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Imprimitive Distance-Transitive Graphs with Primitive Core of Diameter at Least 3

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In memory of Donald G. Higman

1. Introduction

A distance-transitive graph G is one upon which the automorphism group acts transitively on ordered pairs of vertices at every fixed distance. Only connected graphs need to be considered. Those of diameter 2 are the rank-3 graphs, whose careful study was initiated by Donald G. Higman in his breakthrough paper [16].

A huge amount of effort has gone into the classification of all finite distance-transitive graphs. The classification naturally breaks into two parts, primitive and imprimitive. The main part of the problem is the classification of all finite distance-transitive graphs with primitive automorphism group, and it appears that this classification is nearly finished. For the imprimitive case, Smith [24] showed that the possibilities for nontrivial blocks of imprimitivity are severely limited and that a given imprimitive distance-transitive graph can in a sense be reduced to a primitive distance-transitive graph. Van Bon and Brouwer [5] and Hemmeter [14; 15] carried through the reverse of Smith's theorem, classifying for most of the known primitive distance-transitive graphs any associated imprimitive distance-transitive graphs they might have.

In [3] the present authors gave a precise version of Smith's theorem which implies that any unknown imprimitive distance-transitive graph must arise from a primitive distance-transitive graph of diameter at least 2 and valency at least 3. In the present paper, we return to the work of van Bon and Brouwer [5] and Hemmeter [14; 15] and show that, starting from each of the known distance-transitive graphs of diameter and valency at least 3, there are no surprises—the only associated imprimitive graphs are ones already known and in the literature (see [7]).

The terminology and results will be made precise in the next section. In particular we give a precise version of Smith's theorem (following [3]) and describe how the present results fit into the general problem of classifying all distance-transitive graphs. In Section 3 we give various general results about the parameters of a distance-regular graph, particularly those that are imprimitive in one of the two ways specified by Smith's theorem. Section 4 discusses various of the geometries, designs, and codes often used in constructing and describing the graphs under consideration. Of particular import are the gamma spaces introduced by Higman.

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The final Section 5 is the heart of the paper. It describes all the primitive distance-transitive graphs of diameter at least 3 (that are known to us) and in each case describes all imprimitive graphs that have those graphs at their core.

This paper is based upon the thesis [1] of the first author, which was written under the supervision of the second author. Our general reference is the book of Brouwer, Cohen, and Neumaier [7]. All graphs and groups that we consider are finite.

2. Results

2.1. The Basics of Distance-Regular Graphs

Let G be a connected graph of diameter d. Denote by $G_i(x)$ the set of vertices of G at distance i from the vertex x in G.

For $y \in G_i(x)$ set

$$a_i^{x,y} = |G_i(x) \cap G_1(y)|;$$
$$b_i^{x,y} = |G_{i+1}(x) \cap G_1(y)|;$$
$$c_i^{x,y} = |G_{i-1}(x) \cap G_1(y)|.$$

The graph G is a *distance-regular graph* if, for all $0 \leq i \leq d$, each of the parameters $a_i^{x,y}$, $b_i^{x,y}$, and $c_i^{x,y}$ depends not on the choice of x and y but only on i. In that case we will write

$$a_i^{x,y} = a_i = a_i(G);$$
$$b_i^{x,y} = b_i = b_i(G);$$
$$c_i^{x,y} = c_i = c_i(G).$$

Often one writes $\lambda = a_1$ and $\mu = c_2$. Trivially $a_0 = c_0 = b_d = 0$ and $c_1 = 1$. We let $k_i = |G_i(x)|$ (a constant) so that $k_0 = 1$. Set $k = k_1(G) = b_0$, the *valency* of G. Then $k = a_i + b_i + c_i$ for $0 \leq i \leq d$.

The sequence

$$\mathrm{i}(G) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$$

is called the *intersection array* of G and contains all the necessary parameter information (see Proposition 3.1).

A connected diameter-d graph G is *distance-transitive* if it admits a group of automorphisms that is transitive on all ordered pairs (x, y) with $y \in G_i(x)$ for all $0 \leq i \leq d$. Clearly a distance-transitive graph is distance-regular, but the converse is in general false.

We could also consider disconnected distance-regular and distance-transitive graphs. Any connected component of such a graph is itself distance-regular or distance-transitive (respectively). Indeed for a distance-transitive graph all the connected components are isomorphic, whereas for a distance-regular graph all such components must at least have identical intersection arrays.

A distance-transitive graph is, of course, vertex-transitive (setting $i = 0$), so we may ask whether the automorphism group of the graph is vertex primitive or imprimitive. As with distance-regularity, there is a purely combinatorical counterpart.

Let us write $x \sim_i y$ if $y \in G_i(x)$. (So, e.g., $x \sim_2 y$ if and only if x and y are at distance 2 in G.) The distance-regular graph G is then *imprimitive* if there is a subset I of indices with $\{0\} \subset I \subset \{0, 1, \ldots, d\}$ such that the union of the relations $\sim_i$ for $i \in I$ is an equivalence relation on the vertex set of G. (The cases $I = \{0\}$ and $I = \{0, 1, \ldots, d\}$ give trivial equivalence relations.)

2.2. Smith's Theorem

A famous theorem of Derek Smith [4; 24] shows that a connected, imprimitive distance-regular graph of valency at least 3 is antipodal, bipartite, or both.

The graph G of diameter d is *antipodal* if the property of being at distance 0 or d is an equivalence relation on the vertices of G. We write $\mathrm{A}G$ for the quotient graph induced on the antipodal classes of an antipodal graph G. In the bipartite case, the *halves* are the connected components equipped with the adjacency relation $\sim_2$ in G. We write B^-G and B^+G for the two halves.

We have two elementary but fundamental observations (see [7, Prop. 4.2.2] and [4; 24]).

PROPOSITION 2.1. *Let G be a connected, antipodal distance-regular graph of valency k and diameter $d > 1$. Then $\mathrm{A}G$ is a connected distance-regular graph of diameter $\lfloor d/2 \rfloor$. For $0 \le i \le \lfloor d/2 \rfloor$, two vertices (classes) α and β are at distance i in $\mathrm{A}G$ if and only if each pair $a \in \alpha$ and $b \in \beta$ is at distance i or $d - i$ in G. If $d \ge 3$ then $\mathrm{A}G$ also has valency k. If G is a distance-transitive graph, then $\mathrm{A}G$ is also a distance-transitive graph.*

PROPOSITION 2.2. *Let G be a connected, bipartite distance-regular graph of valency k and diameter $d > 1$. Then $(\mathrm{B}^\varepsilon G, \sim_2)$ is a connected distance-regular graph of valency $k(k-1)/c_2$ and diameter $\lfloor d/2 \rfloor$. For $0 \le i \le \lfloor d/2 \rfloor$, two vertices are at distance i in $\mathrm{B}^\varepsilon G$ if and only if they are at distance $2i$ in G. If G is a distance-transitive graph, then $\mathrm{B}^\varepsilon G$ is also a distance-transitive graph.*

A precise version of Smith's theorem [24] appears as [3, Thm. 2.9] and is reproduced here as Theorem 2.3.

THEOREM 2.3. *Let G be a connected distance-regular graph of diameter d and valency k. Set $\mu = c_2$ and $k' = k(k-1)/\mu$. Then one of the following statements holds:*

(1) *G is primitive of diameter $d \ge 2$ and valency $k \ge 3$;*
(2) *$k = 2$, $d = \lfloor n/2 \rfloor$, and G is a cycle C_n for some $n \ge 3$;*
(3) *$d \le 1$, and G is a complete graph K_{k+1};*
(4) *$d = 2$, and G is a complete multipartite graph $K_{r,\ldots,r}$ with $1 + k/r$ parts of size $r \ge 2$;*
(5) *$d = 3$, and G is the bipartite incidence graph of a nontrivial symmetric design with block size $k \ge 3$ and index μ;*
(6) *$d = 3$, and G is an antipodal cover of K_{k+1} with $k \ge 3$;*
(7) *$d = 4$, G is antipodal and bipartite, $\mathrm{A}G$ is $K_{k,k}$ with $k \ge 3$, and $\mathrm{B}^\varepsilon G$ is complete multipartite;*

(8) *$d = 6$, G is antipodal and bipartite,* $\mathrm{A}G$ *is bipartite of diameter 3,* $\mathrm{B}^\varepsilon G$ *is antipodal of diameter 3, and the graphs* $\{\mathrm{B}^-\mathrm{A}G, \mathrm{B}^+\mathrm{A}G\} = \{\mathrm{AB}^-G, \mathrm{AB}^+G\}$ *are* $K_{k'+1}$ *for* $k' \geq k \geq 3$;
(9) *$d \geq 4$, G is antipodal but not bipartite, and* $\mathrm{A}G$ *is primitive of diameter* $\lfloor d/2 \rfloor \geq 2$ *and valency* $k \geq 3$;
(10) *$d \geq 4$, G is bipartite but not antipodal, and* $\mathrm{B}^\varepsilon G$ *is primitive of diameter* $\lfloor d/2 \rfloor \geq 2$ *and valency* $k' \geq k \geq 3$;
(11) *odd $d = 2e+1 \geq 5$, G is antipodal and bipartite, all antipodal classes have size 2,* $\mathrm{A}G$ *is primitive of diameter* $e \geq 2$ *and valency* $k \geq 3$, *and* $\mathrm{B}^\varepsilon G$ *is primitive of diameter* $e \geq 2$ *and valency* $k' \geq k \geq 3$;
(12) *even $d = 2e \geq 8$, G is antipodal and bipartite,* $\mathrm{A}G$ *is bipartite of diameter e,* $\mathrm{B}^\varepsilon G$ *is antipodal of diameter e, and the graphs* $\{\mathrm{B}^-\mathrm{A}G, \mathrm{B}^+\mathrm{A}G\} = \{\mathrm{AB}^-G, \mathrm{AB}^+G\}$ *are primitive of diameter* $\lfloor e/2 \rfloor \geq 2$ *and valency* $k' \geq k \geq 3$.

2.3. Some Terminology and Notation

Recall that the connected distance-regular graph G of diameter d is *antipodal* if the property of being at distance 0 or d is an equivalence relation on the vertices of G. We write $\mathrm{A}G$ for the quotient graph induced on the antipodal classes of an antipodal graph G. We then say that G is an *antipodal cover* or an A-*cover* of $\mathrm{A}G$ and even an *antipodal r-cover* or *r-fold* A-*cover,* where r is the common cardinality of the antipodal classes. The antipodal quotient $\mathrm{A}G$ is usually called a *folded* graph. One often finds $\mathrm{A}G$ denoted $\bar{G}$ (see e.g. [7, p. 140]).

If the connected distance-regular graph G is bipartite, then the *halves* are uniquely determined as the only connected components under the adjacency relation $\sim_2$ given by having distance 2 in G. The two halves are denoted B^-G and B^+G, but we often write $\mathrm{B}G$ for either one of the halves. This is somewhat ambiguous since the two graphs $\mathrm{B}^\varepsilon G = (\mathrm{B}^\varepsilon G, \sim_2)$ need not be isomorphic, but the abuse will not cause any problems for us. The graphs $(\mathrm{B}^\varepsilon G, \sim_2)$ have the same intersection array (in particular, the same number of vertices and valency), and in a distance-transitive graph they are indeed isomorphic.

The graph $(\mathrm{B}^\varepsilon G, \sim_2)$ is a *halved graph* of G, and G is a B-*double* of $\mathrm{B}^\varepsilon G$. In some places G is called a *doubled graph* (or *doubling*) of $\mathrm{B}^\varepsilon G$ (although this may cause confusion in some situations; see Section 3.4). One often finds $\mathrm{B}^\varepsilon G$ denoted $\frac{1}{2}G^\varepsilon$ or even $\frac{1}{2}G$ [7, p. 140] or G' [14; 15].

An A-cover of a B-double of the graph H will be called an AB-*cover* of H. In view of Theorem 2.3, such a graph G will also be a B-double of an A-cover of the same graph—that is, a BA-*double* of H. In this case, one also finds $H = \mathrm{AB}G = \mathrm{BA}G$ denoted as $\frac{1}{2}\bar{G}$ or $\bar{G}'$.

Note that, according to our definitions, an A-cover, B-double, AB-cover, or BA-double is always a connected distance-regular graph.

2.4. Results

Consider the various cases of Theorem 2.3 (Smith's theorem) when we restrict our attention to distance-transitive graphs.

Under the seven exceptional cases, all distance-transitive graphs are known. The specific graphs described in parts (2)–(4) of Theorem 2.3 are all distance-transitive. The classification of distance-transitive antipodal covers of complete graphs, as in Theorem 2.3(6), and of complete bipartite graphs, as in Theorem 2.3(7), has been completely settled in [13] and [19], respectively. The bipartite, diameter-3 distance-regular graphs of Theorem 2.3(5) are exactly the incidence graphs of non-trivial symmetric designs. As such, those that are distance-transitive were classified by Kantor [21]. (See also [15, Lemma 2; 7, Sec. 7.6.A; 3, Thm. 3.2].) The present authors showed in [3, Thm. 3.3] that the 6-cube $H(6,2)$ is the unique distance-transitive graph coming under Theorem 2.3(8). (This confirmed a conjecture of Brouwer, Cohen, and Neumaier [7, p. 416] for distance-regular graphs in the case of distance-transitive graphs.)

This leaves us with the five generic cases, one primitive and four imprimitive. A great deal of work has been done on the primitive case, and it seems likely that the known list of primitive distance-transitive graphs is complete (see e.g. [18] and [22]).

By Theorem 2.3, for distance-regular graphs H coming under one of the imprimitive cases (9)–(12), at least one of $\mathrm{A}H$, $\mathrm{B}H$, or $\mathrm{AB}H = \mathrm{BA}H$ is a primitive distance-regular graph G of valency at least 3 and diameter at least 2. We call such a graph G the primitive *core* of H. The core is uniquely determined except in Theorem 2.3(11) where there are two cores, $\mathrm{A}H$ and $\mathrm{B}H$. (But $\mathrm{B}H$ is the distance-2 graph of $\mathrm{A}H$; for detailed discussion of the case (11), see Section 3.4.)

This paper addresses the four generic imprimitive cases under the assumption that the primitive classification is complete. Specifically, we find all distance-transitive imprimitive graphs H whose primitive core G is one of the known primitive distance-transitive graphs of diameter at least 3 and valency at least 3. Indeed, we do something slightly stronger. We find all distance-regular H whose primitive core G is a known distance-transitive graph with diameter at least 3. It must be emphasized that a great deal of the needed work on these imprimitive cases has been done already by van Bon and Brouwer [5] for antipodal distance-regular graphs and by Hemmeter [14; 15] for bipartite distance-regular graphs.

THEOREM 2.4. *Let G be one of the distance-transitive graphs of diameter $d \geq 3$ and valency at least 3 given in the first column of Table I, subject to the restrictions* (*if any*) *of the third column of the table. Then all imprimitive graphs H with G as core are indicated* (*as described in what follows*) *in the three columns* A, B, *and* AB *of the table.*

If, in the row of the primitive graph G, the column A, B, or AB has the entry "$\times$" then G has no A-cover, B-double, or AB-cover (hence BA-double), respectively. If the entry is "$\surd$" then there is an appropriate H and it is unique up to isomorphism. The two "$\surd\surd$" entries indicate that for these graphs G there are, up to isomorphism, two distinct graphs H with $G = \mathrm{A}H$.

