\widehat{W_f} := \{\hat{\theta} \mid \theta \in W_f\}$ and $\widehat{U_f} := \{\hat{\theta} \mid \theta \in U_f\}$.

REMARK 3.15. Let θ be an element of U_f, let μ_1 be the restriction of $\bar{\theta}$ to the 3-space $\langle \bar{w}_1 + \delta\bar{v}_1, \bar{w}_2 + \delta\bar{v}_2, \bar{w}_3 + \delta\bar{v}_3\rangle$ of V', and let μ_2 be the restriction of $\bar{\theta}$ to the 3-space $\langle \bar{w}_1 - \delta\bar{v}_1, \bar{w}_2 - \delta\bar{v}_2, \bar{w}_3 - \delta\bar{v}_3\rangle$ of V'. Then $1 = \det(\bar{\theta}) = \det(\mu_1)\cdot\det(\mu_2)$.

Now let $\bar{x}$ be an arbitrary vector of $\langle \bar{w}_1 + \delta\bar{v}_1, \bar{w}_2 + \delta\bar{v}_2, \bar{w}_3 + \delta\bar{v}_3\rangle$. Since $\bar{x} + \bar{x}^q \in V$, we have $\bar{y} := \bar{\theta}(\bar{x} + \bar{x}^q) = \bar{\theta}(\bar{x}) + \bar{\theta}(\bar{x}^q) \in V$. Also, $\bar{y} = \bar{y}^q = [\bar{\theta}(\bar{x}^q)]^q + [\bar{\theta}(\bar{x})]^q$. Since there exist unique $\bar{y}_1 \in \langle \bar{w}_1 + \delta\bar{v}_1, \bar{w}_2 + \delta\bar{v}_2, \bar{w}_3 + \delta\bar{v}_3\rangle$ and $\bar{y}_2 \in \langle \bar{w}_1 - \delta\bar{v}_1, \bar{w}_2 - \delta\bar{v}_2, \bar{w}_3 - \delta\bar{v}_3\rangle$ such that $\bar{y} = \bar{y}_1 + \bar{y}_2$, we necessarily have $\bar{\theta}(\bar{x}^q) = [\bar{\theta}(\bar{x})]^q$. Hence $\mu_2(\bar{x}^q) = \bar{\theta}(\bar{x}^q) = [\bar{\theta}(\bar{x})]^q = [\mu_1(\bar{x})]^q$.

By the previous paragraph, $\det(\mu_2) = [\det(\mu_1)]^q$. If $\det(\mu_1) = a + b\delta$, then $\det(\mu_2) = a - b\delta$ and, since $\det(\mu_1)\cdot\det(\mu_2) = 1$, we have $a^2 - b^2 d = 1$.

Conversely, let $a, b \in \mathbb{F}_q$ such that $a^2 - b^2 d = 1$. Then the element of $\mathrm{GL}(V)$ determined by

$$\bar{v}_1 \mapsto a\cdot\bar{v}_1 + b\cdot\bar{w}_1, \quad \bar{w}_1 \mapsto bd\cdot\bar{v}_1 + a\cdot\bar{w}_1,$$
$$\bar{v}_2 \mapsto \bar{v}_2, \quad \bar{w}_2 \mapsto \bar{w}_2, \quad \bar{v}_3 \mapsto \bar{v}_3, \quad \bar{w}_3 \mapsto \bar{w}_3$$

determines an element of U_f for which the corresponding value of $\det(\mu_1)$ is equal to $a + b\delta$.

LEMMA 3.16. *Let* $a_1, a_2, b_1, b_2 \in \mathbb{F}_q$ *such that* $(a_1, a_2) \neq (0,0) \neq (b_1, b_2)$. *Then the* 1*-spaces* $\langle a_1 p_{21} + a_2 p_{20}\rangle$ *and* $\langle b_1 p_{21} + b_2 p_{20}\rangle$ *belong to the same* $\widehat{U_f}$*-orbit if and only if* $\left(a_1^2 - \frac{a_2^2}{d}\right)\left(b_1^2 - \frac{b_2^2}{d}\right)$ *is a square.*

