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Root Systems and Optimal Block Designs

Peter J. Cameron

Dedicated to the memory of Donald G. Higman

Let A be a symmetric $n \times n$ matrix with entries 0, $+1$, and -1, with zero diagonal and constant row sums, and having least eigenvalue greater than -2. The aim of this paper is to describe such matrices. The work is motivated by a question of C.-S. Cheng on optimal block designs; the background is described briefly in the paper.

Of course, we may assume that the matrix is *connected*—that is, not permutation-equivalent to one of the form $\begin{pmatrix} B & O \\ O & C \end{pmatrix}$. We also note that the constant row sum c is an eigenvalue and so $c \geq -1$.

Such a matrix is represented by a set of vectors in a spherical root system, and hence (apart from finitely many examples represented in the exceptional root systems E_6, E_7, and E_8) by either a tree with oriented edges or a unicyclic graph with edges either signed or oriented. We give a test for recognizing when such a graph represents a matrix satisying the conditions of the question. There are many examples. All matrices occurring in the exceptional root systems are determined.

1. Least Eigenvalue −1

As a warm-up, I consider the case where the least eigenvalue is -1.

Let A be such a matrix. Then $A + I$ is positive semidefinite and so is the Gram matrix of inner products of n vectors $v_1, \dots, v_n$ in Euclidean space. Thus $\|v_i\| = 1$ and $v_i \cdot v_j \in \{0, +1, -1\}$ for all i, j. Clearly such vectors consist of an orthonormal set of vectors and their negatives, with each vector possibly repeated. Connectedness implies that there is just one vector v in the set. So if we use only the vector v then all inner products are $+1$ and we have $A = J - I$, where J is the all-1 matrix; if we use both v and $-v$, then $A = \begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$.

Note that the first type has constant row sum $n - 1$. The second type has constant row sum -1 if and only if all the blocks are square; that is, iff v and $-v$ occur equally often in the vector representation. Each matrix has the property that the entry 0 does not occur (except in the diagonal). We call these the *trivial* examples.

2. Root Systems

For the initial analysis we ignore the "constant row sum" condition, and also for a while we assume only that the least eigenvalue is not smaller than -2.

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Let A be such a matrix. Then $A+2I$ is positive semidefinite and thus is the matrix of inner products of a set of vectors $v_1, \ldots, v_n$, where $\|v_i\| = \sqrt{2}$ and $v_i \cdot v_j \in \{0, +1, -1\}$ for all $i \neq j$. Also, the matrix A can be assumed to be connected, so there is no orthogonal decomposition of the Euclidean space such that the vectors all lie in the summands. (This means that the graph with vertices $v_1, \ldots, v_n$, with two vertices adjacent if they are not orthogonal, is connected.) We can assume that the vectors span the space.

According to the main result of [1], these vectors form a subset of a root system of type A_n, D_n, E_6, E_7, or E_8. Here is a brief account.

A *root system* is a finite set of nonzero vectors (called *roots*) in Euclidean space $\mathbb{R}^n$ that is closed under reflection in the hyperplane perpendicular to each of its elements. It is *indecomposable* if it is not contained in the union of two nonzero perpendicular subspaces. Indecomposable root systems arose in the classification of simple Lie algebras; they were determined by Cartan and Killing.

We are interested in the root systems with all roots of the same length. Here is the list in this case; there are two infinite families and three sporadic examples.

A_n: Let $e_1, \ldots, e_{n+1}$ be an orthonormal basis for $\mathbb{R}^{n+1}$. The vectors of A_n are all $e_i - e_j$ for $1 \leq i, j \leq n+1$ with $i \neq j$. (These vectors lie in a hyperplane of $\mathbb{R}^{n+1}$ and thus span a space of dimension n.)

D_n: The vectors of D_n are all $\pm e_i \pm e_j$ for $1 \leq i < j \leq n$, where $e_1, \ldots, e_n$ form an orthonormal basis for $\mathbb{R}^n$.

E_n: These are three specific sets in $\mathbb{R}^n$ for $n = 6, 7, 8$. Several descriptions of them can be found in [1].

Thus our matrix is represented by a subset of one of these root systems, where we choose at most one out of each pair $v, -v$.

Now let us assume that the least eigenvalue of A is strictly greater than -2. Then $A+2I$ is positive definite, so the representing vectors $v_1, \ldots, v_n$ are linearly independent and form a basis for $\mathbb{R}^n$.

We therefore say that an $n \times n$ matrix A is *admissible* if:

- A is real symmetric and has entries $0, +1$, and -1 only;
- the diagonal entries are all 0;
- A is connected; and
- the smallest eigenvalue of A is greater than -2.

