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On $\mathbb{C}$-fibrations over Projective Curves

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1. Introduction

Rational affine surfaces (i.e., affine surfaces birationally equivalent to a plane) represent an interesting and abundant class of surfaces worthy of investigation. One of the tools used for classification of such objects is the so-called ML invariant ($\mathrm{ML}(S)$) of a surface S, which is a characteristic subring of the ring of regular functions of S; it consists of the regular functions that are invariant under all possible $\mathbb{C}^+$-actions on S. Any $\mathbb{C}^+$-action on a surface induces a $\mathbb{C}$-fibration over an affine curve. The invariant answers the question of how many fibrations of this kind the surface admits.

Naturally enough, the fibrations over a projective base have been less studied (but see [DR; GMaMiR; GMi; KiKo; Zai]).

The goal of this paper is to present a modified version of the ML invariant that not only takes into account projective rulings but also allows a further stratification of rational surfaces. Of course, introducing an invariant is much easier than computing it in a particular case. We still do not know how to compute the ML invariant for a given surface, though some techniques are available (see [KM-L1; KM-L2]).

Unfortunately, computation of the modified version of the invariant is even more involved. Nevertheless, we present a nontrivial example where we were able to complete the computation. It is hoped that further techniques will be developed in due course.

Let us recall the definition of the ML invariant. For a ring R, we denote by $\mathrm{DER}(R)$ the set of derivations on R; by $\mathrm{LND}(R) \subset \mathrm{DER}(R)$ the set of locally nilpotent derivations (lnd); and by $F(R)$ the field of fractions of R. For a derivation $\partial \in \mathrm{DER}(R)$, we denote by R^∂ and $F(R)^\partial$ the kernel of ∂ in R and $F(R)$, respectively. Let R be the ring of regular functions of an affine algebraic variety V. Then $\mathrm{ML}(R) = \mathrm{ML}(V) = \bigcap_{\partial \in \mathrm{LND}(R)} R^\partial$.

Here is the modified version. Take an element $f \in F(R)$ and consider the ring $R[f] \subset F(R)$—that is, the extension of R by polynomial functions of f. Call

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$\partial \in \mathrm{DER}(F(R))$ a generalized locally nilpotent derivation (glnd) of R if it is locally nilpotent on $R[f]$ and if $\partial(f) = 0$. Define $\mathrm{GML}(R) = \bigcap_{\partial \in \mathrm{GLND}(R)} F(R)^{\partial}$, where $\mathrm{GLND}(R)$ is the set of all generalized locally nilpotent derivations of R.

If $R = \mathcal{O}(S)$, the ring of regular functions on a surface S, then we will denote $F(R)$ by $F(S)$. Of course, $F(S)^{\partial}$ is the algebraic closure of $\mathbb{C}(f)$ in $F(S)$ when $\partial \in \mathrm{GLND}(R)$. Therefore, $\mathrm{GML}(R)$ is either

- $F(R)$ when the only element of $\mathrm{GLND}(R)$ is the zero derivation; or
- a field of rational functions of a curve C when nonzero lnd are possible on $R[f]$ only for $f \in \mathbb{C}(u)$, where u is a fixed element of $F(R)$; or
- $\mathbb{C}$ when there are at least two substantially different possible choices of f.

If S is rational, then $C \cong \mathbb{P}^1$.

Geometrically speaking, if $R = O(S)$ where S is a surface, then a nonzero glnd of R that is not equivalent to an lnd of R corresponds to a $\mathbb{C}$-fibration of S over a projective curve. Therefore, S contains a cylinder-like subset. By a result of Miyanishi and Sugie [MiS], it is equivalent to $\bar{\kappa} = -\infty$, where $\bar{\kappa}$ is the logarithmic Kodaira dimension of S. We can think of $\mathrm{LND}(R)$ as a subset of $\mathrm{GLND}(R)$ (just take $f = 1$). So in the case of surfaces, the logarithmic Kodaira dimension of S is $-\infty$ if and only if $\mathrm{GLND}(R)$ contains a nonzero derivation.

In Section 2 we give some definitions and demonstrate the first properties of GML. In Section 3 we give an example of computing the GML invariant for a "rigid" surface: a smooth affine rational surface with $\bar{\kappa} = -\infty$ admitting no $\mathbb{C}^+$-actions. In Section 4 we apply the GML invariant to computing the ML invariant of some threefolds.

It appears that the GML invariant of a surface S is closely connected to the ML invariant of line bundles over S. Namely, let $\mathcal{L} = (L, \pi, S)$ be a line bundle over S and let $\partial \in \mathrm{LND}(\mathcal{O}(L))$. Then there exists a $\partial' \in \mathrm{GML}(S)$ such that $\partial f = 0$ for any $f \in \pi^*(F(S)^{\partial'})$ (see Proposition 1). On the other hand, for any $\partial \in \mathrm{GML}(S)$ there is a line bundle $\mathcal{L} = (L, \pi, S)$ and an lnd $\partial' \in \mathrm{LND}(\mathcal{O}(L))$ such that $\partial' f = 0$ for any $f \in \pi^*(F(S)^{\partial})$ (Lemma 10).

This is why the GML invariant is useful for understanding whether the ML invariant of a surface is stable under reasonable geometric constructions. In our previous work, the cylinder over a surface played the part of a "reasonable" geometric construction. Here we are replacing the cylinder by an algebraic line bundle.

It is not always possible to generalize the results known for the cylinders to this setting. For example, for "rigid" surfaces we have

$$\mathrm{ML}(S \times \mathbb{C}) = \mathcal{O}(S),$$

but if $\mathrm{GML}(S)$ is not trivial then it is possible to construct a nontrivial line bundle $\mathcal{L} = (L, \pi, S)$ with $\mathrm{ML}(L) \neq \mathcal{O}(S)$. In Corollary 3 we describe the line bundles for which the equality nevertheless holds.

We shall denote by $\mathbb{C}^n_{x_1,\dots,x_n}$ the n-dimensional complex affine space with coordinates $x_1, \dots, x_n$; for an irreducible subvariety C of codimension 1, we denote by $[C]$ the effective divisor with this support and coefficient 1. As usual, supp(G) and Cl(G) stand for the support and the class of divisor G, respectively, and $(C_1, C_2) = ([C_1], [C_2])$ is the intersection number of two curves (resp. divisors) on a surface. We use $\bar{A}$ to denote a closure of A. For a rational function f, we denote by $(f), (f)_0, (f)_\infty$ the divisors of f, of its zeros, and of its poles, respectively. If $\mathcal{L} = (L, \pi, S)$ is a line bundle over a smooth surface S, then D_L stands for the Weil divisor on S (since S is smooth, we do not distinguish between Weil and Cartier divisors) associated to $\mathcal{L}$. Two $\mathbb{C}^+$-actions are *equivalent* if they have the same general orbit.

The main information on the properties of LND(R) may be found in [KM-L2]. Our reference for affine surfaces with fibrations is the book by Miyanishi [Mi2].

2. Properties of GML

Let S be a smooth affine complex surface, let $R = \mathcal{O}(S)$ be the ring of regular functions on S, and let $F(S) = \mathrm{Frac}(\mathcal{O}(S))$ stand for the field of fractions of $\mathcal{O}(S)$. Every locally nilpotent derivation ∂ on R corresponds via exponentiation to a $\mathbb{C}^+$-action, which is a morphism $\alpha \colon \mathbb{C} \times S \to S$ such that:

- for a fixed $t \in \mathbb{C}$, the restriction $\alpha_t = \alpha|_{t \times S}$ is an automorphism of S;
- $\alpha_{t_1} \circ \alpha_{t_2} = \alpha_{t_1 + t_2}$.

The ring R^∂ is a ring of the functions invariant under the corresponding $\mathbb{C}^+$-action α. If $\partial \neq 0$, then a general orbit of this $\mathbb{C}^+$-action is isomorphic to $\mathbb{C}$ and is a fiber of a morphism $S \to C$, where C is a smooth affine curve and $\mathcal{O}(C) \cong R^\partial$.

