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Higman's Criterion Revisited

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In memory of Donald Higman

Let V be a finite-dimensional k-vector space endowed with an action of a finite group G and hence endowed with the structure of a kG-module. According to Higman's criterion, that module is projective if and only if there exists a k-linear endomorphism α of V such that $\sum_{g\in G} g \cdot \alpha \cdot g^{-1} = \mathrm{Id}_V$. This paper presents a generalization of that criterion to the more general context of symmetric algebras. Having in mind some functors used in the representation theory of finite reductive groups, we then generalize the appropriate version of Higman's criterion applied to relative projectivity to a situation where induction–restriction are replaced by functors induced by pairs of "exact bimodules".

On our way, we present a rather self-contained introduction to the methods used for representation theory of symmetric algebras.

0. Introduction

Although induction and restriction functors have been (and still are) the building blocks of the theory of modular representations of finite groups, recent developments of the theory have shown the pertinence and the importance of other functors like the Harish–Chandra induction–truncation or, more generally, the Rickard functors (see e.g. [Br; R1]), which cover the case of the Deligne–Lusztig functors.

Moreover, the theory of representations of finite reductive groups has led to the study of representation of Iwahori–Hecke algebras (see e.g. [G]), which, like finite group algebras, are *symmetric* algebras. Calabi–Yau algebras have also revitalized interest in symmetric algebras.

For all those reasons, it seemed reasonable to revisit some of the basic tools of representation theory of finite groups from a more general point of view: to replace the group algebra by a symmetric algebra, replace the induction–restriction functors by a pair of bi-adjoint functors, and generalize the notion of relative projectivity and its main criterion (the Higman criterion)—and to do this in such a way that the machinery applies not only to module categories but also to triangulated categories (and hence to derived categories of module categories).

Such are the aims of this paper. We have made the choice of not considering the compatibility of our functors with local structures of finite groups. We certainly hope that the present approach will soon be extended to the more general context

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of exact pairs of functors induced by splendid complexes [R1] between derived bounded categories of group algebras.

It must be noted that ways of generalizing Higman's original criterion had been opened half a century ago by Higman himself (see [H2], where he proved the "relative version" of his criterion, and also [H3]) and by Ikeda (see [I], which considers Frobenius algebras over fields).

Apart from basic facts about adjunctions (for the elementary notions of categories used here, we refer the reader to [J; K1; M]), the paper is mostly self-contained. For the convenience of the reader (and for our own consistency), we devote Section 1 to classical notation, convention, and definitions about modules over noncommutative algebras. Basic definitions and properties of symmetric algebras are developed in Section 2.

1. Conventions on Modules and Bimodules

The notions and results of this section are classical (see e.g. [Bo, Chap. II; J]). They have been put here to fix convention and notation and also for the convenience of the reader.

All the rings that we consider are unitary. The ring morphisms must be unitary. Let R be a commutative unitary ring, and let A be an R-algebra—that is, a ring A endowed with a ring morphism from R into its center ZA.

Left Modules, Left Representations

An A-module, or a left representation of A, is a pair (X, λ_X) where

- X is an R-module and
- $\lambda_X \colon A \to \mathrm{End}_R(X)$ is a morphism of R-algebras.

The morphism λ_X is called the *structural morphism*.

Note that, when speaking of "modules", one often omits the structural morphism (and then only X is called the module) by writing

$$ax := \lambda_X(a)(x) \quad \text{for } a \in A,\ x \in X.$$

We denote by ${}_A\mathbf{Mod}$ the category of A-modules; it is R-linear and abelian. We denote by ${}_A\mathbf{mod}$ the full subcategory of finitely generated left A-modules.

CONVENTION. For X and X' A-modules, we let the morphisms from X to X' act on the right, so that the commutation with the elements of A becomes just an associativity property. Thus, for $\varphi \colon X \to X'$ with $a \in A$ and $x \in X$, we have

$$(ax)\varphi = a(x\varphi).$$

If $X, X' \in {}_A\mathbf{Mod}$, then $\mathrm{Hom}_A(X, X')$ denotes the R-module of A-homomorphisms from X to X'. If $X \in {}_A\mathbf{Mod}$, then $E_A X := \mathrm{End}_A(X)$ denotes the set of A-endomorphisms of M.

The Opposite Algebra and Right Modules

The opposite algebra A^{op} is by definition the R-module A where the multiplication is defined as $(a, a') \mapsto a'a$. A module-A, or a right representation of A, is by definition an A^{op}-module.

Let Y be a module-A. Letting the elements of A (which are the elements of A^{op}) act on the right of Y, we get a structural morphism

$$\rho_Y : A \to \mathrm{End}_R(Y)^{\mathrm{op}}$$

(where $\mathrm{End}_R(Y)^{\mathrm{op}}$ acts on the right of Y). We then set

$$ya := (y)\rho_Y(a),$$

thereby justifying the name "module-A".

Convention. For Y and Y' modules-A, we let the morphisms from Y to Y' act on the left, so that the commutation with the elements of A becomes just an associativity property. Thus, for $\varphi : Y \to Y'$ with $a \in A$ and $y \in Y$, we have

$$\varphi(ya) = (\varphi y)a.$$

We denote by $\mathrm{Hom}(Y, Y')_A$ the R-module of morphisms of modules-A from Y to Y'. We set $EY_A := \mathrm{End}(Y)_A$.

We denote by $\mathbf{Mod}_A$ the (R-linear abelian) category of modules-A, which is also ${}_{A^{\mathrm{op}}}\mathbf{Mod}$. We denote by $\mathbf{mod}_A$ the full subcategory of finitely generated modules-A.

Bimodules

Let A and B be two R-algebras. We denote by $A \otimes_R B$ the algebra defined on the tensor product by the multiplication $(a_1 \otimes b_1)(a_2 \otimes b_2) := a_1a_2 \otimes b_1b_2$. (In what follows, whenever the ring controlling the tensor product is not specified, it means that the tensor product is over R.)

An (A, B)-bimodule, also called an A-module-B, is by definition an $(A \otimes_R B^{\mathrm{op}})$-module. Let M be an A-module-B. For $a \in A$, $b \in B^{\mathrm{op}}$, and $m \in M$, we set

$$amb := (a \otimes b)m,$$

thus justifying the name "A-module-B".

Note. With the preceding notation, one must consider that the elements of R act the same way on both sides of M: for $\lambda \in R$ and $m \in M$, we have

$$\lambda m = m\lambda.$$

Observe that an A-module-B is naturally a B^{op}-module-A^{op} (in other words, a module-$(A^{\mathrm{op}} \otimes_R B)$).

Convention. The question of where the morphisms of bimodules act is solved by the following convention: a morphism of A-modules-B is treated as a morphism of $(A \otimes_R B^{\mathrm{op}})$-modules—that is, it acts on the right.

We set

$$\mathrm{Hom}_A(M,M')_B := \mathrm{Hom}_{A\otimes_R B^{\mathrm{op}}}(M,M').$$

Given this convention, many natural structures follow from associativity. We list just a few of them:

$$\begin{aligned}
X \in {}_A\mathbf{Mod} \quad &\Longrightarrow \quad X \in {}_A\mathbf{Mod}_{E_A X},\\
Y \in \mathbf{Mod}_A \quad &\Longrightarrow \quad Y \in {}_{EY_A}\mathbf{Mod}_A,\\
\left.\begin{matrix} M \in {}_A\mathbf{Mod}_B \\ N \in {}_A\mathbf{Mod}_C \end{matrix}\right\} &\Longrightarrow \quad \mathrm{Hom}_A(M,N) \in {}_B\mathbf{Mod}_C,\\
\left.\begin{matrix} M \in {}_B\mathbf{Mod}_A \\ N \in {}_A\mathbf{Mod}_C \end{matrix}\right\} &\Longrightarrow \quad M \otimes_A N \in {}_B\mathbf{Mod}_C.
\end{aligned}$$

Note that for $\alpha \in \mathrm{Hom}_A(M,N)$ we have $m(b\alpha c) := ((mb)\alpha)c$. Notice also the following natural isomorphisms:

$$\begin{aligned}
\lambda_A &: A \xrightarrow{\sim} EA_A,\\
\rho_A &: A \xrightarrow{\sim} E_A A,\\
\lambda_A &: ZA \xrightarrow{\sim} E_A A_A.
\end{aligned}$$

Isomorphisme cher à Cartan

Let M be an (A,B)-bimodule. Let X (resp. Y) be an A-module (resp. a B-module). The following fundamental result is the "isomorphisme cher à Henri Cartan" (see e.g. [Il]).

THEOREM 1.1. *We have natural isomorphisms*

$$\mathrm{Hom}_A(M\otimes_B Y, X) \simeq \mathrm{Hom}_B(Y, \mathrm{Hom}_A(M,X))$$

through the maps

$$\begin{cases} (\alpha\colon M\otimes_B Y \to X) \mapsto (\hat\alpha\colon y \mapsto (m \mapsto \alpha(m\otimes y))),\\ (\beta\colon Y \to \mathrm{Hom}_A(M,X)) \mapsto (\hat\beta\colon m\otimes y \mapsto \beta(y)(m)). \end{cases}$$

The preceding isomorphisms express the fact that the pair of functors

$$(M\otimes_B \bullet\,, \mathrm{Hom}_A(M,\bullet))$$

between ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$ is an adjoint pair.

Bimodules

Let M be an object of ${}_A\mathbf{Mod}_A$. We set

$$\begin{aligned}
\mathrm{H}^0(A,M) &:= M^A := \{m \in M \mid (\forall a \in A)\ (am = ma)\},\\
\mathrm{H}_0(A,M) &:= M/[A,M],
\end{aligned}$$

where $[A,M]$ denotes the R submodule of M generated by all the elements $[a,m] := am - ma$ for $a \in A$ and $m \in M$.

Then we have natural isomorphisms

$$\mathrm{H}^0(A, M) = \mathrm{Hom}_A(A, M)_A,$$
$$\mathrm{H}_0(A, M) = A \otimes_{(A\otimes_R A^{\mathrm{op}})} M.$$

Let us denote by $M^* := \mathrm{Hom}_R(M, R)$ the R-dual of M, an A-module-A.

LEMMA 1.2. *There is a natural isomorphism*

$$H_0(M)^* \simeq H^0(M^*).$$

Proof. Indeed, by the isomorphisme cher à Cartan (Theorem 1.1) applied to the algebras $A \otimes_R A^{\mathrm{op}}$ and R, we have

$$\mathrm{Hom}_R(A \otimes_{A\otimes_R A^{\mathrm{op}}} M, R) \simeq \mathrm{Hom}_{A\otimes_R A^{\mathrm{op}}}(A, \mathrm{Hom}_R(M, R)). \qquad \square$$

Quadrimodules

Let $M \in {}_A\mathbf{Mod}_B$ and $N \in {}_B\mathbf{Mod}_A$.

- We have a natural structure of $(A \otimes_R A^{\mathrm{op}})$-module-$(B \otimes_R B^{\mathrm{op}})$ on $M \otimes_R N$ defined by
$$(a \otimes a')(m \otimes n)(b \otimes b') := amb \otimes b'na'.$$
- That structure is also a natural structure of $(A \otimes_R B^{\mathrm{op}})$-module-$(A \otimes_R B^{\mathrm{op}})$ on $M \otimes_R N$:
$$(a \otimes b)(m \otimes n)(a' \otimes b') := amb \otimes b'na'.$$

Let us state a few formal properties of these structures and introduce some more notation.

PROPERTY QM1. We have

$$\mathrm{H}^0(A \otimes_R B^{\mathrm{op}}, M \otimes_R N)$$
$$= \Big\{ \sum_{i\in I} m_i \otimes n_i \in M \otimes_R N \mid (\forall a \in A,\ b \in B) \sum_{i\in I} am_i b \otimes n_i = \sum_{i\in I} m_i \otimes bn_i a \Big\}.$$

We define the respective centralizers in $M \otimes_R N$ of A and B by

$$C_A(M \otimes_R N) := \Big\{ \sum_i m_i \otimes n_i \in M \otimes_R N \mid (\forall a) \sum_i am_i \otimes n_i = \sum_i m_i \otimes n_i a \Big\},$$
$$C(M \otimes_R N)_B := \Big\{ \sum_i m_i \otimes n_i \in M \otimes_R N \mid (\forall b) \sum_i m_i b \otimes n_i = \sum_i m_i \otimes bn_i \Big\}.$$

Thus we have

$$\mathrm{H}^0(A \otimes_R B^{\mathrm{op}}, M \otimes_R N) = C_A(M \otimes_R N) \cap C(M \otimes_R N)_B.$$

We also set

$$(M \otimes_R N)^A := C_A(M \otimes_R N) \quad \text{and} \quad {}^B(M \otimes_R N) := C(M \otimes_R N)_B.$$

PROPERTY QM2. The R-module $\mathrm{H}_0(A \otimes_R B^{\mathrm{op}}, M \otimes_R N)$ is naturally identified with the R-module ${}_A\overset{M}{\underset{N}{\otimes}}{}_B$ defined as a "cyclic" tensor product "$M \otimes_B N \otimes_A$", where the last A comes under the first M.

It is clear, by definition of H_0, that

$$\mathrm{H}_0(A \otimes_R B^{\mathrm{op}}, M \otimes_R N) = \mathrm{H}_0(A, \mathrm{H}_0(B^{\mathrm{op}}, M \otimes_R N)).$$

Since $\mathrm{H}_0(B^{\mathrm{op}}, M \otimes_R N) = M \otimes_B N$, it follows that

$$\mathrm{H}_0(A \otimes_R B^{\mathrm{op}}, M \otimes_R N) = \mathrm{H}_0(A, M \otimes_B N).$$

Thus we have proved the following lemma.

LEMMA 1.3. *Let $M \in {}_A\mathbf{Mod}_B$ and $N \in {}_B\mathbf{Mod}_A$. Then*

$$\mathrm{H}_0(A \otimes_R B^{\mathrm{op}}, M \otimes_R N) = {}_A\overset{M}{\underset{N}{\otimes}}{}_B = \mathrm{H}_0(A, M \otimes_B N) \simeq \mathrm{H}_0(B, N \otimes_A M).$$

PROPERTY QM3. Whenever Y is a B-module-B, we have a natural isomorphism

$$\begin{cases} (M \otimes_R N) \otimes_{(B \otimes B^{\mathrm{op}})} Y \xleftrightarrow{\sim} M \otimes_B Y \otimes_B N, \\ (m \otimes_R n) \otimes_{(B \otimes_R B^{\mathrm{op}})} y \longmapsto m \otimes_B y \otimes_B n, \\ (m \otimes_R n) \otimes_{(B \otimes_R B^{\mathrm{op}})} y \longleftarrow\!\shortmid\, m \otimes_B y \otimes_B n. \end{cases}$$

In particular, we have natural isomorphisms that the reader is invited to describe:

$$(M \otimes_R N) \otimes_{(B \otimes_R B^{\mathrm{op}})} B \xrightarrow{\sim} M \otimes_B N;$$

$$(M \otimes_R N) \otimes_{(B \otimes_R B^{\mathrm{op}})} (N \otimes_R M) \xrightarrow{\sim} (M \otimes_B N) \otimes_R (M \otimes_B N).$$

Characterization of Finitely Generated Projective Modules

LEMMA 1.4. *Let X, Y, and M be A-modules.*

(1) *The image of*

$$\mathrm{Hom}_A(X, M) \otimes_R \mathrm{Hom}_A(M, Y) \longrightarrow \mathrm{Hom}_A(X, Y)$$

consists of those morphisms $X \to Y$ that factorize through M^n for some natural integer n.

(2) *If M is an A-module-B, then the preceding map factorizes through a map*

$$\mathrm{Hom}_A(X, M) \otimes_B \mathrm{Hom}_A(M, Y) \longrightarrow \mathrm{Hom}_A(X, Y).$$

Proof. Let

$$x = \sum_{i=1}^{n} \alpha_i \otimes \beta_i \in \mathrm{Hom}_A(X, M) \otimes_R \mathrm{Hom}_A(M, Y).$$

The image of x in $\mathrm{Hom}_A(X, Y)$ is $\sum_{i=1}^{n} \alpha_i \beta_i$. The maps α_i $(1 \le i \le n)$, respectively β_i $(1 \le i \le n)$, describe a unique map $\alpha\colon X \to M^n$, respectively $\beta\colon M^n \to Y$. Their composition $\alpha\beta$ is equal to $\sum_{i=1}^{n} \alpha_i \beta_i$, which proves assertion (1). The proof of (2) is left to the reader. $\square$

The A-dual of an A-module X is the module-A defined by

$$X^\vee := \mathrm{Hom}_A(X, A).$$

We define the map $\tau_{X,Y}$ as the composition

$$\tau_{X,Y}\colon X^\vee \otimes_A Y \longrightarrow \mathrm{Hom}_A(X, Y).$$

We also set

$$\tau_X := \tau_{X,X}.$$

Applying Lemma 1.4 to the particular case where $M = A$ yields the following.

LEMMA 1.5. *The image of $\tau_{X,Y}$ consists of those morphisms that factorize through A^n for some n.*

DEFINITION 1.6. The elements of the image of $\tau_{X,Y}$ are called the *projective maps* from X to Y. We denote the set of all projective maps from X to Y by $\mathrm{Hom}_A^{\mathrm{pr}}(X, Y)$.

By Lemma 1.5 we see that $\mathrm{Hom}_A^{\mathrm{pr}}(\cdot, \cdot)$ is a two-sided ideal in $\mathrm{Hom}_A(\cdot, \cdot)$. In other words, all the $\mathrm{Hom}_A^{\mathrm{pr}}(X, Y)$ are abelian groups; and, whenever $f \in \mathrm{Hom}_A^{\mathrm{pr}}(X, Y)$, $g \in \mathrm{Hom}_A(Y, Z)$, and $h \in \mathrm{Hom}_A(W, X)$, we have $fg \in \mathrm{Hom}_A^{\mathrm{pr}}(X, Z)$ and $hf \in \mathrm{Hom}_A^{\mathrm{pr}}(W, Y)$.

The notation $X' \mid X$ ("X' is a summand of X") means that X' is a submodule of X and there exists a submodule X'' of X such that $X = X' \oplus X''$.

The following omnibus theorem is classical.

THEOREM 1.7. *A finitely generated A-module M is called a* projective module *if it satisfies one of the following, equivalent conditions.*

(i) *If φ is a surjective morphism from the A-module X onto the A-module Y and if ψ is a morphism of M to Y, then there exists a morphism ρ of M to X such that $\rho\varphi = \psi$.*

(ii) *The functor* $\mathrm{Hom}_A(M, \cdot)\colon {}_A\mathbf{Mod} \to {}_{\mathrm{End}_A(M)}\mathbf{Mod}$ *is an exact functor.*

(iii) *Any A-linear surjection with image M is split.*

(iv) *M is a direct summand of a free module; that is, $M \mid A^n$ for some integer n.*

(v) *The map $\tau_M\colon M^\vee \otimes_A M \to \mathrm{Hom}_A(M, M)$ is onto.*

(vi) *The map $\tau_{X,M}\colon X^\vee \otimes_A M \to \mathrm{Hom}_A(X, M)$ is an isomorphism for all A-modules X.*

(vii) *The map $\tau_{M,X}\colon M^\vee \otimes_A X \to \mathrm{Hom}_A(M, X)$ is an isomorphism for all A-modules X.*

(viii) *The map τ_M is an isomorphism.*

Sketch of proof. (i) $\Rightarrow$ (ii). Condition (i) implies that the functor $\mathrm{Hom}_A(M, \cdot)$ is right exact. Since it is always left exact, it must be exact.

(ii) $\Rightarrow$ (iii). Apply the functor $\mathrm{Hom}_A(M, \cdot)$ and use a preimage of 1_M to define a splitting.

(iii) $\Rightarrow$ (iv). Because M is finitely generated over A, it is an image (and hence a summand) of A^n for some n.

(iv) $\Rightarrow$ (v). Since $M \mid A^n$, we know that Id_M is in the image of τ_M. Furthermore, τ_M is a map in ${}_{\mathrm{End}_A(M)}\mathbf{Mod}_{\mathrm{End}_A(M)}$ and consequently is onto.

