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Jacquet Modules and the Langlands Classification

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1. Introduction

The Langlands classification is a fundamental result in representation theory and the theory of automorphic forms. It gives a bijective correspondence between irreducible admissible representations of a connected reductive group G and triples of Langlands data. It was proved by Langlands for real groups [L]. The proof for p-adic groups is due to Borel and Wallach [BoW], Silberger [Si], and Konno [Ko].

We consider the p-adic case, so let G denote a connected reductive p-adic group. Let (P, ν, τ) be a set of Langlands data in the subrepresentation setting of the Langlands classification. Then $P = MU$ is a standard parabolic subgroup of G, τ is an irreducible tempered representation of M, and $\nu \in (\mathfrak{a}_M)^*_-$ (see Section 2 for definitions). Write $\pi = L(P, \nu, \tau)$ for the irreducible representation of G corresponding to (P, ν, τ). Then π is the unique irreducible subrepresentation of the corresponding standard module—that is, the induced representation $i_{G,M}(\exp \nu \otimes \tau)$. We show that the (normalized) Jacquet module $r_{M,G}(i_{G,M}(\exp \nu \otimes \tau))$ contains $\exp \nu \otimes \tau$ with multiplicity 1 and has no other subquotients with central exponent ν. This is a useful result (e.g., [J2, Lemma 3.4] is essentially a special case) that was expected but for which there seems to be no proof available in the literature. Our main purpose here is to fill that gap.

As an application, we prove a dual version of the Langlands classification, essentially extending the Zelevsinky classification from general linear groups to connected reductive groups (cf. [Z]). An irreducible representation θ with unitary central character is called *anti-tempered* if it satisfies the Casselman criterion for temperedness but with the usual inequalities reversed. Equivalently, $\hat{\theta}$—its dual under the involution of [Au; Ber; SSt]—is tempered. If π is an irreducible admissible representation of G, then there exists a unique triple (Q, μ, θ) with $Q = LU$ a standard parabolic subgroup, $\mu \in (\mathfrak{a}_L)^*_+$, and θ an irreducible anti-tempered representation of L such that π is equivalent to the unique irreducible subrepresentation of $i_{G,L}(\exp \mu \otimes \theta)$ (Theorem 6.3). The growing role of duality in representation theory and its conjectured relation with the Arthur parameterization convinced the authors to include this application, especially as it contains information on the composition series (the existence of a unique irreducible subrepresentation) that is

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not a simple consequence of duality. We note that this is also essentially a known result for which we do not know of a proof in the literature.

In [Ko] it was noted that there is a problem with [Si, Lemma 5.3]. We remark that the main result in this paper can serve as a substitute for Lemma 5.3 in Silberger's proof of the Langlands classification. The main result also gives another proof that the Langlands subrepresentation appears with multiplicity 1 in the corresponding standard module.

Our proof of the Jacquet module result is essentially combinatorial in nature. By a result of [BeZ; Ca], we can calculate $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ or the corresponding exponents that appear. Our argument uses the inequalities that arise from having $\nu \in (\mathfrak{a}_M)^*_-$ and τ tempered (the Casselman criterion) to show that any other exponents appearing in $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ are necessarily different. As a technical remark we note that, in order to carry out this analysis, the various exponents must be converted to exponents in $\mathfrak{a}^*$ (the dual of the Lie algebra associated to the maximal split torus of the minimal parabolic subgroup).

For general linear groups, the Langlands classification and the Zelevinsky classification are related by the Zelevinsky involution (cf. [T]). The duality of [Au; Ber; SSt] generalizes the Zelevinsky involution, and it may be used in a similar fashion to construct the dual Langlands data for an irreducible admissible representation from the (ordinary) Langlands data for its dual. One issue arises in this process: the duality of [Au; Ber; SSt] is at the Grothendieck group level and so does not preserve composition series. To show that π is the unique irreducible subrepresentation of $i_{G,L}(\exp \mu \otimes \theta)$, we note that duality does imply $\exp \mu \otimes \theta$ is the unique irreducible subquotient of $r_{L,G} \circ i_{G,L}(\exp \mu \otimes \theta)$ having its central exponent; the result then follows from Frobenius reciprocity.