The proof of the theorem is accomplished on a case-by-case basis in Section 5, the appropriate section for each case being listed in the final column of the table. By our version (Theorem 2.3) of Smith's theorem we know that, leaving aside the

Table I

Primitive graph of diameter $d \geq 3$	d	Restrictions	A	B	AB	Section
Hamming $\mathrm{H}(n,q)$	n	$n \geq 3$	×	×	×	5.1.1
Quotient n-cube $\mathrm{AH}(n,2) = \bar{\mathrm{H}}(n,2)$	$\lfloor n/2 \rfloor$	$n \geq 6$	√	×	×	5.1.2
Halved n-cube $\mathrm{BH}(n,2) = \frac{1}{2}\mathrm{H}(n,2)$	$\lfloor n/2 \rfloor$	$n \geq 6$	×	√	×	5.1.3
Quotient halved cube $\mathrm{ABH}(2m,2) = \mathrm{BAH}(2m,2) = \frac{1}{2}\bar{\mathrm{H}}(2m,2)$	$\lfloor m/2 \rfloor$	$m \geq 6$	√	√	√	5.1.4
Johnson $\mathrm{J}(n,m)$	m	$2m+1 \neq n \geq 2m \geq 6$	×	×	×	5.2.1
Johnson $\mathrm{J}(2m+1,m)$	m	$m \geq 3$	×	√	×	5.2.1
Quotient Johnson $\mathrm{AJ}(2m,m) = \bar{\mathrm{J}}(2m,m)$	$\lfloor m/2 \rfloor$	$m \geq 6$	√	×	×	5.2.2
Odd O_{m+1}	m	$m \geq 3$	√	×	×	5.2.3
Grassmann $\mathrm{J}_q(n,m)$	m	$2m+1 \neq n \geq 2m \geq 6$	×	×	×	5.3.1
Grassmann $\mathrm{J}_q(2m+1,m)$	m	$m \geq 3$	×	√	×	5.3.1
E_7 $[\mathrm{E}_{7,7}(q)]$	3		×	×	×	5.3.2
Affine E_6 $[\mathrm{AE}_6(q)]$	3		×	×	×	5.3.3
Symplectic dual polar $[\mathrm{C}_m(q)] = [\mathrm{Sp}(2m,q)]$	m	$m \geq 3$	×	×	×	5.4
Orthogonal dual polar $[\mathrm{B}_m(q)] = [\Omega(2m+1,q)]$	m	$m \geq 3$	×	×	×	5.4
Orthogonal dual polar $[{}^2\mathrm{D}_m(q)] = [\Omega^-(2m,q)]$	$m-1$	$m \geq 4$	×	×	×	5.4
Unitary dual polar $[{}^2\mathrm{A}_{m-1}(l)] = [U(m,l^2)]$	$\lfloor m/2 \rfloor$	$m \geq 6$	×	×	×	5.4
Halved orthogonal dual polar $\mathrm{B}[\mathrm{D}_m(q)] = \frac{1}{2}[\mathrm{D}_m(q)] = \frac{1}{2}[\Omega^+(2m,q)]$	$\lfloor m/2 \rfloor$	$m \geq 6$	×	√	×	5.4
Bilinear forms $\mathrm{H}_q(n,m)$	m	$n \geq m \geq 3$	×	×	×	5.5.1
Alternating forms $\mathrm{Alt}(n,q)$	$\lfloor n/2 \rfloor$	$n \geq 6$	×	×	×	5.5.2
Hermitean forms $\mathrm{Her}(n,l^2)$	n	$n \geq 3$, $(n,l) \neq (3,2)$	×	×	×	5.5.3
Hermitean forms $\mathrm{Her}(n,l^2)$	n	$(n,l) = (3,2)$	√√	×	×	5.5.3
Generalized 6-gon $(q,1)$	3		×	×	×	5.6

Generalized 6-gon (q,q)	3		×	√	×	5.6
Generalized 6-gon $(q,q^3),(q^3,q)$	3		×	×	×	5.6
Generalized 8-gon $(q,1)$	4	$q=2^a$	×	×	×	5.6
Generalized 8-gon $(q,q^2),(q^2,q)$	4	$q=2^{2a+1}$	×	×	×	5.6
Generalized 12-gon $(q,1)$	6	$q=3^a$	×	×	×	5.6
Extended ternary Golay $[3^6.2.M_{12}]$	3		×	×	×	5.7.1
Binary Golay $[2^{11}.M_{23}]$	3		√	×	×	5.7.2
Binary Golay distance 2 $[2^{11}.M_{23}]_2$	3		×	√	×	5.7.3
Truncated binary Golay $[2^{10}.M_{22}.2]$	3		√√	×	×	5.7.4
Truncated binary Golay distance 2 $[2^{10}.M_{22}.2]_2$	3		×	√	×	5.7.5
Witt $[M_{24}]$	3		×	×	×	5.7.6
Truncated Witt $[M_{23}]$	3		×	×	×	5.7.7
Doubly truncated Witt $[M_{22}.2]$	4		√	×	×	5.7.8
Coxeter $[PSL(3,2).2]$	4		×	×	×	5.8.1
Sylvester $[P\Gamma L(2,9)]$	3		×	×	×	5.8.2
Doro $[P\Gamma L(2,16)]$	3		×	×	×	5.8.3
Biggs–Smith $[PSL(2,17)]$	7		×	×	×	5.8.4
Perkel $[PSL(2,19)]$	3		×	×	×	5.8.5
Locally Petersen $[P\Sigma L(2,25)]$	3		×	×	×	5.8.6
Hermitean forms distance 3 $Her(3,4)_3$	4		×	×	×	5.8.7
Unitary nonisotropics $[P\Gamma U(3,4^2)]$	3		×	×	×	5.8.8
Hoffman–Singleton line graph $[P\Sigma U(3,5^2)]$	3		×	×	×	5.8.9
Livingstone $[J_1]$	4		×	×	×	5.8.10
Hall–Janko near octagon $[HJ.2]$	4		×	×	×	5.8.11
Patterson $[Suz.2]$	4		×	×	×	5.8.12

known cases (2)–(8), we only need look for all H with primitive G equal to $\mathrm{A}H$, $\mathrm{B}H$, or $\mathrm{AB}H$ $(= \mathrm{BA}H)$. Additionally, in order for there to exist an H with $G = \mathrm{AB}H$, there must exist H_1 with $G = \mathrm{A}H_1$ and H_2 with $G = \mathrm{B}H_2$ (namely, $H_1 = \mathrm{B}H$ and $H_2 = \mathrm{A}H$). Therefore if an appropriate H_1 or H_2 does not exist, then we need not look further for H with $G = \mathrm{AB}H$. This means that we concentrate on finding A-covers and B-doubles, and only in the rare occasions when both exist do we continue and look for AB-covers.

In the end, we encounter this possibility only for the quotient halved n-cube $\mathrm{ABH}(n,2) = \mathrm{BAH}(n,2) = \frac{1}{2}\bar{\mathrm{H}}(n,2)$ with even $n \geq 12$. This is treated in Section 5.1.4 where we find only the expected AB-cover (i.e., BA-double)—namely, the n-cube $\mathrm{H}(n,2)$.

In almost all cases the imprimitive distance-regular graphs we find are in fact distance-transitive. The only exceptional case is that of B-doubles of the generalized 6-gons (see Section 5.6). For each prime power q the incidence graph is a B-double of the distance-transitive $\mathrm{G}_2(q)$ generalized 6-gon of order (q,q), but these B-doubles are distance-transitive themselves if and only if q is a power of 3. Thus our smallest distance-regular imprimitive example that is not distance-transitive is the bipartite incidence graph of the 6-gon of type $(2,2)$ and has 126 vertices.

The table only records results starting from primitive distance-transitive graphs of diameter ≥ 3. In the body of the paper cases with diameter 2 are occasionally treated (particularly if that case was handled without exception in the cited literature), but we have made no effort to consider diameter 2 systematically.

3. The Parameters and Structure of Distance-Regular Graphs

3.1. Basic Parameter Restrictions

The parameters of a distance-regular graph are subject to many simple but still very useful constraints.

PROPOSITION 3.1. *Let G be a distance-regular graph with valency k and diameter d. Then the following hold*:

(a) $k_{i-1}b_{i-1} = k_i c_i$ $(1 \leq i \leq d)$;
(b) *if k_i is odd then a_i is even*;
(c) $1 \leq c_2 \leq \cdots \leq c_d$;
(d) $k \geq b_1 \geq \cdots \geq b_{d-1}$;
(e) *if $i + j \leq d$ then $c_j \leq b_i$*.

Proof. Part (a) counts the edges between $G_{i-1}(x)$ and $G_i(x)$ in two ways. Part (b) is in [7, 4.3.1], and parts (c)–(e) are in [7, 4.1.6]. □

There are many more restrictions on possible intersection arrays of distance-regular graphs, some quite difficult and many given in [7]. In particular, we can make from the parameters $a_i(G)$, $b_i(G)$, and $c_i(G)$ a tridiagonal $d \times d$ matrix whose eigenvalues are those of the adjacency matrix of G. The multiplicities can be calculated and must, of course, be integral; see [7, Thm. 4.1.4]. This is a powerful condition.

We do not use further methods (e.g., eigenvalue calculations) here except indirectly through the references and in particular by quoting Chapter 14 of [7]. In that chapter, Brouwer, Cohen, and Neumaier gave extensive lists of possible intersection arrays for distance-regular graphs, lists found using many results and restrictions, including eigenvalue techniques. In particular the lists contain all possible intersection arrays for distance-regular graphs of diameter at least 4 on at most 4096 vertices. In a number of cases (twelve, to be exact) we have eliminated specific A-covers on at most 4096 vertices by observing that the corresponding intersection array is not one of those listed in [7, Chap. 14] as being possible.

When the intersection array of a possible distance-regular graph of diameter at least 4 on at most 4096 vertices does not appear in the lists of [7, Chap. 14], the array will be termed *not feasible*. In this case, the graph does not exist. In many cases where we could appeal to [7, Chap. 14] we have instead given direct and simple arguments. It is likely that in some of the remaining cases we could again make ad hoc arguments along the lines of those in Propositions 5.42 and 5.47.

If we were to restrict our attention to distance-transitive graphs, then many of the specific cases become easier. For instance, the only difficult case for imprimitive distance-regular graphs corresponding to the Perkel graph is that of 3-fold A-covers; see Section 5.8.5. However, a distance-transitive 3-fold A-cover of the Perkel graph must come from the permutation representation of $3 \times \mathrm{PSL}(2,19)$ on the cosets of a subgroup $\mathrm{Alt}(5)$, and a contradiction follows easily.

3.2. *The Parameters of Antipodal Distance-Regular Graphs*

THEOREM 3.2. *Let G be a connected distance-regular graph of diameter $d \geq 2$ with intersection array*

$$\mathrm{i}(G) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}.$$

Suppose that H is a distance-regular r-fold A*-cover of G with $r \geq 2$. Then either H has diameter $D = 2d$ and*

$$\mathrm{i}(H) = \left\{b_0, b_1, \ldots, b_{d-1}, \frac{r-1}{r}c_d, c_{d-1}, \ldots, c_1; c_1, \ldots, c_{d-1}, \frac{1}{r}c_d, b_{d-1}, \ldots, b_0\right\},$$

or H has diameter $D = 2d + 1$ and

$$\mathrm{i}(H) = \{b_0, b_1, \ldots, b_{d-1}, t(r-1), c_d, c_{d-1}, \ldots, c_1; c_1, \ldots, c_{d-1}, c_d, t, b_{d-1}, \ldots, b_0\}$$

for the integer $t = c_{d+1}(H)$.

Proof. The result is due to Gardiner [12, p. 264]; see also [7, p. 142]. □

This theorem gives some important numerical restrictions, which we will use often to prove nonexistence of A-covers.

COROLLARY 3.3. *Let G and H be as in Theorem 3.2.*

(a) *For $D = 2d$ even, r divides c_d and $r \leq c_d/\max(c_{d-1}, c_d - b_{d-1})$.*
(b) *For $D = 2d + 1$ odd, $t(r - 1) \leq \min(b_{d-1}, a_d)$ and $c_d \leq t$. In particular $c_d \leq \min(b_{d-1}, b_0 - c_d)$.*

Proof. See [7, p. 142]. □

3.3. The Parameters and Structure of Bipartite Distance-Regular Graphs

PROPOSITION 3.4. *Let H be a distance-regular graph with intersection array*

$$\mathrm{i}(H) = \{B_0, B_1, \ldots, B_{D-1}; C_1, C_2, \ldots, C_D\}$$

and diameter D. Then H is bipartite if and only if $B_i + C_i = B_0$ for $i = 1, \ldots, D$.

In this case, each halved graph $\mathrm{B}H$ is distance-regular of diameter $d = \lfloor D/2 \rfloor$ with intersection array

$$\mathrm{i}(\mathrm{B}H) = \{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\},$$

where

$$b_i = \frac{B_{2i}B_{2i+1}}{C_2} \quad \text{for } 0 \leq i \leq d - 1,$$

$$c_j = \frac{C_{2j}C_{2j-1}}{C_2} \quad \text{for } 1 \leq j \leq d.$$

Proof. The second part of this proposition is proven as [14, Lemma] and [15, Lemma 1]. The first part is given in [7, Prop. 4.2.2(i)] and attributed to Biggs and Gardiner [4]. □

The conventions adopted in Proposition 3.4 are used at times throughout the paper. Specifically, if H is a bipartite distance-regular graph with halved graph $G = \mathrm{B}H$, then the parameters $a_i(H)$, $b_i(H)$, and $c_i(H)$ may be written as A_i, B_i, and C_i, while the parameters $a_i(G)$, $b_i(G)$, and $c_i(G)$ may be written as a_i, b_i, and c_i.

Hemmeter [15] showed that the problem of finding B-doubles of a given distance-regular graph is related to the study of maximal cliques in the graph.

LEMMA 3.5. *Let H be a connected bipartite distance-regular graph of diameter at least 4, and let $\mathrm{B}H$ be a halved graph of H having G as its set of vertices. Then for every $y \in H \setminus G$, $H_1(y)$ is a maximal clique in $\mathrm{B}H$. Moreover, if $y_1 \neq y_2$ then $H_1(y_1) \neq H_1(y_2)$.*

Proof. This is proven as [15, Lemma 2]. □

These results imply parameter restrictions that will be very helpful in proving nonexistence of B-doubles.

LEMMA 3.6. *Let H be a connected bipartite distance-regular graph of diameter at least 4, and let $G = \mathrm{B}H$. Set $b_0 = b_0(G)$ and $b_1 = b_1(G)$.*

(a) *There exists a maximal clique of size* m *in* G *such that* b_0 *divides* $m(m-1)$.
(b) *In particular* $b_0 \le (b_0 - b_1)((b_0 - b_1) + 1)$.

Proof. By Lemma 3.5 there is a maximal clique M in G with $H_1(y)$ consisting of the vertices of M. Set $m = |M|$ so that $B_0 = m$ (where $B_i = b_i(H)$ and so forth). Proposition 3.4 gives $b_0 = B_0 B_1 / C_2$. Here $B_0 = A_1 + B_1 + C_1 = 0 + B_1 + 1$ since H is bipartite. Therefore b_0 divides $B_0 B_1 = m(m-1)$ as claimed in (a).

The number of triangles on a given edge of G is $a_1(G)$. Therefore

$$m \le a_1(G) + 2 = (b_0 - b_1 - c_1(G)) + 2 = (b_0 - b_1) + 1.$$

We conclude that $m(m-1) \le ((b_0 - b_1) + 1)(b_0 - b_1)$, giving (b). □

LEMMA 3.7. *Let* H *be a connected bipartite distance-regular graph of diameter at least 4, and let* $G = \mathrm{B}H$. *Set* $b_0 = b_0(G)$, $B_0 = b_0(H)$, *and so forth.*

(a) $1 = C_1 \le C_2 \le C_3 \le C_4 \le c_2$.
(b) *The valency* B_0 *of* H *is a root of the polynomial* $x^2 - x - cb_0$ *for some integer* c *with* $1 \le c \le c_2$.

Proof. By Proposition 3.4 with $j = 2$, $C_4 = c_2 C_2 / C_3$. From Proposition 3.1(c) we have $1 = C_1 \le C_2 \le C_3 \le C_4$. In particular $C_4 \le c_2$. This gives (a).

From Proposition 3.4 with $i = 0$ we find $b_0 C_2 = B_0 B_1$. Since H is bipartite, $A_1 = 0$ and $B_1 = B_0 - A_1 - C_1 = B_0 - 1$. Therefore

$$0 = B_0(B_0 - 1) - b_0 C_2 = B_0^2 - B_0 - b_0 C_2.$$

That is, the valency B_0 of the B-double H is a root of the polynomial $x^2 - x - b_0 c$ where $c = C_2$. Since $1 \le C_2 \le C_4 \le c_2$ by (a), we also have (b). □

3.4. Bipartite Doubles

The next-to-last conclusion (11) of Theorem 2.3 is that of a distance-regular graph H that is antipodal (with class size 2) and bipartite of odd diameter at least 5. It turns out that, given either one of the graphs $G = \mathrm{A}H$ and $J = \mathrm{B}H$, such a graph H can be reconstructed explicitly and uniquely. Furthermore, the distance-regular graphs G and J for which this is possible are completely characterized by conditions on their parameters.

For an arbitrary graph L, its *bipartite double* $2 \times L$ is the graph whose vertex set is $L \times \{+, -\}$ with (v, a) adjacent to (w, b) precisely when v is adjacent to w in L and $a = -b$. If in L the condition $L_1(v) = L_1(w)$ always implies $v = w$ (valid in all distance-regular L except for complete bipartite graphs), then $\mathrm{Aut}(2 \times L) = 2 \times \mathrm{Aut}(L)$. If L is distance-transitive, then so is the bipartite double $2 \times L$.

THEOREM 3.8. (a) *Let the connected distance-regular graph* H *be antipodal and bipartite of odd diameter* $2d + 1 \ge 5$. *Then antipodal classes have size 2 and* H *is isomorphic to the bipartite double of* $G = \mathrm{A}H$. *In this case* G *has diameter* d *with* $a_1(G) = \cdots = a_{d-1}(G) = 0$ *and* $a_d(G) > 0$. *Also,* $J = \mathrm{B}H$ *is isomorphic to the distance-2 graph* G_2 *and has diameter* d *with* $k_d(J) = a_d(J) + 1 > 1$.

(b) *Let G be a connected distance-regular graph of diameter $d \geq 2$ with $a_1(G) = \cdots = a_{d-1}(G) = 0$ and $a_d(G) > 0$. Then the bipartite double H of G is a distance-regular graph as in* (a).

(c) *Let J be a connected distance-regular graph of diameter $d \geq 2$ with $k_d(J) = a_d(J)+1 > 1$. Then the distance-d graph $G = J_d$ is a connected distance-regular graph of diameter $d \geq 2$ with $a_1(G) = \cdots = a_{d-1}(G) = 0$ and $a_d(G) > 0$ as in* (b), *and $J = G_2$.*

Proof. This result can be found in [7, Sec. 4.2.D]. An antipodal and bipartite graph of odd diameter D as in (a) must have antipodal classes of size 2 as otherwise it would have cycles of length $3D$; see Theorem 2.3(11). □

Readers should be wary of the term "bipartite double", which we get from [7, Sec. 1.11]. A distance-regular bipartite double $2 \times G$ is indeed bipartite and is a 2-fold A-cover of G, but $2 \times G$ *is not a* B-*double of* G. Instead $2 \times G$ is a B-double of the distance-2 graph G_2; that is, $\mathrm{B}(2 \times G) = G_2$.