(v) $\Rightarrow$ (vi). We exhibit the inverse of $\tau_{X,M}$. By (v) there exists an element $\sum_{i=1}^n n_i \otimes m_i$ such that $\tau_M\left(\sum_{i=1}^n n_i \otimes m_i\right) = 1_M$. We define the map

$$\psi\colon \mathrm{Hom}_A(X, M) \longrightarrow X^\vee \otimes_A M$$

by $\alpha \mapsto \sum_{i=1}^n \alpha n_i \otimes m_i$. This map ψ satisfies $\psi \circ \tau_{X,M} = \mathrm{Id}_{\mathrm{Hom}_A(X,M)}$ and also satisfies $\tau_{X,M} \circ \psi = \mathrm{Id}_{X^\vee \otimes_A M}$.

(v) $\Rightarrow$ (vii). Using the same element $\sum_{i=1}^n n_i \otimes m_i$ as before, one can give an explicit formula of the inverse of $\tau_{M,X}$:

$$\begin{aligned} \mathrm{Hom}_A(M, X) &\longrightarrow M^\vee \otimes_A X \\ \alpha &\longmapsto \textstyle\sum_{i=1}^n n_i \otimes m_i \alpha. \end{aligned}$$

The implications (vi) $\Rightarrow$ (v) and (vii) $\Rightarrow$ (v) are trivial because $\tau_M = \tau_{M,M}$.

(vii) $\Rightarrow$ (i). Since $M^\vee \otimes_A \bullet$ is a right exact functor, the map φ in (i) induces a surjection

$$M^\vee \otimes_A X \xrightarrow{\varphi_*} M^\vee \otimes_A Y.$$

But $M^\vee \otimes_A X$ and $M^\vee \otimes_A Y$ are respectively isomorphic to $\mathrm{Hom}_A(M, X)$ and $\mathrm{Hom}_A(M, Y)$, so φ induces a surjection

$$\mathrm{Hom}_A(M, X) \xrightarrow{\varphi_*} \mathrm{Hom}_A(M, Y).$$

Now, any preimage of ψ satisfies the condition on ρ in (i).

(vii) $\Rightarrow$ (viii) is trivial, as is (viii) $\Rightarrow$ (v). □

We denote the full subcategory of ${}_A\mathbf{mod}$ consisting of all the projective A-modules by ${}_A\mathbf{proj}$. If M is an (A, B)-bimodule that is projective as an A-module, then we abuse notation and write $M \in {}_A\mathbf{mod}_B \cap {}_A\mathbf{proj}$. Similarly, we denote by $\mathbf{proj}_A$ the category of finitely generated projective right A-modules ("projective modules-A").

Notice also that the R-module of projective maps $\mathrm{Hom}_A^{\mathrm{pr}}(X, Y)$ may be defined as the set of those morphisms from X to Y that factorize through a *projective* A-module.

Projective Modules and Duality

Recall that for an A-module X we denote by $X^\vee$ its A-dual, a module-A. Similarly, if Y is a module-A then we denote by ${}^\vee Y$ its dual-A, an A-module.

If $\varphi\colon X \to X'$ is a morphism in ${}_A\mathbf{Mod}$, then the map

$$\varphi^\vee\colon X'^\vee \to X^\vee, \quad (y'\colon X' \to A) \mapsto (\varphi.y'\colon X \to A)$$

is a morphism in $\mathbf{Mod}_A$. Hence we have a contravariant functor

$${}_A\mathbf{Mod} \to \mathbf{Mod}_A, \quad X \to X^\vee,$$

as well as a contravariant functor

$$\mathbf{Mod}_A \to {}_A\mathbf{Mod}, \quad Y \to {}^\vee Y.$$

We also have a natural morphism of A-modules

$$X \to {}^\vee(X^\vee), \quad x \mapsto (y \mapsto xy).$$

The next proposition follows easily from the fact that finitely generated projective modules are nothing but summands of free modules with finite rank.

PROPOSITION 1.8. (a) *If X is a finitely generated projective A-module* (*resp. Y is a finitely generated projective module-A*), *then $X^\vee$ is a finitely generated projective module-A* (*resp. ${}^\vee Y$ is a finitely generated projective A-module*).

(b) *If $X \in {}_A\mathbf{proj}$, then the natural morphism $X \mapsto {}^\vee(X^\vee)$ is an isomorphism and the functors $X \mapsto X^\vee$ and $Y \mapsto {}^\vee Y$ induce quasi-inverse equivalences between ${}_A\mathbf{proj}$ and $\mathbf{proj}_A$.*

REMARK. For X an A-module, the map

$$\begin{cases} \mathrm{Hom}_A(X,X) \to \mathrm{Hom}(X^\vee, X^\vee)_A, \\ \varphi \mapsto \varphi^\vee \end{cases}$$

is an isomorphism of algebras (because of our conventions about right actions of left morphisms and vice versa).

2. Symmetric Algebras: Definition and First Properties

2.A. Central Forms and Traces on Projective Modules

Central Forms

Let A be an R-algebra. A form $t \in \mathrm{Hom}_R(A,R)$ is said to be *central* if it satisfies the property

$$t(aa') = t(a'a) \quad (\forall a, a' \in A).$$

Thus a central form can be identified with a form on the R-module $A/[A,A]$.

Whenever X is an A-module, we denote by $X^* := \mathrm{Hom}_R(X,R)$ its R-dual viewed as an $E_A X$-module-A.

We denote by $\mathrm{CF}(A,R)$ the R-submodule of A^* consisting of all central forms on A. Then $\mathrm{CF}(A,R)$ is the orthogonal of the submodule $[A,A]$ of A, and hence it is canonically identified with the R-module $(A/[A,A])^*$.

If $t\colon A \to R$ is a central form on A, then we shall still denote by $t\colon A/[A,A] \to R$ the form on $A/[A,A]$ that corresponds to t. More generally, let M be an A-module-A and let L be an R-module. Then an R-linear map $t\colon M \to L$ is *central* if $t(am) = t(ma)$ for all $a \in A$ and $m \in M$. In particular, the central forms on M are the forms defined by the R-dual of $\mathrm{H}^0(A,M) = M/[A,M]$.

Observe that multiplication by elements of the center ZA of A gives the R-module $A/[A,A]$ the natural structure of a ZA-module. Thus $\mathrm{CF}(A,R)$ inherits the structure of a ZA-module defined by $zt := t(z\,\cdot)$ (or $zt(a) = t(za)$ for $a \in A$) for all $z \in ZA$ and $t \in \mathrm{CF}(A,R)$.

Traces on Projective Modules, Characters

Let X be an A-module. Then the R-module $X^\vee \otimes_A X$ is naturally equipped with a linear form

$$\begin{cases} X^\vee \otimes_A X \to A/[A,A] \\ y \otimes x \mapsto xy \bmod [A,A]. \end{cases}$$

In particular, if P is a finitely generated projective A-module then, since $P^\vee \otimes_A P \simeq E_A P$, we obtain an R-linear map (the trace on a projective module) $\mathrm{tr}_{P/A}\colon E_A P \to A/[A,A]$, so

$$\mathrm{tr}_{P/A}\Big(\sum_i (y_i \otimes x_i)\Big) = \sum_i x_i y_i \bmod [A,A].$$

LEMMA 2.1. *Whenever P is a finitely generated projective A-module, the trace*

$$\mathrm{tr}_{P/A}\colon E_A P \to A/[A,A]$$

is central.

Proof. In what follows, we identify $P^\vee \otimes_A P$ with $E_A P$. Let $x, x' \in P$ and $y, y' \in P^\vee$. Then

$$(y \otimes_A x)(y' \otimes_A x') = y(xy') \otimes_A x' = y \otimes_A (xy')x',$$

from which it follows that

$$\mathrm{tr}_{P/A}((y \otimes_A x)(y' \otimes_A x')) = (xy')(x'y) \bmod [A,A];$$

this shows indeed that $\mathrm{tr}_{A/P}$ is central. □

Now if $t\colon A \to R$ is a central form then we deduce (by composition) a central form

$$t_P\colon E_A P \to R, \quad \varphi \mapsto t(\mathrm{tr}_P(\varphi)).$$

In particular, whenever X is a finitely generated projective R-module, we have the trace form

$$\mathrm{tr}_{X/R}\colon E_R X \to R \quad \text{defined by} \quad (y \otimes x) \mapsto xy \ \ (\forall y \in X^*,\ x \in X).$$

DEFINITION 2.2. Let X be an A-module that is a finitely generated projective R-module, and let $\lambda_X\colon A \to E_R X$ denote the structural morphism. The character of the A-module X (or of the representation of A defined by λ_X) is the central form

$$\chi_X\colon A \to R, \quad a \mapsto \mathrm{tr}_{X/R}(\lambda_X(a)).$$

2.B. Symmetric Algebras

Definition and First Examples

A central form $t\colon A \to R$ defines a morphism $\hat{t}$ of A-modules-A as follows:

$$\hat{t}\colon A \to A^*,$$
$$a \mapsto \hat{t}(a)\colon a' \mapsto t(aa')$$

Indeed, for $a, a', x \in A$, we have

$$\hat{t}(axa') = t(axa'\cdot) = t(xa'\cdot a) = a\hat{t}(x)a'.$$

Note that the restriction of $\hat{t}$ to ZA defines a ZA-morphism: $ZA \to \mathrm{CF}(A,R)$.

DEFINITION 2.3. Let A be an R-algebra. We say that A is a *symmetric* algebra if the following conditions are fulfilled:

(S1) A is a finitely generated projective R-module; and
(S2) there exists a central form $t\colon A \to R$ such that $\hat{t}$ is an isomorphism.

If A is a symmetric algebra and if t is a form as in (S2), then we call t a *symmetrizing* form for A.

Examples

1. The trace is a symmetrizing form on the algebra $\mathrm{Mat}_n(R)$.
2. If G is a finite group, then its group algebra RG is a symmetric algebra. The form
$$t\colon RG \to R, \quad \sum_{g\in G} \lambda_g g \mapsto \lambda_1$$
is called the *canonical* symmetrizing form on RG.
3. If k is a field, then we shall see later that the algebra $A := \left(\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right)$ is not a symmetric algebra.

The following example is singled out as a lemma.

LEMMA 2.4. *Let D be a finite-dimensional division k-algebra. Then D is a symmetric algebra.*

Proof. First we prove that $[D, D] \neq D$. It is enough to prove this in the case where D is central (indeed, the ZD-vector space generated by $\{ab - ba \mid (a, b \in A)\}$ contains the k-vector space generated by that set). In this case, we know that $\bar{k} \otimes_k D$ is a matrix algebra $\mathrm{Mat}_m(\bar{k})$ over $\bar{k}$. If $[D, D] = D$, then every element of $\mathrm{Mat}_m(k)$ has trace 0, a contradiction.

Now choose a nonzero k-linear form t on D whose kernel contains $[D, D]$. Thus t is central. Let us check that t is symmetrizing. For this, it is enough to prove that $\hat{t}$ is injective. But if x is a nonzero element of D, then the map $y \mapsto xy$ is a permutation of D; hence there exists a $y \in D$ such that $t(xy) \neq 0$, proving that $\hat{t}(x) \neq 0$. $\square$

LEMMA 2.5. *Let A be a symmetric algebra with symmetrizing form t.*

(1) *The restriction of $\hat{t}$ to ZA,*
$$ZA \to \mathrm{CF}(A, R), \quad z \mapsto t(z\,\cdot),$$
is an isomorphism of ZA-modules. In particular, $\mathrm{CF}(A, R)$ *is a free ZA-module of rank 1.*

(2) *A central form $\hat{t}(z)$ corresponding to an element $z \in ZA$ is a symmetrizing form if and only if z is invertible.*

Proof. Let u be a form on A. By hypothesis, we have that $u = t(a\,\cdot)$ for some $a \in A$ and that u is central if and only if a is central. This shows the surjectivity of the map $\hat{t}\colon ZA \to \mathrm{CF}(A, R)$, and its injectivity results from the injectivity of $\hat{t}$. Finally, this proves that symmetrizing forms are the elements t of $\mathrm{CF}(A, R)$ such that $\{t\}$ is a basis of $\mathrm{CF}(A, R)$ as a ZA-module. $\square$

REMARK. As in the classical literature on symmetric algebras over fields, if t is a symmetrizing form on A then its kernel $\ker(t)$ contains no left (or right) nontrivial ideal of A.

Annihilators and Orthogonals

Let $\mathfrak{a}$ be a subset of the algebra A. The right annihilator of $\mathfrak{a}$ is defined as

$$\mathrm{Ann}(\mathfrak{a})_A := \{x \in A \mid (\mathfrak{a}.x = 0)\}.$$

It is immediate to check that the right annihilator of a subset is a right ideal and that the right annihilator of a right ideal is a two-sided ideal.

Suppose now that A is a symmetric algebra, and choose a symmetrizing form t on A. Whenever $\mathfrak{a}$ is a subset of A, we denote by $\mathfrak{a}^\perp$ its orthogonal for the bilinear form defined by t; that is,

$$\mathfrak{a}^\perp := \{x \in A \mid t(\mathfrak{a}x) = 0\}.$$

Note that if $\mathfrak{a}$ is stable by multiplication by $(ZA)^\times$, then $\mathfrak{a}^\perp$ does not depend on the choice of the symmetrizing form t.

PROPOSITION 2.6. *Assume that A is symmetric.*

(1) *We have* $[A, A]^\perp = ZA$.
(2) *If* $\mathfrak{a}$ *is a left ideal of* A, *then* $\mathfrak{a}^\perp = \mathrm{Ann}(\mathfrak{a})_A$.

Proof. (1) We have

$$t(zab) = t(zba) \iff t(bza) = t(zba),$$

which shows that $z \in [A, A]^\perp$ if and only if $z \in ZA$.

(2) We have

$$\mathfrak{a}x = 0 \iff (\forall y \in A)\ t(y\mathfrak{a}x) = 0 \iff t(\mathfrak{a}x) = 0,$$

which proves (2). □

2.C. Characterizations in Terms of Module Categories

This section follows Rickard [R2]. Assume that A is an R-algebra that is a finitely generated projective R-module. We begin by listing a few elementary properties as follows.

1. Any finitely generated projective A-module is also a finitely generated projective R-module. (Indeed, if A is a summand of R^m, then any summand of A^n is also a summand of R^{mn}.)
2. If X is a finitely generated projective A-module and if Y is a module-A, then $Y \otimes_A X$ is isomorphic to a summand of Y^n for some positive integer n. It follows that if, moreover, Y is a finitely generated projective R-module, then $Y \otimes_A X$ is also a finitely generated projective R-module.

Let us denote by ${}_A\mathbf{proj}$ the full subcategory of ${}_A\mathbf{Mod}$ whose objects are the finitely generated projective A-modules. We define similarly the notation $\mathbf{proj}_A$ and ${}_R\mathbf{proj}$.

PROPOSITION 2.7. *Let A be an R-algebra that is assumed to be a finitely generated projective R-module. Then the following conditions are equivalent.*

(i) *A is symmetric.*

(ii) *A and A^* are isomorphic as A-modules-A.*

(iii) *As (contravariant) functors ${}_A\mathbf{Mod} \to \mathbf{Mod}_A$, we have*

$$\mathrm{Hom}_R(\cdot\,, R) \simeq \mathrm{Hom}_A(\cdot\,, A).$$

(iii′) *As (contravariant) functors $\mathbf{Mod}_A \to {}_A\mathbf{Mod}$, we have*

$$\mathrm{Hom}_R(\cdot\,, R) \simeq \mathrm{Hom}(\cdot\,, A)_A.$$

(iv) *For $P \in {}_A\mathbf{proj}$ and $X \in {}_A\mathbf{Mod} \cap {}_R\mathbf{proj}$, we have natural isomorphisms*

$$\mathrm{Hom}_A(P, X) \simeq \mathrm{Hom}_A(X, P)^*.$$

(iv′) *For $P \in \mathbf{proj}_A$ and $X \in \mathbf{Mod}_A \cap {}_R\mathbf{proj}$, we have natural isomorphisms*

$$\mathrm{Hom}(P, X)_A \simeq \mathrm{Hom}(X, P)^*_A.$$

Proof. It is enough to prove (i) $\Leftrightarrow$ (ii) and that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii). This results from the fact, noted previously, that if t is a central form then $\hat{t}$ is a morphism of bimodules from A to A^*.

(ii) $\Rightarrow$ (i). Assume that $\theta : A \xrightarrow{\sim} A^*$ is a bimodule isomorphism. Set $t := \theta(1)$. Then, for $a \in A$, we have

$$\begin{aligned} t(aa') &= \theta(1)(aa') = (a'\theta(1))(a) = \theta(a')(a) = (\theta(1)a')(a) \\ &= \theta(1)(a'a) = t(a'a), \end{aligned}$$

which shows both that t is central and that $\hat{t} = \theta$.

(ii) $\Rightarrow$ (iii). Let X be an A-module. Since $A \simeq A^*$, we have

$$\mathrm{Hom}_A(X, A) \simeq \mathrm{Hom}_A(X, \mathrm{Hom}_R(A, R)).$$

By the "isomorphisme cher à Cartan" (Theorem 1.1), it follows that $\mathrm{Hom}_A(X, A) \simeq \mathrm{Hom}_R(A \otimes_A X, R)$ and hence that $\mathrm{Hom}_A(X, A) \simeq \mathrm{Hom}_R(X, R)$.

(iii) $\Rightarrow$ (iv). Let P be a finitely generated projective A-module and let X be a finitely generated A-module. Since P is a finitely generated projective R-module, we have $P \simeq \mathrm{Hom}_R(P^*, R)$, and it follows from Theorem 1.1 that $\mathrm{Hom}_A(X, P) \simeq \mathrm{Hom}_R(P^* \otimes_A X, R)$. Then, since $P^* \simeq P^\vee$, we get

$$\mathrm{Hom}_A(X, P) \simeq \mathrm{Hom}_R(P^\vee \otimes_A X, R).$$

Since the projective module-A $P^\vee$ is finitely generated and since X is a finitely generated projective R-module, we see that $P^\vee \otimes_A X$ is also a finitely generated projective R-module; hence $\mathrm{Hom}_A(X, P)^* \simeq P^\vee \otimes_A X$. Since P is a finitely generated projective A-module, we know that $P^\vee \otimes_A X \simeq \mathrm{Hom}_A(P, X)$. Thus we have proved that $\mathrm{Hom}_A(X, P)^* \simeq \mathrm{Hom}_A(P, X)$.

(iv) $\Rightarrow$ (ii). Choose $P = X = A$ (viewed as an A-module). Then the natural isomorphism $\mathrm{Hom}_A(A, A)^* \simeq \mathrm{Hom}_A(A, A)$ is a bimodule isomorphism $A^* \simeq A$. $\square$

Symmetric Algebras and Projective Modules

PROPOSITION 2.8. *Let A be a symmetric R-algebra and let P be a finitely generated projective A-module. Then $E_A P$ is a symmetric R-algebra.*

Proof. Recall that we have an isomorphism of $E_A P$-modules-$E_A P$

$$P^\vee \otimes_A P \xrightarrow{\sim} E_A P.$$

Since P is a finitely generated projective A-module and since $P^\vee$ is a finitely generated R-module, this shows that $E_A P$ is a finitely generated projective R-module.

Moreover, by Proposition 2.7(iii), we then have a natural isomorphism

$$\mathrm{Hom}_A(P, P)^* \simeq \mathrm{Hom}_A(P, P),$$

that is, a bimodule isomorphism

$$E_A P^* \simeq E_A P,$$

which shows that $E_A P$ is symmetric. □

COROLLARY 2.9. *An algebra that is Morita equivalent to a symmetric algebra is itself a symmetric algebra.*

Proof. Indeed, we already know that any algebra Morita equivalent to A is isomorphic to the algebra of endomorphisms of a finitely generated projective A-module. □

Explicit Isomorphisms

We give here explicit formulas for the isomorphisms stated in Proposition 2.7. The reader is invited to check the details.

PROPOSITION 2.10. (1) *Whenever X is an A-module, the morphisms t_X^* and u_X defined by*

$$t_X^* : \begin{cases} \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_R(X, R), \\ \phi \mapsto t \cdot \phi, \end{cases}$$

$$u_X : \begin{cases} u_X : \mathrm{Hom}_R(X, R) \to \mathrm{Hom}_A(X, A) \\ \textit{such that } \psi(ax) = t(a u_X(\psi)(x)) \\ (\forall a \in A,\ x \in X,\ \psi \in \mathrm{Hom}_R(X, R)) \end{cases}$$

are inverse isomorphisms in $\mathbf{Mod}_{E_A X}$.

(2) *If X is an A-module that is a finitely generated projective R-module and if P is a finitely generated projective A-module, then the pairing*

$$\begin{cases} \mathrm{Hom}_A(P, X) \times \mathrm{Hom}_A(X, P) \to R, \\ (\varphi, \psi) \mapsto t_P(\varphi\psi) \end{cases}$$

is an R-duality.