Our problem is to determine the admissible matrices with constant row sum. This is equivalent to choosing a connected subset of a root system whose vectors form a vector space basis for the ambient space and such that the Gram matrix of the subset has constant row sums.

3. Determinant

In this section we demonstrate the following result.

PROPOSITION 3.1. *Let A be an admissible $n \times n$ matrix. Then* $\det(2I + A) = n + 1$ *or* 4 *except possibly if* $n = 6, 7,$ *or* 8, *in which case* $\det(2I + A)$ *may be* 3, 2, *or* 1, *respectively.*

To prove this we need the following result. The *root lattice* associated with a root system is the integer span of the root system. (The term "lattice" here means "discrete spanning subgroup of $\mathbb{R}^n$".) We refer to Humphreys [3] for more information about root systems and root lattices.

PROPOSITION 3.2. *Let S be a connected subset of a root system R that forms a vector space basis for the ambient space. Then S is an integer basis for the corresponding root lattice except possibly if $R = E_7$ or $R = E_8$, in which case S may be an integer basis for the A_7, A_8, or D_8 root lattice.*

Proof. The integer span $L = \langle S \rangle_{\mathbb{Z}}$ is a sublattice of the root lattice. Now $L \cap R$ is a root system. (This requires that, for two vectors v, w in this set with $v \cdot w = \varepsilon = \pm 1$, we have $v - \varepsilon w \in L \cap R$; this holds because R is closed under this operation by definition and L is a lattice.) Moreover, $L \cap R \supseteq S$, so $L \cap R$ is connected. Thus $L \cap R$ is a root system of type A_n, D_n, or E_n and is contained in the given root system. Now the only inclusions among these root systems are $A_7 \subseteq E_7$, $A_8 \subseteq E_8$, and $D_8 \subseteq E_8$. □

Proof of Proposition 3.1. Let L be an integral lattice in $\mathbb{R}^n$ (this means that the inner product of any two of its vectors is an integer). The *dual lattice* L^* consists of all vectors $v \in \mathbb{R}^n$ such that $v \cdot w \in \mathbb{Z}$ for all $w \in L$. Clearly L^* is a lattice containing L, and the index $|L^*/L|$ is finite. This number is the *connection number* of the lattice. If S is any integral basis for L, then the determinant of the Gram matrix of S is equal to the connection number of L.

The connection numbers of A_n, D_n, E_6, E_7, and E_8 are (respectively) $n + 1$, 4, 3, 2, and 1. This finishes the proof. □

Hence, if we determine the admissible matrices then we can decide which root system contains each matrix simply by calculating its determinant. The result also restricts the possible row sum c, since $c + 2$ (which is the row sum of $2I + A$ and thus an eigenvalue of this matrix) must divide the determinant. To see this, add the other rows to the first (this does not change the determinant) and then expand along the first row. Every term in the expansion is a multiple of $c + 2$ and hence so is the determinant.

4. The Case A_n

We have a basis $\{v_1, \ldots, v_n\}$ for the root system A_n. We can represent it as a directed graph on $n + 1$ vertices as follows. If $v_i = e_j - e_k$, we represent v_i as a directed edge from e_k to e_j. This graph contains no circuits, since the sum (with appropriate signs) of the vectors corresponding to the edges in a circuit is zero. Because the vectors form a basis, the graph is a tree.

Now, given a directed tree with n edges, the matrix A is constructed as follows. Entries on the diagonal, or corresponding to a pair of edges with no common vertex, are 0. The entry corresponding to a pair of edges meeting at exactly one vertex is -1 if the edges are "head to tail" there, or $+1$ if they are "head to head" or "tail to tail".

The constant row sum condition on A means that, for any edge, if we calculate the entries as just described for all edges meeting the given edge at exactly one vertex and sum them, the result is a constant c.

For any vertex v of the tree, let $d(v)$ be the degree of v and $s(v)$ the "signed degree" (the number of incoming edges minus the number of outgoing edges). Then the sum of the row of A corresponding to the directed edge $v \to w$ from v to w is

$$(-1)(s(v)+1) + (+1)(s(w)-1) = s(w) - s(v) - 2.$$

Thus we have the following statement.

THEOREM 4.1. *An admissible matrix having row sums c arises (in the manner described here) from an oriented tree T if and only if $s(w) - s(v) = c + 2$ for every directed edge $v \to w$ of T. Reversing the orientation of every edge does not change the matrix.*

We note several consequences as follows.