4. Gamma Spaces and Related Combinatorial Constructions

Many of the geometries and graphs of interest to us here are naturally described in terms of Donald Higman's gamma spaces. See [9] for further discussion of most of the material in this section.

4.1. Partial Linear, Gamma, and Projective Spaces

A *partial linear space* $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a set of *points* $\mathcal{P}$ and a set of *lines* $\mathcal{L}$ equipped with an incidence relation I subject to the following axiom:

There do not exist distinct $p, q \in \mathcal{P}$ and distinct $m, n \in \mathcal{L}$ with $p \,\mathrm{I}\, m \,\mathrm{I}\, q \,\mathrm{I}\, n \,\mathrm{I}\, p$.

The axiom is self-dual in the sense that $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial linear space if and only if $(\mathcal{L}, \mathcal{P}, \mathrm{I})$ is. We usually have the additional (self-dual) nondegeneracy condition saying that every point is incident to at least two distinct lines and every line is incident to at least two distinct points. In a nondegenerate partial linear space we may identify a line with the set of points incident to it.

Essentially equivalent to the partial linear space is its *incidence graph*. This is a bipartite graph whose vertex set has as its two parts the sets $\mathcal{P}$ and $\mathcal{L}$, the two vertices $p \in \mathcal{P}$ and $l \in \mathcal{L}$ being adjacent precisely when the point p is incident to the line l. The axiom has a simple interpretation in this setting: The incidence graph has no 4-cycles. The partial linear space is nondegenerate precisely when all vertices of the incidence graph have valency at least 2. The partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is *connected* when its incidence graph is connected.

There are two other important graphs derived from a partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$. One is the *collinearity graph* (or *point graph*) whose vertex set is the point set $\mathcal{P}$ with two vertices adjacent when there is a line incident to both. The *line graph* has as vertex set the line set $\mathcal{L}$ with two vertices adjacent when there is a point incident to both. If H is the incidence graph of the partial linear space, then the collinearity graph is the graph induced by its distance-2 graph H_2 on $\mathcal{P}$, while the line graph is that induced by H_2 on $\mathcal{L}$.

A *subspace* $(\mathcal{P}_0, \mathcal{L}_0, \mathrm{I})$ of the partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ has $\mathcal{P}_0 \subseteq \mathcal{P}$ and $\mathcal{L}_0 \subseteq \mathcal{L}$ with the property that if a and b are both points (or lines) of the subspace and $a \,\mathrm{I}\, z \,\mathrm{I}\, b$, then z also belongs to the subspace. If we have identified each line with its set of incident points, then a subspace corresponds to a set of points that is line-closed. A subspace is *convex* if it contains all shortest paths between points.

The correspondence with bipartite graphs makes it clear that the class of partial linear spaces is quite broad. Higman noted that a single additional axiom has remarkable power and is valid in most of the classical and building geometries of interest. According to Higman, a partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a *gamma space* provided:

> *For any point p and line l not incident to p, the point p is collinear with zero, one, or all points of l.*

This condition is not self-dual. The disjoint union of two gamma spaces is still a gamma space, so we usually focus our attention on connected gamma spaces.

A particular type of connected gamma space is a *linear space*—that is, a partial linear space in which every pair of points is collinear. This condition is not self-dual, either. It corresponds to every pair of distinct points having distance 2 in the incidence graph. The *rank* of a linear space is one less than the length of the longest chain of nonempty subspaces it contains, so a single point has rank 0 while a line has rank 1.

A *projective space* is a linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ that satisfies the following axiom.

> Pasch's Axiom. *If the lines m_1 and m_2 are incident to a common point p, and if the two lines n_1 and n_2 are not incident to p but are both incident to points of each of m_1 and m_2, then there is a point q incident to both n_1 and n_2.*

The Veblen–Young Theorem [9, 2.1] says that a projective space with all lines on at least three points and of rank at least 3 can be realized as the linear space of 1-spaces (projective points) and 2-spaces (projective lines) of a vector space of dimension one more than the rank over a division ring.

4.2. Polar, Dual Polar, and Parapolar Spaces

A *polar space* is a partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ satisfying:

> Buekenhout–Shult Axiom. *If $p \in \mathcal{P}$ and $l \in \mathcal{L}$, then p is collinear either with all points of l or with exactly one point of l.*

Linear spaces arise when the "exactly one point" possibility fails to occur. A polar space is a special kind of gamma space, since the defining property for gamma spaces is the Buekenhout–Shult axiom but with the possibility of zero collinear points also allowed.

The Buekenhout–Shult Theorem [9, Sec. 3] gives a counterpart for polar spaces of the Veblen–Young Theorem for projective spaces. We do not give the details, but the heart of the theorem is that a typical polar space comes from the totally isotropic or totally singular 1-spaces (points) and totally isotropic or totally singular 2-spaces (lines) of a sesquilinear or quadratic form defined on a vector space.

Let V be the vector space $\mathbb{F}^n$, and let $W = \mathbb{F}^m$. Let σ be an automorphism of order at most 2 of the field $\mathbb{F}$. Then the map $f\colon V \times W \to \mathbb{F}$ is a *σ-sesquilinear form* provided f is bi-additive and

$$f(av, bw) = af(v, w)b^\sigma$$

for all $v \in V$, $w \in W$, and $a, b \in \mathbb{F}$. Of particular interest is when $V = W$. In that case, we will call a subspace U of V *totally isotropic* if f vanishes on $U \times U$.

Again let V be the vector space $\mathbb{F}^n$. Then the map $f\colon V \to \mathbb{F}$ is a *quadratic form* provided $g(x, y) = f(x + y) - f(x) - f(y)$ is bilinear (1-sesquilinear) and $f(av) = a^2 f(v)$ for all $v \in V$ and $a \in \mathbb{F}$. In that case, we will call a subspace U of V *totally singular* if f vanishes on U.

In the polar spaces coming from forms, the maximal linear subspaces are exactly the maximal totally isotropic or totally singular subspaces. In particular they are projective spaces of fixed rank. The form is *nondegenerate* if the intersection of all the maximal linear subspaces is trivial. The associated *dual polar space* is then the partial linear space whose point set consists of these maximal linear subspaces and whose line set is the set of corank-1 totally isotropic or singular subspaces with incidence given by containment.

A parapolar space is a more general kind of gamma space whose study was initiated by Bruce Cooperstein [10], a student of Donald Higman's. A *parapolar space* is a connected gamma space equipped with a collection of nondegenerate polar subspaces called *symplecta* with every symplecton convex (in the collinearity graph) and such that every line is in a symplecton as is every noncollinear pair of points that are commonly collinear with more than one further point. It turns out that most buildings can be associated with a parapolar space in a manner extending the association of the projective space of rank k over $\mathbb{F}$ with the building of type A_k over $\mathbb{F}$.

4.3. Near and Generalized Polygons

A *near polygon* is a connected partial linear space $(\mathcal{P}, \mathcal{L}, \mathrm{I})$ that satisfies the following:

> *For each point $p \in \mathcal{P}$ and line $l \in \mathcal{L}$, there is a unique point incident to l that is closest to p in the collinearity graph.*

In particular, a near polygon is a gamma space in which the "all points" possibility fails to occur. The near polygon is a *near $2d$-gon* if the diameter of its collinearity graph is d. We will call a near polygon *regular* if its collinearity graph is distance-regular. The dual polar spaces mentioned before give examples of regular near polygons.

We have the following general result, which will be of help later.

LEMMA 4.1. *For $d \geq 2$, the collinearity graph of a regular near $2d$-gon has no* A-*covers of diameter $2d + 1$.*

Proof. This is proven as part of [5, Cor. 2.3] and [7, Cor. 4.2.9]. □

A *generalized polygon* (resp. *generalized $2d$-gon*) is a regular near polygon (resp. near $2d$-gon) in which the path connecting the point p and the nearest point on the line l is always unique. Generalized polygons were introduced by Tits, and they arise naturally as gamma spaces associated with buildings of rank 2. A generalized polygon is said to have *order* (s, t) provided each of its lines has $1 + s$ points and each point is on $1 + t$ lines.

The incidence graph of a generalized $2d$-gon of order (t, t) is itself a generalized $4d$-gon of order $(1, t)$, the edges of the graph being the lines of the generalized polygon. The dual of a generalized polygon of order (s, t) is a generalized polygon of order (t, s).

4.4. Other Incidence Systems

A partial linear space is a special sort of incidence system. An *incidence system* $(\mathcal{P}, \mathcal{B}, \mathrm{I})$ is a set of *points* $\mathcal{P}$ and a set of *blocks* $\mathcal{B}$ equipped with an *incidence relation* I. The corresponding *incidence graph H* is the bipartite graph with vertex set $\mathcal{P} \cup \mathcal{B}$, two such vertices being adjacent precisely when they are incident in the incidence system. We no longer proscribe 4-cycles. Indeed in this section we are primarily interested in incidence systems for which the point graph induced by H_2 on $\mathcal{P}$ is complete.

A *symmetric design of index μ* is an incidence system $(\mathcal{P}, \mathcal{B}, \mathrm{I})$ with $|\mathcal{P}| = |\mathcal{B}| = v$ and where the following self-dual condition holds.

> *For distinct points $p, q \in \mathcal{P}$, there are exactly k blocks $b \in \mathcal{B}$ incident to p and exactly μ blocks incident to both p and q; for distinct blocks $b, c \in \mathcal{B}$, there are exactly k points $p \in \mathcal{P}$ incident to b and exactly μ points incident to both b and c.*

The symmetric design is *nontrivial* if $1 < k < v$. Nontrivial symmetric designs of index μ are exactly those incidence systems with incidence graphs that are bipartite distance-regular graphs of diameter 3 as in Theorem 2.3(5).

A *Steiner system* $\mathrm{S}(l, k, v)$ is an incidence system $\{\mathcal{P}, \mathcal{B}, \mathrm{I}\}$ with $|\mathcal{B}| = v$ where the number of points incident to a block is always k and where, for each l-tuple of points, there is exactly one block incident to all members of the l-tuple. Clearly these cannot exist for $k < l$, and for $k \geq l$ we can (and do) identify each block with the set of points incident to it.

Of particular interest are the Steiner systems $\mathrm{S}(5, 6, 12)$, $\mathrm{S}(3, 6, 22)$, $\mathrm{S}(4, 7, 23)$, and $\mathrm{S}(5, 8, 24)$. These are the *Witt designs,* since they were discussed and proved to be unique by Witt [25]. Their automorphism groups are, respectively, the Mathieu groups M_{12}, M_{22}, M_{23}, and M_{24}.

4.5. Coset Graphs of Codes

For us a *code* C is a subspace of $V = \mathbb{F}^n$ for some n, which is called the *length* of C. The code C and vector space V are equipped with the *Hamming metric* under which two vectors $v, w \in V$ are at distance d provided they differ in exactly d coordinate positions. The *weight* of a vector is its distance from the zero vector $\mathbf{0}$. A particularly important parameter for a code is its *minimum distance,* the smallest distance between two distinct codewords. Since C is a subspace, this is also the smallest nonzero weight of a codeword. See [23] for more information about codes.

The monomial matrix group $(\mathbb{F}^*)^n.\mathrm{Sym}(n)$ acts on V preserving the Hamming metric, and for us the *automorphism group* of a code is the subgroup of the monomial group that takes the code to itself.

Given the code C, one can *truncate* (or *puncture*) it by deleting a fixed coordinate position from all codewords. This produces a code with the same dimension as C (except in trivial cases) and length $n - 1$, a code whose minimum distance has remained the same or decreased by one. Another related code is the *shortened* code, which one gets by first choosing all codewords that are 0 in a fixed position and then truncating at that position. Again the length goes down by one, but the shortened code will typically have the same minimum distance as the original code but be of dimension one less. Clearly a shortened code is a subcode of the corresponding truncated code.

We will be particularly interested in the Golay codes. The *extended ternary Golay* code is a dimension-6 code in $\mathbb{F}_3^{12}$ with minimum distance 6. It is (in an appropriate sense) uniquely determined by these parameters and admits the central extension $2.M_{12}$ as automorphism group. Its codewords of weight 6 “carry” the blocks of the Witt design $S(5, 6, 12)$.

The *binary Golay* code in $V = \mathbb{F}_2^{23}$ has dimension 12 and minimum distance 7. Indeed its codewords of minimum nonzero weight are the characteristic vectors of the blocks of the Witt design $S(4, 7, 23)$ and span the code. Its automorphism group is therefore the same as that of the Witt design—namely, M_{23}.

The *truncated binary Golay* is then a code of dimension 12 and length 22 in $\mathbb{F}^{22}$ spanned by the characteristic vectors of the blocks of the Witt design $S(3, 6, 22)$ and having these as its codewords of minimum nonzero weight. Its automorphism group is $M_{22}.2$. The corresponding *shortened binary Golay* code is of dimension 11 and length 22 in $\mathbb{F}^{22}$, having codimension 1 in the truncated binary Golay code. Its automorphism group is also $M_{22}.2$.

The *coset graph* of the code $C \leq V$ has as vertices the cosets of C in V with two cosets adjacent when they have coset representatives at Hamming distance 1. The quotient space V/C acts by translation as a regular group of automorphisms of the coset graph, and the extension of the quotient space by the automorphism group of the code is an automorphism group of the coset graph. In particular, as the various

Golay related codes have large Mathieu groups as automorphism groups, the associated coset graphs also have large automorphism graphs with regular normal subgroups.

All of the code examples just given have minimum distance at least 5; therefore no two vectors of weight ≤ 2 are in the same coset. For the binary codes this means that the cosets at distance 1 from C are exactly the n distinct cosets of weight 1 (the weight of a coset being the minimum weight of a vector in the coset), and at distance 2 from C we find the $\binom{n}{2}$ distinct cosets of weight 2. Since the truncated binary Golay code has codewords of weight 6, it will have cosets of weight 3 containing more than one vector of weight 3 (exactly two, in fact).

The binary Golay code C is a perfect 3-error-correcting code. For us, this means exactly that every coset of C in $\mathbb{F}_2^{23}$ contains a unique vector of weight at most 3.

5. Imprimitivity and the Known Distance-Transitive Graphs of Diameter at Least 3

5.1. The Hamming Family

The *Hamming graph* $\mathrm{H}(n, q)$ has as vertex set all ordered n-tuples from a q-set ($q \geq 2$) with two such adjacent when they differ in exactly one coordinate position. The graph has diameter n. Its automorphism group is the wreath product $\mathrm{Sym}(q) \wr \mathrm{Sym}(n)$ and is distance-transitive. For each $n \geq 2$ the only imprimitive Hamming graph $\mathrm{H}(n, q)$ is the *n-cube* $\mathrm{H}(n, 2)$, which is both bipartite and antipodal.

Two vertices of the n-cube (thought of as $\{0, 1\}$-vectors) are at maximal distance n precisely when they are complements of each other. The *folded* or *quotient n-cube* $\bar{\mathrm{H}}(n, 2)$ (sometimes denoted $\square_n$) is the corresponding antipodal quotient $\mathrm{AH}(n, 2)$. It has diameter $\lfloor n/2 \rfloor$ and is distance-transitive (by Proposition 2.1).

The *halved n-cube* is $\mathrm{BH}(n, 2) = \frac{1}{2}\mathrm{H}(n, 2)$ and is distance-transitive of diameter $\lfloor n/2 \rfloor$ (by Proposition 2.2).

When n is even and at least 8, we have the *quotient halved n-cube*

$$\mathrm{ABH}(n, 2) = \mathrm{BAH}(n, 2) = \frac{1}{2}\bar{\mathrm{H}}(n, 2) = \frac{1}{2}\square_n,$$

which has diameter $\lfloor n/4 \rfloor$ and again is distance-transitive.

5.1.1. Hamming Graphs $\mathrm{H}(n, q)$

PROPOSITION 5.1. *The Hamming graph* $\mathrm{H}(n, q)$ *with* $n \geq 3$ *has no* A*-covers.*

Proof. This is proven as Proposition 5.1 of [5]. □

PROPOSITION 5.2. *The Hamming graph* $\mathrm{H}(n, q)$ *with* $n \geq 3$ *has no* B*-doubles.*

Proof. This is proven as Theorem 2 of [14]. □

5.1.2. Quotient n-cube $\mathrm{AH}(n,2) = \bar{\mathrm{H}}(n,2)$

PROPOSITION 5.3. *The only* A*-cover of the quotient n-cube* $\mathrm{AH}(n,2) = \bar{\mathrm{H}}(n,2)$ *with* $n \geq 6$ *is the n-cube* $\mathrm{H}(n,2)$.

Proof. This is proven as Proposition 5.2 of [5]. □

PROPOSITION 5.4. *The quotient Hamming graph* $\mathrm{AH}(n,2) = \bar{\mathrm{H}}(n,2)$ *with* $n \geq 4$ *has no* B*-doubles.*

Proof. This is proven as Theorem 15 of [15]. □

5.1.3. Halved n-cube $\mathrm{BH}(n,2) = \frac{1}{2}\mathrm{H}(n,2)$

PROPOSITION 5.5. *The halved n-cube* $\mathrm{BH}(n,2) = \frac{1}{2}\mathrm{H}(n,2)$ *with* $n \geq 4$ *has no* A*-covers.*

Proof. This is proven in Proposition 5.3 of [5]. □

PROPOSITION 5.6. *The only* B*-double of the halved n-cube* $\mathrm{BH}(n,2) = \frac{1}{2}\mathrm{H}(n,2)$ *with* $n \geq 5$ *is the n-cube* $\mathrm{H}(n,2)$.