Any morphism f of a surface S to a curve C (projective or affine) we call a $\mathbb{C}$-*fibration* if its general fiber $f^{-1}(c)$, $c \in C$, is isomorphic to $\mathbb{C}$. Existence of such a fibration is equivalent to $\bar{\kappa}(S) = -\infty$ [MiS].

Definition 1. The derivation $\partial \in \mathrm{DER}(R)$ is a generalized locally nilpotent derivation (glnd) if there is an $f \in F(S)$ such that $\partial \in \mathrm{LND}(R[f])$ and $\partial(f) = 0$. The set of all generalized locally nilpotent derivations for the ring R is denoted by GLND(R).

Definition 2. Two elements ∂_1 and ∂_2 in GLND(R) are equivalent if $F(R)^{\partial_1} = F(R)^{\partial_2}$.

Definition 3. The invariant GML(R) (or GML(S) if $R = \mathcal{O}(S)$) is the field $\bigcap_{\partial \in \mathrm{GLND}(R)} F^\partial$.

Definition 4. A smooth affine rational surface S is rigid if the log-Kodaira dimension $\bar{k}(S) = -\infty$ and ML$(S) = \mathcal{O}(S)$.

The invariant GML(S) has the following four properties.

Property 1. $\bar{k}(S) = -\infty$ *if and only if* GML$(S) \neq F(S)$.

Proof. Indeed, by definition, GML$(S) \neq F(S)$ is equivalent to the existence of a cylinder-like subset in S, which by [MiS] is equivalent to $\bar{k}(S) = -\infty$. □

PROPERTY 2. *If there exists a Zariski open affine subset* $U \subseteq S$ *such that* $\mathrm{ML}(U) = \mathbb{C}$, *then* $\mathrm{GML}(S) = \mathbb{C}$.

Proof. Since $\mathrm{ML}(U) = \mathbb{C}$, the surface S is rational. Let $\varphi_1 \colon U \to \mathbb{C}$ and $\varphi_2 \colon U \to \mathbb{C}$ be two $\mathbb{C}$-fibrations on U. Let $\bar{S}$ be an NC-completion of S such that the rational extensions $\bar{\varphi}_1 \colon \bar{S} \to \mathbb{P}^1$ and $\bar{\varphi}_2 \colon \bar{S} \to \mathbb{P}^1$ of φ_1 and φ_2, respectively, are regular. Let $S = \bar{S} - D$, $U = \bar{S} - (D \cup D')$, $D = \bigcup_{k=1}^{m} C_k$, and $D' = \bigcup_{j=1}^{n} B_j$, where C_i and B_i are reduced irreducible components of D and D', respectively. All these components are smooth and rational (see [Mi2, Chap. III, Lemma 1.4.1]).

We denote by D_1^∞ and D_2^∞ the components in $D \cup D'$ such that $\bar{\varphi}_i \colon D_i^\infty \to \mathbb{P}^1$ is an isomorphism, $i = 1, 2$. If $D_i^\infty \subset D'$ for some i, then for a general $a \in \mathbb{P}^1$ the intersection $\bar{\varphi}_i^{-1}(a) \cap (D \cup D') \in D' - D$, since $\bar{\varphi}_i^{-1}(a)$ has only a single point in $\bar{S} - U$. It follows that a compact curve $\bar{\varphi}_i^{-1}(a) \subset S$, so S is not affine. This contradiction shows that $D_i^\infty \subset D$ for $i = 1, 2$ and that $\bar{\varphi}_i|_{B_j} = \text{const.}$ for every $j = 1, \dots, n$. Thus the $\bar{\varphi}_i|_S \colon S \to \mathbb{P}^1$ are nonequivalent $\mathbb{C}$-fibrations, and $\mathrm{GML}(S) = \mathbb{C}$. □

PROPERTY 3 (see [GMaMiR; Zai]). *For a* $\mathbb{Q}$*-homology plane* S,

$$\mathrm{GML}(S) = \mathrm{Frac}(\mathrm{ML}(S)).$$

PROPERTY 4 [GMi, Thm. 4.1]. *If there exist a* $\mathbb{C}$*-fibration* $f \colon S \to B$ *and the curve* $B \cong \mathbb{C}$ (*resp.* $B \cong \mathbb{P}^1$), *if all the fibers of* f *are irreducible, and if there are at least two* (*resp. three*) *multiple fibers, then* $\mathrm{GML}(S) = \mathbb{C}(f)$.

The next lemma is a simple fact about locally nilpotent derivations that was proved in another form in [BM-L1]. We will need it in the sequel.

LEMMA 1. *Let* R *be a finitely generated ring and let* $r \in R$. *Assume there is a nonzero lnd* ∂ *on* $R[r^{-1}]$. *Then there is a nonzero lnd on* R.

Proof. Indeed, let $r_1, \dots, r_n$ be a generating set of R. Then $\partial(r_i) = p_i r^{-d_i}$, where $p_i \in R$ and d_i is a natural number. It is clear that $\partial(r) = 0$ since both r and r^{-1} are in $R[r^{-1}]$.

Take m larger than all d_i. Then $\varepsilon = r^m \partial$ is also an lnd on $R[r^{-1}]$. Since $\varepsilon(r_i) \in R$ for all i, the derivation ε is a derivation of R. Hence ε is an lnd of R. □

REMARK 1. The same consideration works if there is a nonzero lnd ∂ on $R(r)$. Again $\partial(r) = 0$, but instead of r^m take a common denominator of all $\partial(r_i)$ that is a polynomial in r.

3. Example

In this section we compute $\mathrm{GML}(S)$ for the surface $S \subset \mathbb{C}^7$, which was introduced as Example 3 in [BM-L2] and is defined by

$$uv = z(z-1), \tag{1}$$

$$v^2 z = uw, \tag{2}$$

$$z^2(w-1) = xu^2, \tag{3}$$

$$u^2(z-1) = tv, \tag{4}$$

$$(z-1)^2(t-1) = yv^2, \tag{5}$$

$$u^2v^2 = wt, \tag{6}$$

$$yz^2 = u^2(t-1), \tag{7}$$

$$x(z-1)^2 = v^2(w-1), \tag{8}$$

$$v^4x = w^2(w-1), \tag{9}$$

$$u^4y = t^2(t-1), \tag{10}$$

$$v^3 = (z-1)w, \tag{11}$$

$$u^3 = tz, \tag{12}$$

$$xy = (w-1)(t-1). \tag{13}$$

The surface is smooth because the rank of the Jacobi matrix of equations (1)–(13) is maximal everywhere.

The surface S has the following properties.

PROPERTY 5.

(i) $\bar{\kappa}(S) = -\infty$.

(ii) $R = \mathrm{ML}(S) = \mathcal{O}(S)$.

(iii) $\pi_1(S) = \mathbb{Z}/2\mathbb{Z}$.

(iv) $\mathrm{Pic}(S) = \mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$.

(v) *S admits an automorphism*

$$a\colon (u, v, z, t, w, x, y) \to (-v, -u, 1-z, w, t, y, x).$$

(vi) *The morphism $b\colon S \to \mathbb{P}^1$, defined as $b(s) = z/u$ for a point $s \in S$, is a $\mathbb{C}$-fibration, and all the fibers of this fibration are isomorphic to $\mathbb{C}^1$. The fibers $B_0 = b^{-1}(0)$ and $B_\infty = b^{-1}(\infty)$ have multiplicity* 2.