Let us in particular exhibit a symmetrizing form on $E_A P$ from a symmetrizing form on A.

Recall that the isomorphism $P^\vee \otimes_A P \xrightarrow{\sim} E_A P$ allows us to define the trace of the finitely generated projective A-module P,

$$\mathrm{tr}_{P/A} \colon E_A P \to A/[A,A], \quad y \otimes_A x \mapsto xy \mod [A,A],$$

and that composing this morphism with a central form t on A yields a central form

$$t_P \colon E_A P \to R$$

on $E_A P$.

Proposition 2.11. *If P is a finitely generated projective A-module and if t is a symmetrizing form on A, then the form t_P is a symmetrizing form on $E_A P$.*

As noted by Keller, the choice of the form t_P (among many other possible choices for a symmetrizing form on $E_A P$) actually corresponds to a unique "extension of t on the category of all finitely generated projective A-modules", as shown by the next proposition [K2].

Proposition 2.12. (1) *The collection of forms (t_P) (for P a finitely generated projective A-module) satisfies the following property: whenever $\alpha \in \mathrm{Hom}_A(P,Q)$ and $\beta \in \mathrm{Hom}_A(Q,P)$, we have $t_P(\alpha\beta) = t_Q(\beta\alpha)$.*

(2) *Reciprocally, if $(t'_P \colon E_A P \to R)$ is a collection of symmetrizing forms (for P running over the collection of finitely generated projective A-modules) such that $t'_A = t$ and $t'_P(\alpha\beta) = t'_Q(\beta\alpha)$ for all $\alpha \in \mathrm{Hom}_A(P,Q)$ and $\beta \in \mathrm{Hom}_A(Q,P)$, then for every P we have $t'_P = t_P$.*

Example. The identity from R onto R is a symmetrizing form for R. It follows that the trace is a symmetrizing form for the matrix algebra $\mathrm{Mat}_n(R)$.

Remark. A particular case of a projective A-module is given by $P := Ai$, where i is an idempotent of A. The map

$$iai \mapsto (x \mapsto xiai)$$

is then the isomorphism $iAi \xrightarrow{\sim} E_A P$. Through that isomorphism, the form t_P becomes the form

$$iai \mapsto t(iai).$$

Products of Symmetric Algebras

The proof of following result is an immediate consequence of the characterizations given in Proposition 2.7, and its proof is left to the reader.

Proposition 2.13. *Let $A_1, A_2, \ldots, A_n$ be R-algebras that are finitely generated projective R-modules, and let A be an algebra isomorphic to a product $A_1 \times A_2 \times \cdots \times A_n$. Then A is symmetric if and only if each A_i $(i = 1, 2, \ldots, n)$ is symmetric.*

More concretely, we know that an isomorphism $A \simeq A_1 \times A_2 \times \cdots \times A_n$ determines a decomposition of the unit element 1 of A into a sum of mutually orthogonal central idempotents,

$$1 = e_1 + e_2 + \cdots + e_n,$$

corresponding to a decomposition of A into a direct sum of two-sided ideals:

$$A = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_n \quad \text{with } \mathfrak{a}_i = Ae_i = A_i.$$

We can thus make the following statements:

- if $(t_1, t_2, \ldots, t_n)$ is a family of symmetrizing forms on $A_1, A_2, \ldots, A_n$, respectively, then the form defined on A by $t_1 + t_2 + \cdots + t_n$ is symmetrizing;
- if t is a symmetrizing form on A, then its restriction to each $\mathfrak{a}_i = Ae_i$ defines a symmetrizing form in the algebra A_i.

Principally Symmetric Algebras

PROPOSITION 2.14. *Let A be a symmetric R-algebra and let t be a symmetrizing form. Then the following conditions are equivalent.*

(i) *The form $t : A \to R$ is onto.*
(ii) *R is isomorphic to a summand of A in ${}_R\mathbf{Mod}$.*
(iii) *As an R-module, A is a progenerator.*

If these conditions are satisfied then we say that the algebra A is principally symmetric.

Proof. (i) ⇒ (ii). Since $t : A \to R$ is onto and since R is a projective R-module, it follows that t splits and that R is indeed isomorphic to a direct summand of A as an R-module.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i). Since A is generator as an R-module, the ideal of R generated by all the $\langle a, b\rangle$ (for $a \in A$ and $b \in A^*$) is equal to R. But since t is symmetrizing, this ideal is equal to $t(A)$, which shows that t is onto. □

EXAMPLES. 1. If A is principally symmetric and if B is an algebra that is Morita equivalent to A, then B is principally symmetric. In particular, the algebra $\mathrm{Mat}_m(R)$ is principally symmetric and, more generally, if X is a progenerator for R then the algebra $E_R X$ is principally symmetric.

2. If all projective R-modules are free, then all symmetric R-algebras are principally symmetric.

3. The algebra RG (G a finite group) is principally symmetric.

4. If $R = R_1 \times R_2$ (a product of two nonzero rings) and if $A := R_1$, then A is a symmetric R-algebra that is not principally symmetric.

3. The Casimir Element and Its Applications

3.A. Definition of the Casimir Element

Actions on $A \otimes_R A$

Let A be an R-algebra. The module $A \otimes_R A$ is naturally endowed with the following structure of an $(A \otimes_R A^{\mathrm{op}})$-module-$(A \otimes_R A^{\mathrm{op}})$:

$$(a \otimes a')(x \otimes y)(b \otimes b') := axb \otimes b'ya'.$$

REMARK. That structure should be understood as a particular case of the structure of an $(A \otimes_R A^{\mathrm{op}})$-module-$(B \otimes_R B^{\mathrm{op}})$-module that is defined on $M \otimes_R N$ (for $M \in {}_A\mathbf{Mod}_B$ and $N \in {}_B\mathbf{Mod}_A$) by

$$(a \otimes a')(m \otimes n)(b \otimes b') := amb \otimes b'na'.$$

We define the left and right centralizers of A in $A \otimes_R A$ as follows:

$$C_A(A \otimes_R A) := \left\{ \sum_i a_i \otimes a_i' \in A \otimes_R A \mid (\forall a) \sum_i aa_i \otimes a_i' = \sum_i a_i \otimes a_i' a \right\};$$

$$C(A \otimes_R A)_A := \left\{ \sum_i a_i \otimes a_i' \in A \otimes_R A \mid (\forall a) \sum_i a_i a \otimes a_i' = \sum_i a_i \otimes aa_i' \right\}.$$

We set

$$C_A(A \otimes_R A)_A := C_A(A \otimes_R A) \cap C(A \otimes_R A)_A.$$

Notice now that

$$C_A(A \otimes_R A) = (A \otimes_R A)^A = \{\xi \in A \otimes_R A \mid (\forall a)\ (a \otimes 1)\xi = (1 \otimes a)\xi\},$$

$$C(A \otimes_R A)_A = {}^A(A \otimes_R A) = \{\xi \in A \otimes_R A \mid (\forall a)\ \xi(a \otimes 1) = \xi(1 \otimes a)\}.$$

(see Section 1 for the notation M^A).

The algebra $E_R A$ of R-endomorphisms of A has the structure of an $(A \otimes_R A^{\mathrm{op}})$-module-$(A \otimes_R A^{\mathrm{op}})$ inherited from the structure of an $(A \otimes_R A^{\mathrm{op}})$-module on each of the two factors A as follows:

$$(\forall \alpha \in E_R A,\ a, a', b, b' \in A)\ \ (a \otimes a').\alpha.(b \otimes b') := [\xi \mapsto a\alpha(a'\xi b')b].$$

REMARK. That structure should be understood as a particular case of the structure of an $(A \otimes_R A^{\mathrm{op}})$-module-$(B \otimes_R B^{\mathrm{op}})$-module defined on $\mathrm{Hom}_R(M, M)$ (for $M \in {}_A\mathbf{Mod}_B$) by

$$(a \otimes a').\alpha.(b \otimes b') := [\xi \mapsto a\alpha(a'\xi b')b].$$

Case Where A Is Symmetric: The Casimir Element

Now assume that A is symmetric and let t be a symmetrizing form. Since A is a finitely generated projective R-module, we have an isomorphism

$$A \otimes_R A^* \xrightarrow{\sim} E_R(A), \quad x \otimes \varphi \mapsto [\xi \mapsto \varphi(\xi)x].$$

Composing this isomorphism with the isomorphism

$$A \otimes_R A \xrightarrow{\sim} A \otimes A^*, \quad x \otimes y \mapsto x \otimes \hat{t}(y),$$

we get the isomorphism

$$A \otimes_R A \xrightarrow{\sim} E_R(A), \quad a \otimes b \mapsto [\xi \mapsto t(b\xi)a].$$

It is immediate to check that this isomorphism is an isomorphism of $(A \otimes_R A^{\mathrm{op}})$-modules-$(A \otimes_R A^{\mathrm{op}})$.

DEFINITION 3.1. We denote by $c_{A,t}^{\mathrm{pr}}$ (or simply c_A^{pr}) the *Casimir element* of (A,t): the element of $A \otimes A$ corresponding to the identity Id_A of A through the preceding isomorphism.

The following lemma is an immediate consequence of this definition of the Casimir element.

LEMMA 3.2. *Let I be a finite set, and let $(e_i)_{i\in I}$ and $(e'_i)_{i\in I}$ be two families of elements of A that are indexed by I. Then the following properties are equivalent*:
(i) $c_A^{\mathrm{pr}} = \sum_{i\in I} e'_i \otimes e_i$;
(ii) *for all $a \in A$, we have $a = \sum_i t(ae'_i)e_i$.*

Notice that, by the preceding formulas, we have

$$(a \otimes a').\mathrm{Id}_A.(b \otimes b') := [\xi \mapsto aa'\xi b'b]$$

or, in other words,

$$(a \otimes a').\mathrm{Id}_A.(b \otimes b') := \lambda(aa')\rho(b'b),$$

where $\lambda(a)$ is the endomorphism of left multiplication by a and $\rho(a)$ is the endomorphism of right multiplication by a. In particular, we see that

$$(a \otimes 1).\mathrm{Id}_A = (1 \otimes a).\mathrm{Id}_A = \lambda(a) \quad \text{and} \quad \mathrm{Id}_A.(a \otimes 1) = \mathrm{Id}_A.(1 \otimes a) = \rho(a).$$

NOTE. The structure of an $A \otimes_R A^{\mathrm{op}}$-module on $A \otimes_R A$ defined here does *not* provide the structure of an $A \otimes_R A^{\mathrm{op}}$-module on A: the morphism

$$A \otimes_A A^{\mathrm{op}} \to E_R A, \quad a \otimes a' \mapsto \lambda(aa')$$

is not an algebra morphism.

Moreover, we know that the commutant of $\lambda(A)$ (resp. of $\rho(A)$) in $E_R A$ is $\rho(A)$ (resp. $\lambda(A)$).

Through the isomorphism $A \otimes_A A \xrightarrow{\sim} E_R A$ just described, the preceding properties translate as follows.

PROPOSITION 3.3. *Assume $c_A^{\mathrm{pr}} = \sum_i e_i \otimes e'_i$.*
(1) *For all $a, a' \in A$, we have*

$$\sum ae_i a' \otimes e'_i = \sum_i e_i \otimes a'e'_i a.$$

(2) *The map*

$$A \to C_A(A \otimes_R A), \quad a \mapsto \sum_i ae_i \otimes e'_i = \sum_i e_i \otimes e'_i a$$

is an isomorphism of A-modules-A.
(2′) *The map*

$$A \to C(A \otimes_R A)_A, \quad a \mapsto \sum_i e_i a \otimes e'_i = \sum_i e_i \otimes ae'_i$$

is an isomorphism of A-modules-A.

EXAMPLES. (i) If $A = RG$ (G a finite group), we have $c_{RG}^{\mathrm{pr}} = \sum_{g \in G} g^{-1} \otimes g$.

(ii) If $A = \mathrm{Mat}_n(R)$ (and t is the ordinary trace), then $c_A^{\mathrm{pr}} = \sum_{i,j} E_{i,j} \otimes E_{j,i}$ (where $E_{i,j}$ denotes the usual elementary matrix all of whose entries are 0 except for the single 1 entry in the ith row of the jth column).

(iii) Assume that A is free over R. Let $(e_i)_{i \in I}$ be an R-basis of A, and let $(e_i')_{i \in I}$ be the dual basis (defined by $t(e_i e_{i'}') = \delta_{i,i'}$); then $c_A^{\mathrm{pr}} = \sum_{i \in I} e_i' \otimes e_i$.

We also define the *central Casimir element* as the image z_A^{pr} of c_A^{pr} by the multiplication morphism $A \otimes A \to A$. Thus, if $c_A^{\mathrm{pr}} = \sum_{i \in I} e_i' \otimes e_i$ then

$$z_A^{\mathrm{pr}} = \sum_i e_i' e_i.$$

REMARKS.

- For $A = RG$, the central Casimir element is the scalar $|G|$.
- For $A = \mathrm{Mat}_m(R)$, the central Casimir element is the scalar m.

The existence of an element such as c_A^{pr} is a necessary and sufficient condition for a central form t to be centralizing, as shown by the following lemma (whose proof is left to the reader).

LEMMA 3.4. *Let u be a central form on A. Assume that there exists an element $f = \sum_j f_j' \otimes f_j \in A \otimes_R A$ such that $\sum_j u(af_j') f_j = a$ for all $a \in A$. Then u is symmetrizing, and f is its Casimir element.*

From now on, we assume that I is a finite set and that $(e_i)_{i \in I}, (e_i')_{i \in I}$ are two families of elements of A, indexed by I, such that

$$c_A^{\mathrm{pr}} = \sum_{i \in I} e_i' \otimes e_i.$$

Let us denote by $x \mapsto x^\iota$ the involutive automorphism of $A \otimes A$ defined by $(a \otimes a')^\iota := a' \otimes a$.

PROPOSITION 3.5. (1) *We have $(c_A^{\mathrm{pr}})^\iota = c_A^{\mathrm{pr}}$; that is,*

$$\sum_{i \in I} e_i' \otimes e_i = \sum_{i \in I} e_i \otimes e_i'.$$

(2) *For all $a \in A$, we have*

$$a = \sum_i t(ae_i') e_i = \sum_i t(ae_i) e_i' = \sum_i t(e_i') e_i a = \sum_i t(e_i) e_i' a.$$

Proof. Indeed, by Lemma 3.2 we have $e_i' = \sum_j t(e_i' e_j') e_j$; hence

$$\sum_i e_i' \otimes e_i = \sum_{i,j} t(e_i' e_j') e_j \otimes e_i = \sum_j e_j \otimes \sum_i t(e_i' e_j') e_i$$
$$= \sum_j e_j \otimes \sum_i t(e_j' e_i') e_i = \sum_j e_j \otimes e_j'.$$

Assertion (2) is an immediate consequence of (1) and Lemma 3.2. $\square$

Define three maps as follows:

$$\mathrm{BiTr}^A\colon A\otimes A\to A,\quad a\otimes a'\mapsto \sum_i e_i a e_i' a';$$

$$\mathrm{Tr}^A\colon A\to A,\quad a\mapsto \sum_i e_i a e_i' = \mathrm{BiTr}^A(a\otimes 1);$$

$$\mathrm{Tr}_A\colon A\to A,\quad a'\mapsto a' z_A^{\mathrm{pr}} = \sum_i a' e_i e_i' = \mathrm{BiTr}^A(1\otimes a').$$

Then:

- Tr^A is a central morphism of ZA modules,

$$\mathrm{Tr}^A(zaa') = z\,\mathrm{Tr}^A(a'a)\quad (\forall z\in ZA,\ a,a'\in A),$$

  and its image is contained in ZA (and hence is an ideal of ZA);
- $\mathrm{BiTr}^A(a\otimes a') = \mathrm{Tr}^A(a)a' = a'\,\mathrm{Tr}^A(a)$;
- Tr_A is a morphism of A-modules-A.

Separably Symmetric Algebras

PROPOSITION 3.6. *If z_A^{pr} is invertible in ZA, then the multiplication morphism*

$$A\otimes_R A\to A,\quad a\otimes a'\mapsto aa'$$

is split as a morphism of A-modules-A.

Proof. Indeed, the composition of the morphism of A-modules-A defined by

$$A\to A\otimes_R A,\quad a\mapsto ac_A^{\mathrm{pr}}$$

with the multiplication $A\otimes_R A\to A$ is equal to the morphism

$$A\to A,\quad a\mapsto az_A^{\mathrm{pr}}.$$

In other words, if we view c_A^{pr} as an element of the algebra $A\otimes_R A^{\mathrm{op}}$, then $(c_A^{\mathrm{pr}})^2 = z_A^{\mathrm{pr}} c_A^{\mathrm{pr}}$. Thus we see that, if z_A^{pr} is invertible in ZA, then (i) the element $(z_A^{\mathrm{pr}})^{-1}c_A^{\mathrm{pr}}$ is a central idempotent in the algebra $A\otimes_A A^{\mathrm{op}}$ and (ii) the morphism

$$A\to A\otimes_R A^{\mathrm{op}},\quad a\mapsto a(z_A^{\mathrm{pr}})^{-1}c_A^{\mathrm{pr}}$$

is a section of the multiplication morphism, identifying A with a direct summand of $A\otimes_R A$ as an A-module-A. □

REMARK. If t is replaced by another symmetrizing form (e.g., by a form $t(z\,\cdot)$ for z an invertible element of ZA), then z_A^{pr} is replaced by zz_A^{pr}. Hence the invertibility of z_A^{pr} depends only on the algebra A and not on the choice of t.

An algebra A such that the the multiplication morphism

$$A\otimes_R A\to A,\quad a\otimes a'\mapsto aa'$$

is split as a morphism of A-modules-A is called *separable*. A symmetric algebra A such that z_A^{pr} is invertible in ZA is called *symmetrically separable*.

NOTE. A symmetrically separable algebra must be separable, but the converse is not true. For example, a matrix algebra $\mathrm{Mat}_m(R)$ is separable but is *symmetrically* separable if and only if m is invertible in R.

Observe that the previous example shows also that the property of being symmetrically separable is not stable under a Morita equivalence.

The following fundamental example justifies the notation and the name chosen for the map Tr^A.

EXAMPLE 3.7. Let us consider the particular case where $A := E_R X$ for X a finitely generated projective R-module. We identify A with $X^* \otimes_R X$ and set

$$\mathrm{Id}_X = \sum_i f_i \otimes e_i .$$

We know that A is symmetric and that $t := \mathrm{tr}_{X/R}$ is a symmetrizing form.

We leave it as an exercise for the reader to check the following properties.

(1) $c_A^{\mathrm{pr}} = \sum_{i,j} (f_i \otimes e_i) \otimes (f_j \otimes e_j)$.
(2) The map $\mathrm{Tr}^A : A \to ZA$ coincides with $\mathrm{tr}_{X/R} : E_R X \to R$.

3.B. Casimir Element, Trace, and Characters

Throughout this section, A is assumed to be a symmetric R-algebra with symmetrizing form t.

For $\tau : A \to R$ a linear form, we denote by τ^0 the element of A defined by the condition

$$t(\tau^0 h) = \tau(h) \quad \text{for all } a \in A.$$

We know that τ is central if and only if τ^0 is central in A.

It is easy to check the following property.

LEMMA 3.8. *We have $\tau^0 = \sum_i \tau(e_i')e_i = \sum_i \tau(e_i)e_i'$; more generally, for all $a \in A$ we have $\tau^0 a = \sum_i \tau(e_i' a)e_i = \sum_i \tau(e_i a)e_i'$.*

The biregular representation of A is by definition the morphism

$$A \otimes_R A^{\mathrm{op}} \to E_R A, \quad a \otimes a' \mapsto (x \mapsto axa'),$$

which defines the structure of an A-module-A of A.

Composing this morphism with the trace $\mathrm{tr}_{A/R}$, we obtain a linear form on $A \otimes_R A^{\mathrm{op}}$, which is called the *biregular character* of A and is denoted by χ_A^{bireg}.