Proof. This is proven as Theorem 14 of [15]. □

5.1.4. Quotient Halved n-cube $\mathrm{ABH}(n,2) = \mathrm{BAH}(n,2) = \frac{1}{2}\bar{\mathrm{H}}(n,2)$

PROPOSITION 5.7. *The only* A*-cover of the quotient halved n-cube* $\mathrm{ABH}(n,2) = \frac{1}{2}\bar{\mathrm{H}}(n,2)$ *with even* $n \geq 8$ *is* $\mathrm{BH}(n,2) = \frac{1}{2}\mathrm{H}(n,2)$.

Proof. This is proven in Proposition 5.3 of [5]. □

PROPOSITION 5.8. *The only* B*-double of the quotient halved n-cube* $\mathrm{BAH}(n,2) = \frac{1}{2}\bar{\mathrm{H}}(n,2)$ *with even* $n \geq 8$ *is* $\mathrm{AH}(n,2) = \bar{\mathrm{H}}(n,2)$.

Proof. This is proven as Theorem 16 of [15]. □

COROLLARY 5.9. *The only* AB*-cover of the quotient halved n-cube* $\mathrm{ABH}(n,2) = \mathrm{BAH}(n,2) = \frac{1}{2}\bar{\mathrm{H}}(n,2)$ *with even* $n \geq 8$ *is the n-cube* $\mathrm{H}(n,2)$.

Proof. This is immediate from Propositions 5.3 and 5.8 (and from Propositions 5.6 and 5.7). □

5.2. The Johnson Family

The *Johnson graph* $\mathrm{J}(n,m)$ has as vertex set the m-subsets of an n-set with two such adjacent when they intersect in a set of size $m-1$. By complementation $\mathrm{J}(n,m)$ is isomorphic to $\mathrm{J}(n,n-m)$, so without loss we may assume that $n \geq 2m$. With this assumption $\mathrm{J}(n,m)$ has diameter m. The symmetric group $\mathrm{Sym}(n)$ acts distance-transitively.

For each m the Johnson graph $\mathrm{J}(n,m)$ is imprimitive only for $n = 2m$, where antipodal classes consist of complementary pairs of m-sets. The *quotient Johnson graph* is then $\mathrm{AJ}(2m,m) = \bar{\mathrm{J}}(2m,m)$ and is distance-transitive of diameter $\lfloor m/2 \rfloor$.

When $n = 2m + 1$ there is a second distance-transitive graph on the m-subsets of an n-set, namely the distance-m graph $\mathrm{J}(2m+1,m)_m$ where two m-subsets are adjacent when they are disjoint. This graph is usually called the *odd graph* O_{m+1} and is primitive and distance-transitive of diameter m. (The subscript in O_{m+1} indicates the valency of the odd graph. Hemmeter [14; 15] instead used O_m to denote this graph, presumably because it is defined in terms of m-subsets.)

A related graph is the bipartite double $2 \times \mathrm{O}_{m+1}$; see Section 3.4. This graph is usually written $2\mathrm{O}_{m+1}$ and is often called the *doubled odd graph.* (This is terminology that we shall avoid because of possible confusion, as discussed in Section 3.4.) The graph $2\mathrm{O}_{m+1}$ can also be realized as the graph whose vertices are the m- and $(m+1)$-subsets of a $(2m+1)$-set with incidence given by containment. This graph is distance-transitive of diameter $2m+1$ and is clearly bipartite; indeed $\mathrm{B}2\mathrm{O}_{m+1} = (\mathrm{O}_{m+1})_2 = \mathrm{J}(2m+1,m)$. The bipartite double $2\mathrm{O}_{m+1}$ is also antipodal, a class consisting of a complementary m-set and $(m+1)$-set; we have $\mathrm{A}2\mathrm{O}_{m+1} = \mathrm{O}_{m+1}$.

5.2.1. Johnson Graphs $\mathrm{J}(n,m)$

PROPOSITION 5.10. *The Johnson graph* $\mathrm{J}(n,m)$ *with* $n \geq 2m \geq 4$ *has no* A-*covers.*

Proof. This is given in [5, p. 146]. □

PROPOSITION 5.11. *The only* B-*double of the Johnson graph* $\mathrm{J}(n,m)$ *with* $n \geq 2m \geq 4$ *is the bipartite double* $2\mathrm{O}_{m+1}$ *for* $n = 2m+1$.

Proof. This is proven as Theorem 1 of [14]. □

5.2.2. Quotient Johnson Graphs $\mathrm{AJ}(2m,m) = \bar{\mathrm{J}}(2m,m)$

PROPOSITION 5.12. *The only* A-*cover of the quotient Johnson graph* $\mathrm{AJ}(2m,m) = \bar{\mathrm{J}}(2m,m)$ *with* $m \geq 4$ *is the Johnson graph* $\mathrm{J}(2m,m)$.

Proof. This is given in [5, p. 147]. □

PROPOSITION 5.13. *The quotient Johnson graph* $\mathrm{AJ}(2m,m) = \bar{\mathrm{J}}(2m,m)$ *with* $m \geq 4$ *has no* B-*doubles.*

Proof. This is proven as Theorem 6 of [15]. □

5.2.3. Odd Graphs O_{m+1}

PROPOSITION 5.14. *The only* A-*cover of* O_{m+1} *with* $m \geq 3$ *is its bipartite double* $2\mathrm{O}_{m+1}$.

Proof. This is proven as Proposition 4.1 of [5]. See also Ivanov [17, Lemma 5.3]. □

PROPOSITION 5.15. *The odd graph* O_{m+1} *with* $m \geq 2$ *has no* B*-doubles.*

Proof. This is proven as Theorem 9 of [15]. □

5.3. Lie-type Examples

5.3.1. Grassmann Graphs $\mathrm{J}_q(n, m)$

The *Grassmann graph* $\mathrm{J}_q(n, m)$ has as vertex set the m-subspaces of an n-space over $\mathbb{F}_q$ with two such adjacent when they intersect in a subspace of dimension $m-1$. By duality $\mathrm{J}_q(n, m)$ is isomorphic to $\mathrm{J}_q(n, n-m)$, so without loss we may assume that $n \geq 2m$. With this assumption $\mathrm{J}_q(n, m)$ has diameter m. The general linear group $\mathrm{GL}(n, q)$ acts distance-transitively.

A related graph is $2\mathrm{J}_q(2m+1, m)$, whose vertex set consists of m- and $(m+1)$-subspaces of a $(2m+1)$-space over $\mathbb{F}_q$ with incidence given by containment. This graph is distance-transitive of diameter $2m+1$ and is clearly bipartite; indeed $\mathrm{B}2\mathrm{J}_q(2m+1, m) = \mathrm{J}_q(2m+1, m)$.

PROPOSITION 5.16. *The Grassmann graph* $\mathrm{J}_q(n, m)$ *with* $n \geq 2m \geq 4$ *has no* A*-covers.*

Proof. This is proven as Proposition 6.1 of [5]. □

PROPOSITION 5.17. *The only* B*-double of the Grassmann graph* $\mathrm{J}_q(n, m)$ *with* $n \geq 2m \geq 4$ *is* $2\mathrm{J}_q(2m+1, m)$ *for* $n = 2m+1$.

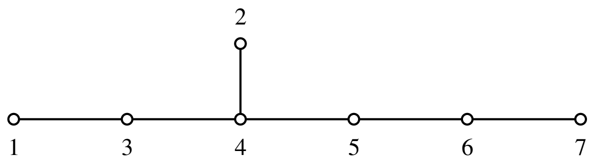
Proof. This is proven as Theorem 8 of [15]. □

5.3.2. E_7 Graph $[\mathrm{E}_{7,7}(q)]$

The E_7 *graph* $[\mathrm{E}_{7,7}(q)]$ defined over $\mathbb{F}_q$ admits $\mathrm{E}_7(q)$ acting distance-transitively. It is primitive of diameter 3 with intersection array

$$\left\{q(q^8+q^4+1)\frac{q^9-1}{q-1}, q^9(q^4+1)\frac{q^5-1}{q-1}, q^{17};\right.$$
$$\left. 1, (q^4+1)\frac{q^5-1}{q-1}, (q^8+q^4+1)\frac{q^9-1}{q-1}\right\}.$$

The graph is the collinearity graph of the parapolar space with lines of size $q+1$ constucted as the shadow space of type 7 for the building of type $\mathrm{E}_7(q)$ where the E_7 diagram is labeled as follows.



See Section 4.2, [7, Sec. 10.7], and [9, Sec. 4.19] for more infomation on parapolar spaces.

PROPOSITION 5.18. *The graph* $[\mathrm{E}_{7,7}(q)]$ *has no* A-*covers.*

Proof. This is proven as Proposition 12.1 of [5]. □

LEMMA 5.19. (a) *The set of lines through a vertex of* $[\mathrm{E}_{7,7}(q)]$ *has the structure of the geometry* $\mathrm{E}_{6,1}(q)$.

(b) *Maximal cliques in* $[\mathrm{E}_{6,1}(q)]$ *are projective spaces over* $\mathbb{F}_q$ *of rank 4 and 5.*

(c) *Maximal cliques in* $[\mathrm{E}_{7,7}(q)]$ *are projective spaces over* $\mathbb{F}_q$ *of rank 5 and 6.*

Proof. Part (a) is clear from the preceding diagram since the parapolar shadow space with collinearity graph $[\mathrm{E}_{6,1}(q)]$ is isomorphic to that for $[\mathrm{E}_{6,6}(q)]$. For (b) see [9, p. 693]. Part (c) then follows immediately from (a) and (b). □

PROPOSITION 5.20. *The graph* $[\mathrm{E}_{7,7}(q)]$ *has no* B-*doubles.*

Proof. By Lemma 5.19 the maximal cliques of $[\mathrm{E}_{7,7}(q)]$ are projective spaces over $\mathbb{F}_q$ of rank 5 or 6. Hence by Lemma 3.5 a B-double H has valency $k = \frac{q^{n+1}-1}{q-1}$ with $n \in \{5, 6\}$. Furthermore by Proposition 3.4 bipartite H has $c_1 = 1$ and $a_1+b_1+c_1 = k$, hence $b_1 = k-1$. Since $[\mathrm{E}_{7,7}(q)]$ has valency $q(q^8+q^4+1)\frac{q^9-1}{q-1}$, Proposition 3.4 with $i = 0$ gives for H

$$c_2 = \frac{k(k-1)(q-1)}{q(q^8+q^4+1)(q^9-1)},$$

which is not an integer. Hence no B-double H exists. □

5.3.3. Affine E_6 *Graph* $[\mathrm{AE}_6(q)]$

If G is the graph $[\mathrm{E}_{7,7}(q)]$ of Section 5.3.2 and ∞ is a vertex of G, then the graph $G_3(\infty)$ is the *affine* E_6 *graph* $[\mathrm{AE}_6(q)]$. It is distance-transitive (via the stabilizer of ∞ in $\mathrm{E}_7(q)$) and primitive of diameter 3 with intersection array

$$\left\{\frac{(q^{12}-1)(q^9-1)}{q^4-1}, q^8(q^4+1)(q^5-1), q^{16}(q-1); 1, q^4(q^4+1), q^8\frac{q^{12}-1}{q^4-1}\right\}.$$

See [7, Sec. 10.8].

PROPOSITION 5.21. *The graph* $[\mathrm{AE}_6(q)]$ *has no* A-*covers.*

Proof. This is proven as Proposition 13.1 of [5]. □

PROPOSITION 5.22. *The graph* $[\mathrm{AE}_6(q)]$ *has no* B-*doubles.*

Proof. By Lemma 5.19 the maximal cliques of the parapolar space $\mathrm{E}_{6,1}(q)$ are projective spaces of rank 4 and 5. The subgraph $[\mathrm{AE}_6(q)](x)$ of the affine E_6 graph $[\mathrm{AE}_6(q)]$ induced on the neighbors of the vertex x is a $(q-1)$-clique extension of the distance-regular graph $[\mathrm{E}_{6,1}(q)]$ (see the remark after Theorem 10.8.1 in [7]). Therefore, the maximal cliques of the graph $[\mathrm{AE}_6(q)]$ have size $(q-1)\frac{q^{n+1}-1}{q-1}+1=q^{n+1}$ with $n=4,5$ (indeed they have the structure of affine spaces).

For the parameters of a B-double Lemma 3.5 gives valency $k=q^{n+1}$ with $n\in\{4,5\}$. Also $a_1=0$ and $c_1=1$, so $a_1+b_1+c_1=k$ leads to $b_1=q^{n+1}-1$. Since $[\mathrm{AE}_6(q)]$ has valency $\frac{(q^{12}-1)(q^9-1)}{q^4-1}$, Proposition 3.4 with $i=0$ gives

$$c_2=\frac{q^{n+1}(q^{n+1}-1)(q^4-1)}{(q^{12}-1)(q^9-1)},$$

which is not an integer. Hence no B-double exists. □

5.4. Dual Polar Graphs

Let V be a vector space of dimension n over the field $\mathbb{F}_q$ equipped with a nondegenerate sesquilinear form or quadratic form. The corresponding *dual polar graph* then has as point set the maximal totally isotropic or totally singular subspaces (as appropriate) with two such adjacent when they intersect in a subspace of codimension 1. If m is the uniform dimension of these subspaces, then the dual polar space has diameter m (and is, in fact, a subgraph of the Grassmann graph $\mathrm{J}_q(n,m)$).

Specifically, we have one of the following:

(i) the symplectic dual polar graph $[\mathrm{Sp}(2m,q)]=[C_m(q)]$ with $n=2m$;
(ii) the orthogonal dual polar graphs $[\Omega(2m+1,q)]=[B_m(q)]$ with $n=2m+1$, $[\Omega^+(2m,q)]=[D_m(q)]$ with $n=2m$, and $[\Omega^-(2m+2,q)]=[{}^2D_{m+1}(q)]$ with $n=2m+2$;
(iii) for $q=l^2$, the unitary dual polar graphs $[U(n,q)]=[{}^2A_{n-1}(l)]$ with $m=\lfloor n/2\rfloor$.

These graphs are primitive except that the hyperbolic orthogonal dual polar graphs $[D_m(q)]$ are bipartite. The halved graph $\mathrm{B}[D_m(q)]=\frac{1}{2}[D_m(q)]$ is distance-transitive of diameter $\lfloor m/2\rfloor$.

PROPOSITION 5.23. *A dual polar graph of diameter at least 3 has no* A-*covers.*

Proof. This is proven as Proposition 7.1 of [5]. □

PROPOSITION 5.24. *The halved graph* $\mathrm{B}[D_m(q)]=\frac{1}{2}[D_m(q)]$ *with* $m\geq 4$ *has no* A-*covers.*

Proof. This is given in [5, p. 151]. □

PROPOSITION 5.25. *A dual polar graph of diameter at least 3 has no* B-*doubles.*

Proof. This is proven as Theorem 11 of [15]. □

PROPOSITION 5.26. *The only* B-*double of the halved graph* $\frac{1}{2}[D_m(q)]$ *with* $m\geq 5$ *is* $[D_m(q)]$.

Proof. This is proven as Theorem 13 in [15] for $m\geq 8$, but the arguments actually are valid for all $m\geq 5$. □

5.5. Sesquilinear Form Graphs

Let V be the vector space $\mathbb{F}^n$, and let $W = \mathbb{F}^m$. Further let $f\colon V \times W \to \mathbb{F}$ be a σ-sesquilinear form.

For a fixed σ, the form f is completely determined by its Gram matrix $\Gamma = (\gamma_{ij})_{ij}$ with $\gamma_{ij} = f(v_i, w_j)$, where $\{v_i \mid 1 \le i \le n\}$ is the canonical basis of V and $\{w_j \mid 1 \le j \le m\}$ that of W. We then have, for all $v \in V$ and $w \in W$, that

$$f(v, w) = v\Gamma w^{\sigma\top}.$$

We are specifically interested in those forms f belonging to one of three sets of forms:

(i) the set of all *bilinear forms* on V and W—that is, those forms with $\sigma = 1$;
(ii) the set of all *alternating forms* f on $V = W$—that is, those forms with $\sigma = 1$ and $f(v, v) = 0$ and $f(v, w) = -f(w, v)$ for all $v, w \in V$;
(iii) the set of all *Hermitean forms* f on $V = W$—that is, those forms with σ of order 2 and $f(v, w) = f(w, v)^\sigma$ for all $v, w \in V$.

The Gram matrix Γ for an alternating form is an *alternating matrix* in that it has 0 diagonal and $\Gamma = -\Gamma^\top$. The Gram matrix for a Hermitean form is a *Hermitean matrix* in that $\Gamma = \Gamma^{\sigma\top}$.

If we have two forms f and g in one of these classes and if $a, b \in \mathbb{F}$, then clearly $af + bg$ is also a form in the same class, so the classes have a natural vector space structure. This corresponds to standard scalar multiplication and matrix addition for Gram matrices. We then can turn the vector space of forms, or equivalently Gram matrices, into a graph by letting two Gram matrices be adjacent precisely when their difference has minimal possible rank. This rank is 1 for bilinear and Hermitean matrices and 2 for alternating matrices.

5.5.1. Bilinear Forms Graphs $\mathrm{H}_q(n, m)$

The *bilinear forms graph* $\mathrm{H}_q(n, m)$ $(n \ge m)$ has as vertex set the $n \times m$ matrices over $\mathbb{F}_q$ with two matrices joined by an edge if and only if their difference has rank 1. The bilinear forms graph is distance-transitive with diameter m.