COROLLARY 4.2. *Let T be an oriented tree that satisfies the preceding conditions. Suppose, without loss of generality, that there is an edge directed out of a leaf x. Then all values of $s(v)$ are congruent to -1 modulo $c+2$, and if the vertices are arranged on levels corresponding to the values of $s(v)$, then each oriented edge goes from a level to the next level above.*

Proof. We lose no generality because we may reverse all orientations. If x is as in the statement then $s(x) = -1$, and the theorem (together with the connectedness of T) shows that all values of $s(v)$ are congruent modulo $c+2$. The last claim follows because $s(w) - s(v) = c + 2$. □

COROLLARY 4.3. *If $c \notin \{-1, 0\}$, then there cannot be both a leaf with an outgoing edge and a leaf with an incoming edge.*

Proof. If x and y are such leaves, then $s(x) = -1$ and $s(y) = 1$. □

COROLLARY 4.4. *If c is even, then all vertices of the tree have odd degree. If c is odd, then the tree is bipartite and the parity of the degrees is even in one bipartite block and odd in the other; in particular, all leaves lie in the same bipartite block.*

Proof. For any edge $v \to w$, we have $s(w) - s(v) = c+2$ and $s(v) \equiv d(v) \bmod 2$, so $d(v) + d(w) \equiv c \bmod 2$.

If c is even then all degrees have the same parity; since there exist leaves, the parity is odd. If c is odd, then the two ends of an edge have degrees of opposite parity and the conclusion of the corollary follows. □

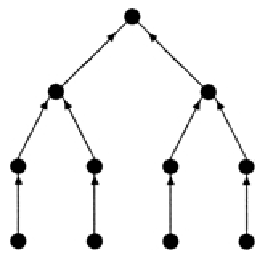
COROLLARY 4.5. *Suppose there is a vertex v that is on an edge toward a leaf and an edge from another leaf. Then the row sum is -1 and the degree of v is even, with half the edges entering v and half of them leaving.*

Proof. Let $v \to x$ and $y \to v$ be the two given edges, where x and y are leaves. Then $s(x) = +1$ and $s(y) = -1$, so the row sums corresponding to (v, x) and

(y, v) are respectively $-s(v) - 1$ and $s(v) - 1$. These are equal, so $s(v) = 0$ and the row sum is -1. $\square$

Now the examples $J - I$ and $\left(\begin{smallmatrix} J-I & -J \\ -J & J-I \end{smallmatrix}\right)$ are realized by stars. In the first case, direct all the edges in to the center; in the second, let the number of edges be even and direct half of them in and half of them out.

There are many examples of oriented trees satisfying our conditions. Here is the smallest one with row sums -1 that is not a star:

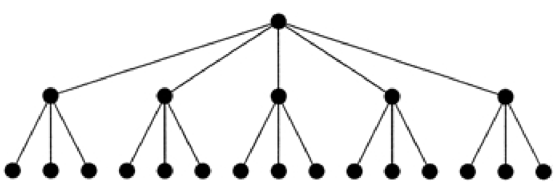


Numbering the edges from left to right and from bottom to top yields the admissible matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ - & 0 & 0 & 0 & 0 & + & 0 & 0 & - & 0 \\ 0 & - & 0 & 0 & + & 0 & 0 & 0 & - & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & - & 0 & 0 & + & 0 & 0 & - \\ 0 & 0 & 0 & 0 & - & - & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & 0 & 0 & - & - & + & 0 \end{pmatrix},$$

whose least eigenvalue, according to Maple, is about -1.860805854.

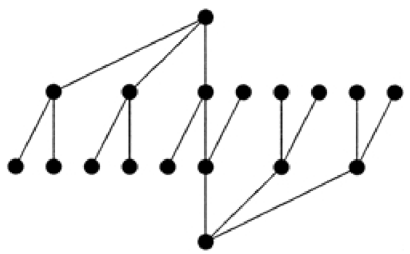
Examples with other values of c are easily produced. For instance, take a star with $2c + 3$ leaves all directed inward. Identify each leaf with the center of a star with $c+2$ leaves all directed inward. This produces a matrix of order $(c+3)(2c+3)$ with row sums c. The case $c = 0$ gives a matrix of order 9, the smallest nontrivial matrix represented in A_n. The following picture shows the case $c = 1$ (directions are not shown: all edges are oriented upward).



One can give constructions for new examples from old. For example, the following is obviously true and gives a construction for infinitely many examples. This example works only in the case where row sums are -1.

PROPOSITION 4.6. *Let T_1 and T_2 be oriented trees giving rise to admissible matrices, and for $i = 1, 2$ let v_i be a vertex of T_i satisfying $s(v_i) = 0$. Then the tree formed from the disjoint union of T_1 and T_2 by identifying v_1 and v_2 also gives an admissible matrix.*

Other recursive constructions are also possible. Rather than formulate general conditions, in the next picture we give an example of two trees glued along an edge. This shows also that the case can occur where row sums are 0 and there are initial and terminal leaves. Again, all edges are oriented upward.