(vii) *The following relations are valid*:

$$z = ub, \quad v = b(ub-1), \quad w = b^3(ub-1)^2;$$

$$x = b^2(b^3(ub-1)^2 - 1), \quad t = \frac{u^2}{b}, \quad y = \frac{u^2 - b}{b^3}.$$

(viii) *The surfaces $S_0 = S - B_\infty$ and $S_\infty = S - B_0$ are isomorphic to the hypersurface $S' = \{\beta^3\gamma = \alpha^2 - \beta\}$. The isomorphisms $\tau_0\colon S_0 \to S'$ and $\tau_\infty\colon S_\infty \to S'$ are defined, respectively, by $\beta = b$, $\alpha = u$, and $\gamma = y$ and by $\beta = 1/b$, $\alpha = v$, and $\gamma = x$. Indeed, $R[b] = \mathbb{C}[u, b, (u^2 - b)/b^3]$ and $R[1/b] = \mathbb{C}[b(ub-1), 1/b, b^2(b^3(ub-1)^2 - 1)]$.*

THEOREM 1. $\mathrm{GML}(S) = \mathbb{C}(b)$.

Proof. The proof is rather long, but the main idea is as follows. If $\mathrm{GML}(S) \neq \mathbb{C}(b)$ then there exists a $\mathbb{C}$-fibration $\varphi \in F(S)$ such that $\mathrm{LND}(R[\varphi]) \neq \{0\}$, where $R = \mathcal{O}(S)$ and φ is algebraically independent of b. We introduce some weights for the generators u, v, z, t, w, x, y and consider the corresponding graded algebras $\hat{R}$ and $\widehat{R[\varphi]}$ (since φ is a rational function, the weight of φ is also defined). The reference for details is [KM-L2]. We will show that, for these weights, $\mathrm{LND}(\hat{R}) = \{0\}$. Then we will prove that the leading forms of the numerator and the denominator of φ are algebraically dependent and, finally, that $\mathrm{LND}(\widehat{R[\varphi]}) = \{0\}$. This will bring us to a contradiction because (as was shown in [KM-L1]; see also [KM-L2]), $\mathrm{LND}(R[\varphi]) \neq \{0\}$ implies $\mathrm{LND}(\widehat{R[\varphi]}) \neq \{0\}$.

Let us specify the weights (ω) by $\omega(u) = 4$ and $\omega(b) = -1 + \rho$, where $\rho \ll 1$ is an irrational number. Then $\omega(z) = 3 + \rho$, $\omega(v) = 2 + 2\rho$, $\omega(w) = 3 + 5\rho$, $\omega(x) = 1 + 7\rho$, $\omega(t) = 9 - \rho$, and $\omega(y) = 11 - 3\rho$.

LEMMA 2. $\mathrm{LND}(\hat{R}) = \{0\}$.

Proof. Let $\partial \in \mathrm{LND}(\hat{R})$ be a nonzero derivation. The system

$$uv = z^2, \quad v^2 z = uw, \quad z^2 w = xu^2, \quad u^2 z = tv, \tag{14}$$

$$z^2 t = yv^2, \quad u^2 v^2 = wt, \quad yz^2 = u^2 t, \quad xz^2 = v^2 w, \tag{15}$$

$$v^4 x = w^3, \quad u^4 y = t^3, \quad v^3 = zw, \quad u^3 = tz, \quad xy = wt \tag{16}$$

defines a reduced (the rank of a Jacobian matrix is maximal in a Zariski open subset $\{uvz \neq 0\}$) and irreducible surface. The latter follows because each fiber of a rational function $k = u/z = z/v$ is irreducible.

According to [KM-L2, Lemma 6.2], the system (14)–(16) defines $\hat{R}$; thus,

$$\hat{R} = \mathbb{C}[u, z, u^{-1}z^2, u^{-3}z^5, u^{-5}z^7, u^3 z^{-1}, u^5 z^{-3}].$$

We want to show that this ring does not have a nonzero locally nilpotent derivation. For our choice of weights we will show that the induced nonzero locally nilpotent derivation $\hat{\partial}$ also belongs to $\mathrm{LND}(\hat{R})$ because $\hat{R}$ is a graded algebra relative to these weights. The weights are not commensurable, which is why both $\hat{\partial}(u)$ and $\hat{\partial}(z)$ are monomials and why $\hat{\partial}$ of any monomial is a monomial.

We present monomials in u, z of $\hat{R}$ by points of a two-dimensional integer lattice. The set A of points $(1,0)$, $(0,1)$, $(-1,2)$, $(-3,5)$, $(-5,7)$, $(3,-1)$, $(5,-3)$, which corresponds to the generating set of $\hat{R}$, is located on the plane (r,s) inside the angle between the lines $L_1 = \{3r + 5s = 0\}$ and $L_2 = \{7r + 5s = 0\}$ containing the first quadrant. The points $(5,-3)$ and $(-5,7)$ belong to L_1 and L_2, respectively. There is an involution $\hat{a}\colon (u,z) \to (u^{-1}z^2, z)$ of the ring $\hat{R}$, which changes the roles of lines L_1 and L_2.

Since $\hat{\partial}$ is locally nilpotent and nonzero, it implies that there is a monomial $f \in \ker(\hat{\partial}) - \mathbb{C} \subset \hat{R}$. This means that $\ker(\hat{\partial})$ is generated by a monomial, say f. There is also a monomial $g \in \hat{R}$ for which $\hat{\partial}(g) \neq 0$ and $\hat{\partial}^2(g) = 0$. It is known (see [KM-L2]) that $\hat{R} \subset \mathbb{C}(f)[g]$.

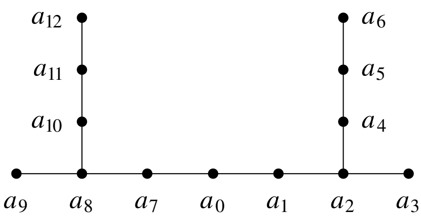
Let t_1 and t_2 be the vectors on the plane (r, s) that represent f and g. These vectors are, of course, not collinear. Hence $\hat{R} \subset \mathbb{C}(f)[g]$ implies that the set of points corresponding to $\hat{R}$ belongs to the half-plane spanned by t_1, $-t_1$, and t_2. Therefore t_1 must belong to a boundary line—that is, either to L_1 or to L_2. Because of the involution we may assume that $t_1 \in L_1$ and $\hat{\partial}(u^5 z^{-3}) = 0$.

As in [FLN], we let "deg" denote the degree function induced by $\hat{\partial}$. Then $5 \deg u - 3 \deg z = 0$; that is, $\deg u = 3n$ and $\deg z = 5n$ for some $n \in \mathbb{N}$. Since $\hat{\partial}(u)$ is a monomial, n should divide $3n - 1$ and so $n = 1$.

Now $\deg(g) = 1$, so one of the monomials in $\hat{R}$ has degree 1. But $\deg u = 3$, $\deg z = 5$, $\deg u^{-1}z^2 = 7$, $\deg u^{-3}z^5 = 16$, $\deg u^{-5}z^7 = 20$, $\deg u^3 z^{-1} = 4$, and $\deg u^5 z^{-3} = 0$, and since g is a product of these monomials it cannot have degree equal to 1. □

The next step is computation of the leading form $\hat{\varphi}$ of the function φ. We need several lemmas. We will denote by the same letter the function u on S, its extension to $\bar{S}$, and its lift to any blow-up $\tilde{S}$ of S.

Lemma 3. *The map b can be extended to a morphism $\bar{b}\colon \bar{S} \to \mathbb{P}^1$ to a closure $\bar{S}$ such that the divisor $D = \bar{S} - S$ has the following graph Γ.*



Here vertex a_i, $0 \le i \le 12$, represents a component A_i of divisor D. Moreover, the components have the following properties.

Property 6.

(i) $A_i^2 = -2$ *for* $i > 0$.
(ii) $A_6 \cap B_0 \neq \emptyset$ *and* $A_{12} \cap B_\infty \neq \emptyset$.
(iii) $F_0 = \bar{b}^{-1}(0) = A_1 + A_3 + 2A_2 + 2A_4 + 2A_5 + 2A_6 + 2B_0$ *and* $F_\infty = \bar{b}^{-1}(\infty) = A_7 + A_9 + 2A_8 + 2A_{10} + 2A_{12} + 2A_{11} + 2B_\infty$.
(iv) $u|_{\bigcup_2^6 A_i} = u|_{B_0} = u|_{B_\infty} = 0$ *and* $u|_{A_1}$ *is linear*; $v|_{\bigcup_8^{12} A_i} = v|_{B_0} = v|_{B_\infty} = 0$ *and* $v|_{A_7}$ *is linear.*

Proof of Lemma 3. By Property 5(v) and Property 5(viii) it is sufficient to analyze the structure of the closure of the surface $S_0 = S - B_\infty$ and to prove only parts (i)–(iv) of Property 6. A detailed description of the graph of the divisor $D_0 = \overline{S_0} - S_0$ is given in [MaMi2] and [tD], together with the proof of parts (i)–(iii) of Property 6.