Proposition 5.27. *The bilinear forms graph $\mathrm{H}_q(n, m)$ with $n \ge m \ge 2$ has no* A*-covers.*

Proof. This is proved as Proposition 8.1 of [5]. □

Proposition 5.28. *The bilinear forms graph $\mathrm{H}_q(n, m)$ with $n \ge m \ge 2$ has no* B*-doubles.*

Proof. This is proved as Theorem 18 of [15]. □

5.5.2. Alternating Forms Graphs $\mathrm{Alt}(n, q)$

The *alternating forms graph* $\mathrm{Alt}(n, q)$ has as vertex set the $n \times n$ alternating matrices over $\mathbb{F}_q$—that is, all $n \times n$ matrices $(a_{ij})_{ij}$ with $a_{ij} = -a_{ji}$ for $1 \le i, j \le n$ and $a_{ii} = 0$ for all i. Alternating matrices always have even rank, and two such matrices are joined by an edge if and only if their difference has rank 2. The alternating forms graph $\mathrm{Alt}(n,q)$ is distance-transitive of diameter $\lfloor n/2 \rfloor$.

PROPOSITION 5.29. *The alternating forms graph* $\mathrm{Alt}(n,q)$ *with* $n \geq 4$ *has no* A-*covers.*

Proof. This is proven as Propositions 9.1 and 9.2 of [5]. □

PROPOSITION 5.30. *The alternating forms graph* $\mathrm{Alt}(n,q)$ *with* $n \geq 4$ *has no* B-*doubles.*

Proof. This is proven as Theorem 20 of [15]. □

5.5.3. Hermitean Forms Graphs $\mathrm{Her}(n,l^2)$

The finite field $\mathbb{F}_q$ has a nontrivial automorphism σ of order 2 if and only if $q = l^2$ is a square prime power, in which case we have the Frobenius automorphism $\sigma : \alpha \mapsto \alpha^l$.

The *Hermitean forms graph* $\mathrm{Her}(n,l^2)$ has as vertex set the $n \times n$ Hermitean matrices over $\mathbb{F}_{l^2}$—that is, all $n \times n$ matrices $(a_{ij})_{ij}$ with $a_{ij} = a_{ji}^l$ for $1 \leq i, j \leq n$. Two such matrices are joined by an edge if and only if their difference has rank 1. The Hermitean forms graph $\mathrm{Her}(n,l^2)$ is distance-transitive of diameter n.

PROPOSITION 5.31. *The Hermitean forms graph* $\mathrm{Her}(n,l^2)$ *with* $n \geq 3$ *has no* A-*covers except when* $(n,l) = (3,2)$. *In the expectional case it has unique 2- and 4-fold* A-*covers, and they are both distance-transitive of diameter 6.*

Proof. This is proven as Propositions 10.1 and 10.2 of [5] except for uniqueness of the A-covers, which is given in [5, Sec. 14] and proven in [7, p. 365]. □

PROPOSITION 5.3.2. *The Hermitean forms graph* $\mathrm{Her}(n,l^2)$ *with* $n \geq 2$ *has no* B-*doubles.*

Proof. This is proven as Theorem 21 of [15]. □

5.6. Generalized Polygons

Generalized polygons were introduced in Section 4.3. The classification of finite generalized polygons seems remote. Nevertheless, in Propositions 5.34 and 5.36 we are able to say something about A-covers of odd diameter and B-doubles of arbitrary finite generalized polygons.

The finite distance-transitive generalized polygons were classified by Buekenhout and Van Maldeghem [8]. Their full result is too long to give here, but we present the parts of specific interest to us. The collinearity graph of a generalized $2d$-gon with $s = 1$ is bipartite.

THEOREM 5.33 [8]. (a) *A generalized* $2d$-*gon of order* (s,t) *with* $s > 1$ *and* $d \geq 3$ *whose collinearity graph is distance-transitive is one of*:

(i) *the generalized 6-gon of order* $(q,1)$ *associated with* $\mathrm{PSL}(3,q)$;
(ii) *a generalized 6-gon of order* (q,q) *associated with* $\mathrm{G}_2(q)$;

(iii) *the generalized 6-gon of order* (q, q^3) *or its dual of order* (q^3, q), *both associated with* ${}^3\mathrm{D}_4(q)$;
(iv) *the generalized 8-gon of order* $(q, 1)$ *associated with* $\mathrm{Sp}_4(q)$ *for* $q = 2^a$;
(v) *the generalized 8-gon of order* (q, q^2) *or its dual of order* (q^2, q), *both associated with* ${}^2\mathrm{F}_4(q)$ *for* $q = 2^{2a+1}$;
(vi) *the generalized 12-gon of order* $(q, 1)$ *associated with* $\mathrm{G}_2(q)$ *for* $q = 3^a$.

(b) *A generalized 4-gon of order* (q, q) *with* $q > 1$ *whose collinearity graph is distance-transitive is one of a dual pair of generalized 4-gons associated with* $\mathrm{Sp}_4(q)$.

PROPOSITION 5.34. *Generalized* $2d$*-gons with diameter* $d \geq 2$ *have no* A*-covers of odd diameter.*

Proof. Since a generalized $2d$-gon is a special type of near $2d$-gon, this is immediate from Lemma 4.1. □

PROPOSITION 5.35. *The distance-transitive finite generalized* $2d$*-gons with diameter* $d \geq 3$ *listed in Theorem 5.33(a) have no* A*-covers of even diameter.*

Proof. This is from [2]. □

PROPOSITION 5.36. *Let* G *be the collinearity graph of a finite generalized* $2d$*-gon with diameter* $d \geq 2$ *and order* (s, t). *Then there is a* B*-double of* G *if and only if* $s = t$. *In that case, the* B*-double is uniquely determined as the incidence graph of the generalized* $2d$*-gon.*

Proof. Let H be a B-double of G, the collinearity graph of a generalized $2d$-gon of order (s, t) and diameter $d \geq 2$. Also let $B_i = b_i(H)$, $b_i = b_i(G)$, and so forth.

The maximal cliques of G are the lines of the generalized $2d$-gons of order (s, t) and have size $s + 1$. Therefore by Lemma 3.5 we have $B_0 = s + 1$ and, since two points are incident to at most one common line, $C_2 = 1$.

As $A_1 = 0$ in bipartite H and $B_1 + C_1 = B_1 + 1 = B_0$, we have $B_1 = s$. Since $b_0 = s(t + 1)$, Proposition 3.4 with $i = 0$ gives

$$1 = C_2 = \frac{B_0 B_1}{b_0} = \frac{s + 1}{t + 1};$$

hence $s = t$.

As in Lemma 3.5, the point set of H has bipartition $G \cup Y$ with $|G| = |Y|$, and each $H_1(y)$, $y \in Y$, is a maximal clique of $G = \mathrm{B}H$, which can only be a line of the generalized polygon. As $s = t$, there are the same number of points and lines. Therefore for every line l there is a unique y in Y with $l = H_1(y)$. That is, H is the incidence graph of generalized $2d$-gon, as claimed. □

COROLLARY 5.37. *Let* G *be the collinearity graph of a finite distance-transitive generalized* $2d$*-gon of order* (s, t) *with* $d \geq 2$ *and* $st > 1$. *Then there is a* B*-double of* G *if and only if* G *is a generalized 4-gon of type* $\mathrm{Sp}_4(l)$ *or* G *is a generalized*

6-gon of type $G_2(q)$*. In all cases, the unique* B*-double is the incidence graph of* G *and so is a generalized 8-gon or 12-gon (respectively) with order* $(1, q)$.

The B*-double is distance-transitive if and only if* l *is a power of 2 and* q *is a power of 3, respectively.*

Proof. By Theorem 5.33 the only distance-transitive finite generalized $2d$-gons with $s = t$ are polygons (i.e., $st = 1$), the $\mathrm{Sp}_4(l)$ 4-gons, and the $G_2(q)$ 6-gons. The previous lemma now gives the first paragraph.

The corresponding bipartite 8-gons and 12-gons are distance-transitive precisely when these 4-gons and 6-gons are isomorphic to their duals. This happens if and only if l is a power of 2 and q is a power of 3, respectively (see e.g. [7, Sec. 6.5]). □

REMARK. The generalized 6-gons of type (q, q) with q not a power of 3 give our only examples of distance-transitive primitive graphs of diameter at least 3 with imprimitive covers that are distance-regular but not distance-transitive.

5.7. The Mathieu Family

The various graphs in the Mathieu family have combinatorial descriptions in terms of the Witt Steiner systems and Golay codes, but they all are distance-transitive because of the action of automorphism groups related to the Mathieu groups.

5.7.1. Coset Graph $[3^6.2.M_{12}]$ *of the Extended Ternary Golay Code*

Let C be the extended ternary Golay code inside $V = \mathbb{F}_3^{12}$. The coset graph $[3^6.2.M_{12}]$ of the extended ternary Golay code has as vertex set the $3^6 = 729$ cosets of C in V, two such adjacent when they contain vectors that differ in exactly one coordinate position. It is distance-transitive with automorphism group $3^6.2.M_{12}$ and intersection array

$$\{24, 22, 20; 1, 2, 12\}.$$

PROPOSITION 5.38. *The coset graph* $[3^6.2.M_{12}]$ *of the extended ternary Golay code has no* A*-covers.*

Proof. This is given in [5, p. 163].

Let H be an r-fold A-cover of the coset graph $[3^6.2.M_{12}]$ of the extended ternary Golay code having diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{24, 22, 20, \frac{r-1}{r}(12), 2, 1; 1, 2, \frac{1}{r}(12), 20, 22, 24\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{24, 22, 20, t(r-1), 12, 2, 1; 1, 2, 12, t, 20, 22, 24\}.$$

Case 1: $D = 6$. By Corollary 3.3(a), r divides $c_d = 12$ and

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{12}{\max(2, 12 - 20)}.$$

Thus we have three possibilities of r—namely 2, 3, and 6.

Therefore we have one of:

(i) $r = 2$, $\mathrm{i}(H) = \{24, 22, 20, 6, 2, 1; 1, 2, 6, 20, 22, 24\}$, $|H| = r|G| = 1458$;
(ii) $r = 3$, $\mathrm{i}(H) = \{24, 22, 20, 8, 2, 1; 1, 2, 4, 20, 22, 24\}$, $|H| = r|G| = 2187$;
(iii) $r = 6$, $\mathrm{i}(H) = \{24, 22, 20, 10, 2, 1; 1, 2, 2, 20, 22, 24$, $|H| = r|G| = 2374$.

In [7, Chap. 14] we find that none of these intersection arrays are feasible, and therefore no such A-covers exist.

Case 2: $D = 7$. This graph is a near 6-gon, so by Lemma 4.1 there are no A-covers of odd diameter. See also [5, p. 163]. □

PROPOSITION 5.39. *The coset graph* $[3^6.2.M_{12}]$ *of the extended ternary Golay code has no* B*-doubles.*

Proof. The graph has $b_0 = 24$ and $b_0 - b_1 = 2$. Therefore

$$24 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 2 \cdot 3 = 6,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.7.2. Coset Graph $[2^{11}.M_{23}]$ *of the Binary Golay Code*

Let C be the binary Golay code inside $V = \mathbb{F}_2^{23}$. The coset graph $[2^{11}.M_{23}]$ of the binary Golay code has as vertex set the $2^{11} = 2048$ cosets of C in V, two such adjacent when they contain vectors that differ in exactly one coordinate position. It is distance-transitive with automorphism group $2^{11}.M_{23}$ and intersection array

$$\{23, 22, 21; 1, 2, 3\}.$$

The graph $[2^{11}.M_{23}]$ has a 2-fold A-cover, namely its bipartite double $2 \times [2^{11}.M_{23}]$, which is distance-transitive under the action of its automorphism group $2 \times 2^{11}.M_{23}$. It has intersection array

$$\{23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23\}.$$

PROPOSITION 5.40. *The coset graph* $[2^{11}.M_{23}]$ *of the binary Golay code has no* A*-covers of even diameter.*

Proof. Let H be an r-fold A-cover of $G = [2^{11}.M_{23}]$ of even diameter D. Then by Theorem 3.2 we have $D = 6$ and

$$\mathrm{i}(H) = \left\{23, 22, 21, \frac{r-1}{r}(3), 2, 1; 1, 2, \frac{1}{r}(3), 21, 22, 23\right\}.$$

By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{3}{\max(2, 3 - 21)}.$$

Thus $r = 1$, and hence no such A-covers exist. □

LEMMA 5.41. *Let S be a nonempty subset of the set of blocks of the Steiner system* $\mathrm{S}(4, 7, 23)$ *with the property that every triple of points is in exactly f of the members of S. Then $f = 5$ and S consists of all blocks of* $\mathrm{S}(4, 7, 23)$.

Proof. Count the pairs $(O, \{a, b, c\})$ with $\{a, b, c\} \subset O \in S$ in two ways. We have

$$|S| \cdot \binom{7}{3} = f \cdot \binom{23}{3}.$$

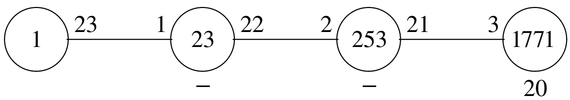
Therefore $|S| = 253 \cdot f/5$. In particular 5 divides f and $|S| \geq 253$. As there are only 253 blocks in $\mathrm{S}(4, 7, 23)$, we conclude that $|S| = 253$ and $f = 5$. □

PROPOSITION 5.42. *The only odd-diameter* A-*cover of the coset graph* $[2^{11}.M_{23}]$ *of the binary Golay code is its bipartite double* $2 \times [2^{11}.M_{23}]$, *which is a distance-transitive* 2-*fold* A-*cover.*

Proof. Let H be an r-fold A-cover of $G = [2^{11}.M_{23}]$ of odd diameter D. Then by Theorem 3.2 we have $D = 7$ and

$$\mathrm{i}(H) = \{23, 22, 21, t(r-1), 3, 2, 1; 1, 2, 3, t, 21, 22, 23\}.$$

We have the following distribution diagram for G.



Since the binary Golay code C is a perfect 3-error-correcting code, each coset of C in V has a unique coset representative of weight at most 3. Viewing V as the set of characteristic vectors for $X = \{1, 2, \ldots, 23\}$, we identify each coset with the unique small subset that represents it: $\emptyset$ for C itself, $\{i \mid i \in X\}$ for the cosets of weight 1, $\{ij \mid i, j \in X\}$ for the cosets of weight 2, and $\{ijk \mid i, j, k \in X\}$ for the cosets of weight 3. (We write ijk for $\{i, j, k\}$ and so forth.)

As indicated in the distribution diagram, most of the adjacencies are clear. The neighborhood $G_1(\emptyset)$ of $\emptyset = \binom{X}{0}$ is $\binom{X}{1} = \{i \mid i \in X\}$. Next, $G_2(\emptyset) = \binom{X}{2}$ and $G_3(\emptyset) = \binom{X}{3}$. The adjacencies in

$$G = \binom{X}{0} \cup \binom{X}{1} \cup \binom{X}{2} \cup \binom{X}{3}$$

are then given by set containment except within $\binom{X}{3}$ where ijk and mno are adjacent if and only if they are disjoint and there is a block of the Steiner system $\mathrm{S}(4, 7, 23)$ containing them both.

Let R be a set $\{p, q, \ldots\}$ of size r. The distribution diagram for the r-fold A-cover H of diameter 7 can be thought of as a disjoint union of r copies

$$G^{[u]} = \binom{X}{0}^{[u]} \cup \binom{X}{1}^{[u]} \cup \binom{X}{2}^{[u]} \cup \binom{X}{3}^{[u]}$$

of the preceding diagram for $u \in R$. (For J a subset of G and S a subset of R, we let $J^{[S]}$ be $\{h^{[u]} \mid h \in J, u \in S\}$.)

Here the r vertices $\emptyset^{[u]}$ form a single antipodal class in H; and, except for within

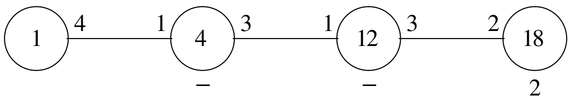
$$L = \binom{X}{3}^{[R]} = \bigcup_{u \in R} \binom{X}{3}^{[u]},$$

the adjacencies are precisely those inherited from G. That is, the vertices $v^{[p]}$ and $w^{[q]}$, not both in L, are adjacent if and only if $p = q$ and v and w are adjacent in G.

The subgraph L is an r-fold cover of the subgraph $\binom{X}{3}$ of the G described previously. Each vertex of $\binom{X}{3}^{[p]}$ is adjacent to exactly t vertices of $\binom{X}{3}^{[q]}$ for $q \neq p$. Thus the subgraph $H_3(\emptyset^{[p]}) = \binom{X}{3}^{[p]}$ has degree $a_3(G) - (r-1)t = 20 - (r-1)t$. We further have $H_4(\emptyset^{[p]}) = \binom{X}{3}^{[R \setminus \{p\}]}$.

The subgraphs $J = \binom{X}{3}$ and L are not distance-regular. In particular, $c_2(G) = 2$ but there are distance-2 pairs x, y in J with $c_2^{x,y}(J) = 1$ and others with $c_2^{x,y}(J) = 2$. Nevertheless $\binom{X}{3}$ and $L = J^{[R]}$ have nice structure inherited from G. Indeed it is precisely the pairs of vertices with $c_2^{x,y}(J) = 1$ that are at the heart of our proof.

Distinct vertices ijk and ijl of J are both adjacent to ij and so are at distance 2 in G. The unique vertex of J adjacent to both is mno where $O = ijklmno$ is the unique block on $ijkl$. Indeed, the subgraph $\binom{O}{3}$ of J induced by O is the graph of all 3-subsets of a 7-set with two adjacent when disjoint; that is, $\binom{O}{3}$ is a copy of the odd graph O_4. This has the following distribution diagram.