5. The Case D_n

In this case our vectors are of the form $\pm e_i \pm e_j$ for $i \neq j$. Again the set can be represented by a connected graph, this time with n vertices and n edges. Note that a connected graph with equally many vertices and edges is *unicyclic* (consisting of a cycle with trees attached at some of its vertices).

There are several kinds of edges. A vector $e_i - e_j$ can be represented by a directed edge from j to i. A vector $e_i + e_j$ can be represented by an undirected edge carrying a $+$ sign (i.e., positive at both ends), and a vector $-e_i - e_j$ can be regarded as an undirected edge carrying a $-$ sign. We define $s(v)$ similarly to before: it is the number of positive undirected or incoming directed edges at v minus the number of negative undirected or outgoing edges at v. Now we must take some care to ensure that the vectors are linearly independent. The result is as follows.

THEOREM 5.1. *Let G be a unicyclic graph with each edge signed or directed. Then G corresponds to an admissible matrix having row sums c if and only if the following statements hold*:

(a) *the cycle of G contains an odd number of undirected edges*;

(b) *if $v \to w$ then $s(w) - s(v) = c + 2$, and if $\{v, w\}$ is undirected with sign ε then $\varepsilon(s(v) + s(w)) = c + 2$.*

Proof. Condition (a) guarantees that the edges in the cycle correspond to linearly independent vectors. It is clear that there are no other possible dependencies. Condition (b) is, as in the previous case, the translation of the "row sum c" condition. □

This time, however, we have the freedom of changing the signs of the basis vectors arbitrarily. For example, changing e_i to $-e_i$ will change a directed edge leaving i to an undirected edge with sign $+$ (and vice versa) and will also change a directed edge entering i to an undirected edge with sign $-$ (and vice versa). We can exploit this freedom as follows.

LEMMA 5.2. *A connected set of vectors forming a basis for D_n can be represented by a unicyclic graph in which all edges* (*except perhaps one*) *are undirected. If the unique cycle has odd length, then all edges are undirected*; *if it has even length, then there is a directed edge contained in the cycle.*

Proof. Temporarily remove an edge from the cycle to leave a tree. Working from a leaf of the tree, change signs of basis vectors so that each edge of the tree is undirected. Now, since the vectors in the cycle are linearly independent, it is easy to see that the remaining edge is undirected or directed according as the cycle has odd or even length. □

Note that examples do exist, as our next result shows.

PROPOSITION 5.3. *Suppose that the graph is a cycle of length n.*

(a) *If n is odd, then the row sums are $+2$ and all the signs can be taken to be $+$. If $n = 2r + 1$, then the eigenvalues of A are $2\cos(2j\pi/(2r+1))$ for $j = 0, \ldots, 2r$, with the smallest occurring when $j = r$.*
(b) *If n is even then $n \equiv 2 \bmod 4$, the row sums are 0, and the undirected edges have signs $(++--++\cdots--++)$ while the directed edge points from vertex 1 to vertex n. Finally, if $n = 4r + 2$, then the eigenvalues of A are $\pm 2\sin(2j\pi/(2r+1))$ for $j = 0, \ldots, 2r$.*

Proof. (a) Suppose that n is odd, so that all the edges are undirected. Clearly the row sums are either 0 or 2. If they are 0, then the two edges meeting a given edge have different signs, so the signs are $(++--++\cdots)$, which is not possible for odd n. Hence the row sums must be 2 and all edges have the same sign, which we can assume is $+$. Now the matrix A is the adjacency matrix of the n-cycle, whose least eigenvalue is as claimed. Note that these matrices do not contain the entry -1. For $n = 5$, the smallest eigenvalue is $-(\sqrt{5}+1)/2 = -1.618033988$.

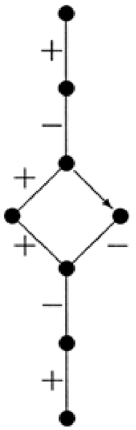
(b) Suppose n is even so that there is a single directed edge, which we may suppose is between vertices 1 and n. If the row sums are 2, then all undirected edges have the same sign, which implies that the directed edge has row sum 0—a contradiction. Hence the row sums are 0. Now, as in the preceding paragraph, each undirected edge other than $\{1, 2\}$ and $\{n-1, n\}$ has neighbors of opposite sign, and the neighbors of the directed edge have the same sign. This implies that $n \equiv 2 \bmod 4$ and so the signs are as claimed.