PROPOSITION 3.9. *We have*

$$\chi_A^{\mathrm{bireg}}(a \otimes a') = t(\mathrm{BiTr}^A(a \otimes a'))$$

or, in other words,

$$\chi_A^{\mathrm{bireg}}(a \otimes a') = \sum_i t(a' e_i a e_i) = t(\mathrm{Tr}^A(a)a') = t(a\,\mathrm{Tr}^A(a')).$$

Proof. We know by Proposition 3.5(2) that

$$axa' = \sum_i t(axa'e_i')e_i,$$

which shows that the endomorphism of A defined by $a \otimes a'$ corresponds to the element

$$\sum_i \hat{t}(a'e_i a) \otimes e_i \in A^* \otimes A,$$

whose trace is

$$\sum_i t(a'e_i a e_i) = t(\mathrm{Tr}^A(a)a'). \qquad \square$$

Let χ_{reg} denote the character of the (left) regular representation of A—that is, the linear form on A defined by

$$\chi_{\mathrm{reg}}(a) := \mathrm{tr}_{A/R}(\lambda_A(a))$$

where $\lambda_A(a)\colon A \to A$, $x \mapsto ax$, is the left multiplication by a.

COROLLARY 3.10. *For all $a \in A$ we have $\chi_{\mathrm{reg}}(a) = t(az_A^{\mathrm{pr}})$; that is,*

$$\chi_{\mathrm{reg}}^0 = z_A^{\mathrm{pr}}.$$

COROLLARY 3.11. *Let i be an idempotent of A, and let χ_{Ai} denote the character of the (finitely generated projective) A-module Ai. Then*

$$\chi_{Ai}^0 = \mathrm{Tr}^A(i).$$

Indeed, we have

$$\mathrm{tr}_{Ai/R}(a) = \mathrm{tr}_{A/R}(a \otimes i) = t(a\,\mathrm{Tr}^A(i)).$$

3.C. Projective Center, Higman's Criterion

The Projective Center of an Algebra

Let A be an R-algebra, and let M be an A-module-A. We know (see Section 1) that the morphism

$$\mathrm{Hom}_A(A, M)_A \to M^A, \quad \varphi \mapsto \varphi(1)$$

is an isomorphism. In particular, we have

$$\mathrm{Hom}_A(A, A \otimes_R A^{\mathrm{op}})_A = (A \otimes A)^A.$$

The module $\mathrm{Hom}_A^{\mathrm{pr}}(A, M)_A$ consisting of projective morphisms (see Definition 1.6) from A to M is the image of the map

$$\mathrm{Hom}_A(A, A \otimes_R A^{\mathrm{op}})_A \otimes M \to \mathrm{Hom}_A(A, M)_A, \quad \varphi \otimes m \mapsto (a \mapsto (a\varphi)m).$$

Through the previous isomorphism, this translates to

$$(A \otimes_R A^{\mathrm{op}})^A \otimes M \to M^A, \quad x \otimes m \mapsto xm;$$

that is, we have a natural isomorphism

$$(A \otimes_R A^{\mathrm{op}})^A M = \mathrm{Hom}_A^{\mathrm{pr}}(A, M)_A.$$

PROPOSITION 3.12. *The module*

$$(A \otimes_R A^{\mathrm{op}})^A . A = \left\{ \sum_i a_i a a_i' \mid (a \in A) \left(\sum_i a_i \otimes a_i' \in (A \otimes_R A)^A \right) \right\}$$

is called the projective center *of A and is denoted by $Z^{\mathrm{pr}}A$. This is an ideal in ZA, and the map*

$$Z^{\mathrm{pr}}A \to \mathrm{Hom}_A(A, A)_A, \quad z \mapsto (a \mapsto az)$$

induces an isomorphism of ZA-modules from $Z^{\mathrm{pr}}A$ onto $\mathrm{Hom}_A^{\mathrm{pr}}(A, A)_A$.

When A Is Symmetric

If A is symmetric and if $c_A^{\mathrm{pr}} = \sum_i e_i' \otimes e_i$, then it follows from Proposition 3.3 that

$$(A \otimes_R A)^A = \left\{ \sum_i e_i' a \otimes e_i \mid (a \in A) \right\}.$$

Thus we have

$$(A \otimes_R A)^A M = \left\{ \sum_i e_i' m e_i \mid (m \in M) \right\},$$

which makes the next result obvious.

PROPOSITION 3.13. *The module* $\mathrm{Hom}_A^{\mathrm{pr}}(A, M)_A$ *is naturally isomorphic to the image of the map*

$$\mathrm{Tr}^A \colon M \to M^A, \quad m \mapsto c_A^{\mathrm{pr}} m = \sum_i e_i' m e_i.$$

In particular, Z_A^{pr} is the image of the map $\mathrm{Tr}^A \colon A \to A$.

Note that since $c_A^{\mathrm{pr}} \in C(A \otimes_R A)_A$, the map Tr^A factorizes through $[A, M]$ and so defines a map

$$\mathrm{Tr}^A \colon H_0(A, M) \to H^0(A, M).$$

EXAMPLE. If $A = RG$ (G a finite group), then $Z^{\mathrm{pr}}RG$ is the image of

$$\mathrm{Tr}^{RG} \colon RG \to ZRG, \quad x \mapsto \sum_{g \in G} g x g^{-1}.$$

We denote by $\mathrm{Cl}(G)$ the set of conjugacy classes of G, and for $C \in \mathrm{Cl}(G)$ we define a central element by

$$\mathcal{S}C := \sum_{g \in C} g.$$

Then it is immediate to check that

$$Z^{\mathrm{pr}}RG = \bigoplus_{C \in \mathrm{Cl}(G)} \frac{|G|}{|C|} \mathcal{S}C.$$

Higman's Criterion
If X and X' are A-modules, then applying what precedes to the case where $M := \mathrm{Hom}_R(X, X')$ yields a map

$$\mathrm{Tr}^A \colon \mathrm{Hom}_R(X, X') \to \mathrm{Hom}_A(X, X'), \quad \alpha \mapsto \left[x \mapsto \textstyle\sum_i (e_i \alpha(e_i' x))\right].$$

For an A-module X, let us describe in terms of the Casimir element the inverse of the isomorphism

$$t_X^* \colon \begin{cases} \mathrm{Hom}_A(X, A) \to \mathrm{Hom}_R(X, R), \\ \phi \mapsto t \cdot \phi. \end{cases}$$

By the formula given in Proposition 2.10(1) we see that, for all $x \in X$ and $\psi \in \mathrm{Hom}_R(X, R)$,

$$u_X(\psi)(x) = \widehat{\psi(\bullet x)}.$$

By Lemma 3.8, we then have the following property.

LEMMA 3.14. *For any A-module X, the morphism*

$$\begin{cases} \mathrm{Hom}_R(X, R) \to \mathrm{Hom}_A(X, A), \\ \psi \mapsto \left[x \mapsto \sum_i \psi(e_i' x) e_i = \sum_i \psi(e_i x) e_i'\right] \end{cases}$$

is the inverse of the isomorphism t_X^*.

Let X and X' be A-modules such that X or X' is a finitely generated projective R-module. Then Lemma 3.14 implies that the natural morphism $\mathrm{Hom}_A(X, A) \otimes_R X' \to \mathrm{Hom}_A(X, X')$ factorizes as follows:

$$\begin{aligned} \mathrm{Hom}_A(X, A) \otimes_R X' &\xrightarrow{\sim} \mathrm{Hom}_R(X, R) \otimes_R X' \\ &\xrightarrow{\sim} \mathrm{Hom}_R(X, X') \xrightarrow{\mathrm{Tr}^A} \mathrm{Hom}_A(X, X'). \end{aligned}$$

The next lemma is now an immediate consequence of the characterization of finitely generated projective modules.

LEMMA 3.15. *Let X and X' be A-modules such that X is a finitely generated projective R-module.*

(1) *The submodule* $\mathrm{Hom}_A^{\mathrm{pr}}(X, X')$ *of* $\mathrm{Hom}_A(X, X')$ *consisting of maps factorizing through a finitely generated projective A-module coincides with the image of the map*

$$\mathrm{Tr}^A \colon \begin{cases} \mathrm{Hom}_R(X, X') \to \mathrm{Hom}_A(X, X'), \\ \alpha \mapsto \left[x \mapsto \sum_{i \in I} e_i' \alpha(e_i x)\right]. \end{cases}$$

(2) *The image of the map*

$$\mathrm{Tr}^A \colon E_R X \to E_A X$$

is a two-sided ideal of $E_A X$.

The following proposition is a consequence of Lemma 3.15. It is known, in the case where $A = RG$, as *Higman's criterion* (see [H1]).

PROPOSITION 3.16. *Let A be a symmetric R-algebra that has Casimir element $\sum_i e_i' \otimes e_i$. Let X be an A-module that is a finitely generated projective R-module. Then X is a projective A-module if and only if there exists an R-endomorphism α of X such that*

$$(\forall x \in X) \quad \sum_i e_i' \alpha(e_i x) = x.$$

REMARK. For symmetric algebras over fields, Higman's criterion is also a necessary and sufficient condition for X to be injective. In Theorem 6.8 that property will be addressed (and generalized) in a more general context.

4. Schur Elements

The notion of Schur element of an absolutely irreducible representation of a symmetric algebra (as well as the application to orthogonality relations between characters) was first introduced by M. Geck ([G]; see also [GRo]). We present here a slight generalization of that notion.

Quotients of Symmetric Algebras

Let A and B be two symmetric algebras, and let $\lambda\colon A \twoheadrightarrow B$ be a *surjective* algebra morphism. The morphism λ defines a morphism

$$A \otimes_R A^{\mathrm{op}} \to B \otimes_R B^{\mathrm{op}}, \quad a \otimes a' \mapsto \lambda(a) \otimes \lambda(a');$$

hence λ defines the structure of an A-module-A on B.

REMARK. We shall apply the results in this section to the following context. Let A be a finite-dimensional algebra over a (commutative) field k, let X be an irreducible A-module, let $D := E_A X$ (a division algebra), and let $B := EX_D$. We know that B is a symmetric algebra, and by the double centralizer property we know that the structural morphism $\lambda_X\colon A \to B$ is onto.

Let t be a symmetrizing form on A and let u be a symmetrizing form on B. Let $c_A^{\mathrm{pr}} = \sum_i e_i \otimes e_i'$ and $c_B^{\mathrm{pr}} = \sum_j f_j \otimes f_j'$ be the corresponding Casimir elements for A and B, respectively.

The form $u \cdot \lambda$ is a central form on A, so there exists an element $(u \cdot \lambda)^0 \in ZA$ whose image under $\hat{t}$ is $u \cdot \lambda$. Because λ is onto, the element $s_\lambda := \lambda((u \cdot \lambda)^0)$ belongs to ZB.

DEFINITION 4.1. The element s_λ is called the *Schur element* of the (surjective) morphism λ.

PROPOSITION 4.2. *We have*

$$(\lambda \otimes \lambda)(c_A^{\mathrm{pr}}) = s_\lambda c_B^{\mathrm{pr}} \quad \text{and} \quad \lambda(z_A^{\mathrm{pr}}) = s_\lambda z_B^{\mathrm{pr}}.$$

Proof. Let us set $c_A^{\mathrm{pr}} = \sum_i e_i' \otimes e_i$. For all $a \in A$ we have

$$(u.\lambda)^0 a = \sum_i t((u.\lambda)^0 a e_i') e_i; \quad \text{hence} \quad (u.\lambda)^0 a = \sum_i u(\lambda(a)\lambda(e_i')) e_i,$$

from which we deduce

$$s_\lambda \lambda(a) = \sum_i u(\lambda(ae'_i))\lambda(e_i).$$

Since λ is surjective, it follows that for all $b \in B$ we have

$$s_\lambda b = \sum_i u(b\lambda(e'_i))\lambda(e_i).$$

This shows that, through the isomorphism $B \otimes_R B \xrightarrow{\sim} E_R B$ defined by $\hat{u}$, the element $\sum_i \lambda(e'_i) \otimes \lambda(e_i)$ corresponds to $s_\lambda \operatorname{Id}_B$. Therefore,

$$\sum_i \lambda(e'_i) \otimes \lambda(e_i) = s_\lambda c_B^{\mathrm{pr}}. \qquad \square$$

REMARK. Choose $A = B$ and $\lambda := \operatorname{Id}_A$. Now if t and u are two symmetrizing forms on A, we have $u = t(u^0 \,\boldsymbol{\cdot})$. The formula of Proposition 4.2 can be written (with obvious notation) as

$$c_{A,t}^{\mathrm{pr}} = u^0 c_{A,u}^{\mathrm{pr}}.$$

The structure of the A-module-A on B defined by λ allows us to define, for N any B-module-B, the trace map

$$\operatorname{Tr}^A \colon N \to N^A, \quad n \mapsto c_A^{\mathrm{pr}}.n = \sum_i \lambda(e_i) n \lambda(e'_i).$$

The following property is an immediate consequence of Proposition 4.2.

COROLLARY 4.3. *If N is a B-module-B, then*

$$\operatorname{Tr}^A(n) = s_\lambda \operatorname{Tr}^B(n).$$

We now characterize the situation where the Schur element is invertible.

PROPOSITION 4.4. *The following properties are equivalent.*

(i) *The Schur element s_λ is invertible in ZB.*
(ii) *The morphism $\lambda \colon A \twoheadrightarrow B$ is split as a morphism of A-modules-A.*
(iii) *B is a projective A-module.*
(iv) *Any projective B-module is a projective A-module.*

If these properties are fulfilled, then the map

$$\sigma : \begin{cases} B \to A, \\ b \mapsto \sum_i u(s_\lambda^{-1} b \lambda(e'_i)) e_i \end{cases}$$

is a section of λ as a morphism of A-modules-A.

Proof. (i) $\Rightarrow$ (ii). Since

$$\sum_i \lambda(e'_i) \otimes \lambda(e_i) = s_\lambda c_B^{\mathrm{pr}}$$

and since s_λ is invertible, we have

$$c_B^{\mathrm{pr}} = s_\lambda^{-1} \sum_i \lambda(e_i') \otimes \lambda(e_i).$$

It follows that

$$\lambda(\sigma(b)) = \sum_i u(s_\lambda^{-1} b\lambda(e_i'))\lambda(e_i) = \sum_j u(bf_j') f_j = b,$$

which proves that σ is a section of λ.

Put $\tilde{s} := (u.\lambda)^0$, and let us choose a preimage $\tilde{s}'$ of s_λ^{-1} in A. If we choose a preimage $\tilde{b}$ of b through λ then

$$\begin{aligned}\sum_i u(s_\lambda^{-1} b\lambda(e_i'))e_i &= \sum_i u(\lambda(\tilde{s}'\tilde{b}e_i'))e_i = \sum_i t(\tilde{s}\tilde{s}'\tilde{b}e_i')e_i = \tilde{s}\tilde{s}'\tilde{b} \\ &= \sum_i u(bs_\lambda^{-1}\lambda(e_i'))e_i = \sum_i u(\lambda(\tilde{b}\tilde{s}'e_i'))e_i = \sum_i t(\tilde{b}\tilde{s}\tilde{s}'\tilde{b}e_i')e_i \\ &= \tilde{b}\tilde{s}\tilde{s}',\end{aligned}$$

which makes it obvious that σ commute with the two-sided action of A.

(ii) $\Rightarrow$ (iii). Because λ is split as a morphism of A-modules, we see that B is projective as an A-module.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). Since B is a finitely generated projective A-module, Higman's criterion (see Proposition 3.16) shows that there is a $\beta \in E_R B$ such that $\mathrm{Tr}^A(\beta) = \mathrm{Id}_B$. By Corollary 4.3, we then see that

$$s_\lambda \, \mathrm{Tr}^B(\beta) = \mathrm{Id}_B.$$

Since $\mathrm{Tr}^B(\beta) \in \mathrm{Hom}_B(B, B) = B$, this equality shows that s_λ is invertible in B and hence is invertible in ZB. □

Remark. Since σ is a morphism of A-modules-A, it follows that for

$$e_\lambda := \sigma(1) = \sum_i u(s_\lambda^{-1} b\lambda(e_i'))e_i$$

we have

$$\sigma(bb') = aea'$$

whenever $\lambda(a) = b$ and $\lambda(a') = b'$; hence, in particular, e is a central idempotent of A. Thus we may view (B, λ, σ) as

$$\begin{cases} B = Ae_\lambda, \\ \lambda\colon A \to Ae_\lambda, \quad a \mapsto ae_\lambda, \\ \sigma\colon Ae_\lambda \to A, \quad ae_\lambda \mapsto ae_\lambda. \end{cases}$$

Schur Elements of Split Irreducible Modules

When $R = k$ for k a (commutative) field, the next definition coincides with the definition of a *split irreducible module*. The reader may keep this example in mind.

DEFINITION 4.5. An A-module X is called *split quasi-irreducible* if:

(1) X is a generator and a finitely generated projective R-module (i.e., a "progenerator" for ${}_R\mathbf{Mod}$); and
(2) the morphism $\lambda_X : A \to E_R X$ is onto.

If X is split quasi-irreducible then X induces a Morita equivalence between R and $E_R X$ and so, in particular, the map

$$R \to E_R X, \quad \lambda \mapsto \lambda\, \mathrm{Id}_X$$

is an isomorphism from R onto the center $Z(E_R X)$ of $E_R X$. Thus the restriction of λ_X to ZA induces an algebra morphism

$$\omega_X : ZA \to R.$$

We denote by χ_X the character of the A-module X, that is, the central form on A defined by

$$\chi_X(a) = \mathrm{tr}_{X/R}(\lambda_X(a)).$$

The next result is an immediate application of the definition.

LEMMA 4.6. *Let X be a split quasi-irreducible A-module. Then the Schur element of X is the element $s_X \in R$ defined by*

$$s_X := \omega_X(\chi_X^0).$$

EXAMPLE. Assume $R = \mathbb{C}$ and $A = \mathbb{C}G$ (G a finite group). Let χ be the character of an irreducible $\mathbb{C}G$-module. Then the Schur element of this module is the scalar $s_\chi := |G|/\chi(1)$.

PROPOSITION 4.7. *If X is a split quasi-irreducible A-module with character $\chi := \chi_X$, then*:

(1) $s_X \chi(1) = \sum_i \chi(e'_i)\chi(e_i)$;
(2) $s_X \chi(1)^2 = \chi\big(\sum_i e'_i e_i\big)$.

Proof. The trace of the central element $s_X = \omega_X(\chi_X^0)$ is $\chi(1)s_X = \chi(1)\chi(\chi_X^0)$, and since $\chi^0 = \sum_i \chi(e'_i)e_i$ we see that $\chi(1)s_X = \sum_i \chi(e'_i)\chi(e_i)$. Assertion (2) is a consequence of the following lemma.

LEMMA 4.8. *Whenever $\alpha \in E_R X$, the central element* $\mathrm{Tr}^A(\alpha)$ *is the scalar multiplication by* $s_X\, \mathrm{tr}_{X/R}(\alpha)$.

Proof. This is an immediate application of the results of Example 3.7 and Corollary 4.3. □

Let us also give a "direct" proof as an exercise.

Since for all $a \in A$ we have $a\chi^0 = \sum_i \chi(ae'_i)e_i$, it follows that

$$\lambda_X(a\chi^0) = \sum_i \chi(ae'_i)\lambda_X(e_i),$$

and if $\alpha = \lambda_X(a)$ then

$$\alpha\lambda_X(\chi^0) = s_X\alpha = \sum_i \mathrm{tr}_{X/R}(\alpha\lambda_X(e_i'))\lambda_X(e_i).$$

Hence, through the isomorphism $E_R X \xrightarrow{\sim} X^* \otimes X$, the action of $s_X\alpha$ on X corresponds to the element

$$\sum_i \mathrm{tr}_{X/R}(\lambda_X(e_i')\alpha(\bullet)) \otimes \lambda_X(e_i)$$

and its trace is

$$\sigma_X \,\mathrm{tr}_{X/R}(\alpha) = \sum_i \mathrm{tr}_{X/R}(\lambda_X(e_i')\alpha(\lambda_X(e_i))),$$

completing the proof.

Proposition 4.4 has the following important consequence.

PROPOSITION 4.9. *Let X be a split quasi-irreducible A-module. Then the following properties are equivalent.*

(i) *Its Schur element s_X is invertible in R.*
(ii) *The structural morphism $\lambda_X \colon A \twoheadrightarrow E_R X$ is split as a morphism of A-modules-A.*
(iii) *X is a projective A-module.*

If these properties are satisfied, then the map

$$\sigma : \begin{cases} E_R X \to A, \\ \alpha \mapsto \sum_i \mathrm{tr}_{X/R}(s_X^{-1}\alpha e_i')e_i \end{cases}$$

is a section of λ as a morphism of A-modules-A.

REMARK. The last formula of Proposition 4.9 is what Serre calls the "Fourier inversion formula" in the case where A is the group algebra of a finite group over the field of complex numbers (see [Se, 6.2, Prop. 11]).