In order to obtain S_0, one must consider the open set $U \cong \mathbb{C}^2_{b,u}$ of a Hirzebruch surface and then repeatedly blow up the point $b = 0$, $u = 0$ of the fiber $B = \{b = 0\}$. This is why Property 6(iv) is valid: $u = 0$ on all exceptional components A_i $(1 \le i \le 6)$ of this process, and u is linear along the proper transform A_1 of B. The equality $u|_{B_\infty} = 0$ follows from equation (1) in the definition of the surface. □

Any fibration $\varphi\colon S \to \mathbb{P}^1$, $\varphi \in F(S)$, that is nonequivalent to b has the following properties.

Property 7.

(i) *Every fiber* $\Phi_q = \varphi^{-1}(q)$ *is isomorphic to* $\mathbb{C}$ (*since* $\operatorname{rank}\operatorname{Pic}_{\mathbb{Q}}(S) = 1$) [Mi2, Chap. 3, 2.4.3.1].

(ii) *There are precisely two values* $q_0, q_1 \in \mathbb{P}^1$ *such that the fibers* Φ_{q_0}, Φ_{q_1} *have multiplicity* 2; *all other fibers are of multiplicity* 1 [Fu, 4.19, 4.20, 5.9].

(iii) φ *is not a function of* b (*because they define nonequivalent fibrations*).

(iv) *There is a* $\partial \in \mathrm{LND}(R[\varphi])$ *such that* $\partial \ne \{0\}$ *and* $\partial(\varphi) = 0$.

Lemma 4. *There is no* $p \in \mathbb{P}^1$ *such that* φ *is constant along the fiber* $B_p = b^{-1}(p)$.

Proof. If such a p exists, then the affine surface $S'' = S - B_p$ admits two nonequivalent $\mathbb{C}$ fibrations over $\mathbb{C}$; that is, $\mathrm{ML}(S'') = \mathbb{C}$.

If $p \ne 0, \infty$, then $b|_{S''}$ has two singular fibers; thus $\mathrm{ML}(S'') \ne \mathbb{C}$ [Be; Gi]. If $p = 0$ or ∞, then $S'' \cong S'$ (see Property 5(viii)). But $\mathrm{ML}(S') = \mathbb{C}[\beta] \ne \mathbb{C}$ as well [MaMi2, Thm. 2.3]. Thus, neither case is possible. □

Lemma 5. *The extension* $\bar{\varphi}$ *of* φ *to* $\bar{S}$ *is not regular and has only one singular point.*

Proof. Assume that $\bar{\varphi}$ is a morphism of $\bar{S}$ onto $\mathbb{P}^1$. Then, for one of the components A_i of divisor $D = \bar{S} - S$ (see Lemma 3), the restriction $\varphi|_{A_i}\colon A_i \to \mathbb{P}^1$ is an isomorphism. Given the existence of the automorphism a of S (see Property 5(v)), we may assume that $0 \le i \le 6$.

Case 1: $i = 0$. Then a general fiber $\bar{\Phi}_q = \bar{\varphi}^{-1}(q) = \overline{\varphi^{-1}(q)} \cong \mathbb{P}^1$ of $\bar{\varphi}$ intersects A_0 transversely. Since the function u is linear along a general fiber of b [BM-L2, Ex. 3], it has a simple pole along A_0. Since this is the sole puncture of Φ_q and since $u \in \mathcal{O}(S)$, it follows that the restriction $u|_{\bar{\Phi}_q}$ has its only simple pole at the point $A_0 \cap \bar{\Phi}_q$. But it has zero at every point of intersections $\bar{\Phi}_q \cap B_0 \ne \emptyset$ and $\bar{\Phi}_q \cap B_\infty \ne \emptyset$. Hence the number of zeros is at least two. This contradiction shows that $i \ne 0$.

Case 2: $0 < i \le 6$. In this case, $\bar{\Phi}_q$ intersects D only at a point of A_i, and u is finite at the intersection point for a general q (see Property 6(iv)). Therefore u is finite everywhere in $\bar{\Phi}_q$ and hence constant. Since the curve $\{u = \text{const.}\} \ncong \mathbb{C}$ in S, this is impossible.

Thus $\bar{\varphi}$ is regular on S but is not a morphism of $\bar{S}$. In other words, the singular point of $\bar{\varphi}$ is at the puncture of the general fiber Φ_q (or, which is the same, at the intersection of general fibers $\bar{\Phi}_q$). Since Φ_q has only one puncture, there is only one singular point $s \in \bar{S}$. □

Let $\bar{b}(s) = p_0$. We may assume that $p_0 \neq 0$ (because of the involution a, we may always change the roles of 0 and ∞).

Let $\pi : \tilde{S} \to \bar{S}$ be a resolution of $\bar{\varphi}$. The morphism π is an isomorphism outside $\pi^{-1}(s)$. Moreover, by the assumption $p_0 \neq 0$, no blow-ups in π occur on $A_1, \dots, A_6$. Let $\pi^{-1}(s) = \bigcup_0^k E_j$ (where E_j are exceptional components in $\tilde{D} = \tilde{S} - S$), let $\tilde{\varphi} = \bar{\varphi} \circ \pi$ and $\tilde{b} = \bar{b} \circ \pi$, let $\tilde{A}_i$ be proper transforms of A_i, and let $\tilde{\varphi}|_{E_0}$ be an isomorphism. Then $\tilde{\varphi}$ must be constant along each connected component of $\tilde{D} - E_0$.

Let $\tilde{\Phi}_q = \tilde{\varphi}^{-1}(q)$ and $\tilde{B}_q = \tilde{b}^{-1}(q)$ for a point $q \in \mathbb{P}^1$. As before, $\bar{\Phi}_q = \overline{\varphi^{-1}(q)}$ and $\tilde{\Phi}_q = \bar{\Phi}_q$ for a general q. Consider the connected component R of $\tilde{D} - E_0$ containing the proper transform $\tilde{A}_0$ of A_0. If $\tilde{\varphi}|_R = \kappa \in \mathbb{P}^1$, then $\tilde{\Phi}_\kappa = \tilde{\varphi}^{-1}(\kappa) = R \cup C$, where $C = \bar{\Phi}_\kappa$ is the closure of Φ_κ (this means that C is the only component of $\tilde{\Phi}$ that intersects S).

Lemma 6. $\tilde{b}(s_1) \neq 0$, *where* $s_1 = R \cap C$.

Proof. Assume that $s_1 \in \tilde{b}^{-1}(0)$. We recall that π is an isomorphism in the neighborhood of $\tilde{b}^{-1}(0)$. Point s_1 cannot be the intersection point of $\tilde{A}_0$ and $\tilde{b}^{-1}(0)$, because three components $(C, \tilde{A}_0, \tilde{A}_1)$ of the fiber of $\tilde{\varphi}$ cannot intersect at a point [Mi2, Chap. 3, 1.4.1]. Thus, $s_1 \in \left(\bigcup_1^6 \tilde{A}_i\right) - (\tilde{A}_0 \cap \tilde{A}_1)$ and $u(s_1)$ is finite. But then u is finite at every point of C, which is impossible. □

Lemma 7. *The fiber* Φ_κ *has multiplicity* 2 *in the fibration* φ.