The proof of Proposition 5.42 now proceeds in five steps.

STEP 1. *Let $ijk^{[p]}$ and $mno^{[q]}$ be adjacent in L. Then ijk and mno are disjoint and there is a unique block O with $ijkmno \subset O$. In this case, $\{ijk, mno\}^{[R]}$ is a disjoint union of r edges. The subgraph $\binom{O}{3}^{[R]}$ has valency 4 and is an r-fold cover of* O_4.

Proof. This is clear. □

STEP 2. *Let $ijk^{[p]}$ and $mno^{[q]}$ be adjacent in L as in Step 1. Then one of the following statements holds*:

(i) *$p = q$ and the connected component of this edge in $\binom{O}{3}^{[R]}$ is $\binom{O}{3}^{[p]}$ and isomorphic to* O_4;

(ii) *$p \neq q$ and the connected component of this edge in $\binom{O}{3}^{[R]}$ is $\binom{O}{3}^{[p,q]}$ and isomorphic to the bipartite double* $2O_4$.

Proof. Let $O = ijklmno$. Since O_4 is connected of valency 4, by Step 1 it is enough to show that each of $ijk^{[p]}$, $ijl^{[p]}$, $ikl^{[p]}$, and $jkl^{[p]}$ is adjacent to $mno^{[q]}$. This is true by hypothesis for $ijk^{[p]}$, and by symmetry we need only consider $ijl^{[p]}$.

The vertex $ij^{[p]}$ is the unique vertex of $\binom{X}{2}^{[p]} = H_2(\emptyset^{[p]})$ adjacent to both $ijk^{[p]}$ and $ijl^{[p]}$. Hence there must be another vertex of $H_3(\emptyset^{[p]}) \cup H_4(\emptyset^{[p]}) = L$ adjacent to both. As L is a cover of $\binom{X}{3}$, this vertex can only be $mno^{[u]}$ for some $u \in R$. But $ijk^{[p]}$ is already adjacent to $mno^{[q]}$, so by Step 1 we must have $q = u$. Therefore $ijl^{[p]}$ is adjacent to $mno^{[q]}$, as desired. □

STEP 3. *The case (i) of Step 2 does not occur.*

Proof. Suppose otherwise that for some block O, the subgraph $\binom{O}{3}^{[p]}$ is a copy of O_4. Call a block F *flat* if $\binom{F}{3}^{[p]}$ is isomorphic to O_4. Each edge comes from a unique block, and O_4 has valency 4; so an arbitrary vertex $ijk^{[p]}$ has valency in $\binom{X}{3}^{[p]}$ equal to $4f(ijk)$ where $f(ijk)$ is the number of flat blocks on the triple ijk. Since $\binom{X}{3}^{[p]}$ has valency $20 - (r-1)t$, the numbers $f(ijk)$ are equal to the constant f determined by $4f = 20 - (r-1)t$. That is, the set of flat blocks is nonempty and has the property that every triple of $\binom{X}{3}$ is in exactly $f = (20 - (r-1)t)/4$ flat blocks. By Lemma 5.41 every block is flat and $f = 5$. But then $5 = (20 - (r-1)t)/4$ and $0 = (r-1)t$, which is not the case. The contradiction proves that there are no flat blocks, as claimed. □

STEP 4. *We have $r = 2$ and $t = 20$.*

Proof. By Step 3, we have $0 = 20 - (r-1)t$. Also since we are always in case (ii) of Step 2, the parameter r must be even. Therefore the only possibilities are $(r, t) = (2, 20)$ and $(r, t) = (6, 4)$.

Suppose that $(r, t) = (6, 4)$. Since $t = 4$ is the valency of $2O_4$, for each triple ijk there is a unique block O on ijk with $\binom{O}{3}^{[p,q]}$ isomorphic to $2O_4$. Call such a block *special*. Then the set of special blocks is a subset of all blocks with every triple ijk in exactly $f = 1$ special block. This contradicts Lemma 5.41, so (r, t) is not $(6, 4)$. □

STEP 5. *H is the bipartite double of G.*

Proof. We now have

$$\mathrm{i}(H) = \{23, 22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22, 23\}.$$

In particular, by Theorem 3.2 the graph H is bipartite as well as antipodal of diameter $D = 7$. Therefore H is isomorphic to the bipartite double of G by Theorem 3.8(a). □

This completes the proof of the five steps and thus of the proposition. □

PROPOSITION 5.43. *The coset graph $[2^{11}.M_{23}]$ of the binary Golay code has no* B*-doubles.*

Proof. The coset graph $[2^{11}.M_{23}]$ of the binary Golay code has $b_0 = 23$ and $b_0 - b_1 = 1$. Therefore

$$23 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.7.3. Distance-2 Graph $2^{11}.M_{23}]_2$ *of the Coset Graph of the Binary Golay Code*
The distance-2 graph $[2^{11}.M_{23}]_2$ of the coset graph of the binary Golay code is $\mathrm{B}(2 \times [2^{11}.M_{23}])$ and so remains distance-transitive under the action of $2^{11}.M_{23}$. Its intersection array is

$$\{253, 210, 3; 1, 30, 231\}.$$

PROPOSITION 5.44. *The distance-2 graph* $[2^{11}.M_{23}]_2$ *of the coset graph of the binary Golay code has no* A*-covers.*

Proof. Let H be an r-fold A-cover of $[2^{11}.M_{23}]_2$ of diameter $D = 6$ or 7.

Case 1: $D = 6$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{231}{\max(30, 231 - 3)}.$$

Thus $r = 1$, and no such A-covers exist.

Case 2: $D = 7$. Since $231 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(3, 22)$, no such A-cover exists by Corollary 3.3(b). □

LEMMA 5.45. *If* M *is a clique with* $|M| > 4$ *in the distance-2 graph* $G = [2^{11}.M_{23}]_2$ *of the coset graph* $G^+ = [2^{11}.M_{23}]$ *of the binary Golay code, then there is a vertex* x *with* M *in the neighborhood* $G_1^+(x)$ *of size 23.*

Proof. Let $V = \mathbb{F}_2^{23}$, and view the vectors of V as the characteristic vectors of subsets of $X = \{1, 2, \ldots, 23\}$. Let $C \leq V$ be the binary Golay code, and set $\bar{V} = V/C$, the vertex set of G and G^+.

Assume (by transitivity) that $\bar{\emptyset} \in M$. We have $G_1^+(\bar{\emptyset}) = \{\bar{1}, \ldots, \overline{23}\}$ and $G_2^+(\bar{\emptyset}) = \{\overline{ij} \mid i \neq j \in X\}$ (writing $\bar{i}$ for $\overline{\{i\}}$ and $\overline{ij}$ for $\overline{\{i, j\}}$). Set $M_0 = M \cap G_2^+(\bar{\emptyset}) = M \setminus \{\bar{\emptyset}\}$.

For distinct $\overline{ij}, \overline{kl} \in M_0$, we must have $\overline{kl} \in G_2^+(\overline{ij})$; hence $\overline{ij} + \overline{kl} \in G_2^+(\bar{\emptyset})$. (Translation by elements of V induces an automorphism group of G, where the kernel is C.) Hence for some $a, b \in X$ we have $\overline{ij} + \overline{kl} = \overline{ab}$, and $\{i, j\} + \{k, l\} + \{a, b\} \in C$. Since C has minimum distance 7, we must have $\{i, j\} + \{k, l\} + \{a, b\} = \mathbf{0}$ in V. That is, $\{i, j, k, l, a, b\} = \{r, s, t\}$ with each appearing twice. In particular $\{i, j\} \cap \{k, l\}$ is nonempty, and this is true for any distinct pair of elements from M_0.

As $|M| > 4$, we have $|M_0| > 3$. Therefore there is an $i \in X$ with $i \in \{k, l\}$ for all $\overline{kl} \in M_0$. If we set $x = \bar{i}$, then $M \subseteq G_1^+(x)$ as claimed. □

PROPOSITION 5.46. *A* B*-double of the distance-2 graph* $[2^{11}.M_{23}]_2$ *of the coset graph of the binary Golay code is isomorphic to the bipartite double* $2 \times [2^{11}.M_{23}]$ *of the coset graph of the binary Golay code.*

Proof. Let H be a B-double of $G = [2^{11}.M_{23}]_2$ with $G = \mathrm{B}H$ one of the parts of H. By Lemmas 3.5 and 3.6, for each $h^- \in H \setminus G$, the neighborhood $H_1(h^-)$ is a maximal clique M of G having size m with $253 = m(m-1)/c_2(H)$. In particular $m > 4$, so by Lemma 5.45 there is a vertex h^+ of G^+ (as in the lemma) with $M = G_1^+(h^+)$ (as a set). Since there are exactly $|G|$ such h^+, there are exactly $|G|$ such cliques. Therefore each occurs as $H_1(h^-)$ exactly once, and H is revealed as the bipartite double of the graph G^+. □

5.7.4. Coset Graph $[2^{10}.M_{22}.2]$ *of the Truncated Binary Golay Code*

Let C be the truncated (punctured) binary Golay code inside $V = \mathbb{F}_2^{22}$. The coset graph $[2^{10}.M_{22}.2]$ of the truncated binary Golay code has as vertex set the $2^{10} = 1024$ cosets of C in V, two such adjacent when they contain vectors that differ in exactly one coordinate position. It is distance-transitive with automorphism group $2^{10}.M_{22}.2$ and intersection array

$$\{22, 21, 20; 1, 2, 6\}.$$

The graph has A-doubles. Its bipartite double $2 \times [2^{10}.M_{22}.2]$ is a 2-fold A-cover of diameter 7 with intersection array

$$\{22, 21, 20, 16, 6, 2, 1; 1, 2, 6, 16, 20, 21, 22\}$$

and automorphism group $2 \times 2^{10}.M_{22}.2$ acting distance-transitively.

Since the shortened binary Golay code has codimension 1 in the truncated binary Golay code, its coset graph $[2^{11}.M_{22}.2]$ is a 2-fold A-cover of $[2^{10}.M_{22}.2]$ having diameter 6. It is distance-transitive with automorphism group $2^{11}.M_{22}.2$ and intersection array

$$\{22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22\}.$$

PROPOSITION 5.47. *The only* A*-covers of the coset graph* $[2^{10}.M_{22}.2]$ *of the truncated binary Golay code are its bipartite double* $2 \times [2^{10}.M_{22}.2]$ *and the coset graph* $[2^{11}.M_{22}.2]$ *of the shortened binary Golay code.*

Proof. Let H be an r-fold A-cover of $G = [2^{10}.M_{22}.2]$ of diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{22, 21, 20, \frac{r-1}{r}(6), 2, 1; 1, 2, \frac{1}{r}(6), 20, 21, 22\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{22, 21, 20, t(r-1), 6, 2, 1; 1, 2, 6, t, 20, 21, 22\}.$$

Case 1: $D = 6$. By Corollary 3.3(a), r divides $c_d = 6$ and

$$r \le \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{6}{\max(2, 6-20)}.$$

Thus we have two possibilities for r, namely 2 and 3.

For $r = 2$,

$$\mathrm{i}(H) = \{22, 21, 20, 3, 2, 1; 1, 2, 3, 20, 21, 22\}$$

and $|H| = 2|G| = 2048$. This is the intersection array for the coset graph of the shortened binary Golay code. In [7, p. 365] it is shown that the coset graph is uniquely determined by its parameters.

For $r = 3$,

$$\mathrm{i}(H) = \{22, 21, 20, 4, 2, 1; 1, 2, 2, 20, 21, 22\}$$

and $|H| = 3|G| = 3072$. In [7, Chap. 14] we find that this intersection array is not feasible, and therefore no such A-cover exists.

Case 2: $D = 7$. By Corollary 3.3(b)

$$t(r-1) \leq \min(b_{d-1}, b_0 - c_d) = \min(20, 22-6) = 16$$

and $c_d = 6 \leq t$. Thus either $r = 2$ and $6 \leq t \leq 16$ or $r = 3$ and $6 \leq t \leq 8$.

Since r is 2 or 3, we have $|H| = r|G|$ equal to 2048 or 3072; so we can check feasibility for each of these fourteen cases in [7, Chap. 14] and discover that only the case $(r, t) = (2, 16)$ survives.

Rather than invoke [7, Chap. 14] thirteen times, we can argue as in Proposition 5.42 to get down to the case $(r, t) = (2, 16)$. We sketch the argument.

Let X be the set $\{1, 2, \ldots, 22\}$, and view $V = \mathbb{F}_2^{22}$ as the space of characteristic vectors of subsets of X. The graph $G = [2^{10}.M_{22}.2]$ is

$$\binom{X}{0} \cup \binom{X}{1} \cup \binom{X}{2} \cup \frac{1}{2}\binom{X}{3}.$$

Here $\binom{X}{0}$ is the empty set $\emptyset$ (and represents the coset that is the truncated Golay code itself), and $G_1(\emptyset) = \binom{X}{1}$ and $G_2(\emptyset) = \binom{X}{2}$ are the 1- and 2-subsets of X. Finally $G_3(\emptyset) = \frac{1}{2}\binom{X}{3}$ is a partition of $\binom{X}{3}$ into $770 = \frac{1}{2}\binom{22}{3}$ pairs of 3-subsets with $\{\{a, b, c\}, \{d, e, f\}\} = abc|def$ in $\frac{1}{2}\binom{X}{3}$ precisely when $\{a, b, c, d, e, f\} = abcdef$ is a block of $S(3, 6, 22)$—that is, a codeword of weight 6 in the truncated Golay code C. Except within $\frac{1}{2}\binom{X}{3}$, adjacency in G is given by containment, where ab and de of $\binom{X}{2}$ are adjacent to $abc|def$ of $\frac{1}{2}\binom{X}{3}$, but cd is not.

The vertices $abc|def$ and $ijk|lmn$ are adjacent if and only if the coset representatives abc and ijk have $abc + ijk$ in a coset of weight 1. That is, there is an s with $abc + ijk + s$ in C. This happens exactly when $abcijks$ is a codeword of weight 7 in C, in which case $defijks$, $abclmns$, and $deflmns$ are also codewords of weight 7. There are 352 codewords O of weight 7, and (as in Proposition 5.42) the subgraph induced by $\binom{O}{3}$ within $\frac{1}{2}\binom{X}{3}$ is a copy of the odd graph O_4. In particular, for the codeword $O = abcijks$, the vertices $abc|\cdots$ and $abs|\cdots$ have distance 2 in G, the vertices between them being $ijk|\cdots$ of the O_4 and ab of $\binom{X}{2} = G_2(\emptyset)$.

As in Proposition 5.42, we let R be a set of size r and we write $H = G^{[R]} = \bigcup_{u \in R} G^{[u]}$ with $G^{[u]} = \binom{X}{0}^{[u]} \cup \binom{X}{1}^{[u]} \cup \binom{X}{2}^{[u]} \cup \frac{1}{2}\binom{X}{3}^{[u]}$. The r vertices $\emptyset^{[u]}$ form a single antipodal class in H; and, except for within $L = \frac{1}{2}\binom{X}{3}^{[R]}$, adjacencies are only those induced by G within each separate $G^{[u]}$.

The subgraph L is an r-fold cover of $\frac{1}{2}\binom{X}{3}$. Again by keeping track of pairs of vertices at distance 2 in L, we conclude that the connected components of $\binom{O}{3}^{[R]}$ are either $\binom{O}{3}^{[p]}$ and isomorphic to O_4, or $\binom{O}{3}^{[p,q]}$ and isomorphic to $2O_4$.

For $O = abcijks$ a codeword of weight 7, suppose that $\binom{O}{3}^{[p]}$ is isomorphic to O_4. (This is the "flat" case.) Then the vertices $abc|\cdots^{[p]}$ and $ijk|\cdots^{[p]}$ form an edge in this subgraph. If the full names of these two vertices are $abc|def$ and $ijk|lmn$, then they also are the vertices $abc|\cdots^{[p]}$ and $lmn|\cdots^{[p]}$ of an edge in the subgraph $\binom{P}{3}^{[p]}$ for $P = abclmns$, a codeword of weight 7 that therefore must also be flat. That is, if $abcijks$ is a flat codeword, then any codeword $abclmns$ matching it in exactly four coordinate positions is also flat. Since the relation of matching in exactly four positions turns the codewords of weight 7 into a connected graph, we see that all codewords of weight 7 must be flat. But then $\frac{1}{2}\binom{X}{3}^{[p]}$ has valency 16 and $(r-1)t = 0$, a contradiction.

Therefore no codewords of weight 7 are flat and $(r-1)t = 16$. Additionally all connected components of $\binom{O}{3}^{[R]}$ are isomorphic to $2O_4$; hence r must be even. So again $(r,t) = (2,16)$ is the only case that must be considered further.