Now a bit of rearranging shows that A can be written in the form

$$\begin{pmatrix} O & C - C^\top \\ C^\top - C & O \end{pmatrix},$$

where C is the matrix of a directed $(2r+1)$-cycle. In this form it is not hard to calculate the eigenvalues. For $n = 6$, the smallest eigenvalue is $-\sqrt{3} = -1.732050808$. In general, the smallest occurs when j is nearest to $(2r+1)/4$ or $3(2r+1)/4$. $\square$

It is possible for examples other than cycles to occur. Here is one with least eigenvalue -1.813606504:



$$\begin{pmatrix}
0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\
- & 0 & - & + & 0 & 0 & 0 & 0 \\
0 & - & 0 & - & + & 0 & 0 & 0 \\
0 & + & - & 0 & 0 & - & 0 & 0 \\
0 & 0 & + & 0 & 0 & - & - & 0 \\
0 & 0 & 0 & - & - & 0 & + & 0 \\
0 & 0 & 0 & 0 & - & + & 0 & - \\
0 & 0 & 0 & 0 & 0 & 0 & - & 0
\end{pmatrix}$$

Some analogues of results in the A_n case hold. In particular, if the row sum c is even then all vertex degrees have the same parity (which is even if the graph consists of a single cycle, as in the proposition, and is odd otherwise, since leaves will exist); if c is odd, then the graph is bipartite (so the unique cycle has even length) and vertices in different bipartite blocks have degrees of opposite parity.

6. The Case E_n for $n = 6, 7, 8$

Now we determine all admissible matrices with constant row sum of order at most 8; this includes all matrices that generate exceptional root lattices.

The search strategy is as follows. If A is admissible with constant row sums, then the unsigned version of A is the adjacency matrix of a connected graph in which all vertex degrees have the same parity. By the handshaking lemma, odd parity can arise only when the number n of vertices is even, in which case the complement of the graph has all vertices of even parity. So we begin with a list of the Eulerian graphs (with even parity), include also their complements if n is even, and then select just the connected graphs from the list. The Eulerian graphs on small numbers of vertices are available from Brendan McKay's Web page [4].

Now we take each such graph, sign the edges in all possible ways, and test to see whether the resulting matrix is admissible and has constant row sums. This is done with a GAP program [2]. We also test isomorphism using the GAP package DESIGN [6].

As explained earlier, we determine the root system for any such matrix by calculating its determinant. We find that the number of such matrices in the exceptional root systems E_n are $2, 4, 12$ for $n = 6, 7, 8$, respectively. The output also includes the matrices in A_n and D_n, as a check on our earlier results. Matrices that generate the exceptional root systems are listed in the Appendix (Section 9).

The table that follows shows counts of the admissible matrices of order n with constant row sums for $n \leq 8$, classified by the type of root lattice they generate. The trivial types (in A_n) are $J - I$ and (for n even) $\begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$. We see that this accounts for all matrices in A_n for $n \leq 8$; the smallest nontrivial example has $n = 9$. The matrices in D_n are the cycles for $n = 5, 6, 7$ as well as three (including the example given in Section 5) for $n = 8$. The matrices are given in the Appendix.

n	3	4	5	6	7	8
A_n	1	2	1	2	1	2
D_n		0	1	1	1	3
E_n				2	4	12

An alternative search strategy would be to examine all the bases for the root systems E_6, E_7, and E_8. Bray (personal communication) has done this; his results agree with those reported in this paper.

7. Which Entries May Be Missing?

We allow the entries 0, +1, and −1 in the matrix A. What happens if not all of these entries occur?

In this section I will ignore the exceptional root systems of type E_6, E_7, and E_8. In principle they can contribute at most a finite number of counterexamples to the assertion of the following result; in fact, the results in the Appendix show that there are none.

PROPOSITION 7.1. *Suppose that A is admissible yet does not contain all the entries* $0, +1, -1$ *among its nondiagonal entries. Then $A = J - I$, or $A = \begin{pmatrix} J-I & -J \\ -J & J-I \end{pmatrix}$ with square blocks, or A is the adjacency matrix of an odd cycle.*

Proof. We examine three cases according to the missing value.

Case 1: -1 *does not occur.* In this case, A is the usual adjacency matrix of the line graph of the graph formed by the representing vectors in the root system A_n or D_n. (Directions can be ignored.)

Now, if the line graph of a graph G is regular, then either G is regular or G is semiregular bipartite (this means that the degrees are constant within each bipartite block).

Subcase: G is a tree. One bipartite block must consist of all the leaves, so every edge goes from a leaf to a nonleaf. Thus G is a star and $A = J - I$.

Subcase: G is unicyclic. If there are leaves then they form a bipartite block, which is clearly impossible. Hence our graph G is a cycle. But the line graph of an even cycle has eigenvalue -2. So G is an odd cycle and is isomorphic to its line graph.