Proof. Let $\tilde{\Phi}_\kappa = \bigcup_0^6 \tilde{A}_i \cup C \cup R_1$, where R_1 is the union of other components of R. Let the corresponding divisor G of the fiber $\tilde{\Phi}_\kappa = \tilde{\varphi}^{-1}(\kappa)$ be $G = \sum_0^6 k_i \tilde{A}_i + \varepsilon C + H$, where $\operatorname{supp} H = R_1$.

We want to prove that $\varepsilon \neq 1$. We have:

$(\tilde{A}_6, G) = 0$, which implies $-2k_6 + k_5 = 0$;
$(\tilde{A}_5, G) = 0$, which implies $-2k_5 + k_6 + k_4 = 0$;
$(\tilde{A}_4, G) = 0$, which implies $-2k_4 + k_5 + k_2 = 0$;
$(\tilde{A}_3, G) = 0$, which implies $-2k_3 + k_2 = 0$;
$(\tilde{A}_2, G) = 0$, which implies $-2k_2 + k_3 + k_4 + k_1 = 0$;
$(\tilde{A}_1, G) = 0$, which implies $-2k_1 + k_2 + k_0 = 0$; and
$(\tilde{A}_0, G) = 0$, which implies $k_0(A_0^2) + k_1 + (A_0, \varepsilon C + H) = 0$.

From these equalities it follows that $k_1 = \frac{3}{2}k_0$ and $k_0\left(A_0^2 + \frac{3}{2}\right) + (A_0, \varepsilon C + H) = 0$. Since $(A_0, \varepsilon C + H) > 0$ and $k_0 > 0$, we have $A_0^2 \neq -1$. Along all components of G except A_0 and C, the map $\tilde{b}$ is constant. If any of them were a (-1)-curve, then it would be possible to contract it. The new divisor still would have normal crossings because it was obtained by the blow-up process from the normal crossing divisor. Hence we may assume that the only (-1)-curve in G is C. But then it cannot be of multiplicity 1 [Mi2, Chap. 3, 1.4.1]. According to Property 7(ii), the multiplicity of C should be 2. □

LEMMA 8. *$\hat{\varphi} = \hat{y}^k$ for some $k \in \mathbb{Z}$.*

Proof. By a bilinear transformation of φ we may always achieve that $q_0 = \kappa = 0, q_1 = \infty$ (see Property 7(ii)). According to Lemma 3 and Property 5(viii),

$$S - B_\infty = S_0 = \{b^3 y = u^2 - b\}. \tag{17}$$

Since $\operatorname{Pic}(S_0) = \mathbb{Z}/2\mathbb{Z}$, divisors $2[\Phi_0 \cap S_0] \cong 0$ and $2[\Phi_\infty \cap S_0] \cong 0$. This implies that there exist polynomials $P(u,b,y)$ and $Q(u,b,y)$ such that

$$2[\Phi_0 \cap S_0] = (P(u,b,y))_0 \cap S_0,$$
$$2[\Phi_\infty \cap S_0] = (Q(u,b,y))_0 \cap S_0,$$

and

$$\varphi|_{S_0} = \frac{P(u,b,y)}{Q(u,b,y)}.$$

On the other hand, $\varphi|_{B_\infty} \neq \text{const}$. It follows that, in S,

$$\varphi = \frac{P(u,b,y)}{Q(u,b,y)}. \tag{18}$$

We may replace u^2 by $b^3 y - b$ in polynomials P and Q to obtain

$$P(u,b,y) = P_1(y) + uP_2(y,b) + bP_3(y,b), \tag{19}$$
$$Q(u,b,y) = Q_1(y) + uQ_2(y,b) + bQ_3(y,b). \tag{20}$$

Along B_0, the function y is linear and $u = b = 0$; along a general fiber B_p, we have that $b = p$, u is linear, and $y = (u^2 - p)/p^3$.

For two general fibers $B_p = b^{-1}(p) \subset S$ and $\Phi_q = \varphi^{-1}(q) \subset S$, we denote by $|B_p, \Phi_q|$ the number of points in their intersection $B_p \cap \Phi_q$ counted with multiplicities. We consider B_p and Φ_q as reduced curves isomorphic to $\mathbb{C}$. Recall that $\bar{B}_p$ and $\bar{\Phi}_q$ are the closures in $\tilde{S}$ of B_p and Φ_q, respectively. For general p, the fiber $\bar{B}_p$ is irreducible and the only point of $\bar{B}_p - B_p$ belongs to $\tilde{A}_0$. For general q, the fiber $\bar{\Phi}_q$ is irreducible and the only point of $\bar{\Phi}_q - \Phi_q$ belongs to E_0. Thus, for general p, q, the fibers $\bar{B}_p$ and $\bar{\Phi}_q$ intersect only inside S.

For two general points $p, q \in \mathbb{P}^1$, let

$$|B_p, \Phi_q| = (\bar{B}_p, \bar{\Phi}_q) = N. \tag{21}$$

For $q \neq 0, \infty$,

$$|B_0, \Phi_q| = (\bar{B}_0, \bar{\Phi}_q) = \frac{N}{2}. \tag{22}$$

Let r be the multiplicity of zero of function $\tilde{\varphi}$ along $\tilde{A}_0$. For general p we have

$$|B_p, \Phi_0| = (\bar{B}_p, \bar{\Phi}_0) = \frac{N - r}{2}, \tag{23}$$

$$|B_p, \Phi_\infty| = (\bar{B}_p, \bar{\Phi}_\infty) = \frac{N}{2}, \tag{24}$$

$$|B_0, \Phi_\infty| = (\bar{B}_0, \bar{\Phi}_\infty) = \frac{N}{4}. \tag{25}$$

In order to compute $|B_0, \Phi_0|$, we denote by $B = 2B_0 + \sum_1^6 n_i \tilde{A}_i$ the divisor of the zero fiber, $\tilde{b}^{-1}(0)$. By Lemma 6, $\tilde{B}_0$ intersects $\bar{\Phi}_0$ inside the surface S only; thus, for general p,

$$|B_0, \Phi_0| = (\bar{B}_0, \bar{\Phi}_0) = \frac{1}{2}(B_p, \bar{\Phi}_0) = \frac{1}{2}(\bar{B}_p, \bar{\Phi}_0) = \frac{N-r}{4}. \tag{26}$$

Combining (23), (26), and (19), we get

$$\deg P_1(y) = |B_0, \Phi_0| = \frac{N-r}{4} \quad \text{and} \tag{27}$$

$$2\deg_y P + \deg_u P = |B_p, \Phi_0| = \frac{N-r}{2}, \tag{28}$$

where $\deg_s H$ stands for the degree of a polynomial H relative to an indefinite s. Combining (24), (25), and (20) yields

$$\deg Q_1(y) = |B_0, \Phi_\infty| = \frac{N}{4} \quad \text{and} \tag{29}$$

$$2\deg_y Q + \deg_u Q = |B_p, \Phi_\infty| = \frac{N}{2}. \tag{30}$$

For our weights $\omega(u) = 1$, $\omega(b) = -1 + \rho$, and $\omega(y) = 11 - 3\rho$, this gives

$$\hat{P} = \hat{P}_1 = \hat{y}^{(N-r)/4}, \quad \hat{Q} = \hat{Q}_1 = \hat{y}^{N/4}, \quad \hat{\varphi} = \hat{y}^{-r/4}. \qquad \square$$

Proof of Theorem 1 (cont.). If there were a fibration φ then there would be a nonzero locally nilpotent derivation on $\widehat{R[\varphi]}$. Since the system defining $\hat{R}$ together with the equation $\hat{\varphi} y^{r/4} = 1$ again defines a reduced irreducible surface, we may conclude that $\widehat{R[\varphi]} = \hat{R}[\hat{\varphi}]$. But that is impossible by Lemma 2 and Lemma 1. $\square$

REMARK 2. The curve $\{y = 0\} \subset S$ contains two rational curves. As we have just proved, neither can be included in a $\mathbb{C}$-fibration (cf. [GMaMiR], where such curves are called *anomalous*).