By whatever means, we arrive at the case $r = 2$ and $t = 16$, giving the intersection array

$$\mathrm{i}(H) = \{22, 21, 20, 16, 6, 2, 1; 1, 2, 6, 16, 20, 21, 22\}$$

with $|H| = 2|G| = 2048$. By Proposition 3.4, we see that such a graph H is bipartite as well as antipodal. Since $D = 7$, H is uniquely determined as the bipartite double of G by Theorem 3.8(a). □

Proposition 5.48. *The coset graph* $[2^{10}.M_{22}.2]$ *of the truncated binary Golay code has no* B-*doubles.*

Proof. The coset graph of the truncated binary Golay code has $b_0 = 22$ and $b_0 - b_1 = 1$. Therefore

$$22 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.7.5. Distance-2 Graph $[2^{10}.M_{22}.2]_2$ *of the Truncated Golay Graph*

The distance-2 graph $[2^{10}.M_{22}.2]_2$ of the coset graph of the truncated binary Golay code is $\mathrm{B}(2 \times [2^{10}.M_{22}.2])$ and so remains distance-transitive under the action of $2^{10}.M_{22}.2$. Its intersection array is

$$\{231, 160, 6; 1, 48, 210\}.$$

Proposition 5.49. *The distance-2 graph* $[2^{10}.M_{22}.2]_2$ *of the coset graph of the truncated binary Golay code has no* A-*covers.*

Proof. Let H be an r-fold A-cover of $G = [2^{10}.M_{22}.2]_2$ of diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{231, 160, 6, \frac{r-1}{r}(210), 48, 1; 1, 48, \frac{1}{r}(210), 6, 160, 231\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{231, 160, 6, t(r-1), 210, 48, 1; 1, 48, 210, t, 6, 160, 231\}.$$

Case 1: $D = 6$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{210}{\max(48, 210-6)}.$$

Hence $r = 1$, and no such A-cover exists.

Case 2: $D = 7$. Since $210 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(6, 21)$, no such A-cover exists by Corollary 3.3(b). □

LEMMA 5.50. *If M is a clique with $|M| > 16$ in the distance-2 graph $G = [2^{10}.M_{22}.2]_2$ of the coset graph $G^+ = [2^{10}.M_{22}.2]$ of the truncated binary Golay graph, then there is a vertex x with M in the neighborhood $G_1^+(x)$ of size 22.*

Proof. The proof is similar to that of Lemma 5.45, but it is more complicated since the truncated binary Golay code contains codewords of weight 6.

As before let $V = \mathbb{F}_2^{22}$, and view the vectors of V as the characteristic vectors of subsets of $X = \{1, 2, \ldots, 22\}$. Let $C \leq V$ be the truncated binary Golay code, and set $\bar{V} = V/C$.

Assume (by transitivity) that $\bar{\emptyset} \in M$. We have $G_1^+(\bar{\emptyset}) = \{\bar{1}, \ldots, \overline{22}\}$ and $G_2^+(\bar{\emptyset}) = \{\overline{ij} \mid i \neq j \in X\}$. Set $M_0 = M \cap G_2^+(\bar{\emptyset}) = M \setminus \{\bar{\emptyset}\}$.

For distinct $\overline{ij}, \overline{kl} \in M_0$, we must have $\overline{kl} \in G_2^+(\overline{ij})$; hence $\overline{ij} + \overline{kl} \in G_2^+(\bar{\emptyset})$. Thus for some $a, b \in X$ we have $\overline{ij} + \overline{kl} = \overline{ab}$, and $\{i, j\} + \{k, l\} + \{a, b\} \in C$. There are two possibilities: either $\{i, j\} + \{k, l\} + \{a, b\} = \mathbf{0}$ in V, in which case $\{i, j\}$ and $\{k, l\}$ meet nontrivially; or $\{i, j, k, l, a, b\}$ is a codeword of weight 6 in the truncated Golay code C and is uniquely determined by any of its 3-subsets.

If always $\{i, j\} \cap \{k, l\} = \emptyset$, then $|M_0| \leq 22/2 = 11$, which is not the case. So we can assume that $\overline{ij}$ and $\overline{ik}$ are in M_0. Let $\{i, j, k, a, b, c\}$ be the unique word of C containing $\{i, j, k\}$.

We claim that if $\overline{gh} \in M_0$ then either $g, h \in \{i, j, k, a, b, c\}$ or $i \in \{g, h\}$. In proving this, suppose $i \notin \{g, h\}$. Then the 3- or 4-sets $\{i, j, g, h\}$ and $\{i, k, g, h\}$ are contained in unique codewords of weight 6. Indeed, since i, g, and h are distinct, they are in the same codeword. But this then contains i, j, and k and so must be $\{i, j, k, a, b, c\}$. Therefore $g, h \in \{i, j, k, a, b, c\}$, completing the claim.

If for all $\overline{gh} \in M_0$ we have $g, h \in \{i, j, k, a, b, c\}$, then $|M_0| \leq 15$, which is not the case. Thus, by the claim, there is $\overline{im} \in M_0$ with $m \notin \{i, j, k, a, b, c\}$. Let $\{i, j, m, q, r, s\}$ be the word of weight 6 in the code that contains $\{i, j, m\}$. By the claim, for any $\overline{gh} \in M_0$ we must have either $g, h \in \{i, j, k, a, b, c\} \cap \{i, j, m, q, r, s\} = \{i, j\}$ or $i \in \{g, h\}$. Thus, for all $\overline{gh} \in M_0$, we have $i \in \{g, h\}$. With $x = \bar{i}$, we have M_0 and hence M in the neighborhood $G_1^+(x)$, as desired. □

PROPOSITION 5.51. *A* B-*double of the distance-2 graph* $[2^{10}.M_{22}.2]_2$ *of the coset graph of the truncated binary Golay code is isomorphic to the bipartite double* $2 \times [2^{10}.M_{22}.2]$ *of the coset graph of the truncated binary Golay code.*

Proof. Let H be a B-double of $G = [2^{10}.M_{22}.2]_2$ with $G = \mathrm{B}H$ one of the parts of H. By Proposition 3.4 and Lemma 3.5, for each $h^- \in H \setminus G$, the neighborhood

$H_1(h^-)$ is a maximal clique M of G having size m with $231 = m(m-1)/c_2(H)$. Therefore $m \geq 16$. Indeed, since $16(16-1)/231$ is not integral, we have $m > 16$.

By Lemma 5.50 there is a vertex h^+ of G^+ (as in the lemma) with $M = G_1^+(h^+)$ (as a set). Since there are exactly $|G|$ such h^+, there are exactly $|G|$ such cliques. Therefore each occurs as $H_1(h^-)$ exactly once, and H is revealed as the bipartite double of the graph G^+. □

5.7.6. Witt Graph $[M_{24}]$

The *large Witt graph* has as vertex set the 759 blocks of the Steiner system $S(5, 8, 24)$ with two such adjacent when they are disjoint. It has intersection array

$$\{30, 28, 24; 1, 3, 15\}$$

and admits M_{24} acting primitively and distance-transitively.

PROPOSITION 5.52. *The large Witt graph* $[M_{24}]$ *has no* A-*covers.*

Proof. This is proven as Proposition 14.1 of [5]. □

PROPOSITION 5.53. *The large Witt graph* $[M_{24}]$ *has no* B-*doubles.*

Proof. The large Witt graph has $b_0 = 30$ and $b_0 - b_1 = 2$. Therefore

$$30 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 2 \cdot 3 = 6,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.7.7. Truncated Witt Graph $[M_{23}]$

The *truncated large Witt graph* $[M_{23}]$ is the subgraph of the large Witt graph induced by the 506 blocks of $S(5, 8, 24)$ that miss a fixed symbol. It is itself distance-transitive with automorphism group M_{23} and intersection array

$$\{15, 14, 12; 1, 1, 9\}.$$

PROPOSITION 5.54. *The truncated large Witt graph* $[M_{23}]$ *has no* A-*covers.*

Proof. See [5, p. 162] and [20, Cor. 4.1].

Let H be an r-fold A-cover of $[M_{23}]$ of diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{15, 14, 12, \frac{r-1}{r}(9), 1, 1; 1, 1, \frac{1}{r}(9), 12, 14, 15\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{15, 14, 12, t(r-1), 9, 1, 1; 1, 1, 9, t, 12, 14, 15\}.$$

Case 1: $D = 6$. By Corollary 3.3(a), r divides $c_d = 9$. Thus we have two possibilities for r, namely 3 and 9. For $r = 3$,

$$\mathrm{i}(H) = \{15, 14, 12, 6, 1, 1; 1, 1, 3, 12, 14, 15\};$$

for $r = 9$,

$$\mathrm{i}(H) = \{15, 14, 12, 8, 1, 1; 1, 1, 1, 1, 12, 14, 15\}.$$

Ivanov and Shpectorov [20, Cor. 4.1] showed that neither A-cover exists.

Case 2: $D = 7$. Since $9 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(12, 15 - 9)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.55. *The truncated large Witt graph* $[M_{23}]$ *has no* B*-doubles.*

Proof. The truncated large Witt graph has $b_0 = 15$ and $b_0 - b_1 = 1$. Therefore

$$15 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.7.8. Doubly Truncated Witt Graph $[M_{22}.2]$

The *doubly truncated large Witt graph* $[M_{22}.2]$ is the subgraph of the large Witt graph induced by the 330 blocks of S(5, 8, 24) that miss two fixed symbols. It is itself distance-transitive with automorphism group $M_{22}.2$ and intersection array

$$\{7, 6, 4, 4; 1, 1, 1, 6\}.$$

The graph has an A-cover. The Faradjev–Ivanov–Ivanov [11] 3-fold A-cover $[3.M_{22}.2]$ has diameter 8 and 990 vertices. It is distance-transitive with automorphism group $3.M_{22}.2$ and intersection array

$$\{7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7\}.$$

PROPOSITION 5.56. *The only* A*-cover of the doubly truncated large Witt graph* $[M_{22}.2]$ *is the Faradjev–Ivanov–Ivanov graph* $[3.M_{22}.2]$.

Proof. See [5, p. 163], [6], and [11].

Let H be an r-fold A-cover of $[M_{22}.2]$ of diameter D. Then by Theorem 3.2 either $D = 8$ and

$$\mathrm{i}(H) = \left\{7, 6, 4, 4, \frac{r-1}{r}(6), 1, 1, 1; 1, 1, 1, \frac{1}{r}(6), 4, 4, 6, 7\right\}$$

or $D = 9$ and

$$\mathrm{i}(H) = \{7, 6, 4, 4, t(r-1), 6, 1, 1, 1; 1, 1, 1, 6, t, 4, 4, 6, 7\}.$$

Case 1: $D = 8$. By Corollary 3.3(a), r divides $c_d = 6$ and

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{6}{\max(1, 6-4)}.$$

Thus we have two possibilities for r, namely 2 and 3.

For $r = 2$, $\mathrm{i}(H) = \{7, 6, 4, 4, 3, 1, 1, 1; 1, 1, 1, 3, 4, 4, 6, 7\}$. Brouwer [6] showed that no such 2-fold A-cover exists.

For $r = 3$, $\mathrm{i}(H) = \{7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7\}$. Faradjev, Ivanov, and Ivanov [11] constructed such a distance-transitive 3-fold A-cover admitting $3.M_{22}.2$, and Brouwer [6] showed that this graph is uniquely determined by its parameters.

Case 2: $D = 9$. Since $6 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(4, 7-6)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.57. *The doubly truncated large Witt graph has no* B*-doubles.*

Proof. The doubly truncated large Witt graph has $b_0 = 7$ and $b_0 - b_1 = 1$. Therefore

$$7 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8. Other Sporadic Examples

5.8.1. Coxeter Graph [PSL(3, 2).2]

The *Coxeter graph* [PSL(3, 2).2] has as vertices the conjugacy class of 28 elements with order 2 in [PSL(3, 2).2] corresponding to the transpose-inverse automorphism of PSL(3, 2) = GL(3, 2). Two such are adjacent when they commute. Its intersection array is

$$\{3, 2, 2, 1; 1, 1, 1, 2\}.$$

PROPOSITION 5.58. *The Coxeter graph* [PSL(3, 2).2] *has no* A*-covers.*

Proof. Let H be an r-fold A-cover of $G = [\mathrm{PSL}(3,2).2]$ of diameter D. Then by Theorem 3.2 either $D = 8$ and

$$\mathrm{i}(H) = \left\{3, 2, 2, 1, \frac{r-1}{r}(2), 1, 1, 1; 1, 1, 1, \frac{1}{r}(2), 1, 2, 2, 3\right\}$$

or $D = 9$ and

$$\mathrm{i}(H) = \{3, 2, 2, 1, t(r-1), 2, 1, 1, 1; 1, 1, 1, 2, t, 1, 2, 2, 3\}.$$

Case 1: $D = 8$. By Corollary 3.3(a), r divides $c_d = 2$. Thus 2 is the only possible value of r and

$$\mathrm{i}(H) = \{3, 2, 2, 1, 1, 1, 1, 1; 1, 1, 1, 1, 1, 2, 2, 3\}$$

with $|H| = r|G| = 56$. In [7, Chap. 14] we find that this intersection array is not feasible, and therefore no such A-cover exists.

Case 2: $D = 9$. Since $2 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(1, 1)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.59. *The Coxeter graph* [PSL(3, 2).2] *has no* B*-doubles.*

Proof. The Coxeter graph has $b_0 = 3$ and $b_0 - b_1 = 1$. Therefore

$$3 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.2. Sylvester Graph [PΓL(2, 9)]

The *Sylvester graph* [PΓL(2, 9)] has as vertices the conjugacy class of 36 elements with order 2 in PΓL(2, 9) corresponding to nontrivial field automorphisms. Two such are adjacent when they commute. The graph has intersection array

$$\{5, 4, 2; 1, 1, 4\}$$

and is distance-transitive with automorphism group PΓL(2, 9).

PROPOSITION 5.60. *The Sylvester graph* [PΓL(2, 9)] *has no* A-*covers.*

Proof. Let H be an A-cover of $G = [\mathrm{P\Gamma L}(2,9)]$ of diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{5, 4, 2, \frac{r-1}{r}(4), 1, 1; 1, 1, \frac{1}{r}(4), 2, 4, 5\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{5, 4, 2, t(r-1), 4, 1, 1; 1, 1, 4, t, 2, 4, 5\}.$$

Case 1: $D = 6$. By Corollary 3.3(a), r divides $c_d = 4$ and

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{4}{\max(1, 4-2)}.$$

Thus 2 is the only possible value of r.

Hence $\mathrm{i}(H) = \{5, 4, 2, 2, 1, 1; 1, 1, 2, 2, 4, 5\}$ with $|H| = r|G| = 72$. In [7, Chap. 14] we find that this intersection array is not feasible, and therefore no such A-cover exists.

Case 2: $D = 7$. Since $4 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(2, 1)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.61. *The Sylvester graph* [PΓL(2, 9)] *has no* B-*doubles.*

Proof. The Sylvester graph has $b_0 = 5$ and $b_0 - b_1 = 1$. Therefore

$$5 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.3. Doro Graph [PΓL(2, 16)]

The *Doro graph* [PΓL(2, 16)] has as vertices the conjugacy class of 68 elements with order 2 in PΓL(2, 16) corresponding to nontrivial field automorphisms. Two such are adjacent when they have product of order 3. The graph has intersection array

$$\{12, 10, 3; 1, 3, 8\}$$

and is distance-transitive with automorphism group PΓL(2, 16).

PROPOSITION 5.62. *The Doro graph* [PΓL(2, 16)] *has no* A-*covers.*

Proof. Let H be an r-fold A-cover of $[\mathrm{P\Gamma L}(2,16)]$ of diameter D. Then by Theorem 3.2 either $D = 6$ or $D = 7$.

Case 1: $D = 6$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{8}{\max(3, 8-3)}.$$

Thus $r = 1$, and no such A-cover exists.

Case 2: $D = 7$. Since $8 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(3, 12-8)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.63. *The Doro graph* $[\mathrm{P\Gamma L}(2,16)]$ *has no* B*-doubles.*

Proof. The Doro graph has $b_0 = 12$ and $b_0 - b_1 = 2$. Therefore

$$12 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 2 \cdot 3 = 6,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.4. Biggs–Smith Graph $[\mathrm{PSL}(2,17)]$

The *Biggs–Smith graph* $[\mathrm{PSL}(2,17)]$ has as vertices the conjugacy class of 102 subgroups $\mathrm{Sym}(4)$ in $\mathrm{PSL}(2,17)$. The graph has intersection array

$$\{3,2,2,2,1,1,1;1,1,1,1,1,1,3\}$$

and is distance-transitive with automorphism group $\mathrm{PSL}(2,17)$.

PROPOSITION 5.64. *The Biggs–Smith graph* $[\mathrm{PSL}(2,17)]$ *has no* A*-covers.*

Proof. Let H be an r-fold A-cover of $[\mathrm{PSL}(2,17)]$ of diameter D. Then by Theorem 3.2 either $D = 14$ or $D = 15$.

Case 1: $D = 14$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{3}{\max(1, 3-1)}.$$

Thus $r = 1$, and no such A-cover exists.

Case 2: $D = 15$. Since $3 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(1, 3-3)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.65. *The Biggs–Smith graph* $[\mathrm{PSL}(2,17)]$ *has no* B*-doubles.*

Proof. The Biggs–Smith graph has $b_0 = 3$ and $b_0 - b_1 = 1$. Therefore

$$3 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.5. Perkel Graph $[\mathrm{PSL}(2,19)]$

The *Perkel graph* $[\mathrm{PSL}(2,19)]$ has as vertices a conjugacy class of 57 subgroups $\mathrm{Alt}(5)$ in $\mathrm{PSL}(2,19)$. The graph has intersection array

$$\{6, 5, 2; 1, 1, 3\}$$

and is distance-transitive with automorphism group PSL(2, 19).