Case of a Symmetric Algebra over a Field

Let k be a field and let A be a finite-dimensional symmetric k-algebra.

For X an irreducible A-module, we recall that the algebra $D_X := E_A X$ is a division algebra, that the algebra $B := EX_{D_X}$ is symmetric, and that the structural morphism $\lambda \colon A \to B$ is onto. Thus each irreducible A-module has a Schur element $s_X \in ZD_X$, and since ZD_X is a field it follows that the Schur element s_X is invertible if and only if it is nonzero.

PROPOSITION 4.10. *Let k be a field, and let A be a finite-dimensional symmetric k-algebra. Then the following assertions are equivalent.*

(i) *A is semisimple.*
(ii) *If $X \in \mathrm{Irr}(A)$ then $s_X \neq 0$.*

Proof. This follows because a finite-dimensional k-algebra is semisimple if and only if all its irreducible modules are projective. $\square$

Now assume that the algebra $\bar{A} := A/\operatorname{Rad}(A)$ is split. In other words, assume that

$$(\forall S \in \operatorname{Irr}(A)) \quad \operatorname{End}_A(S) = k\operatorname{Id}_S; \quad \text{hence} \quad \bar{A} \xrightarrow{\sim} \prod_{S \in \operatorname{Irr}(A)} \operatorname{End}_k(S).$$

We denote by $a \mapsto \bar{a}$ the canonical epimorphism from A onto $\bar{A}$.

Let $S \in \operatorname{Irr}(A)$. By a slight abuse of notation, we consider that the structural morphism defining the structure of an A-module of S is defined by the composition

$$A \longrightarrow \bar{A} \xrightarrow{\lambda_S} \operatorname{End}_k(S).$$

Denote by e_S the corresponding central idempotent of $\bar{A}$, and let us choose an element $\tilde{e}_S \in A$ whose image modulo $\operatorname{Rad}(A)$ is e_S.

We have

$$\chi_S(a) = t(\chi_S^0 a) = \operatorname{tr}_{S/k}(\lambda_S(\bar{a})).$$

Hence, for all $S, T \in \operatorname{Irr}(A)$,

$$t(\chi_S^0 \tilde{e}_T a) = \operatorname{tr}_{S/k}(\lambda_S(e_T \bar{a})) = \delta_{S,T}\chi_S(a)$$

and so

$$\chi_S^0 \tilde{e}_T = \delta_{S,T}\chi_S^0.$$

This equality allows us to prove the following orthogonality relation between characters of absolutely irreducible modules.

PROPOSITION 4.11. *Let A be a symmetric algebra such that $A/\operatorname{Rad}(A)$ is split. Let $c^{\mathrm{pr}} = \sum_i e_i' \otimes e_i$ be the Casimir element of A. For all $S, T \in \operatorname{Irr}(A)$, we have*

$$\sum_i \chi_S(e_i')\chi_T(e_i) = \begin{cases} s_S\chi_S(1) & \text{if } S = T, \\ 0 & \text{if } S \neq T. \end{cases}$$

Symmetric Split Semisimple Algebras

PROPOSITION 4.12. *Let k be a field, and let A be a finite-dimensional symmetric k-algebra with symmetrizing form t. Assume that A is split semisimple. For each irreducible character χ of A, let e_χ be the primitive idempotent of the center ZA associated with χ, and let s_χ denote its Schur element.*

(1) *We have*

$$s_\chi \neq 0 \quad \text{and} \quad \chi^0 = s_\chi e_\chi.$$

(2) *We have*

$$t = \sum_{\chi \in \operatorname{Irr}(A)} \frac{1}{s_\chi}\chi.$$

Proof. (1) Since for all $a \in A$ we have $\chi(e_\chi h) = \chi(h)$, we see that $t(\chi^0 e_\chi h) = t(\chi^0 h)$, which proves that $\chi^0 = \chi^0 e_\chi$. The desired equality then follows because, for all $z \in ZA$, we have $z = \sum_{\chi \in \operatorname{Irr}(FA)} \omega_\chi(z) e_\chi$.

(2) Through the isomorphism between A and its dual, the equality

$$t = \sum_{\chi \in \mathrm{Irr}(FA)} \frac{1}{s_\chi} \chi$$

is equivalent to

$$1 = \sum_{\chi \in \mathrm{Irr}(FA)} \frac{1}{s_\chi} \chi^0,$$

which is obvious by part (1) of the proposition. □

5. Parabolic Subalgebras

Definition and First Properties

The following definition covers the case of subalgebras such as RH (H a subgroup of G) of a group algebra RG, as well as the case of the so-called parabolic subalgebras of Hecke algebras.

DEFINITION 5.1. Let A be a symmetric R-algebra, and let t be a symmetrizing form on A. A subalgebra B of A is called *parabolic* (relative to t) if the following two conditions are satisfied.

(Pa1) Viewed as a B-module through left multiplication, A is projective.
(Pa2) The restriction of t to B is a symmetrizing form for B.

REMARKS. 1. Condition (Pa1) is equivalent to:

(Pa1′) Viewed as a module-B through right multiplication, A is projective.

Indeed, A is a projective B-module if and only if A^* is a projective module-B, hence (since A^* is isomorphic to A) if and only if A is a projective module-B.

2. Condition (Pa2) is equivalent to:

(Pa2′) $B \cap B^\perp = 0$.

PROPOSITION 5.2. *Let A be a symmetric algebra with a symmetrizing form t, and let B be a subalgebra of A such that A is a projective B-module.*

(1) *The subalgebra B is parabolic if and only if $B \oplus B^\perp = A$, and then the corresponding projection of A onto B is the morphism of B-modules-B*

$$\mathrm{Br}_B^A : A \to B \quad \textit{such that } t(\mathrm{Br}_B^A(a)b) = t(ab) \textit{ for all } a \in A \textit{ and } b \in B.$$

(2) *If part* (1) *is satisfied, then $B^\perp$ is the B-submodule-B of A characterized by the following two properties*:

(a) $A = B \oplus B^\perp$ (*as B-modules-B*);
(b) $B^\perp \subseteq \ker(t)$.

EXAMPLE. Assume $A = RG$ and $B = RH$ (G a finite group, H a subgroup of G). Then the map Br_{RH}^{RG} is defined as follows:

$$\mathrm{Br}_{RH}^{RG}(g) = \begin{cases} g & \text{if } g \in H, \\ 0 & \text{if } g \notin H. \end{cases}$$

Hence we shall call Br_B^A the *Brauer morphism* from A to B.

NOTE. The subalgebra $R.1$ is not necessarily a parabolic subalgebra. Indeed, the symmetrizing forms on R are the forms τ such that $\tau(1) \in R^\times$. Thus $R.1$ is parabolic if and only if $t(1)$ is invertible in R.

This is not always the case, since for $A := \mathrm{Mat}_m(R)$ and $t := \mathrm{tr}$ we have $t(1) = m$. This example shows as well that the property of $R.1$ being parabolic is not stable under Morita equivalence.

REMARKS.

- If $R.1$ is parabolic, then we may wish to normalize the form t by assuming that $t(1) = 1$.
- If $R.1$ is parabolic, then A is principally symmetric (see 2.14).

However, an algebra may be principally symmetric without $R.1$ being parabolic, as shown by the example $A := \mathrm{Mat}_m(R)$ when m is not invertible in R.

6. Exact Bimodules and Associated Functors

6.A. Self-dual Pairs of Exact Bimodules

In what follows, we use A and B to denote two symmetric R-algebras. Assume that we have chosen two symmetrizing forms t and u on (respectively) A and B.

DEFINITION 6.1. An A-module-B M is called *exact* if M is finitely generated projective both as an A-module and as a module-B.

If M is exact, then the functors

$$M \otimes_B \cdot : {}_B\mathbf{Mod} \to {}_A\mathbf{Mod} \quad \text{and} \quad \cdot \otimes_A M : \mathbf{Mod}_A \to \mathbf{Mod}_B$$

defined by M are exact.

DEFINITION. A *self-dual pair of exact bimodules* for A and B is a pair (M, N), where M is an exact A-module-B and where N is an exact B-module-A endowed with an R-duality of bimodules

$$M \times N \to R, \quad (m, n) \mapsto \langle m, n \rangle,$$

that is, an R-bilinear map such that

$$\langle amb, n \rangle = \langle m, bna \rangle \quad (\forall a \in A,\ b \in B,\ m \in M,\ n \in N),$$

which induces (bimodules) isomorphisms

$$M \xrightarrow{\sim} N^* \quad \text{and} \quad N \xrightarrow{\sim} M^*.$$

EXAMPLES. 1. Take $B = R$, $M = {}_AA_R$ (i.e., A viewed as an object in ${}_A\mathbf{mod}_R$), $N = {}_RA_A$ (i.e., A viewed as an object in ${}_R\mathbf{mod}_A$), and $\langle a, b \rangle := t(ab)$. Then $({}_AA_R, {}_RA_A)$ is an exact pair of bimodules for A and R and is called the *trivial pair* for A.

2. Let G be a finite group, and let U be a subgroup of G whose order is invertible in R. Let $N_G(U)$ denote the normalizer of U in G, and set $H := N_G(U)/U$.

Then the set G/U is naturally endowed with a left action of G and a right action of H, and the set $U\backslash G$ is naturally endowed with a left action of H and a right action of G.

Take $A := RG$, $B := RH$ (both induced with the canonical symmetrizing forms of group algebras), $M := R[G/U]$ (the R-free module with basis G/U), $N := R[U\backslash G]$, and

$$\langle gU, Ug' \rangle := \begin{cases} 1 & \text{if } Ug' = (gU)^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Then the pair $(R[G/U], R[U\backslash G])$ is an exact pair of bimodules for RG and RH.

The functor defined by M is known as the Harish–Chandra *induction*: take an RH-module Y, view it as an $RN_G(U)$-module, and induce it up to RG. The adjoint functor defined by N is the Harish–Chandra *restriction* (or *truncation*): take an RG-module X and view its fixed points under U as an RH-module.

3. This example is a generalization of the previous two. Let B be a parabolic subalgebra of A. Let e be a central idempotent of A, and let f be a central idempotent of B. Let us choose

$$M := eAf, \quad N := fAe, \quad \langle m, n \rangle := t(mn).$$

Then the functor induced by M is the induction truncated by e,

$$Y \mapsto e.\mathrm{Ind}_B^A Y,$$

while the functor induced by N is the restriction truncated by f,

$$X \mapsto f.\mathrm{Res}_B^A X.$$

Let (M, N) be a self dual pair of exact bimodules.

A. The isomorphism $N \xrightarrow{\sim} M^*$, when composed with the isomorphism

$$M^* \xrightarrow{\sim} M^\vee = \mathrm{Hom}_A(M, A)$$

given by Proposition 2.10, gives an isomorphism $N \xrightarrow{\sim} M^\vee$ of B-modules-A, which is described as follows.

6.2. *The element $n \in N$ defines the A-linear form $m \mapsto mn$ on M such that*

$$t(mn) = \langle m, n \rangle.$$

Similarly, we have an isomorphism $M \xrightarrow{\sim} N^\vee$ of A-modules-B, which is described as follows.

6.3. *The element $m \in M$ defines the B-linear form $n \mapsto nm$ on N such that*

$$u(nm) = \langle m, n \rangle.$$

B. The isomorphism $M \xrightarrow{\sim} N^\vee$ just described induces isomorphisms

$$M \otimes_B N \xrightarrow{\sim} N^\vee \otimes_B N \xrightarrow{\sim} E_B N \xrightarrow{\sim} EM_B.$$

We know (see Proposition 2.11) that there is a symmetrizing form u_N on the algebra $E_B N$. Transporting the algebra structure and the form u_N through the preceding isomorphisms gives the following property.

PROPOSITION 6.4. (1) *The rule*

$$(m \otimes_B n)(m' \otimes_B n') := m \otimes_B (nm')n$$

provides $M \otimes_B N$ with the structure of an algebra that is isomorphic to $E_B N$ (and EM_B).

(2) *The form*

$$t_{M,N} \colon M \otimes_B N \to R, \quad m \otimes_B n \mapsto \langle m, n\rangle$$

is a symmetrizing form on the algebra $M \otimes_B N$.

Similarly, we have an algebra structure on $N \otimes_A M$ and a symmetrizing form

$$t_{N,M} \colon N \otimes_A M \to R, \quad n \otimes_A m \mapsto \langle m, n\rangle.$$

We denote by $c_{M,N}$ the unity of $M \otimes_B N$ (that is, the "(M,N)-Casimir element"). Thus, if $c_{M,N} = \sum_i m_i \otimes_B n_i$ for all $m \in M$ and $n \in N$, then

$$\sum_i m \otimes_B (nm_i)n_i = \sum_i m_i \otimes_B (n_i m)n = m \otimes_B n.$$

We likewise denote by $c_{N,M}$ the unity of the algebra $N \otimes_A M$.

The Case of the Trivial Pair

Let us consider the trivial pair $({}_A A_R, {}_R A_A)$ for A. The following statements hold.

- The algebra $A \otimes_A A$ is isomorphic to A and its symmetrizing form is the form t.
- The algebra $A \otimes_R A$ is isomorphic to $E_R A$ and its symmetrizing form is defined by $a \otimes a' \mapsto t(aa')$.

NOTE. The algebra $A \otimes_R A$ is not, in general, isomorphic to $A \otimes A^{\mathrm{op}}$. Observe also that the multiplication in the algebra $A \otimes_R A$ is defined by the rule

$$(a \otimes a')(b \otimes b') := a \otimes t(a'b)b'$$

and that, by its very definition, c_A^{pr} is the unity of this algebra.

Adjunctions

Let (M,N) be a self-dual pair of exact bimodules for A and B. Because $M \simeq N^\vee$ and $N \simeq M^\vee$, the pair $(M \otimes_B \cdot, N \otimes_A \cdot)$ is a pair of *bi-adjoint functors*—in other words, a pair of functors that are left and right adjoint to each other.

The isomorphisms $N \xrightarrow{\sim} M^\vee$ and $M \xrightarrow{\sim} N^\vee$ described in 6.2 and 6.3, together with the adjunctions defined by the "isomorphisme cher à Cartan" (Theorem 1.1), define the following set of four adjunctions (described in terms of morphisms of bimodules):

$$\varepsilon_{M,N} \colon \begin{cases} M \otimes_B N \to A, \\ m \otimes_B n \mapsto mn, \end{cases} \qquad \eta_{M,N} \colon \begin{cases} B \to N \otimes_A M, \\ b \mapsto bc_{N,M}; \end{cases}$$

$$\varepsilon_{N,M} \colon \begin{cases} N \otimes_A M \to B, \\ n \otimes_B m \mapsto nm, \end{cases} \qquad \eta_{N,M} \colon \begin{cases} A \to M \otimes_B N, \\ a \mapsto ac_{M,N}. \end{cases}$$

PROPOSITION 6.5. *The morphisms*

$$\varepsilon_{M,N} : M \otimes_B N \to A \quad \textit{and} \quad \eta_{N,M} : A \to M \otimes_B N$$

are adjoint to each other relative to the bilinear forms defined on A and $M \otimes_B N$ by (respectively) t and $t_{M,N}$; that is,

$$t(\varepsilon_{M,N}(x)a) = t_{M,N}(x\eta_{N,M}(a)) \quad (\forall x \in M \otimes_B N,\ a \in A).$$

6.B. Relative Projectivity, Relative Injectivity

Let us generalize the preceding situation, first replacing ${}_A\mathbf{Mod}$ and ${}_B\mathbf{Mod}$ by two arbitrary *R-linear triangulated* or *abelian* categories $\mathfrak{A}$ and $\mathfrak{B}$ and then considering two functors $M : \mathfrak{B} \to \mathfrak{A}$ and $N : \mathfrak{A} \to \mathfrak{B}$ such that (M, N) is a bi-adjoint pair.

As in the "concrete" situation considered previously, let $(\varepsilon_{M,N}, \eta_{M,N})$ (resp. $(\varepsilon_{N,M}, \eta_{N,M})$) be a co-unit and a unit associated with an adjunction for the pair (M, N) (resp. (N, M)).

NOTATION. We say that an object X' of such a (R-linear triangulated) category $\mathfrak{A}$ is *isomorphic to a direct summand* of an object X if there exist two morphisms

$$\iota : X' \to X \quad \text{and} \quad \pi : X \to X' \quad \text{with} \quad \pi \circ \iota = \mathrm{Id}_X.$$

This is indeed equivalent (see [BS, Lemma 1.8]) to the existence of an object X'' and an isomorphism

$$X \xrightarrow{\sim} X' \oplus X''.$$

DEFINITION 6.6. For X and X' in $\mathfrak{A}$ we denote by $\mathrm{Tr}_N^M(X, X')$, and call *relative trace,* the map

$$\mathrm{Tr}_N^M(X, X') : \mathrm{Hom}_{\mathfrak{B}}(NX, NX') \to \mathrm{Hom}_{\mathfrak{A}}(X, X')$$

defined by

$$\mathrm{Tr}_N^M(X, X')(\beta) := \varepsilon_{M,N}(X') \circ M(\beta) \circ \eta_{N,M}(X),$$

that is,

$$\begin{tikzcd} X \arrow[d, "\eta_{N,M}"'] & X' \\ MNX \arrow[r, "M\beta"] & MNX'. \arrow[u, "\varepsilon_{M,N}"'] \end{tikzcd}$$

When it is clear from the context what the domain and the codomain of β are, we will write $\mathrm{Tr}_N^M(\beta)$ instead of $\mathrm{Tr}_N^M(X, X')(\beta)$. Furthermore, $\mathrm{Tr}_N^M(X)$ stands for $\mathrm{Tr}_N^M(X, X)$. Notice that the map Tr_M^N is also defined.

The following example is fundamental.

EXAMPLE (Induction and restriction from R). Let A be a symmetric R-algebra with symmetrizing form t and Casimir element $c_A^{\mathrm{pr}} = \sum_i e_i \otimes e'_i$. We take $\mathfrak{A} = {}_A\mathbf{Mod}$ and $\mathfrak{B} = {}_R\mathbf{Mod}$ and then consider the pair of bi-adjoint functors defined by the module A, viewed as an object of ${}_A\mathbf{Mod}_R$ and as an object of ${}_R\mathbf{Mod}_A$. In other words, the functors are the induction Ind_R^A and the restriction Res_R^A. Let us set

$$\mathrm{Tr}_R^A := \mathrm{Tr}_{\mathrm{Res}_R^A}^{\mathrm{Ind}_R^A} \quad \text{and} \quad \mathrm{Tr}_A^R := \mathrm{Tr}_{\mathrm{Ind}_R^A}^{\mathrm{Res}_R^A}.$$

Verification of the following two statements is left to the reader.

1. For $X, X' \in {}_A\mathbf{Mod}$, the map

$$\mathrm{Tr}_R^A \colon \mathrm{Hom}_R(X, X') \to \mathrm{Hom}_A(X, X')$$

is defined by

$$\mathrm{Tr}_R^A(\beta)(x) = \sum_i e_i \beta(e_i' x) = \mathrm{Tr}^A(x);$$

in other words, we thus have

$$\mathrm{Tr}_R^A = \mathrm{Tr}^A.$$

2. For $Y, Y' \in {}_R\mathbf{Mod}$, the map

$$\mathrm{Tr}_A^R \colon \mathrm{Hom}_A(A \otimes_R Y, A \otimes_R Y') \to \mathrm{Hom}_R(Y, Y')$$

is defined in the following way. Let α be an element of $\mathrm{Hom}_A(A \otimes_R Y, A \otimes_R Y')$ and let $y \in Y$. If $\alpha(1 \otimes y) = \sum_i a_i \otimes y_i$, then the relative trace is given by the formula

$$\mathrm{Tr}_A^R(\alpha)(y) = \sum_i t(a_i) y_i.$$

PROPOSITION 6.7. *Given three morphisms*

$$\beta \colon NX \to NX', \quad \alpha \colon X_1 \to X, \quad \alpha' \colon X' \to X_1',$$

it follows that

$$\alpha' \circ \mathrm{Tr}_N^M(\beta) \circ \alpha = \mathrm{Tr}_N^M(N(\alpha') \circ \beta \circ N(\alpha)).$$

In particular, the image of Tr_N^M *is a two-sided ideal in* $\mathrm{Hom}_{\mathfrak{A}}(\bullet, \bullet)$.