CONJECTURE 1. *Let S be a rigid surface that admits a morphism $b\colon S \to \mathbb{P}^1$. Consider the divisor D at infinity, built as in Lemma 3. Let D have a graph Γ' that differs from graph Γ (see Lemma 3) only by the number of vertices in the vertical components of the graph. We conjecture that Theorem 1 remains valid in this case.*

4. ML Invariant of a Line Bundle over a Rigid Surface

In this section we establish a connection between the GML invariant of a surface and the ML invariant of the total space of a line bundle over the surface. Computing the ML invariant is often a highly involved matter even for surfaces and cylinders over surface, which is why we find it interesting to compute the invariant for threefolds of another type. Here we consider line bundles over rigid surfaces. The information on $\mathrm{GML}(S)$ appears to be very helpful.

Let us now recall some notions and notation that will be used in this section. The triple $\mathcal{L} = (L, \pi, X)$, where L, X are affine varieties and $\pi : L \to X$ is a morphism, defines a line bundle if there is a covering of X by Zariski open affine subsets U_α such that $L_\alpha = \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}_{t_\alpha}$ and, in the intersection $L_\alpha \cap L_\beta$, the function $g_{\alpha\beta} = t_\alpha/t_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)$ and does not vanish.

Assume that there are functions $h_\alpha \in F(U_\alpha)$ (i.e., rational in U_α) such that $g_{\alpha\beta} = h_\alpha/h_\beta$. Let the divisor D_L be such that $D_L \cap U_\alpha = (h_\alpha) \cap U_\alpha$ (recall that (h_α) is the divisor of h_α). We say that divisor D_L (and its class $[D_L]$) and line bundle $\mathcal{L}$ are associated. (Because the surface is smooth, we do not differentiate between Cartier and Weil divisors.) If $h_\alpha \in \mathcal{O}(U_\alpha)$ then divisor D_L is effective.

The set of functions $f_\alpha \in \mathcal{O}(U_\alpha)$, such that $f_\alpha/h_\alpha = f_\beta/h_\beta$ is a globally defined rational function $f \in F(X)$, defines the section of $\mathcal{L}$ by $t_\alpha(u) = f_\alpha(u)$ for a point $u \in U_\alpha$. If D_L is effective, then it has a section $t_\alpha(u) = h_\alpha(u)$ that vanishes (intersects a zero section) at D_L. The quotient of two sections is a rational function on X. The sheaf $\mathcal{F}$ of germs of sections of the line bundle is coherent, and the global sections $\Gamma(\mathcal{F})$ form a projective module over $\mathcal{O}(X)$, which generates $\mathcal{F}_x$ at every point $x \in X$ as an $\mathcal{O}_{x,X}$-module (see [Se, Sec. 45, Thm. 2; Se, Sec. 41, Prop. 5]). It follows from [Se] that one can choose a covering U_α $(\alpha = 0, \dots, K)$ of X and functions $\rho_\alpha \in \mathcal{O}(X)$ such that the ρ_α do not vanish in U_α and $g_{\alpha\beta}\rho_\alpha \in \mathcal{O}(X)$ for any $\beta = 0, \dots, K$. Hence the ring $\mathcal{O}(L) = \mathcal{O}(X)[tr_0, tr_1, \dots, tr_K]$, where $r_\alpha \in \mathcal{O}(X)$ and t is rational on L. (One can take $t = t_0$ and $r_\alpha = g_{\alpha 0}\rho_\alpha$.)

This ring naturally admits an lnd $\partial_\pi \in \mathrm{LND}(\mathcal{O}(L))$ such that $\partial_\pi f = 0$ if $f \in \mathcal{O}(X)$ and $\partial_\pi t = 1$. The $\mathbb{C}^+$-action ψ_π corresponding to ∂_π acts along the fibers of π.

LEMMA 9. *Let X be a smooth affine variety admitting a $\mathbb{C}^+$-action $\phi\colon \mathbb{C} \times X \to X$. Let $\mathcal{L} = (L, \pi, X)$ be an algebraic line bundle over X. Then the total space L of $\mathcal{L}$ admits a $\mathbb{C}^+$-action $\phi'\colon \mathbb{C} \times L \to L$ such that the image $\pi(\Phi')$ of the general orbit Φ' of the action ϕ' is the general orbit of the action ϕ.*

Proof. Since the action ϕ corresponds to a $\partial \in \mathrm{lnd}(R)$ that is nonzero, we can find an element $r \in R = \mathcal{O}(X)$ such that $\partial(r) = p \neq 0$ and $\partial(p) = 0$. Put $A = R^\partial[r]$ and $B = \mathrm{Frac}(R^\partial)[r] = \mathrm{Frac}(R)^\partial[r]$ (cf. [M-L, Lemma 1 of O. Hadas]).

As we know, $r_i \in B$. Consider the ideal generated by r_i, $i = 1, \dots, K$, in B. Since B is a principal ideal domain, this ideal is generated by some element q. So we can write $r_i = q\rho_i$ $(\rho_i \in B)$. The polynomials $\rho_0, \dots, \rho_K$ are relatively prime. Thus we can find $\varsigma_0, \dots, \varsigma_K \in B$ for which $\sum_i \rho_i\varsigma_i = 1$. Because all elements in B are elements of A divided by elements from R^∂, we can find elements $\tilde{\varsigma}_i$ $(i = 1, \dots, K)$ in A such that $\sum_i r_i\tilde{\varsigma}_i = q\Delta$, where $\Delta \in R^\partial$. Therefore $tq\Delta \in \mathcal{O}(L)$. Next, $tr_i = tq\rho_i$. Let $\delta \in R^\partial$ be a common denominator for the coefficients of all ρ_i. We can now define $\hat{\partial}$ by $\hat{\partial}(tq) = \delta$, $\hat{\partial}(r) = \delta\Delta$, and $\hat{\partial}(r') = 0$ for every function $r' \in R^\partial$. □

COROLLARY 1. *If* $\mathrm{ML}(X) = \mathbb{C}$ *and if* $\mathcal{L} = (L, \pi, X)$ *is an algebraic line bundle over* X, *then* $\mathrm{ML}(L) = \mathbb{C}$.

Our main object of interest is rigid surfaces.

DEFINITION 5. If a general orbit of a $\mathbb{C}^+$-action $\varphi\colon \mathbb{C}_\lambda \times L \to L$ on the total space of a line bundle $\mathcal{L} = (L, \pi, S)$ over a smooth affine surface S is not contained in a fiber of π, then we will call φ a *skew* $\mathbb{C}^+$-action.

EXAMPLE 1. Define the projection $\pi : \mathbb{C}^9 \to \mathbb{C}^7$ by

$$\pi(u, v, z, w, x, t, y, s, r) = (u, v, z, w, x, t, y) \tag{31}$$

and define the affine variety $L \subset \mathbb{C}^9$ by equations (1)–(13) and the following two:

$$su = rz, \tag{32}$$

$$s(z-1) = rv. \tag{33}$$

Then $\mathcal{L} = (L, \pi, S)$ is a line bundle over the surface S defined in Section 3 by (1)–(13).

Indeed, in the notation of Section 3, $S = S_0 \cup S_\infty$ and

$$\pi^{-1}(S_0) \cong S_0 \times \mathbb{C}^1_r, \quad s = rb;$$
$$\pi^{-1}(S_\infty) \cong S_\infty \times \mathbb{C}^1_s, \quad r = sb^{-1}.$$

There is a $\partial \in \mathrm{LND}(L)$ defined as follows:

$$\partial s = \partial r = 0, \quad \partial b = 0, \quad \partial u = s^m r^{n-m}, \quad \partial z = s^{m+1} r^{n-m-1},$$
$$\partial v = s^{m+2} r^{n-m-2}, \quad \partial w = 2vs^{m+3} r^{n-m-3}, \quad \partial x = 2vs^{m+5} r^{n-m-5},$$
$$\partial t = 2us^{m-1} r^{n-m+1}, \quad \partial y = 2us^{m-3} r^{n-m+3}.$$

For any $m \geq 3$ and $n \geq m+5$, this lnd is well-defined and provides a skew $\mathbb{C}^+$-action. Note that this line bundle has a section $Z = \{r = u,\ s = z\} \subset L$. The divisor D of the intersection of Z with the zero section Z_0 is the divisor associated to $\mathcal{L}$. Let $C = \{u = 0,\ b \neq 0,\ b \neq \infty\}$ and let F be a fiber $b =$ const. $\neq 0, \infty$. Then $D = C + B_0$ and, since $(u)_0 = C + B_0 + 2B_\infty \sim 0$, we have $D \sim -2B_\infty \sim -F$.