Case 2: 0 *does not occur* (except in the diagonal). In this case, the graph has the property that any two edges meet. Therefore, it must be a star (in case A_n) or a 3-cycle (in case D_n). We thus have the "trivial" examples of the first section.

Case 3: $+1$ *does not occur.* In this case, the inner products of the basis vectors are all nonpositive. For type A_n, an easy argument shows that the bases are

$$e_1 - e_2, e_2 - e_3, \dots, e_n - e_{n+1},$$

which gives a Gram matrix with constant row sums only in the trivial cases $n = 2$ and $n = 3$. In the case D_n, we cannot have a cycle of length greater than 2 because each basis vector would occur twice with opposite signs in the representing vectors, whose sum would then be zero. So, without loss of generality, we have $e_{n-1} \pm e_n$, from which it is easy to see that the basis is

$$e_1 - e_2, e_2 - e_3, \dots, e_{n-2} - e_{n-1}, e_{n-1} - e_n, e_{n-1} + e_n,$$

which never gives a constant row sum for $n > 2$.

Observe that the bases here are the standard bases for the root systems as used in the theory of Lie algebras and elsewhere.

8. Optimal Block Designs

I conclude by describing the background in optimal design theory of the question of Cheng that motivates this research. For further details, see [5].

A *block design* here means a 1-design, or a binary proper equireplicate block design. Thus there are v points, each block is a set of k points, and each point is contained in r blocks.

The *incidence matrix* N of a block design D is the $v \times b$ matrix (where b is the number of blocks) whose i, j entry is 1 if the ith point is contained in the jth block and is 0 otherwise. The *concurrence matrix* $\Lambda = NN^\top$ is the $v \times v$ matrix whose i, j entry is the number of blocks containing the ith and jth points. The *information matrix* L is given by $L = rI - \Lambda/k$. The information matrix has a trivial eigenvalue 0 corresponding to the all-1 eigenvector.

Several notions of optimality of block designs have been proposed. A block design D is *A-optimal* (in the class of all block designs with given v, k, r) if it maximizes the harmonic mean of the nontrivial eigenvalues; it is *D-optimal* if it maximizes the geometric mean of the nontrivial eigenvalues; and it is *E-optimal* if it maximizes the smallest nontrivial eigenvalue. (We stress that the letters A, D, E here have no relation to the names of the root systems.)

If a balanced design (a 2-design) exists, then it is optimal in all three senses. But if the parameters are such that no balanced design exists then the question of optimality is more subtle, and there may be no design that is optimal on all criteria. Cheng's question was motivated by a search for E-optimal designs. The idea is that for a balanced design we have $\Lambda = (r-\lambda)I + \lambda J$, where J is the all-1 matrix; so it is reasonable to search for designs whose concurrence matrix is almost of this form—that is, of the form $(r-t)I + tJ - A$, where A is a symmetric integer matrix with small entries (say, $-1, 0, 1$) and constant row sums. To maximize the least eigenvalue of L, we should make the least eigenvalue of A as large as possible (say, greater than -2). This gives precisely the problem addressed in this paper, with $v = n$.

Two questions remain. First, given a matrix A of order v, can we find a block design with concurrence matrix $(r-t)I + tJ - A$? Second, is such a design actually E-optimal?

The first question can be readily answered by using the DESIGN software. Having chosen the block size k, we first choose r and t such that

$$\begin{aligned} t(v-1) - c &= r(k-1), \\ k &\mid vr, \end{aligned}$$

where c is the row sum of A. Then we can calculate Λ and use DESIGN to find a block design with the given v, k, r and with concurrence matrix Λ. For example, if A is one of the matrices in the root system E_6 (see the Appendix) and if $k = 3$, then $c = -1$ and so $2r = 5t + 1$. The smallest solution has $t = 1$ and $r = 3$; we find that there are no solutions for either matrix. However, for $t = 3$ and $r = 8$, the second matrix gives us a unique design with block set

$$\{123, 125, 125, 134, 136, 136, 146, 156, 234, 245, 246, 246, 256, 345, 345, 356\}.$$

The twelve matrices in E_8 all have $c = -1$. For $k = 3$, the smallest feasible values are $r = 18$ and $t = 5$, where designs exist for each of the twelve matrices. For $k = 4$, the smallest feasible values $r = 5$ and $t = 2$ cannot be realized; however, for the next values $r = 12$ and $t = 5$, once again designs exist for each of the twelve matrices.

I have not investigated the second question.

9. Appendix: Matrices of Order at Most 8

We list here the admissible matrices with constant row sums having order at most 8.

9.1. Matrices in A_n

We have only the trivial matrices $J - I$ and (for n even) $\left(\begin{smallmatrix} J-I & -J \\ -J & J-I \end{smallmatrix}\right)$.