PROPOSITION 5.66. *The Perkel graph* [PSL(2, 19)] *has no* A-*covers.*

Proof. Let H be an r-fold A-cover of $G = [\mathrm{PSL}(2,19)]$ of diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{6, 5, 2, \frac{r-1}{r}(3), 1, 1; 1, 1, \frac{1}{r}(3), 2, 5, 6\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{6, 5, 2, t(r-1), 3, 1, 1; 1, 1, 3, t, 2, 5, 6\}.$$

Case 1: $D = 6$. By Corollary 3.3(a), r divides $c_d = 3$ and so 3 is the only possible value of r. Therefore $\mathrm{i}(H) = \{6, 5, 2, 2, 1, 1; 1, 1, 1, 2, 5, 6\}$ with $|H| = r|G| = 171$. In [7, Chap. 14] we find that this intersection array is not feasible, and therefore no such A-cover exists.

Case 2: $D = 7$. Since $3 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(2, 6-3)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.67. *The Perkel graph* [PSL(2, 19)] *has no* B-*doubles.*

Proof. The Perkel graph has $b_0 = 6$ and $b_0 - b_1 = 1$. Therefore

$$6 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.6. Locally Petersen Graph [PΣL(2, 25)]

The *locally Petersen graph* [PΣL(2, 25)] has as vertices the conjugacy class of 65 elements with order 2 in PΣL(2, 25) corresponding to nontrivial field automorphisms. Two such are adjacent when they commute. The graph has intersection array

$$\{10, 6, 4; 1, 2, 5\}$$

and is distance-transitive with automorphism group PΣL(2, 25).

PROPOSITION 5.68. *The locally Petersen graph* [PΣL(2, 25)] *has no* A-*covers.*

Proof. Let H be an r-fold A-cover of [PΣL(2, 25)] of diameter D. Then by Theorem 3.2 either $D = 6$ or $D = 7$.

Case 1: $D = 6$. By Corollary 3.3(a), r divides $c_d = 5$ and

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{5}{\max(2, 5-4)}.$$

Thus $r = 1$, and no such A-cover exists.

Case 2: $D = 7$. Since $5 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(4, 10-5)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.69. *The locally Petersen graph* $[\mathrm{P\Sigma L}(2,25)]$ *has no* B*-doubles.*

Proof. The Petersen graph has no triangles, so maximal cliques in the locally Petersen graph $[\mathrm{P\Sigma L}(2,25)]$ have size 3. As $b_0 = 10 > 3(3-1) = 6$, Lemma 3.6(a) implies that the graph has no B-doubles. □

5.8.7. Distance-3 Hermitean Forms Graph $\mathrm{Her}(3,4)_3$

The distance-3 graph $\mathrm{Her}(3,4)_3$ of the Hermitean forms graph $\mathrm{Her}(3,4)$ is itself distance-transitive with 280 vertices and intersection array

$$\{9, 8, 6, 3; 1, 1, 3, 8\}.$$

Its automorphism group is $\mathrm{P\Gamma L}(3,4).2$.

PROPOSITION 5.70. *The distance-3 graph* $\mathrm{Her}(3,4)_3$ *of the Hermitian forms graph has no* A*-covers.*

Proof. Let H be an r-fold A-cover of $\mathrm{Her}(3,4)_3$ of diameter D. Then by Theorem 3.2 either $D = 8$ or $D = 9$.

Case 1: $D = 8$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{8}{\max(3, 8-3)}.$$

Thus $r = 1$, and no such A-cover exists.

Case 2: $D = 9$. Since $8 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(3, 9-8)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.71. *The distance-3 graph* $\mathrm{Her}(3,4)_3$ *of the Hermitian forms graph has no* B*-doubles.*

Proof. The graph $\mathrm{Her}(3,4)_3$ has $b_0 = 9$ and $b_0 - b_1 = 1$. Therefore

$$9 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.8. Unitary Nonisotropics Graph $[\mathrm{P\Gamma U}(3,4^2)]$

Let $V = \mathbb{F}_{16}^3$, and let f be a Hermitean form on V of full rank 3. There are 208 1-spaces $\mathbb{F}_{16}x$ with $f(x,x) \neq 0$. (Such a 1-space is *nonisotropic.*) The *unitary nonisotropics graph* $[\mathrm{P\Gamma U}(3,4^2)]$ has these 1-spaces as vertices with $\mathbb{F}_{16}x_1$ and $\mathbb{F}_{16}x_2$ adjacent when $f(x_1,x_2) = 0$. Its automorphism group $\mathrm{P\Gamma U}(3,4^2)$ is distance-transitive with intersection array

$$\mathrm{i}(G) = \{12, 10, 5; 1, 1, 8\}.$$

PROPOSITION 5.72. *The unitary nonisotropics graph* $[\mathrm{P\Gamma U}(3,4^2)]$ *has no* A-*covers.*

Proof. Let H be an r-fold A-cover of $G = [\mathrm{P\Gamma U}(3,4^2)]$ of diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{12, 10, 5, \frac{r-1}{r}(8), 1, 1; 1, 1, \frac{1}{r}(8), 5, 10, 12\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{12, 10, 5, t(r-1), 8, 1, 1; 1, 1, 8, t, 5, 10, 12\}.$$

Case 1: $D = 6$. By Corollary 3.3(a),

$$r \le \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{8}{\max(1, 8-5)}.$$

Thus 2 is the only possible value of r. Therefore $\mathrm{i}(H) = \{12, 10, 5, 4, 1, 1; 1, 1, 4, 5, 10, 12\}$, and so $|H| = r|G| = 416$. In [7, Chap. 14] we find that this intersection array is not feasible, and therefore no such A-cover exists.

Case 2: $D = 7$. Since $8 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(5, 4)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.73. *The unitary nonisotropics graph* $[\mathrm{P\Gamma U}(3,4^2)]$ *has no* B-*doubles.*

Proof. The graph $[\mathrm{P\Gamma U}(3,4^2)]$ has $b_0 = 12$ and $b_0 - b_1 = 2$. Therefore

$$12 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 2 \cdot 3 = 6,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.9. Hoffman–Singleton Line Graph $[\mathrm{P\Sigma U}(3,5^2)]$

The *Hoffman–Singleton graph* has as vertex set a conjugacy class of 50 subgroups $\mathrm{Aut(Sym(6))} = \mathrm{Sym(6).2}$ in the group $\mathrm{P\Sigma U}(3,5^2)$. It is distance-transitive of diameter 2 and has intersection array

$$\{7, 6; 1, 1\}.$$

Its line graph $[\mathrm{P\Sigma U}(3,5^2)]$ has as vertex set the 175 edges of the Hoffman–Singleton graph, two such adjacent when they share a Hoffman–Singleton vertex. The line graph has intersection array

$$\{12, 6, 5; 1, 1, 4\}$$

and is distance-transitive with automorphism group $\mathrm{P\Sigma U}(3,5^2)$.

PROPOSITION 5.74. *The line graph* $[\mathrm{P\Sigma U}(3,5^2)]$ *of the Hoffman–Singleton graph has no antipodal* A-*covers.*

Proof. Let H be an r-fold A-cover of $G = [\mathrm{P\Sigma U}(3,5^2)]$ of diameter D. Then by Theorem 3.2 either $D = 6$ and

$$\mathrm{i}(H) = \left\{12, 6, 5, \frac{r-1}{r}(4), 1, 1; 1, 1, \frac{1}{r}(4), 5, 6, 12\right\}$$

or $D = 7$ and

$$\mathrm{i}(H) = \{12, 6, 5, t(r-1), 4, 1, 1; 1, 1, 4, t, 5, 6, 12\}.$$

Case 1: $D = 6$. By Corollary 3.3(a), r divides $c_d = 4$. Thus we have two possibilities for r, namely 2 and 4.

For $r = 2$,

$$\mathrm{i}(H) = \{12, 6, 5, 2, 1, 1; 1, 1, 2, 5, 6, 12\}$$

with $|H| = r|G| = 350$; for $r = 4$,

$$\mathrm{i}(H) = \{12, 6, 5, 3, 1, 1; 1, 1, 1, 5, 6, 12\}$$

with $|H| = r|G| = 700$. In [7, Chap. 14] we find that these intersection arrays are not feasible, and therefore no such A-covers exist.

Case 2: $D = 7$. By Corollary 3.3(b)

$$t(r-1) \le \min(b_{d-1}, b_0 - c_d) = \min(5, 8)$$

and $c_d = 4 \le t$. Thus we have $r = 2$ and $t \in \{4, 5\}$ and

$$\mathrm{i}(H) = \{12, 6, 5, t, 4, 1, 1; 1, 1, 4, t, 5, 6, 12\}$$

with $|H| = r|G| = 350$. In [7, Chap. 14] we find that these intersection arrays are not feasible, and therefore no such A-covers exist. □

PROPOSITION 5.75. *The line graph* $[\mathrm{P\Sigma U}(3, 5^2)]$ *of the Hoffman–Singleton graph has no* B*-doubles.*

Proof. The Hoffman–Singleton graph J has

$$a_1(J) = b_0(J) - b_1(J) - c_1(J) = 7 - 6 - 1 = 0;$$

that is, it has no triangles. Therefore, for any clique M in its line graph $[\mathrm{P\Sigma U}(3, 5^2)]$, there must be a vertex x of J with the J-edges in M all containing x. This implies that the only maximal cliques of $[\mathrm{P\Sigma U}(3, 5^2)]$ are the 50 cliques of size 7 induced by the 50 vertex neighborhoods in the Hoffman–Singleton graph J. By Lemma 3.5, for the line graph $[\mathrm{P\Sigma U}(3, 5^2)]$ to have a B-double it must have at least $175 = |[\mathrm{P\Sigma U}(3, 5^2)]|$ distinct maximal cliques. Therefore, the line graph has no B-double. □

5.8.10. Livingstone Graph $[\mathrm{J}_1]$

The *Livingstone graph* $[\mathrm{J}_1]$ has as vertices a conjugacy class of 266 subgroups $\mathrm{PSL}(2, 11)$ in the sporadic Janko group J_1. The graph has intersection array

$$\{11, 10, 6, 1; 1, 1, 5, 11\}$$

and is distance-transitive with automorphism group J_1.

PROPOSITION 5.76. *The Livingstone graph* $[\mathrm{J}_1]$ *has no* A*-covers.*

Proof. Let H be an r-antipodal A-cover of [J_1] of diameter D. Then by Theorem 3.2 either $D = 8$ or $D = 9$.

Case 1: $D = 8$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{11}{\max(5, 11-1)}.$$

Thus $r = 1$, and hence no A-cover exists.

Case 2: $D = 9$. Since $11 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(1, 11 - 11)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.77. *The Livingstone graph* [J_1] *has no* B*-doubles.*

Proof. The Livingstone graph [J_1] has $b_0 = 11$ and $b_0 - b_1 = 1$. Therefore

$$11 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 1 \cdot 2 = 2,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.11. Hall–Janko Near Octagon [HJ.2]

The *Hall–Janko near octagon graph* [HJ.2] has as vertices a conjugacy class of 315 elements with order 2 in the sporadic group HJ. Two such are adjacent when they commute. The graph has intersection array

$$\{10, 8, 8, 2; 1, 1, 4, 5\}$$

and is distance-transitive with automorphism group Aut(HJ) = HJ.2. The graph is the incidence graph of a near 8-gon with three points per line.

PROPOSITION 5.78. *The Hall–Janko near octagon graph* [HJ.2] *has no* A*-covers.*

Proof. Let H be an r-fold A-cover of G of diameter D. Then by Theorem 3.2 either $D = 8$ or $D = 9$.

Case 1: $D = 8$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{5}{\max(4, 5-2)}.$$

Thus $r = 1$, and no such A-cover exists.

Case 2: $D = 9$. Since $5 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(2, 10 - 5)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.79. *The Hall–Janko near octagon graph* [HJ.2] *has no* B*-doubles.*

Proof. The graph [HJ.2] has $b_0 = 10$ and $b_0 - b_1 = 2$. Therefore

$$10 = b_0 > (b_0 - b_1)((b_0 - b_1) + 1) = 2 \cdot 3 = 6,$$

and by Lemma 3.6(b) the graph has no B-doubles. □

5.8.12. Patterson Graph [Suz.2]

The *Patterson graph* [Suz.2] has as vertices a conjugacy class of 22,880 subgroups with order 3 in the sporadic Suzuki group Suz. Two such are adjacent when they commute. The graph has intersection array

$$\{280, 243, 144, 10; 1, 8, 90, 280\}$$

and is distance-transitive with automorphism group $\mathrm{Aut}(\mathrm{Suz}) = \mathrm{Suz}.2$.

PROPOSITION 5.80. *The Patterson graph* [Suz.2] *has no* A*-covers.*

Proof. Let H be an r-fold A-cover of [Suz.2] of diameter D. Then by Theorem 3.2 either $D = 8$ or $D = 9$.

Case 1: $D = 8$. By Corollary 3.3(a),

$$r \leq \frac{c_d}{\max(c_{d-1}, c_d - b_{d-1})} = \frac{280}{\max(90, 280 - 10)}.$$

Thus $r = 1$, and hence no such A-cover exists.

Case 2: $D = 9$. Since $280 = c_d > \min(b_{d-1}, b_0 - c_d) = \min(10, 280 - 280)$, no such A-cover exists by Corollary 3.3(b). □

PROPOSITION 5.81. *The Patterson graph* [Suz.2] *has no* B*-doubles.*

Proof. By Lemma 3.7 the valency $B_0(H)$ of the B-double H of $G = $ [Suz.2] is a root of the polynomial

$$x^2 - x - 280c,$$

where $c = C_2(H)$ satisfies $1 \leq c \leq c_2(G) = 8$. This polynomial has discriminant $\sqrt{1 + 1120c}$. However, for integral c with $1 \leq c \leq 8$, this discriminant is never an integer. The contradiction proves that no such B-double H exists. □

References

[1] M. R. Alfuraidan, *Imprimitive distance-transitive graphs,* Ph.D. thesis, Michigan State University, March 2004.

[2] M. R. Alfuraidan, A. Gardiner, and J. I. Hall, untitled manuscript (in preparation).

[3] M. R. Alfuraidan and J. I. Hall, *Smith's theorem and a characterization of the* 6*-cube as distance-transitive graph,* J. Algebraic Combin. 24 (2006), 195–207.

[4] N. L. Biggs and A. Gardiner, *The classification of distance transitive graphs,* unpublished manuscript, 1974.

[5] J. T. M. van Bon and A. E. Brouwer, *The distance-regular antipodal covers of classical distance-regular graphs,* Combinatorics (Eger, 1987), pp. 141–166, Colloq. Math. Soc. János Bolyai, 52, North-Holland, Amsterdam, 1988.

[6] A. E. Brouwer, *Uniqueness and nonexistence of some graphs related to* M_{22}, Graphs Combin. 2 (1986), 21–29.

[7] A. E. Brouwer, A. M. Cohen, and A. Neumaier, *Distance-regular graphs,* Ergeb. Math. Grenzgeb. (3), 13, Springer-Verlag, Berlin, 1989.

[8] F. Buekenhout and H. van Maldeghem, *Finite distance-transitive generalized polygons,* Geom. Dedicata 52 (1994), 41–51.

[9] A. M. Cohen, *Point-line spaces related to buildings,* Handbook of incidence geometry (F. Buekenhout, ed.), pp. 647–737, North-Holland, Amsterdam, 1995.

[10] B. N. Cooperstein, *Some geometries associated with parabolic representations of groups of Lie type,* Canad. J. Math. 28 (1976), 1021–1031.

[11] I. A. Faradjev, A. A. Ivanov, and A. V. Ivanov, *Distance-transitive graphs of valency 5, 6 and 7,* European J. Combin. 7 (1986), 303–319.

[12] A. Gardiner, *Antipodal covering graphs,* J. Combin. Theory Ser. B 16 (1974), 255–273.

[13] C. D. Godsil, R. A. Liebler, and C. E. Praeger, *Antipodal distance transitive covers of complete graphs,* European J. Combin. 19 (1998), 455–478.

[14] J. Hemmeter, *Halved graphs, Johnson and Hamming graphs,* Utilitas Math. 25 (1984), 115–118.

[15] ———, *Distance-regular graphs and halved graphs,* European J. Combin. 7 (1986), 119–129.

[16] D. G. Higman, *Finite permutation groups of rank* 3, Math. Z. 86 (1964), 145–156.

[17] A. A. Ivanov, *Distance-transitive representations of the symmetric groups,* J. Combin. Theory Ser. B 41 (1986), 255–274.

[18] ———, *Distance-transitive graphs and their classification,* Investigations in the algebraic theory of combinatorial objects (I. A. Faradzev, A. A. Ivanov, M. H. Klin, A. J. Woldar, eds.), pp. 283–378, Kluwer, Dordrecht, 1994.

[19] A. A. Ivanov, R. A. Liebler, T. Penttila, and C. E. Praeger, *Antipodal distance-transitive covers of complete bipartite graphs,* European J. Combin. 18 (1997), 11–33.

[20] A. A. Ivanov and S. V. Shpectorov, *The P-geometry for* M_{23} *has no nontrivial 2-coverings,* European J. Combin. 11 (1990), 373–379.

[21] W. M. Kantor, *Classification of* 2*-transitive symmetric designs,* Graphs Combin. 1 (1985), 165–166.

[22] M. W. Liebeck, *The affine permutation groups of rank three,* Proc. London Math. Soc. (3) 54 (1987), 477–516.

[23] J. H. van Lint, *Introduction to coding theory,* 3rd ed., Grad. Texts in Math., 86, Springer-Verlag, Berlin, 1999.

[24] D. H. Smith, *Primitive and imprimitive graphs,* Quart. J. Math. Oxford Ser. (2) 22 (1971), 551–557.

[25] E. Witt, *Über Steinersche Systeme,* Abh. Math. Sem. Univ. Hamburg 12 (1938), 265–275.

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