Proof. Because $\eta_{N,M}$ is a natural transformation, the diagram

$$\begin{tikzcd} X_1 \arrow[r, "\eta_{N,M}(X_1)"] \arrow[d, "\alpha"] & MN(X_1) \arrow[d, "MN(\alpha)"] \\ X \arrow[r, "\eta_{N,M}(X)"] & MN(X) \end{tikzcd}$$

commutes; that is,

$$MN(\alpha) \circ \eta_{N,M}(X_1) = \eta_{N,M}(X) \circ \alpha.$$

Similarly, we have

$$\varepsilon_{M,N}(X_1') \circ MN(\alpha') = \alpha' \circ \varepsilon_{M,N}(X').$$

Using these equations, we obtain

$$\begin{tikzcd}
X_1 \arrow[r, "\alpha"] \arrow[d, "\eta_{N,M}(X_1)"'] & X \arrow[d, "\eta_{N,M}(X)"'] & X' \arrow[r, "\alpha'"] & X'1 \\
MNX_1 \arrow[r, "MN\alpha"] & MNX \arrow[r, "M\beta"] & MNX' \arrow[r, "MN\alpha'"] \arrow[u, "\varepsilon_{M,N}(X')"] & MNX_1' \arrow[u, "\varepsilon_{M,N}(X_1')"]
\end{tikzcd}$$

and

$$\begin{aligned}
\alpha' \circ \mathrm{Tr}_N^M(\beta) \circ \alpha &= \alpha' \circ \varepsilon_{M,N}(X') \circ M(\beta) \circ \eta_{N,M}(X) \circ \alpha \\
&= \varepsilon_{M,N}(X_1') \circ M(N(\alpha') \circ \beta \circ N(\alpha)) \circ \eta_{N,M}(X_1) \\
&= \mathrm{Tr}_N^M(N(\alpha') \circ \beta \circ N(\alpha)).
\end{aligned}$$

□

The following theorem extends to our general context the classical and relative Higman criteria (see resp. [H1] and [H2]) and also extends to our context the equivalence of injectivity and projectivity for modules over a symmetric algebra over a field. See the subsequent examples and in particular the next section, "Relatively Projective Modules and Projective Modules".

THEOREM 6.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be R-linear triangulated or abelian categories, and let $M\colon \mathfrak{B} \to \mathfrak{A}$ and $N\colon \mathfrak{A} \to \mathfrak{B}$ be two exact functors such that (M, N) is a bi-adjoint pair. Then, for an object X in $\mathfrak{A}$, the following statements are equivalent.*

(i) *X is isomorphic to a direct summand of $MN(X)$.*
(ii) *X is isomorphic to a direct summand of $M(Y)$ for some object Y in $\mathfrak{B}$.*
(iii) *The morphism Id_X is in the image of $\mathrm{Tr}_N^M(X)$.*
(iv) *The morphism $\eta_{N,M}(X)\colon X \to MN(X)$ has a left inverse.*
(v) *The morphism $\varepsilon_{M,N}(X)\colon MN(X) \to X$ has a right inverse.*
(vi) *Relative projectivity of X:*

$$N(X'') \underset{\beta}{\overset{N(\pi)}{\rightleftarrows}} N(X'), \qquad \begin{tikzcd}
 & X \arrow[dl, dashed, "\tilde{\alpha}"'] \arrow[d, "\alpha"] \\
X'' \arrow[r, "\pi"] & X'.
\end{tikzcd}$$

Given morphisms $\alpha\colon X \to X'$ and $\pi\colon X'' \to X'$ such that there exists a morphism $\beta\colon N(X') \to N(X'')$ with $N(\pi) \circ \beta = \mathrm{Id}_{N(X')}$, there then exists a morphism $\hat{\alpha}\colon X \to X''$ with $\pi \circ \hat{\alpha} = \alpha$.

(vii) *Relative injectivity of X:*

$$N(X') \underset{\beta}{\overset{N(\iota)}{\rightleftarrows}} N(X''), \qquad \begin{tikzcd}
X & \\
X' \arrow[u, "\alpha"] \arrow[r, "\iota"] & X''. \arrow[ul, dashed, "\tilde{\alpha}"']
\end{tikzcd}$$

Given morphisms $\alpha\colon X' \to X$ and $\iota\colon X' \to X''$ such that there exists a morphism $\beta\colon N(X'') \to N(X')$ with $\beta \circ N(\iota) = \mathrm{Id}_{N(X')}$, there then exists a morphism $\hat{\alpha}\colon X'' \to X$ with $\hat{\alpha} \circ \iota = \alpha$.

Proving this theorem requires the following lemma.

LEMMA 6.9. *We have* $\mathrm{Tr}_N^M(M(Y))(\eta_{M,N}(Y) \circ \varepsilon_{N,M}(Y)) = \mathrm{Id}_{M(Y)}$.

Proof. By definition, we have

$$\begin{tikzcd}
MY \arrow[d, "\eta_{N,M}M"'] & & MY \\
MNMY \arrow[dr, "M\varepsilon_{N,M}"'] & & MNMY \arrow[u, "\varepsilon_{M,N}M"'] \\
& MY \arrow[ur, "M\eta_{M,N}"'] &
\end{tikzcd}$$

and

$$\begin{aligned}\mathrm{Tr}_N^M&(M(Y))(\eta_{M,N}(Y) \circ \varepsilon_{N,M}(Y)) \\ &= \varepsilon_{M,N}(M(Y)) \circ M(\eta_{M,N}(Y)) \circ M(\varepsilon_{N,M}(Y)) \circ \eta_{N,M}(M(Y)).\end{aligned}$$

It is a classical property of adjunctions (see [M] or [J]) that $\varepsilon_{M,N}(M(Y)) \circ M(\eta_{M,N}(Y))$ and $M(\varepsilon_{N,M}(Y)) \circ \eta_{N,M}(M(Y))$ are the identity on $M(Y)$. $\square$

Proof of Theorem 6.8. We prove the implications

$$(\mathrm{i}) \Rightarrow (\mathrm{ii}) \Rightarrow (\mathrm{iii}) \Rightarrow \begin{cases} (\mathrm{iv}) \Rightarrow (\mathrm{i}) \\ (\mathrm{v}) \Rightarrow (\mathrm{i}) \end{cases}$$

and

$$(\mathrm{ii}) \Rightarrow \begin{cases} (\mathrm{vi}) \Rightarrow (\mathrm{v}) \\ (\mathrm{vii}) \Rightarrow (\mathrm{iv}). \end{cases}$$

(i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). We may assume that $X = M(Y)$, for if X is a direct summand of $M(Y)$ then we have two morphisms $p\colon M(Y) \to X$ and $i\colon X \to M(Y)$ such that $p \circ i = \mathrm{Id}_X$. Hence, if $\mathrm{Tr}_N^M(M(Y))(\beta)$ is the identity morphism on $M(Y)$, then the identity morphism on X is given by $p \circ \mathrm{Tr}_N^M(M(Y))(\beta) \circ i$; then using Proposition 6.7 yields

$$\mathrm{Id}_X = \mathrm{Tr}_N^M(N(p) \circ \beta \circ N(i)).$$

For $X = M(Y)$ the assertion follows from Lemma 6.9.

(iii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (v). These implications follow from the definition of the relative trace, since we have

$$\mathrm{Id}_X = \mathrm{Tr}_N^M(X)(\beta) = \varepsilon_{M,N}(X) \circ M(\beta) \circ \eta_{N,M}(X).$$

(iv) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i) are self-evident.

(ii) $\Rightarrow$ (vi). We may assume that $X = M(Y)$. Let φ be an adjunction for the pair (M, N). Given a morphism $\alpha\colon M(Y) \to X'$, we must construct a morphism $\hat{\alpha}\colon M(Y) \to X''$ such that $\pi \circ \hat{\alpha} = \alpha$. Using the adjunction, we obtain a morphism $\varphi_{Y,X'}(\alpha)\colon Y \to N(X')$, which we compose with β to obtain a morphism from Y to $N(X'')$. We claim that if

$$\hat{\alpha} := \varphi_{Y,X''}^{-1}(\beta \circ \varphi_{Y,X'}(\alpha))$$

then $\hat{\alpha}$ has the desired property. Since the adjunction is natural, we have

$$\pi \circ \varphi_{Y,X''}^{-1}(\beta \circ \varphi_{Y,X'}(\alpha)) = \varphi_{Y,X'}^{-1}(N(\pi) \circ \beta \circ \varphi_{Y,X'}(\alpha)).$$

By assumption, $N(\pi) \circ \beta = \mathrm{Id}_{N(X')}$, from which it follows that $\pi \circ \hat{\alpha} = \alpha$.

The proof of the implication (ii) $\Rightarrow$ (vii) is analogous to the previous one.

(vi) $\Rightarrow$ (v). Let us choose $\alpha := \mathrm{Id}_X$ and $\pi := \varepsilon_{M,N}(X)$. We must check that the morphism $N(\varepsilon_{M,N}(X))$ splits: but this follows from the properties of an adjunction, since $N(\varepsilon_{M,N}(X)) \circ \eta_{M,N}(N(X))$ is the identity on $N(X)$ (see e.g. [M]).

The proof of the implication (vii) $\Rightarrow$ (iv) is similar to the previous one. $\square$

Definition 6.10. An object X that is of the category $\mathfrak{A}$ and that satisfies one of the conditions in Theorem 6.8 is called *M-split* (or *relatively M-projective,* or *relatively M-injective*).

Notice that any object isomorphic to $M(Y)$ (for $Y \in \mathfrak{B}$) is M-split.

Example (Induction–restriction with R). Let A be a symmetric algebra over R, and consider the categories

$$\mathfrak{A} = {}_A\mathbf{Mod} \quad \text{and} \quad \mathfrak{B} = {}_R\mathbf{Mod}.$$

We have already seen that the functors $M := \mathrm{Ind}_R^A$ and $N := \mathrm{Res}_R^A$ build a biadjoint pair. We shall prove and then generalize the following set of properties.

- The relative trace Tr_R^A is the trace Tr^A defined in the previous paragraph—that is, multiplication by the Casimir element.
- The split modules are the relatively R-projective modules.
- For X a finitely generated A-module, the following conditions are equivalent:
  (i) X is a projective A-module; and
  (ii) X is a projective R-module and a split module (a relatively projective R-module).

For $R = k$ a field, the A-split modules are exactly the projective modules and the projective modules coincide with the injective modules.

Relatively Projective Modules and Projective Modules

Consider the following particular situation.

- B is a symmetric subalgebra of A such that A is a projective B-module (and hence, as already noted, A is a projective module-B); we choose a symmetrizing form t on A and a symmetrizing form u on B.
- We choose $M := A$ (viewed as an object of ${}_A\mathbf{Mod}_B$) and $N := A$ (viewed as an object of ${}_B\mathbf{Mod}_A$); the pairing $A \times A \to R$ is defined by $(a, a') \mapsto t(aa')$.

Thus the functor $M \otimes_B \cdot$ coincides with the induction

$$\mathrm{Ind}_B^A \colon {}_B\mathbf{Mod} \to {}_A\mathbf{Mod},$$

while the functor $N \otimes_A \cdot$ coincides with the restriction

$$\mathrm{Res}_B^A : {}_A\mathbf{Mod} \to {}_B\mathbf{Mod}.$$

We then say that an A-module X is *relatively B-projective* when it is split for the pair (M, N) just defined.

We now construct in this context the analogue of the Casimir element. Because A is a (finitely generated) projective B-module, the natural morphism

$$\mathrm{Hom}_B(A, B) \otimes_B A \to E_B A$$

is an isomorphism. Since B is symmetric, its chosen symmetrizing form u induces a natural isomorphism

$$\mathrm{Hom}_B(A, B) \xrightarrow{\sim} A^*,$$

and since A is symmetric, its chosen symmetrizing form t induces an isomorphism $A \xrightarrow{\sim} A^*$. Thus we have an isomorphism (of $(A \otimes A^{\mathrm{op}})$-modules-$(E_B A \otimes E_B A^{\mathrm{op}})$)

$$A \otimes_B A \xrightarrow{\sim} E_B A.$$

We call the *relative Casimir element* (and denote by c_B^A) that element of $A \otimes_B A$ corresponding to Id_A through the preceding isomorphism.

Let X be an A-module. The relative trace may be viewed as a morphism

$$\mathrm{Tr}_B^A : \mathrm{Hom}_B(X, X') \to \mathrm{Hom}_A(X, X').$$

This morphism is nothing but the multiplication by the relative Casimir element c_B^A: If $c_B^A = \sum_i a_i \otimes_B a_i'$ and if Y is any A-module-A, then

$$\mathrm{Tr}_B^A : \begin{cases} Y^B \to Y^A, \\ y \mapsto c_B^A . y = \sum_i a_i y a_i'. \end{cases}$$

EXAMPLE. This example is precisely the case of Higman's criterion for relative projectivity [H1].

Assume $A = RG$ and $B = RH$ (G a finite group, H a subgroup of G). Then

$$c_{RH}^{RG} = \sum_{g \in [G/H]} g \otimes_{RH} g^{-1},$$

where $[G/H]$ denotes a complete set of representatives of the left cosets of G modulo H. Thus, whenever Y is an RG-module-RG and $y \in Y^H$, we have

$$\mathrm{Tr}_{RH}^{RG}(y) = \sum_{g \in [G/H]} g y g^{-1}.$$

In such a situation, projectivity and relative projectivity are connected by the following property.

PROPOSITION 6.11. *Let B be a symmetric subalgebra of A such that A is a projective B-module, and let X be a finitely generated A-module. Then the following conditions are equivalent.*

(i) *X is a projective A-module.*
(ii) *X is relatively B-projective and* $\mathrm{Res}_B^A X$ *is a projective B-module.*

Proof. (i) $\Rightarrow$ (ii). Since A is a projective B-module, any projective A-module is also (by restriction) a projective B-module. Moreover, if a morphism $X'' \to X'$ gets a right inverse after restriction to B then it is onto, so every morphism $X \to X'$ can be lifted to a suitable morphism $X \to X''$. We have

$$\mathrm{Res}_B^A(X'') \underset{\beta}{\overset{\mathrm{Res}_B^A(\pi)}{\rightleftarrows}} \mathrm{Res}_B^A(X'), \qquad \begin{tikzcd} & X \arrow[dl, dashed, "\tilde{\alpha}"'] \arrow[d, "\alpha"] \\ X'' \arrow[r, "\pi"] & X'. \end{tikzcd}$$

(ii) $\Rightarrow$ (i). Since X is relatively projective, we may choose an endomorphism

$$\iota\colon \mathrm{Res}_B^A(X) \to \mathrm{Res}_B^A(X) \quad \text{such that } \mathrm{Tr}_B^A(\iota) = \mathrm{Id}_X.$$

Suppose there exist a surjective morphism $X'' \xrightarrow{\pi} X'$ and a morphism $X \xrightarrow{\alpha} X'$. Then, since $\mathrm{Res}_B^A X$ is projective, there must exist a morphism $\gamma\colon \mathrm{Res}_B^A X \to \mathrm{Res}_B^A X''$ such that the following triangle commutes:

$$\begin{tikzcd} & \mathrm{Res}_B^A X \arrow[dl, dashed, "\gamma"'] \arrow[d, "\alpha\iota"] \\ \mathrm{Res}_B^A X'' \arrow[r, "\pi"] & \mathrm{Res}_B^A X'; \end{tikzcd}$$

that is, $\pi\gamma = \alpha\iota$. Applying Tr_B^A to this equality yields

$$\pi.\mathrm{Tr}_B^A(\gamma) = \alpha\,\mathrm{Tr}_B^A(\iota) = \alpha,$$

and this shows that the morphism α has indeed been lifted to a suitable morphism $X \to X''$. $\square$

Harish–Chandra Functors

The relative trace introduced previously may be computed in terms of generalized Casimir elements (see Section 6.E for the definition of $c_{M,N}$) by generalizing the element C_{RH}^{RG} already defined.

Consider, for example, the case of Harish–Chandra induction–restriction as defined in Examples 2 and 3 of Section 6.A. Set

$$A := RG,\ B := RN_G(U),\ H := N_G(U)/U,\ M := R[G/U],\ N := R[U\backslash G],$$

$$e := 1, \quad f := e(U) := \frac{1}{|U|}\sum_{u \in U} u.$$

We denote by R_H^G the functor defined by M (the Harish–Chandra induction). Then the relative Casimir element is

$$c_{M,N} := \sum_{g \in [G/N_G(U)]} ge(U) \otimes_B e(U)g^{-1},$$

and the generalized relative Higman criterion becomes as follows.

PROPOSITION 6.12. *Let X be an RG-module. Then X is a summand of $R_H^G(Y)$ for some RH-module Y if and only if there exists an endomorphism β of the RH-module X^U such that*

$$\sum_{g \in [G/N_G(U)]} g\beta g^{-1} = \mathrm{Id}_X.$$

6.C. The M-stable Category

Generalities

What follows could be written in the general context of triangulated categories (and we hope this will be done soon). Nonetheless, for the comfort of readers unfamiliar with triangles, we shall assume now that $\mathfrak{A}$ and $\mathfrak{B}$ are two R-linear *abelian* categories and, as before, we denote by (M, N) a pair of bi-adjoint functors for $(\mathfrak{A}, \mathfrak{B})$.

We denote by $\mathrm{Hom}_{\mathfrak{A}}^M(X, X')$ the image of $\mathrm{Tr}_N^M(X, X')$ in $\mathrm{Hom}_{\mathfrak{A}}(X, X')$ and call these morphisms the *M-split morphisms.* By definition, the M-split objects are those objects whose identity is M-split (i.e., such that all endomorphisms are M-split).

Because the M-split morphism functor $\mathrm{Hom}_{\mathfrak{A}}^M(\cdot\,,\cdot)$ is an ideal (see Proposition 6.7), we have the following property.

LEMMA 6.13. *A morphism $X \to X'$ in $\mathfrak{A}$ is M-split if and only if it factorizes through an M-split object of $\mathfrak{A}$.*

DEFINITION 6.14. The category ${}_M\mathbf{Stab}(\mathfrak{A})$ (or, by abuse of notation, $\mathbf{Stab}(\mathfrak{A})$) is defined as follows.

(1) The objects of $\mathbf{Stab}(\mathfrak{A})$ are the objects of $\mathfrak{A}$.
(2) The morphisms in $\mathbf{Stab}(\mathfrak{A})$, which we denote by $\mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(\cdot\,,\cdot)$, are the morphisms in $\mathfrak{A}$ modulo the M-split morphisms; that is,

$$\mathrm{Homst}_{\mathfrak{A},M}(X, X') := \mathrm{Hom}_{\mathfrak{A}}(X, X')/\mathrm{Hom}_{\mathfrak{A}}^M(X, X').$$

Let A be an R-algebra. If $\mathfrak{A} = {}_A\mathbf{Mod}$ and $\mathfrak{B} = {}_R\mathbf{Mod}$ and if the bi-adjoint pair of functors is given by $(\mathrm{Ind}_R^A, \mathrm{Res}_R^A)$, then we denote the corresponding stable category by ${}_A\mathbf{Stab}$.

REMARKS. 1. For $R = k$ a field, the category ${}_A\mathbf{Stab}$ coincides with the usual notion of the stable category—that is, the module category "modulo the projectives". In general, however, our category ${}_A\mathbf{Stab}$ is not the quotient of ${}_A\mathbf{Mod}$ modulo the projective A-modules.

2. $\mathbf{Stab}(\mathfrak{A})$ is an R-linear additive category (but in general not an abelian category; we leave its triangulated structure to further work).

From the way we defined the M-stable category, it is clear that there is a natural functor $\mathrm{St}\colon \mathfrak{A} \to \mathbf{Stab}(\mathfrak{A})$.

PROPOSITION 6.15. *If X is an object in $\mathfrak{A}$, then $\mathrm{St}(X) \simeq 0$ if and only if X is M-split.*

Proof. If $\mathrm{St}(X) \simeq 0$, then the identity on X is in the image of the relative trace $\mathrm{Tr}_N^M(X)$, which is equivalent to saying that X is M-split.

If X is M-split, then the identity on X is an M-split homomorphism and thus is zero in $\mathbf{Stab}(\mathfrak{A})$. Hence $\mathrm{St}(X) \simeq 0$. □

The Heller Functor on $\mathbf{Stab}(\mathfrak{A})$

Whenever $\alpha \in \mathrm{Hom}_{\mathfrak{A}}(X, X')$, we denote by α^{st} its image in $\mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(X, X')$.