Proof. By Lemma 3.12,

$$a_1 p_{21} + a_2 p_{20} = \left(\frac{a_1}{2} + \frac{a_2}{2d}\delta\right)\cdot(\bar{w}_1 + \delta\bar{v}_1)\wedge(\bar{w}_2 + \delta\bar{v}_2)\wedge(\bar{w}_3 + \delta\bar{v}_3)$$
$$+ \left(\frac{a_1}{2} - \frac{a_2}{2d}\delta\right)\cdot(\bar{w}_1 - \delta\bar{v}_1)\wedge(\bar{w}_2 - \delta\bar{v}_2)\wedge(\bar{w}_3 - \delta\bar{v}_3)$$

and

$$b_1 p_{21} + b_2 p_{20} = \left(\frac{b_1}{2} + \frac{b_2}{2d}\delta\right)\cdot(\bar{w}_1 + \delta\bar{v}_1)\wedge(\bar{w}_2 + \delta\bar{v}_2)\wedge(\bar{w}_3 + \delta\bar{v}_3)$$
$$+ \left(\frac{b_1}{2} - \frac{b_2}{2d}\delta\right)\cdot(\bar{w}_1 - \delta\bar{v}_1)\wedge(\bar{w}_2 - \delta\bar{v}_2)\wedge(\bar{w}_3 - \delta\bar{v}_3).$$

By Remark 3.15, the 1-spaces $\langle a_1 p_{21} + a_2 p_{20}\rangle$ and $\langle b_1 p_{21} + b_2 p_{20}\rangle$ belong to the same $\widehat{U_f}$-orbit if and only if there exist a $\lambda \in \mathbb{F}_q^*$ and $c_1, c_2 \in \mathbb{F}_q$ satisfying $c_1^2 - c_2^2 d = 1$ such that $\left(\frac{a_1}{2} + \frac{a_2}{2d}\delta\right)\cdot(c_1 + c_2\delta)\cdot\lambda = \frac{b_1}{2} + \frac{b_2}{2d}\delta$. If c_1' and c_2' are the unique elements of $\mathbb{F}_q$ such that $\left(\frac{a_1}{2} + \frac{a_2}{2d}\delta\right)(c_1' + c_2'\delta) = \frac{b_1}{2} + \frac{b_2}{2d}\delta$, then one readily verifies that $b_1^2 - \frac{b_2^2}{d} = \left(a_1^2 - \frac{a_2^2}{d}\right)((c_1')^2 - (c_2')^2 d)$. It now follows that

$\langle a_1 p_{21} + a_2 p_{20}\rangle$ and $\langle b_1 p_{21} + b_2 p_{20}\rangle$ belong to the same $\widehat{U_f}$-orbit if and only if $\big(a_1^2 - \frac{a_2^2}{d}\big)\big(b_1^2 - \frac{b_2^2}{d}\big)$ as a square. □

LEMMA 3.17. *There are two $\widehat{U_f}$-orbits on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.*

Proof. If $a_1 = 1$ and $a_2 = 0$, then $a_1^2 - \frac{a_2^2}{d} = 1$ is a square.

Now, choose $a_1 \in \mathbb{F}_q^*$. Then there exist $a_2, a_3 \in \mathbb{F}_q^*$ such that $da_1^2 = a_2^2 + a_3^2$. Then $a_1^2 - \frac{a_2^2}{d} = \frac{a_3^2}{d}$ is a nonsquare.

The claim now follows from Lemma 3.16. □

Next we will construct a particular element $\widehat{\theta^*}$ of $\widehat{W_f} \setminus \widehat{U_f}$. Let $A, B \in \mathbb{F}_q^*$ such that $\big(\frac{A}{B}\big)^2 + \big(\frac{1}{B}\big)^2 = d$ (hence $A^2 - B^2 d = -1$), and consider the following map θ^* of S_f:

$$\begin{cases} \bar{v}_1 \mapsto A\cdot\bar{v}_1 + B\cdot\bar{w}_1, \\ \bar{w}_1 \mapsto -Bd\cdot\bar{v}_1 - A\cdot\bar{w}_1, \\ \bar{v}_2 \mapsto A\cdot\bar{v}_2 - B\cdot\bar{w}_2, \\ \bar{w}_2 \mapsto Bd\cdot\bar{v}_2 - A\cdot\bar{w}_2, \\ \bar{v}_3 \mapsto A\cdot\bar{v}_3 + B\cdot\bar{w}_3, \\ \bar{w}_3 \mapsto -Bd\cdot\bar{v}_3 - A\cdot\bar{w}_3. \end{cases}$$

Then one readily verifies that $\theta^* \in W_f \setminus U_f$. Moreover, $\widehat{\theta^*}(p_{21}) = Ap_{21} + Bdp_{20}$.