9.2. Matrices in D_n for $n \geq 4$

For $n \leq 7$ we have the odd cycles C_5 and C_7 with all signs $+$. For $n = 6$, we have the cycle signed as described in Proposition 5.3(b); this yields the matrix $\left(\begin{smallmatrix} O & A \\ A^\top & O \end{smallmatrix}\right)$, where

$$A = \begin{pmatrix} 0 & - & + \\ + & 0 & - \\ - & + & 0 \end{pmatrix}.$$

For $n = 8$, there are three matrices:

$$\begin{pmatrix} 0 & + & 0 & 0 & - & 0 & 0 & 0 \\ + & 0 & 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & - & 0 \\ 0 & 0 & 0 & 0 & 0 & + & 0 & - \\ - & - & 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & + & + & 0 & 0 & - & - \\ 0 & 0 & - & 0 & + & - & 0 & + \\ 0 & 0 & 0 & - & + & - & + & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & + & 0 & 0 & - & - \\ 0 & 0 & + & - & 0 & - & 0 & 0 \\ 0 & + & 0 & - & - & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & + & - & 0 & - \\ 0 & 0 & - & + & 0 & 0 & - & 0 \\ 0 & - & 0 & - & 0 & + & 0 & 0 \\ 0 & + & - & 0 & - & 0 & 0 & 0 \\ + & 0 & 0 & - & 0 & - & 0 & 0 \\ - & 0 & + & 0 & - & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

9.3. Matrices in E_6

$$\begin{pmatrix} 0 & - & + & + & - & - \\ - & 0 & - & - & + & + \\ + & - & 0 & 0 & 0 & - \\ + & - & 0 & 0 & - & 0 \\ - & + & 0 & - & 0 & 0 \\ - & + & - & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & - & + & 0 & - \\ 0 & 0 & + & - & - & 0 \\ - & + & 0 & - & 0 & 0 \\ + & - & - & 0 & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

9.4. Matrices in E_7

$$\begin{pmatrix} 0 & 0 & 0 & - & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & 0 & - & + \\ - & 0 & 0 & 0 & + & + & - \\ 0 & - & 0 & + & 0 & + & - \\ 0 & 0 & - & + & + & 0 & - \\ + & + & + & - & - & - & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & + & 0 & 0 & 0 & 0 & - \\ + & 0 & 0 & 0 & 0 & 0 & - \\ 0 & 0 & 0 & 0 & - & 0 & + \\ 0 & 0 & 0 & 0 & 0 & - & + \\ 0 & 0 & - & 0 & 0 & + & 0 \\ 0 & 0 & 0 & - & + & 0 & 0 \\ - & - & + & + & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & - & 0 & 0 & + & 0 & 0 \\ - & 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & - & + \\ 0 & 0 & 0 & 0 & - & 0 & + \\ + & 0 & 0 & - & 0 & + & - \\ 0 & + & - & 0 & + & 0 & - \\ 0 & 0 & + & + & - & - & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & - & - & + & + & 0 & 0 \\ - & 0 & + & - & 0 & + & 0 \\ - & + & 0 & 0 & - & 0 & + \\ + & - & 0 & 0 & 0 & - & + \\ + & 0 & - & 0 & 0 & + & - \\ 0 & + & 0 & - & + & 0 & - \\ 0 & 0 & + & + & - & - & 0 \end{pmatrix}.$$