A similar example may be constructed over any rigid surface S.

LEMMA 10. *Let S be a rigid surface and let $\partial \in \mathrm{GML}(S)$. Then there exists a line bundle $(\mathcal{L}, \pi, S)$ and $\partial' \in \mathrm{LND}(\mathcal{O}(L))$ such that $\partial' f = 0$ for any $f \in \pi^*(F(S)^\partial)$* (*as mentioned in the Introduction*).

Proof. Consider a $\mathbb{C}$-fibration $f\colon S \to \mathbb{P}^1$ on S induced by ∂ and a nonsingular fiber $F = f^{-1}(\infty)$. Consider the line bundle $(\mathcal{L}, \pi, S)$ associated to the divisor $-mF$. Let $U_1 = S - F$ and $U_2 = S - F'$, where $F' = \{f = 0\}$ is another nonsingular fiber. (We may always assume that the fibers F and F' are nonsingular.) Then $L = L_1 \cup L_2$, where $L_1 = \pi^{-1}(U_1) \cong U_1 \times \mathbb{C}_{t_1}$ and $L_2 = \pi^{-1}(U_2) \cong U_2 \times \mathbb{C}_{t_2}$ and where $t_2 = f^m t_1$. The function $\tau = t_1 = f^{-m} t_2 \in \mathcal{O}(L)$ has zero of order m along F because f has a simple pole there. The divisor $(\tau) = Z_0 + m\pi^* F$. Thus, $\mathcal{O}(L) = \mathcal{O}(S)[\tau, \tau\omega_1^*, \ldots, \tau\omega_n^*]$, where the ω_i are rational functions on S such that $(\omega_i) \geq -mF$. Since $f(U_1)$ is an affine curve, there exists an lnd $\partial_1 \in \mathrm{LND}(\mathcal{O}(U_1))$ such that $\partial_1 f = 0$. Let N be bigger than the order of poles of $\partial_1\omega_i$ along F for all $i = 1, \dots, n$. One can define an lnd $\partial' \in \mathcal{O}(L)$ by $\partial'\tau = \partial' f = 0$ and $\partial' u = \tau^N \partial_1 u$ for $u \in \mathcal{O}(S)$. □

Take now any morphism $f\colon S \to \mathbb{P}^1$ of a rigid surface S onto $\mathbb{P}^1$ such that the general fiber of f is isomorphic to $\mathbb{C}$. The Picard group of S is generated by the divisor $[F]$ of the general fiber F and the divisors $[E_{i,j}]$ of the irreducible components $E_{i,j}$ of the singular fibers F_i: $[F_i] = \sum_1^{n_i} \alpha_{i,j}[E_{i,j}]$, $i = 1, \dots, n$, with relations reflecting that all the fibers are equivalent.

The group $\mathrm{Pic}(S) \otimes \mathbb{Q} \cong \mathbb{Q}^{\oplus N}$, where $N = \left(\sum n_i\right) - n + 1$. This group is generated by $[F]$ and $[E_{i,j}]$, $j > 1$ [Mi2, Chap. 3, Lemma 2.4.3.1].

Any element $l \in \mathrm{Pic}(S)$ may be represented uniquely as

$$l = m[F] + \sum_1^n \sum_1^{n_i} m_{i,j}[E_{i,j}],$$

where

(a) $m_{i,j} < \alpha_{i,j}$ for any i, $1 \leq i \leq n$, and any j, $1 \leq j \leq n_i$;
(b) $m_{i,j} \geq 0$ for at least one j, $1 \leq j \leq n_i$, for any i, $1 \leq i \leq n$.

DEFINITION 6. We will call the representation with properties (a) and (b) *standard* for the fibration f. We will call the element $l \in \mathrm{Pic}(S)$ *positive* relative to fibration f if, in the standard representation, $m \geq 0$.

The crucial fact for Lemma 10 and Example 1 is that the line bundles are associated to the nonpositive (relative to a given fibration) element of the Picard group. The following example presents the line bundle associated to a positive divisor.

EXAMPLE 2. Define the same projection $\pi : \mathbb{C}^9 \to \mathbb{C}^7$ by

$$\pi(u, v, z, w, x, t, y, s, r) = (u, v, z, w, x, t, y) \tag{34}$$

and the affine variety $L \subset \mathbb{C}^9$ by equations (1)–(13) and the following three:

$$su = rv, \tag{35}$$

$$st = ru(z - 1), \tag{36}$$

$$svz = rw. \tag{37}$$

Then $\mathcal{L} = (L, \pi, S)$ is a line bundle over the surface S defined in Section 3 by (1)–(13). In the notation of Section 3, we have

$$\pi^{-1}(S_0 - C) \cong (S_0 - C) \times \mathbb{C}^1_r, \quad s = rv/u;$$

$$\pi^{-1}(S_\infty - C_1) \cong (S_\infty - C_1) \times \mathbb{C}^1_s, \quad r = su/v.$$

Here $C_1 = \{v = 0, b \neq 0, b \neq \infty\}$ does not intersect $C = \{u = 0, b \neq 0, b \neq \infty\}$.

The divisor associated to $\mathcal{L}$ is the intersection divisor of the section $Z_1 = \{r = u, s = v\} \subset L$, and the zero section Z_0 is $B_0 + B_\infty$. Therefore $\mathcal{L}$ is associated to a positive (relative to the fibration) divisor. We will show that there are no skew actions on L.

Proposition 1. *Let $\mathcal{L} = (L, \pi, S)$ be an algebraic line bundle over a rigid surface S. Assume that L admits a skew $\mathbb{C}^+$-action $\alpha\colon \mathbb{C} \times L \to L$. Then there exists a skew $\mathbb{C}^+$-action $\beta\colon \mathbb{C} \times L \to L$, $\partial' \in \mathrm{GML}(S)$, and a morphism $g\colon S \to \mathbb{P}^1$ of S induced by ∂' such that the following statements hold.*

(i) *The general fiber of g is $\mathbb{C}$.*
(ii) *$g(\pi(O))$ is a point for a general orbit O of β.*
(iii) *There is no nonzero section Z of $\mathcal{L}$ over an open subset $U \subset S$ such that*:
  (a) *$g(U) = \mathbb{P}^1$;*
  (b) *the components of $g^{-1}(p) \cap U$ are isomorphic to $\mathbb{C}$ for each $p \in \mathbb{P}^1$;*
  (c) *$g(\pi(Z \cap Z_0))$ is a finite set in $\mathbb{P}^1$.*

Proof. We proceed by first demonstrating three preliminary results.