PROPOSITION 3.18. (i) *If -1 is a nonsquare, then $\widehat{W_f}$ has one orbit on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.*

(ii) *If -1 is a square, then $\widehat{W_f}$ has two orbits on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.*

Proof. Since $\widehat{U_f}$ is a normal index-2 subgroup of $\widehat{W_f}$, we can conclude as follows.

- If $\langle p_{21}\rangle$ and $\langle\widehat{\theta^*}(p_{21})\rangle$ belong to the same $\widehat{U_f}$-orbit, then $\widehat{\theta^*}$ stabilizes the two $\widehat{U_f}$-orbits; in this case, $\widehat{W_f}$ has two orbits on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.
- If $\langle p_{21}\rangle$ and $\langle\widehat{\theta^*}(p_{21})\rangle$ belong to different $\widehat{U_f}$-orbits, then $\widehat{\theta^*}$ interchanges the two $\widehat{U_f}$-orbits; in this case, $\widehat{W_f}$ has one orbit on the set of 1-spaces of $\langle p_{20}, p_{21}\rangle$.

Now $\langle p_{21}\rangle$ and $\langle\widehat{\theta^*}(p_{21})\rangle$ belong to the same $\widehat{U_f}$-orbit if and only if

$$\left(1^2 - \frac{0}{d^2}\right)\left(A^2 - \frac{(Bd)^2}{d}\right) = A^2 - B^2 d = -1$$

is a square. The proposition follows. □

4. Proof of the Fusion Theorem

Since $\widehat{S_f}$ is normal in $\widehat{\widehat{G_f}}$, it follows that if two $\widehat{S_f}$-orbits were to fuse via $\widehat{\sigma^*}$ or $\widehat{T_\gamma}$, $\gamma \in \mathrm{Aut}(\mathbb{F}_q)$, then they must have the same size. When -1 is a nonsquare there are no such possibilities. When -1 is a square it could be that the orbits with representatives P_{18} and P_{19} fuse and that the orbits with representatives P_{20} and P_{21} also fuse. We show that this is indeed the case.

Suppose then that -1 is a square. Now $\widehat{\sigma^*}(p_1 + p_8) = p_1 + d^3 p_8$ and d^3 is a nonsquare. The points $P_{19} = \langle p_1 + dp_8\rangle$ and $\langle p_1 + d^3 p_8\rangle$ are in the same $\widehat{S_f}$-orbit. So, in this case we get the fusion of the $\hat{S}_f$-orbits with representatives P_{18} and P_{19}. We also show that the orbits with representatives P_{20} and P_{21} fuse. Before doing so, observe that the points $P_{21} = \langle p_8 + dp_2 + dp_3 + dp_5\rangle$ and $\langle p_1 + dp_4 + dp_6 + dp_7\rangle$ are in the same $\widehat{S_f}$-orbit. Let $\sigma(\bar{v}_i) = \bar{w}_i$ and $\sigma(\bar{w}_i) = -\bar{v}_i$ for $i = 1,2,3$; then $\hat{\sigma}(p_1 + dp_4 + dp_6 + dp_7) = p_8 + dp_2 + dp_3 + dp_5$, from which the claim follows. Now $\widehat{\sigma^*}(p_{20}) = \widehat{\sigma^*}(dp_1 + p_4 + p_6 + p_7) = dp_1 + d^2 p_4 + d^2 p_6 + d^2 p_7 = d(p_1 + dp_4 + dp_6 + dp_7)$ and therefore $\widehat{\sigma^*}(P_{20}) = \langle p_1 + dp_4 + dp_6 + dp_7\rangle$ is in the $\widehat{S_f}$-orbit of P_{21}. This completes the proof of the Fusion Theorem.

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