PROPOSITION 6.16 (Schanuel's lemma). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two R-linear abelian categories and let (M, N) be a bi-adjoint pair of functors on $\mathfrak{A}$ and $\mathfrak{B}$. Assume that*

$$0 \longrightarrow X_1' \xrightarrow{\iota_1} P_1 \xrightarrow{\pi_1} X_1 \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow X_2' \xrightarrow{\iota_2} P_2 \xrightarrow{\pi_2} X_2 \longrightarrow 0$$

are short exact sequences in $\mathfrak{A}$ such that:

(1) *their images through N are split; and*

(2) *P_1 and P_2 are M-split objects.*

Then there exists an isomorphism

$$\begin{aligned} \mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(X_1, X_2) &\xrightarrow{\sim} \mathrm{Hom}^{\mathrm{st}}_{\mathfrak{A},M}(X_1', X_2'), \\ \alpha^{\mathrm{st}} &\longmapsto \alpha'^{\mathrm{st}} \end{aligned}$$

that is determined, for $\alpha \in \mathrm{Hom}_{\mathfrak{A}}(X_1, X_2)$ and $\alpha' \in \mathrm{Hom}_{\mathfrak{A}}(X_1', X_2')$, by the following condition: There exists a $u \in \mathrm{Hom}_{\mathfrak{A}}(P_1, P_2)$ such that the diagram

$$\begin{tikzcd} X_1' \arrow[r, "\iota_1"] \arrow[d, "\alpha'"'] & P_1 \arrow[r, "\pi_1"] \arrow[d, "u"'] & X_1 \arrow[d, "\alpha"'] \\ X_2' \arrow[r, "\iota_1"] & P_2 \arrow[r, "\pi_2"] & X_2 \end{tikzcd}$$

commutes.

Proof. We may assume that α is given. Then, since $N(\pi_2)$ splits and since P_1 is an M-split object, there exists a map u and a map α' such that the preceding diagram commutes. It suffices to verify that α^{st} is zero if and only if α'^{st} is zero.

If α^{st} is zero, then α factorizes through the object P_2. Let us say $\alpha = \pi_2 \circ h$, where $h\colon X_1 \to P_2$. The map $u - h \circ \pi_1$ is a map from P_1 to the kernel of π_2. Therefore, if we set $h' = u - h \circ \pi_1$, then $\alpha' = h' \circ \iota_1$ (i.e., the map α' factorizes through an M-split object). The converse implication can be verified similarly. □

REMARK. It follows from the proof of Schanuel's lemma that (α', u, α) defines a single homotopy class of morphisms from

$$0 \longrightarrow X_1' \xrightarrow{\iota_1} P_1 \xrightarrow{\pi_1} X_1 \longrightarrow 0 \quad \text{to} \quad 0 \longrightarrow X_2' \xrightarrow{\iota_2} P_2 \xrightarrow{\pi_2} X_2 \longrightarrow 0.$$

This is a particular case of a more general lemma about projective resolution that will not be addressed here.

COROLLARY 6.17. *Assume that*

$$0 \longrightarrow X_1' \longrightarrow P_1 \longrightarrow X \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow X_2' \longrightarrow P_2 \longrightarrow X \longrightarrow 0$$

are short exact sequences in $\mathfrak{A}$ *such that*:

(1) *their images through* N *are split*; *and*
(2) P_1 *and* P_2 *are* M*-split objects.*

Then there exists an isomorphism

$$\varphi^{\mathrm{st}}\colon X_1' \xrightarrow{\sim} X_2' \ \textit{in } \mathbf{Stab}(\mathfrak{A})$$

characterized by the following condition: *There exists* $u \in \mathrm{Hom}_{\mathfrak{A}}(P_1, P_2)$ *such that the diagram*

$$\begin{tikzcd} X_1' \arrow[r, "\iota_1"] \arrow[d, "\varphi"'] & P_1 \arrow[r, "\pi_1"] \arrow[d, "u"'] & X \arrow[d, "\mathrm{Id}_X"'] \\ X_2' \arrow[r, "\iota_1"] & P_2 \arrow[r, "\pi_2"] & X \end{tikzcd}$$

commutes.

Corollary 6.17 allows us to define a functor $\Omega_M : \mathbf{Stab}(\mathfrak{A}) \to \mathbf{Stab}(\mathfrak{A})$, the Heller functor. It is given by $\Omega_M(X) := X_1'$.

Similarly, we have a functor $\Omega_M^{-1} : \mathbf{Stab}(\mathfrak{A}) \to \mathbf{Stab}(\mathfrak{A})$. It can be checked that the functors Ω_M and Ω_M^{-1} induce reciprocal equivalences of $\mathbf{Stab}(\mathfrak{A})$.

The Case of ${}_A\mathbf{Stab}$: *The Heller Bimodules*

Let again A be a symmetric R-algebra. From now on, we assume that $\mathfrak{A} = {}_A\mathbf{Mod}$ and $\mathfrak{B} = {}_R\mathbf{Mod}$ and that the modules inducing the bi-adjoint pair of functors are $M \in {}_A\mathbf{Mod}_R$ and $N \in {}_R\mathbf{Mod}_A$. We proceed to give another definition of the Heller functors Ω_A and Ω_A^{-1}.

The *Heller bimodule,* denoted Ω_A, is the kernel of the multiplication morphism

$$A \otimes_R A \to A\,.$$

Thus we have

$$\Omega_A = \Big\{ \sum_i a_i \otimes b_i \;\Big|\; \sum_i a_i b_i = 0 \Big\}.$$

Viewing Ω_A as a left ideal in $A \otimes_R A^{\mathrm{op}}$, we see that if $\sum_i a_i \otimes b_i \in \Omega_A$ then

$$\sum_i a_i \otimes b_i = \sum_i (a_i \otimes b_i - 1 \otimes a_i b_i) = \sum_i (1 \otimes b_i)(a_i \otimes 1 - 1 \otimes a_i);$$

hence Ω_A is the left ideal of $A \otimes_R A^{\mathrm{op}}$ generated by $\{a \otimes 1 - 1 \otimes a \mid (a \in A)\}$.

Since $(A \otimes_R A)^A$ is by definition the right annihilator in $A \otimes_R A^{\mathrm{op}}$ of the set $\{a \otimes 1 - 1 \otimes a \mid (a \in A)\}$, it follows that

$$(A \otimes_R A)^A = \mathrm{Ann}(\Omega_A)_{(A \otimes_R A^{\mathrm{op}})}.$$

If A is symmetric and t is a symmetrizing form, then the form

$$t^{\mathrm{en}}\colon \begin{cases} A \otimes_R A^{\mathrm{op}} \to R, \\ a \otimes a' \mapsto t(a)t(a') \end{cases}$$

is a symmetrizing form on $A \otimes_R A^{\mathrm{op}}$. This fact and what precedes imply that

$$(A \otimes_R A)^A = \Omega_A^{\perp},$$

where the orthogonal is relative to the form t^{en}.

The *inverse Heller bimodule* Ω_A^{-1} is defined as the quotient

$$\Omega_A^{-1} := (A \otimes_R A)/(A \otimes_R A)^A.$$

Thus we see that the form t^{en} induces an isomorphism of A-modules-A:

$$\Omega_A^{-1} \xrightarrow{\sim} \Omega_A^{*}.$$

Taking the dual (relative to the forms t and t^{en}) of the short exact sequence

$$0 \longrightarrow \Omega_A \longrightarrow A \otimes_R A^{\mathrm{op}} \longrightarrow A \longrightarrow 0,$$

we obtain the short exact sequence

$$0 \longrightarrow A \longrightarrow A \otimes_R A^{\mathrm{op}} \longrightarrow \Omega_A^{-1} \longrightarrow 0.$$

PROPOSITION 6.18. (1) *The A-modules-A Ω_A and Ω_A^{-1} are exact.*

(2) *The bimodules $\Omega_A \otimes_A \Omega_A^{-1}$ and $\Omega_A^{-1} \otimes_A \Omega_A$ are both isomorphic to A in the category ${}_A\mathbf{Stab}_A$.*

COROLLARY 6.19. *The functors*

$$\Omega_A, \Omega_A^{-1}\colon {}_A\mathbf{Mod} \to {}_A\mathbf{Mod}$$

induce reciprocal self-equivalences on ${}_A\mathbf{Stab}$.

Proof of Proposition 6.18. (1) Since A is projective on both sides, we see that

$$0 \longrightarrow \Omega_A \longrightarrow A \otimes_R A \xrightarrow{\mu} A \longrightarrow 0$$

is a split short exact sequence in ${}_A\mathbf{Mod}$ as well as in $\mathbf{Mod}_A$. In particular, it is R-split. Taking the dual with respect to the bilinear forms defined previously yields the R-split short exact sequence

$$0 \longrightarrow A \xrightarrow{\mu^*} A \otimes_R A \longrightarrow \Omega_A^{-1} \longrightarrow 0.$$

The relative injectivity of A implies that this sequence splits in ${}_A\mathbf{Mod}$ and in $\mathbf{Mod}_A$. Thus, we have shown that Ω_A and Ω_A^{-1} are in ${}_A\mathbf{proj} \cap \mathbf{proj}_A$.

(2) Since we want the isomorphism from $\Omega_A \otimes_A \Omega_A^{-1}$ to A to be in ${}_A\mathbf{Stab}_A$, the symmetric algebra to consider here is $(A \otimes_R A^{\mathrm{op}})$. We shall apply Schanuel's lemma to the short exact sequences

$$0 \longrightarrow \Omega_A \otimes_A \Omega_A^{-1} \longrightarrow A \otimes_R \Omega_A^{-1} \xrightarrow{\mu \otimes \mathrm{Id}_{\Omega_A^{-1}}} \Omega_A^{-1} \longrightarrow 0,$$

$$0 \longrightarrow A \xrightarrow{\mu^*} A \otimes_R A \longrightarrow \Omega_A^{-1} \longrightarrow 0.$$

These sequences split as sequences in ${}_A\mathbf{Mod}$, since Ω_A^{-1} is an A-projective module. In particular, they split when restricted to R. Thus, by Schanuel's lemma, it is enough to check that $A \otimes_R A$ and $A \otimes_R \Omega_A^{-1}$ are both relatively $(A \otimes_R A^{\mathrm{op}})$-projective, and for this it is enough to remark that they are projective $(A \otimes_R A^{\mathrm{op}})$-modules.

Similarly, one shows that $\Omega_A^{-1} \otimes_A \Omega_A$ is isomorphic to A in the category ${}_A\mathbf{Stab}_A$. $\square$

DEFINITION 6.20. For $X, X' \in {}_A\mathbf{Mod}$ and $n \in \mathbb{N}$, we set

$$\mathrm{Ext}_A^n(X, X') := \mathrm{Hom}_{{}_A\mathbf{Stab}}(\Omega_A^n(X), X').$$

Note that we have also

$$\mathrm{Ext}_A^n(X, X') = \mathrm{Hom}_{{}_A\mathbf{Stab}}(X, \Omega^{-n}(X')).$$

PROPOSITION 6.21. *Let A and B be symmetric R-algebras, and let $M \in {}_A\mathbf{Mod}_B$ be an exact bimodule. Then the functor $M \otimes_B \bullet$ commutes with $\Omega_\bullet$; that is,*

$$\Omega_A \otimes_A M \xrightarrow{\sim} M \otimes_B \Omega_B \ \text{ in } {}_A\mathbf{Stab}_B.$$

Proof. The module M induces a functor on the stable category.

Consider the two short exact sequences

$$0 \longrightarrow \Omega_A \longrightarrow A \otimes_R A \xrightarrow{\mu_A} A \longrightarrow 0, \qquad 0 \longrightarrow \Omega_B \longrightarrow B \otimes_R B \xrightarrow{\mu_B} B \longrightarrow 0.$$

Tensoring the first one over A with M and the second one over B with M, we obtain the two short exact sequences

$$0 \longrightarrow \Omega_A \otimes_A M \longrightarrow A \otimes_R M \longrightarrow M \longrightarrow 0,$$
$$0 \longrightarrow M \otimes_B \Omega_B \longrightarrow M \otimes_R B \longrightarrow M \longrightarrow 0.$$

Both sequences split as sequences over R. Since M is in $\mathbf{proj}_B$, it follows that $A \otimes_R M$ is a projective $(A \otimes_R B^{\mathrm{op}})$-module. Similarly, one shows that $M \otimes_R B$ is a projective $(A \otimes_R B^{\mathrm{op}})$-module. Thus, we can apply Schanuel's lemma to obtain the isomorphism

$$\Omega_A \otimes_A M \xrightarrow{\sim} M \otimes_B \Omega_B \ \text{ in } {}_A\mathbf{Stab}_B. \qquad \square$$

The following corollary is an application of Proposition 6.21.

COROLLARY 6.22 (Schapiro's lemma). *Let (M, N) be a self-dual pair of exact bimodules for the algebras A and B. Then $(\Omega_A \otimes_A M, \Omega_B^{-1} \otimes_A N)$ is also a self-dual pair of exact bimodules for the algebras A and B.*

In particular, for all $n \in \mathbb{N}$, we have

$$\mathrm{Ext}_A^n(M(Y), X) \simeq \mathrm{Ext}_B^n(Y, N(X)).$$

Proof. We will show only that $\Omega_A M$ is left adjoint to $\Omega_B^{-1} N$. We know that both (M, N) and $(\Omega_A, \Omega_A^{-1})$, are bi-adjoint pairs. Hence the functor $\Omega_A M$ is left adjoint to the functor $N\Omega_A^{-1}$. By Proposition 6.21, $N\Omega_A^{-1}$ is naturally equivalent to the functor $\Omega_B^{-1} N$. $\square$

6.D. Stable Equivalences of Morita Type

Let (M, N) be a self-dual exact pair of bimodules for A and B. Since the functors $M \otimes_B \cdot$ and $N \otimes_A \cdot$ factorize through the functors

$$\mathrm{St}_A : {}_A\mathbf{Mod} \to {}_A\mathbf{Stab} \quad \text{and} \quad \mathrm{St}_B : {}_B\mathbf{Mod} \to {}_B\mathbf{Stab},$$

it follows that the bimodules M and N induce two functors

$$M \otimes_B \cdot : {}_B\mathbf{Stab} \to {}_A\mathbf{Stab} \quad \text{and} \quad N \otimes_A \cdot : {}_A\mathbf{Stab} \to {}_B\mathbf{Stab}.$$

Because the functors $M \otimes_B \cdot$ and $N \otimes_A \cdot$ are bi-adjoint, the induced functors on the stable categories are also bi-adjoint. The associated adjunctions are the images in the stable categories of the adjunctions of M and N on the module category level.

These preliminaries suggest the following definition of a stable equivalence of Morita type.

Definition 6.23. Let M and N be bimodules as before. We say that M and N *induce a stable equivalence of Morita type* between A and B if

$$M \otimes_B N \simeq A \ \text{ in } {}_A\mathbf{Stab}_A \quad \text{and} \quad N \otimes_A M \simeq B \ \text{ in } {}_B\mathbf{Stab}_B$$

through the co-units and the units of the adjunctions.

Remark. Observe (cf. [M]) that we need not specify which co-units and units provide the isomorphisms just defined. If the elements of one appropriate pair of them are isomorphisms, then all of them will be isomorphisms.

Definition 6.24. The stable center of the symmetric algebra A, denoted by $Z^{\mathrm{st}}A$, is the quotient $ZA/Z^{\mathrm{pr}}A$.

Remark. If we view A as an object in the category ${}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Mod}$, then the center of A is isomorphic to $\mathrm{End}_{(A\otimes_R A^{\mathrm{op}})}(A)$. It follows from the definition of the stable category ${}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Stab}$ and the definition of projective endomorphisms of A considered as an $(A \otimes_R A^{\mathrm{op}})$-module that the stable center of A is isomorphic to $\mathrm{End}_{{}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Stab}}(A)$.

Proposition 6.25. *A stable equivalence of Morita type between the symmetric algebras A and B induces an algebra isomorphism*

$$Z^{\mathrm{st}}A \simeq Z^{\mathrm{st}}B.$$

Proof. Let ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ denote the full subcategory of ${}_A\mathbf{stab}_A$ whose objects are the exact A-modules-A. Assume that (M, N) induces a stable equivalence of Morita type between A and B. Then the pair $(M \otimes_R N, N \otimes_R M)$ (where, as before, $M \otimes_R N$ is viewed as an $(A \otimes_R A^{\mathrm{op}})$-module-$(B \otimes_R B^{\mathrm{op}})$ and $N \otimes_R M$ is viewed as a $(B\otimes_R B^{\mathrm{op}})$-module-$(A\otimes_R A^{\mathrm{op}})$) induces inverse equivalences between ${}_A\mathbf{stab}_A^{\mathrm{pr}}$ and ${}_B\mathbf{stab}_B^{\mathrm{pr}}$ that exchange A and B. The assertion then follows because $Z^{\mathrm{st}}(A)$ is the algebra of endomorphisms of A in ${}_A\mathbf{stab}_A^{\mathrm{pr}}$. □

6.E. (M, N)-split Algebras

More on Exact Pairs

We retain the notation introduced in Section 6.D.

By the isomorphisme cher à Cartan and by projectivity of the B-module N, we have

$$(M \otimes_B N)^* = H_0(B, M \otimes_R N)^* \simeq H^0(B, N \otimes_R M) \simeq (N \otimes_R M)^B \simeq M \otimes_B N.$$

It follows that the pairing

$$\begin{cases} (M \otimes_B N) \times (M \otimes_B N) \to R, \\ (m \otimes n, m' \otimes n') \mapsto \langle m, n' \rangle \langle m', n \rangle \end{cases}$$

defines a duality between $M \otimes_B N$ and itself; therefore, $(M \otimes_B N, M \otimes_B N)$ is an exact pair between the algebra A and itself. Similarly, $(N \otimes_A M, N \otimes_A M)$ is an exact pair between the algebra B and itself.

Let us now compute the pairing $(N \otimes_A M) \times (N \otimes_A M) \to B$ associated to the previous pairing and to the chosen symmetrizing form on B. We do this through the following series of isomorphisms (a series that uses the projectivity of the B-module $N \otimes_A M$ and the isomorphisme cher à Cartan):

$$\begin{aligned} N \otimes_A M &\xrightarrow{\sim} \mathrm{Hom}_A(M, M) \\ &\xrightarrow{\sim} \mathrm{Hom}_A(M, \mathrm{Hom}_B(N, B)) \xrightarrow{\sim} \mathrm{Hom}_B(N \otimes_A M, B). \end{aligned}$$

We have

$$n \otimes_A m \mapsto (x \mapsto (xn)m) \mapsto (x \mapsto (y(xn)m)) \mapsto (y \otimes_A x \mapsto (y(xn)m)).$$

Thus we have proved the following lemma.

LEMMA 6.26. (1) *The pairing $(N \otimes_A M) \times (N \otimes_A M) \to B$ is given as follows: for $n \otimes_A m$ and $n' \otimes_A m'$ in $N \otimes_A M$, we have*

$$((n \otimes_A m)(n' \otimes_A m')) = (n(mn')m').$$

(2) *The pairing $(M \otimes_B N) \times (M \otimes_B N) \to A$ is given as follows: for $m \otimes_B n$ and $m' \otimes_B n'$ in $M \otimes_B N$, we have*

$$((m \otimes_B n)(m' \otimes_B n')) = (m(nm')n').$$

Notice the natural isomorphisms of R-algebras

$$N \otimes_A M \xrightarrow{\sim} \mathrm{Hom}_A(M, M), \quad n \otimes_A m \mapsto (x \mapsto (xn)m);$$

$$N \otimes_A M \xrightarrow{\sim} \mathrm{Hom}(N, N)_A, \quad n \otimes_A m \mapsto (y \mapsto n(my)).$$

(Here, as seen before, the structure of the algebra on $N \otimes_A M$ is defined by $(n \otimes_A m)(n' \otimes_A m') := (n \otimes_A (mn')m')$.) Similarly,

$$M \otimes_B N \xrightarrow{\sim} \mathrm{Hom}_B(N, N), \quad m \otimes_B n \mapsto (y \mapsto (ym)n);$$

$$M \otimes_B N \xrightarrow{\sim} \mathrm{Hom}(M, M)_B, \quad m \otimes_B n \mapsto (x \mapsto m(ny)).$$

We shall now describe the inverses of the preceding isomorphisms. Let us denote by

$$c_{M,N} := \sum_\alpha \mu_\alpha \otimes_B \nu_\alpha$$

the Casimir element of $M \otimes_B N$, that is, the element such that

$$(\forall n \in N)\ \sum_\alpha (n\mu_\alpha)\nu_\alpha = n \quad \text{and} \quad (\forall m \in M)\ \sum_\alpha \mu_\alpha(\nu_\alpha m) = m$$

(see Lemma 3.2 for a particular case). Then the inverse of the isomorphism $M \otimes_B N \xrightarrow{\sim} \mathrm{Hom}(M,M)_B$ is given by

$$\varphi \mapsto \sum_\alpha \varphi\mu_\alpha \otimes_B \nu_\alpha .$$

We leave it as an exercise for the reader to write down similar formulas for the other isomorphisms described here.