Lemma 11. *Let R be an affine ring and let $Q = R[t, t^{r_1}\omega_1, \dots, t^{r_k}\omega_k]$, where t is a variable and $\omega_i \in \mathrm{Frac}(R)$. Let $\partial \in \mathrm{LND}(Q)$, which is not identically zero on R. Then there exists a locally nilpotent derivation on Q that is t-homogeneous and is not identically zero on R.*

Proof. Let us introduce a weight function on Q by $w(t) = 1$, $w(r) = 0$ for $r \in R^*$ and $w(0) = -\infty$. Consider the (nonzero) locally nilpotent derivation $\bar{\partial}$ that corresponds to this weight function [KM-L2]. Clearly $\bar{\partial} \in \mathrm{LND}(Q)$ because Q is a graded algebra relative to the introduced weight function. Then $\bar{\partial}(t) = t^{k+1}\varepsilon(t)$ and $\bar{\partial}(r) = t^k\varepsilon(r)$, where $\varepsilon \in \mathrm{DER}(Q)$ is such that $\varepsilon(t), \varepsilon(r) \in \mathrm{Frac}(R)$ if $r \in R$. Since our goal is to produce a locally nilpotent derivation on R, we may assume that $k > 0$ (otherwise ∂ can be restricted on R). It remains to show that $\bar{\partial}$ is not identically zero on R. By way of contradiction, assume that $\bar{\partial}$ is identically zero on R. Then $\bar{\partial}(t) = t^{k+1}\varepsilon(t)$ implies that $\bar{\partial}(t) = 0$, so $\bar{\partial}$ would be identically zero—contrary to the facts. Indeed, if deg is the degree function induced on Q by $\bar{\partial}$ then $\deg(t) - 1 = (k+1)\deg(t) + \deg(\varepsilon(t))$. But since we assumed that $\bar{\partial}$ is identically zero on R, it follows that $\deg(\varepsilon(t)) = 0$ if $\varepsilon(t) \neq 0$. (If $\varepsilon(t) = 0$ then $\deg(\varepsilon(t)) = -\infty$.) So if $\varepsilon(t) \neq 0$ then $\deg(t) - 1 = (k+1)\deg(t)$. Since $k > 0$ we then see that $\deg(t) < 0$, which is impossible. This proves the lemma. □

Corollary 2. *If* $\dim(R) > 1$, *then* $\mathrm{Frac}(Q)^{\bar{\partial}}$ *contains a nonconstant rational function from* $\mathrm{Frac}(R)$.

Proof. Since $\bar{\partial}$ is t-homogeneous, the ring of $\bar{\partial}$-constants is generated by t-homogeneous elements. Because $\dim(Q) > 2$, there exist two algebraically independent homogeneous $\bar{\partial}$-constants, say $f_1 = t^m\omega_1$ and $f_2 = t^n\omega_2$. Then $f_1^n f_2^{-m} \in \mathrm{Frac}(R)$. □

We apply Corollary 2 assuming that $R = \mathcal{O}(S)$ and $Q = \mathcal{O}(L)$ and that $t \in \mathcal{O}(L)$ is any regular function on L that is linear along the general fiber and vanishing at the zero section. Let β be the $\mathbb{C}^+$-action defined by a locally nilpotent derivation $\bar{\partial}$. By construction, all the points of the zero section $Z_0 \subset L$ are fixed by β and there exists a β-invariant function $f = \pi^* g \in \operatorname{Frac}(\mathcal{O}(L))$ with $g \in \operatorname{Frac}(\mathcal{O}(S))$. Using the Stein factorization, we may assume that a general fiber of $g^{-1}(p)$, $p \in \mathbb{P}^1$, is connected (and irreducible).

LEMMA 12. *$g\colon S \to \mathbb{P}^1$ is a morphism.*

Proof. We will identify S with the zero section Z_0 (i.e., $S \subset L$); by construction, S is β-invariant. The function f is the composition of rational maps: $L \xrightarrow{\pi} S \xrightarrow{g} \mathbb{P}^1$. Let p be a point in $\mathbb{P}^1$. Let $C_p = g^{-1}(p) \subset S$ and let $T_p = \pi^{-1}(C_p) = f^{-1}(p)$. Because f is β-invariant, T_p is β-invariant as well and thus consists of β-orbits. Since β is a skew action, these orbits are not mapped to a point by π. Hence $C_p = \pi(T_p) = T_p \cap Z_0 \cong \mathbb{C}$. By construction, T_p is the restriction of our line bundle $\mathcal{L}$ over C_p; thus $T_p \cong \mathbb{C}^2$.

If g were not a morphism then there would be a point $s \in S$ contained in every fiber $C_p = \{g = p\}$. In this case, for every p the set T_p would contain two β-invariant intersecting curves: C_p and $A_s = \pi^{-1}(s)$. But then all the points of T_p for all p would be fixed by β. This contradiction shows that such a point s does not exist and that g is a morphism. □

Proof of Proposition 1 (cont.). Parts (i) and (ii) of the proposition are proved in Lemma 12. Assume now that there exists a section Z as in part (iii).

By (a)–(c) of part (iii), $Z \cong U$ admits a $\mathbb{C}$-fibration over $\mathbb{P}^1$ such that $Z \cap Z_0$ is the union of a finite set of fibers of this fibration. We want to show that Z is β-invariant and that this fibration should be induced by the restriction of β on Z. This would lead to a contradiction, because a $\mathbb{C}^+$-action has an affine base [MaMi1, Lemma 1.1].

In the notation of Lemma 12, part (iii)(c) means that, for general $p \in \mathbb{P}^1$, the curve $B_p = Z \cap T_p$ does not intersect Z_0. In particular, the curves $B_p \subset T_p$ and $C_p = Z_0 \cap T_p \subset T_p$ do not intersect.

Since $C_p = \pi(T_p)$ and since Z is a section, it follows that $B_p = Z \cap T_p$ is a section of the bundle over C_p and that $\pi|_{B_p}\colon B_p \to C_p$ is an isomorphism. Hence $B_p \cong \mathbb{C}$. Thus, in the β-invariant set $T_p \cong \mathbb{C}^2$ we have two rational disjoint curves: C_p is a β-orbit in T_p, so the same should be true for B_p. Therefore, Z is β-invariant and the base of the restriction of the induced fibration should be affine. This contradicts (iii)(a). □

COROLLARY 3. *Let S be a rigid surface, let* $\operatorname{GML}(S) = \mathbb{C}(f)$, *and let $f\colon S \to \mathbb{P}^1$ be the corresponding fibration. Let $\mathcal{L} = (L, \pi, S)$ be a line bundle over S. Then* $\operatorname{ML}(L) = \mathcal{O}(S)$ *if $\mathcal{L}$ is associated to a positive (relative to fibration f) element l of* $\operatorname{Pic}(S)$.

Proof. Let $\psi\colon \mathbb{C} \times L \to \mathbb{C}$ be a skew action on L. According to Proposition 1, ψ will give rise to a $\mathbb{C}$-fibration $g\colon S \to \mathbb{P}^1$. Since $\operatorname{GML}(S) = \mathbb{C}(f)$, the fibrations g and f must be equivalent. Let the element $l \in \operatorname{Pic}(S)$ associated to $\mathcal{L}$ have the standard representation

$$l = m[F] + \sum_1^n \sum_1^{n_i} m_{i,j}[E_{i,j}];$$

let l_+ be the sum of summands with a nonnegative coefficient, and let l_- be the sum of summands with a negative coefficient. Let D_+ and D_- be the union of components appearing in l_+ and l_-, respectively.

Over $U = S - D_- \subset S$, the line bundle $\mathcal{L}$ is associated to the effective divisor and hence has a section Z_U such that $Z_0 \cap Z_U \subset D_+$. Since supp D_+ contains at least one component of every fiber of g, U enjoys all the properties of Proposition 1(iii), which is impossible if ψ is a skew $\mathbb{C}^+$-action. □

Corollary 3 provides a situation in which—similar to the case of the trivial line bundle—the isomorphism $\mathrm{ML}(S) \cong \mathcal{O}(S)$ implies $\mathrm{ML}(L) \cong \mathrm{ML}(S) \cong \mathcal{O}(S)$ [BM-L2].

The following questions remain open.

QUESTIONS. 1. Let S be a rigid surface, let $\mathrm{GML}(S) = \mathbb{C}(f)$, and let $f: S \to \mathbb{P}^1$ be the corresponding fibration. Let $\mathcal{L} = (L, \pi, S)$ be a line bundle over S. Is it possible that $\mathrm{ML}(L) = \mathcal{O}(S)$ if $\mathcal{L}$ is associated to a nonpositive (relative to fibration f) element l of $\mathrm{Pic}(S)$?

2. Assume that S is rigid and that $\mathrm{GML}(S) = \mathbb{C}$. When does $\mathrm{ML}(L) = \mathcal{O}(S)$?

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