9.5. Matrices in E_8

$$\begin{pmatrix} 0 & - & + & + & - & - & + & - \\ - & 0 & - & - & + & + & - & + \\ + & - & 0 & + & - & - & 0 & 0 \\ + & - & + & 0 & - & - & 0 & 0 \\ - & + & - & - & 0 & + & 0 & 0 \\ - & + & - & - & + & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 & - \\ - & + & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & + & - & - & + & 0 & - & 0 \\ + & 0 & - & - & + & - & 0 & 0 \\ - & - & 0 & + & - & + & 0 & 0 \\ - & - & + & 0 & - & 0 & + & 0 \\ + & + & - & - & 0 & 0 & 0 & - \\ 0 & - & + & 0 & 0 & 0 & - & 0 \\ - & 0 & 0 & + & 0 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & - & + & - & - & + & 0 & 0 \\ - & 0 & - & + & + & 0 & 0 & - \\ + & - & 0 & - & 0 & 0 & 0 & 0 \\ - & + & - & 0 & 0 & 0 & 0 & 0 \\ - & + & 0 & 0 & 0 & - & 0 & 0 \\ + & 0 & 0 & 0 & - & 0 & - & 0 \\ 0 & 0 & 0 & 0 & 0 & - & 0 & 0 \\ 0 & - & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & - & - & + & + & - & + & - \\ - & 0 & + & - & 0 & 0 & - & + \\ - & + & 0 & - & - & + & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & - & 0 & 0 \\ - & 0 & + & 0 & - & 0 & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & 0 & - \\ - & + & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & + & - & - & + & - & 0 & 0 \\ + & 0 & - & - & 0 & 0 & 0 & 0 \\ - & - & 0 & + & 0 & 0 & 0 & 0 \\ - & - & + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & - & - & 0 \\ - & 0 & 0 & 0 & - & 0 & + & 0 \\ 0 & 0 & 0 & 0 & - & + & 0 & - \\ 0 & 0 & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & - & - & + & 0 & 0 & + & - \\ - & 0 & + & - & 0 & - & 0 & + \\ - & + & 0 & - & + & 0 & - & 0 \\ + & - & - & 0 & - & + & 0 & 0 \\ 0 & 0 & + & - & 0 & 0 & 0 & - \\ 0 & - & 0 & + & 0 & 0 & - & 0 \\ + & 0 & - & 0 & 0 & - & 0 & 0 \\ - & + & 0 & 0 & - & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & - & - & + & + & 0 & 0 & - \\ - & 0 & + & 0 & - & 0 & 0 & 0 \\ - & + & 0 & - & 0 & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 & - & 0 & 0 \\ + & - & 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 0 & - & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 & 0 & 0 \\ - & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & - & - & 0 & - & + & + & 0 \\ - & 0 & + & + & 0 & - & - & 0 \\ - & + & 0 & 0 & + & 0 & - & - \\ 0 & + & 0 & 0 & - & - & 0 & 0 \\ - & 0 & + & - & 0 & 0 & 0 & 0 \\ + & - & 0 & - & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & - & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & - & 0 & 0 & - & - & + & + \\ - & 0 & + & + & 0 & 0 & - & - \\ 0 & + & 0 & 0 & 0 & - & 0 & - \\ 0 & + & 0 & 0 & - & 0 & - & 0 \\ - & 0 & 0 & - & 0 & + & 0 & 0 \\ - & 0 & - & 0 & + & 0 & 0 & 0 \\ + & - & 0 & - & 0 & 0 & 0 & 0 \\ + & - & - & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & + & - & + & - & 0 & - \\ 0 & 0 & + & - & - & + & - & 0 \\ + & + & 0 & - & 0 & 0 & - & - \\ - & - & - & 0 & 0 & 0 & + & + \\ + & - & 0 & 0 & 0 & - & 0 & 0 \\ - & + & 0 & 0 & - & 0 & 0 & 0 \\ 0 & - & - & + & 0 & 0 & 0 & 0 \\ - & 0 & - & + & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & + & - & - & - & + & 0 \\ 0 & 0 & - & + & - & 0 & 0 & 0 \\ + & - & 0 & - & 0 & 0 & 0 & 0 \\ - & + & - & 0 & 0 & 0 & 0 & 0 \\ - & - & 0 & 0 & 0 & + & 0 & 0 \\ - & 0 & 0 & 0 & + & 0 & - & 0 \\ + & 0 & 0 & 0 & 0 & - & 0 & - \\ 0 & 0 & 0 & 0 & 0 & 0 & - & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & - & + & + & - & - \\ 0 & 0 & - & 0 & + & - & + & - \\ 0 & - & 0 & 0 & - & + & - & + \\ - & 0 & 0 & 0 & - & - & + & + \\ + & + & - & - & 0 & 0 & 0 & - \\ + & - & + & - & 0 & 0 & - & 0 \\ - & + & - & + & 0 & - & 0 & 0 \\ - & - & + & + & - & 0 & 0 & 0 \end{pmatrix}.$$

References

[1] P. J. Cameron, J.-M. Goethals, J. J. Seidel, and E. E. Shult, *Line graphs, root systems and elliptic geometry,* J. Algebra 43 (1976), 305–327.

[2] The GAP Group, *GAP—Groups, algorithms, and programming,* version 4.4, 2006; ⟨http://www.gap-system.org⟩.

[3] J. E. Humphreys, *Reflection groups and Coxeter groups,* Cambridge Stud. Adv. Math., 29, Cambridge Univ. Press, Cambridge, 1990.

[4] B. D. McKay, *Data on graphs,* ⟨http://cs.anu.edu.au/~bdm/data/graphs.html⟩.

[5] K. R. Shah and B. K. Sinha, *Theory of optimal designs,* Lecture Notes in Statist., 54, Springer-Verlag, New York, 1989.

[6] L. H. Soicher, *The DESIGN package for GAP,* ⟨http://designtheory.org/software⟩.

School of Mathematical Sciences
Queen Mary, University of London
London E1 4NS
United Kingdom

p.j.cameron@qmul.ac.uk