The following lemma is a consequence of our results so far.

LEMMA 6.27. (1) *The R-duality functor induces an isomorphism*

$$\mathrm{Hom}_A(M,M)_B \simeq \mathrm{Hom}_B(N,N)_A,$$

which in turn induces the following isomorphism of algebras:

$$\begin{cases} (N \otimes_A M)^B \xrightarrow{\sim} (M \otimes_B N)^A, \\ \sum_i n_i \otimes_A m_i \mapsto \sum_{\alpha,i} (\mu_\alpha n_i) m_i \otimes_B \nu_\alpha . \end{cases}$$

(2) *In particular, the morphism*

$$(M \otimes_B N)^A \to ZA, \quad \sum_j m_j \otimes_B n_j \mapsto \sum_j (m_j n_j)$$

induces the morphism

$$(N \otimes_A M)^B \to ZA, \quad \sum_i n_i \otimes_A m_i \mapsto \sum_{\alpha,i} (\mu_\alpha n_i)(m_i \nu_\alpha).$$

Similarly, we denote by

$$c_{N,M} := \sum_\beta \nu'_\beta \otimes_A \mu'_\beta$$

the Casimir element of $N \otimes_A M$.

Quadrimodules Again

Let us reconsider (cf. Section 1) the objects

$$F := M \otimes_R N \in {}_{(A\otimes_R A^{\mathrm{op}})}\mathbf{Mod}_{(B\otimes_R B^{\mathrm{op}})},$$
$$G := N \otimes_R M \in {}_{(B\otimes_R B^{\mathrm{op}})}\mathbf{Mod}_{(A\otimes_R A^{\mathrm{op}})}.$$

Then the pair (F,G) with

$$\begin{cases} F \times G \to R, \\ (m \otimes n,\, n' \otimes m') \mapsto \langle m, n' \rangle \langle m', n \rangle \end{cases}$$

is an exact pair for the algebras $A \otimes_R A^{\mathrm{op}}$ and $B \otimes_R B^{\mathrm{op}}$.

1. We have

$$\begin{cases} F \otimes_{(B \otimes_R B^{\mathrm{op}})} G \xrightarrow{\sim} (M \otimes_B N) \otimes_R (M \otimes_B N), \\ (m \otimes n) \otimes (n' \otimes m') \mapsto (m \otimes_B n') \otimes_R (m' \otimes_B n). \end{cases}$$

Notice that the structure of an $(A \otimes A^{\mathrm{op}})$-module-$(A \otimes A^{\mathrm{op}})$ on $M \otimes_B N \otimes_R M \otimes_B N$ is given by

$$(a \otimes a^0) \cdot (m \otimes_B n' \otimes_R m' \otimes_B n) \cdot (a' \otimes a'^0) := am \otimes_B n'a' \otimes_R a'^0 m' \otimes_B na^0.$$

Similarly, we have

$$\begin{cases} G \otimes_{(A \otimes_R A^{\mathrm{op}})} F \xrightarrow{\sim} (N \otimes_A M) \otimes_R (N \otimes_A M), \\ (n' \otimes m') \otimes (m \otimes n) \mapsto (n' \otimes_A m) \otimes_R (n \otimes_A m'). \end{cases}$$

2. Through the isomorphisms just described, the co-units are given by:

$$\varepsilon_{F,G} = \begin{cases} (M \otimes_B N) \otimes_R (M \otimes_B N) \to A \otimes_R A^{\mathrm{op}}, \\ (m \otimes n') \otimes (n \otimes m') \mapsto (mn') \otimes_R (m'n); \end{cases}$$

$$\varepsilon_{G,F} = \begin{cases} (N \otimes_A M) \otimes_R (N \otimes_A M) \to B \otimes_R B^{\mathrm{op}}, \\ (n' \otimes_A m) \otimes_R (n \otimes_A m') \mapsto (n'm) \otimes_R (nm'). \end{cases}$$

The units are given by:

$$\eta_{F,G} = \begin{cases} B \otimes_R B^{\mathrm{op}} \to N \otimes_A M \otimes_R N \otimes_A M, \\ 1 \mapsto c_{G,F} = c_{N,M} \otimes_R c_{N,M} = \sum_{\beta,\beta'} \nu'_\beta \otimes_A \mu'_\beta \otimes_R \nu'_{\beta'} \otimes_A \mu'_{\beta'}, \\ b \otimes b^0 \mapsto \sum_{\beta,\beta'} b\nu'_\beta \otimes_A \mu'_\beta \otimes_R \nu'_{\beta'} \otimes_A \mu'_{\beta'} b^0 \\ \qquad\qquad = \sum_{\beta,\beta'} \nu'_\beta \otimes_A \mu'_\beta b \otimes_R b^0 \nu'_{\beta'} \otimes_A \mu'_{\beta'}; \end{cases}$$

$$\eta_{G,F} = \begin{cases} A \otimes_R A^{\mathrm{op}} \to M \otimes_B N \otimes_R M \otimes_B N, \\ 1 \mapsto c_{F,G} = c_{M,N} \otimes_R c_{M,N} = \sum_{\alpha,\alpha'} \mu_\alpha \otimes_B \nu_\alpha \otimes_R \mu_{\alpha'} \otimes_B \nu_{\alpha'}, \\ a \otimes a^0 \mapsto \sum_{\alpha,\alpha'} a\mu_\alpha \otimes_B \nu_\alpha \otimes_R \mu_{\alpha'} \otimes_B \nu_{\alpha'} a^0 \\ \qquad\qquad = \sum_{\alpha,\alpha'} \mu_\alpha \otimes_B \nu_\alpha a \otimes_R a^0 \mu_{\alpha'} \otimes_B \nu_{\alpha'}. \end{cases}$$

3. We have

$$FB \xrightarrow{\sim} M \otimes_B N \in {}_{A \otimes A^{\mathrm{op}}}\mathbf{mod}, \quad (m \otimes_R n) \otimes_{B \otimes B^{\mathrm{op}}} b \mapsto mb \otimes_B n;$$

$$GA \xrightarrow{\sim} N \otimes_A M \in {}_{B \otimes B^{\mathrm{op}}}\mathbf{mod}, \quad (n' \otimes_R m') \otimes_{A \otimes A^{\mathrm{op}}} a \mapsto n'a \otimes_A m'.$$

Let us now compute the relative trace

$$\mathrm{Tr}_G^F(A) \colon \mathrm{Hom}_B(GA, GA)_B \simeq \mathrm{Hom}_B(N \otimes_A M, N \otimes_A M)_B \\ \to ZA \simeq \mathrm{Hom}_A(A, A)_A.$$

Following Lemma 6.26, we have the isomorphism

$$\begin{cases} N \otimes_A M \otimes_B N \otimes_A M \xrightarrow{\sim} \mathrm{Hom}_B(N \otimes_A M, N \otimes_A M), \\ n \otimes_A m \otimes_B n' \otimes_A m' \mapsto ((y \otimes_A x) \mapsto (y(xn)m)n' \otimes_A m'); \end{cases}$$

from this we deduce the isomorphism

$$\begin{cases} (N \otimes_A M \otimes_B N \otimes_A M)^B \xrightarrow{\sim} \mathrm{Hom}_B(N \otimes_A M, N \otimes_A M)_B, \\ \sum_i n_i \otimes_A m_i \otimes_B n_i' \otimes_A m_i' \mapsto \big((y \otimes_A x) \mapsto \sum_i (y(xn_i)m_i)n_i' \otimes_A m_i'\big). \end{cases}$$

The relative trace

$$\mathrm{Tr}_G^F(A)\colon (N \otimes_A M \otimes_B N \otimes_A M)^B \to ZA$$

is computed as follows. For $\xi \in (N \otimes_A M \otimes_B N \otimes_A M)^B$, we denote by $\tilde{\xi}$ the corresponding element of $\mathrm{Hom}_B(N \otimes_A M, N \otimes_A M)_B$. Then $\mathrm{Tr}_G^F(\xi)$ is the image of 1 through the following composition of morphisms:

$$\begin{tikzcd} A \arrow[d, "c_{G,F}(A)"'] & A \\ M \otimes_B N \otimes_A M \otimes_B N \arrow[r, "M \otimes_B \tilde{\xi} \otimes_B N"] & M \otimes_B N \otimes_A M \otimes_B N. \arrow[u, "\varepsilon_{F,G}(A)"'] \end{tikzcd}$$

One finds

$$\mathrm{Tr}_G^F\colon \sum_i n_i \otimes_A m_i \otimes_B n_i' \otimes_A m_i' \mapsto \sum_\alpha \sum_i (\mu_\alpha n_i)(m_i n_i')(m_i' \nu_\alpha).$$

Bicenter and Relative Traces

DEFINITION. The *bicenter* $Z(M, N)$ is the algebra defined by

$$Z(M, N) := \mathrm{Hom}_{A \otimes B^{\mathrm{op}}}(M \otimes_R N, M \otimes_R N)_{A \otimes B^{\mathrm{op}}}.$$

Notice that

$$(F \otimes_{B \otimes B^{\mathrm{op}}} G)^{A \otimes A^{\mathrm{op}}} \simeq (M \otimes_B N \otimes_A M \otimes_B N)^B,$$
$$(G \otimes_{A \otimes A^{\mathrm{op}}} F)^{B \otimes B^{\mathrm{op}}} \simeq (N \otimes_A M \otimes_B N \otimes_A M)^A.$$

Applying Lemma 6.27 (where we replace the pair (M, N) by the pair (F, G) defined in the previous section) yields the following result.

PROPOSITION 6.28. (1) *There are isomorphisms of R-algebras*

$$Z(M, N) \simeq (M \otimes_B N \otimes_A M \otimes_B N)^B \simeq (N \otimes_A M \otimes_B N \otimes_A M)^A.$$

(2) *We have the following diagram involving relative traces*:

$$\begin{tikzcd} & ZA \\ Z(M,N) \simeq (M \otimes_B N \otimes_A M \otimes_B N)^A \simeq (N \otimes_A M \otimes_B N \otimes_A M)^B. \arrow[ur, "\mathrm{Tr}_G^F(A)"] \arrow[dr, "\mathrm{Tr}_F^G(B)"'] & \\ & ZB \end{tikzcd}$$

Here the relative traces can be computed with two formulas as follows:

$$\operatorname{Tr}_G^F(A)\colon \begin{cases} (M\otimes_B N\otimes_A M\otimes_B N)^A \to ZA, \\ \sum_i (m_i\otimes_B n_i\otimes_A m_i'\otimes_B n_i') \mapsto \sum_i (m_i(n_i m_i')n_i); \end{cases}$$

$$\operatorname{Tr}_G^F(A)\colon \begin{cases} (N\otimes_A M\otimes_B N\otimes_A M)^B \to ZA, \\ \sum_i (n_i\otimes_A m_i\otimes_B n_i'\otimes_A m_i') \mapsto \sum_{\alpha,i} (\mu_\alpha n_i)(m_i n_i')(m_i'\nu_\alpha). \end{cases}$$

(*Recall that* $\sum_\alpha \mu_\alpha\otimes_B \nu_\alpha$ *is the Casimir element of* $M\otimes_B N$.)

REMARK. The isomorphism of R-modules $Z(M,N)\simeq (M\otimes_B N\otimes_A M\otimes_B N)^A$ may be written as $Z(M,N)\simeq H^0(A, M\otimes_B N\otimes_A M\otimes_B N)$, which implies that $Z(M,N)^*\simeq H_0(A, M\otimes_B N\otimes_A M\otimes_B N)$. Thus we may see $Z(M,N)^*$ as the following cyclic tensor product.

$$\begin{array}{ccccc} & & M & & \\ & \otimes_A & & {}_B\otimes & \\ N & & & & N \\ & \otimes^B & & {}^A\otimes & \\ & & M & & \end{array}$$

(M,N) *Split Algebras*

The following proposition is an immediate application of Theorem 6.8.

PROPOSITION 6.29. *The following assertions are equivalent.*

(i) *A is isomorphic to a direct summand of $M\otimes_B N$ in ${}_A\mathbf{Mod}_A$.*
(ii) *$\varepsilon_{M,N}$ is a split epimorphism in ${}_A\mathbf{Mod}_A$.*
(iii) *$\eta_{N,M}$ is a split monomorphism in ${}_A\mathbf{Mod}_A$.*
(iv) *The trace map*
$$\operatorname{Tr}_G^F(A)\colon Z(M,N)\to ZA$$
is onto.
(v) *Every A-module is M-split.*

If conditions (i)–(v) *are satisfied, we say that the algebra A is* (M,N)-split (*or, by abuse of language if the context* (M,N) *is clear, we say that A is* B-split).

EXAMPLE (Induction–restriction with R). Choose $B:=R$, $M={}_A A_R$, $N:={}_R A_A$, and $\langle a,a'\rangle := t(aa')$.

1. The following conditions are equivalent:
   (i) A is principally symmetric;
   (ii) R is A-split.
2. The following conditions are equivalent:
   (i) A is separable;
   (ii) A is R-split.

EXAMPLE (Induction–restriction with a parabolic subalgebra). Let B be a parabolic subalgebra for A (cf. Section 5). Choose

$$M:={}_A A_B,\quad N:={}_B A_A,\quad \langle a,a'\rangle := t(aa').$$

Let $B^{\perp}$ be the orthogonal of B in A, so that $A = B \oplus B^{\perp}$ and $\mathrm{Br}_B^A \colon A \to B$ is the projection onto B parallel to $B^{\perp}$.

Then we have the following pairing associated with the preceding scalar product:

$$A \times_A A \to A, \quad a \otimes_A a' \mapsto aa';$$

$$A \times_B A \to B, \quad a \otimes_B a' \mapsto \mathrm{Br}_B^A(aa').$$

It is clear that B is always A-split, whereas A is B-split if and only if A is a summand of $A \otimes_B A$ in ${}_A\mathbf{Mod}_A$.

Let $c_B^A = \sum_i e_i \otimes_B e_i'$ be the relative Casimir element—in other words, the element such that, for all $b \in B$, we have $\sum_i \mathrm{Br}_B^A(be_i)e_i' = b$. Then the *double relative trace* is

$$\mathrm{Tr}_G^F \colon \begin{cases} (A \otimes_B A)^B \to ZA, \\ \sum_j x_j \otimes y_j \mapsto \sum_i e_i\big(\sum_j x_j y_j\big)e_i'. \end{cases}$$

Observe that the element $1 \otimes_B 1$ belongs to $(A \otimes_B A)^B$. Its image by Tr_B^A is the relative projective central element z_B^A. Thus, if z_B^A is invertible in ZA then the algebra A is B-split.

For example if $A = RG$ and $B = RH$, then the corresponding relative trace is $\sum a_i \otimes_B a_i' \mapsto \sum_{g \in [G/H]} g a_i a_i' g^{-1}$ and the relative projective central element is $|G : H|$. It follows that if the index $|G : H|$ is invertible in R, then RG is RH-split.

Example (Induction–restriction with idempotents). This example is, of course, a generalization of the preceding example. We still denote by

- B a parabolic subalgebra of A,
- $\mathrm{Br}_B^A \colon A \to B$ the "Brauer morphism", the projection of A onto B that is parallel to $B^{\perp}$, and
- $c_B^A = \sum_i e_i \otimes_B e_i' \in A \otimes_B A$ the relative Casimir element of A relative to B.

Let e be a central idempotent in A and let f be a central idempotent in B. We shall apply what precedes to the symmetric algebras Ae and Bf.

Choose

$$M := eAf, \quad N := fAe, \quad \langle a, a' \rangle := t(aa').$$

Then we have the following pairing associated with the preceding scalar product:

$$M \times_A M \to Ae, \quad a \otimes_A a' \mapsto aa';$$

$$N \times_B N \to Bf, \quad a \otimes_B a' \mapsto \mathrm{Br}_B^A(aa').$$

The Casimir element $c_{M,N}$ is

$$c_{M,N} = \sum_i ee_i f \otimes_B fe_i'e.$$

The relative traces are computed as follows:

$$\mathrm{Tr}_G^F : \begin{cases} (feAf \otimes_B fAef)^B \to ZAe, \\ \sum_j a_j \otimes_B a'_j \mapsto \sum_i e_i\big(\sum_j a_j a'_j\big)e'_i; \end{cases}$$

$$\mathrm{Tr}_F^G : \begin{cases} (eAf \otimes_B fAef \otimes_B fAe)^A \to ZBf, \\ \sum_j x_j \otimes_B z_j \otimes_B y_j \mapsto \sum_j \mathrm{Br}_B^A(x_j)\,\mathrm{Br}_B^A(z_j)\,\mathrm{Br}_B^A(y_j). \end{cases}$$

As an application of Proposition 6.29 we get the following proposition, a generalization of old results of Fan Yun ([F1; F2]; see also Alperin's take in [A, Sec. 15]).

PROPOSITION 6.30. (1) *The following assertions are equivalent*:

(i) *every Ae-module is a summand of* $\mathrm{Ind}_B^A Y$ *for some Bf-module Y*;

(ii) *the relative trace $(fAef)^B \to ZAe$ is onto.*

(2) *The following assertions are equivalent*:

(i) *every Bf-module is a summand of* $\mathrm{Res}_B^A X$ *for some Ae-module X*;

(ii) *the Brauer morphism* $\mathrm{Br}_B^A : (AefA)^A \to ZBf$ *is onto.*

References

[A] J. L. Alperin, *Local representation theory,* Cambridge Stud. Adv. Math., 11, Cambridge Univ. Press, Cambridge, 1986.

[BS] P. Balmer and M. Schlichting, *Idempotent completion of triangulated categories,* J. Algebra 236 (2001), 819–834.

[Bo] N. Bourbaki, *Éléments de mathématique. Algèbre I,* Hermann, Paris, 1970.

[Br] M. Broué, *Equivalences of blocks of group algebras,* Finite dimensional algebras and related topics (Ottawa, 1992), pp. 1–26, Kluwer, Dordrecht, 1994.

[F1] Y. Fan, *Block covers and module covers of finite groups,* Group theory (Singapore, 1987), pp. 357–366, de Gruyter, Berlin, 1989.

[F2] ———, *Local characterizations of block covers and their applications,* J. Algebra 152 (1992), 397–416.

[G] M. Geck, *Beiträge zur Darstellungstheorie von Iwahori–Hecke–Algebren,* Habilitationsschrift, RWTH Aachen, 1993.

[GRo] M. Geck and R. Rouquier, *Centers and simple modules for Iwahori–Hecke algebras,* Finite reductive groups (Luminy, 1994), Progr. Math., 141, pp. 251–272, Birkhäuser, Boston, 1997.

[H1] D. G. Higman, *Indecomposable representations at characteristics p*, Duke Math. J. 21 (1954), 377–381.

[H2] ———, *Modules with a group of operators,* Duke Math. J. 21 (1954), 369–376.

[H3] ———, *Induced and produced modules,* Canadian J. Math. 7 (1955), 490–508.

[I] M. Ikeda, *On a theorem of Gaschütz,* Osaka Math. J. 5 (1953), 53–58.

[Il] L. Illusie, *Calculs de termes locaux cohomologie ℓ-adique et fonctions L* (SGA 5), Lecture Notes in Math., 589, pp. 138–203, Springer-Verlag, Berlin, 1977.

[J] N. Jacobson, *Basic algebra II,* Freeman, San Francisco, 1980.

[K1] B. Keller, *Introduction to abelian and derived categories,* Representations of reductive groups, pp. 41–61, Cambridge Univ. Press, Cambridge, 1998.

[K2] ———, *Private communication,* 2007.

[M] S. Mac Lane, *Categories for the working mathematician,* Grad. Texts in Math., 5, Springer-Verlag, Berlin, 1971.

[R1] J. Rickard, *The abelian defect group conjecture,* Proceedings of the International Congress of Mathematicians, vol. II (Berlin, 1998), Doc. Math., extra vol. II, pp. 121–128, Bielefeld, 1998.

[R2] ———, *Equivalences of derived categories for symmetric algebras,* J. Algebra 257 (2002), 460–481.

[Se] J.-P. Serre, *Représentations linéaires des groupes finis,* Hermann, Paris, 1971.

[V] J.-L. Verdier, *Des catégories dérivées, des catégories abéliennes,* Astérisque 239 (1996).

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