We now briefly discuss the contents of this paper section by section. In Section 2, we introduce notation and review some background results. In Section 3, we prove a technical lemma that describes the action of the Weyl group on certain elements in the dual Lie algebra $\mathfrak{a}^*$. This result, together with a criterion for temperedness proved in Section 4 (a variation of the Casselman criterion), is the basis for proving the uniqueness of central characters and central exponents in Section 5. In Section 6 we apply these results to obtain the dual Langlands classification; in Section 7 we show that, for general linear groups, it is essentially the same as the Zelevinsky classification.

Before closing the Introduction, we would like to take this opportunity to note an error in the paper [BJ2]: the ordering on $\mathfrak{a}_0^*$ defined in the beginning of Section 3 of that paper should match that in Section XI.1 of [BoW], but it does not. We give the correct definition in Remark 2.3. Note that this does not affect the arguments given in [BJ2]. We also take this opportunity to thank P. Schneider and the referee for their comments, which were helpful in revising the paper.

2. Preliminaries

In this section, we review some background material and notation that will be used in what follows.

Let F be a non-Archimedean local field with finite residue field $\mathbb{F}_q$. We denote by $|\cdot|$ the absolute value on F satisfying $|\varpi| = q^{-1}$ for a uniformizer ϖ. Let G be the group of F-points of a connected reductive algebraic group defined over F. Fix a maximal split torus A in G. We denote by $W = W(G, A)$ the Weyl group of G with respect to A. Let $\Phi = \Phi(G, A)$ be the set of roots. Fix a minimal parabolic subgroup P_0 containing A. The choice of P_0 determines the set of simple roots $\Pi \subset \Phi$ and the set of positive roots $\Phi^+ \subset \Phi$. If $\alpha \in \Phi^+$, we write $\alpha > 0$.

Let $P = MU \subset G$ be a standard parabolic subgroup of G. We denote by $\Pi_M \subset \Pi$ the corresponding set of simple roots. Let A_M be the split component of the center of M and let $X(M)_F$ be the group of F-rational characters of M. The discussion here follows [A2, Sec. 5]. The restriction homomorphism $X(M)_F \to X(A_M)_F$ is injective and has a finite cokernel. Therefore, we have a canonical linear isomorphism

$$\mathfrak{a}_M^* = X(M)_F \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} X(A_M)_F \otimes_{\mathbb{Z}} \mathbb{R}.$$

If L is a standard Levi subgroup such that $L < M$, then

$$A_M \subset A_L \subset L \subset M.$$

The restriction $X(M)_F \to X(L)_F$ is injective and induces a linear injection $\iota_M^L \colon \mathfrak{a}_M^* \to \mathfrak{a}_L^*$. The restriction $X(A_L)_F \to X(A_M)_F$ is surjective and induces a linear surjection $r_M^L \colon \mathfrak{a}_L^* \to \mathfrak{a}_M^*$. Let $(\mathfrak{a}_L^M)^*$ denote the kernel of the restriction r_M^L. Then

$$\mathfrak{a}_L^* = \iota_M^L(\mathfrak{a}_M^*) \oplus (\mathfrak{a}_L^M)^*$$

(see [A2, Sec. 5] for details). In the case of the dual Lie algebra $\mathfrak{a}^* = \mathfrak{a}_A^*$ corresponding to the maximal split torus A of G, we write simply

$$\iota_M \colon \mathfrak{a}_M^* \to \mathfrak{a}^* \quad \text{and} \quad r_M \colon \mathfrak{a}^* \to \mathfrak{a}_M^*.$$

Note that we have $r_M \circ \iota_M = \mathrm{id}$.

There is a homomorphism (cf. [H-C]) $H_M \colon M \to \mathfrak{a}_M$ such that $q^{\langle \chi, H_M(m) \rangle} = |\chi(m)|$ for all $m \in M$ and $\chi \in X(M)_F$. Given $\nu \in \mathfrak{a}_M^*$, let us write

$$\exp \nu = q^{\langle \nu, H_M(\cdot) \rangle}$$

for the corresponding character of M.

Let $R(G)$ denote the Grothendieck group of the category of smooth finite-length representations of G. For a smooth finite-length representation π of G, we denote by $\mathrm{s.s.}(\pi)$ the image of π in $R(G)$. There exists a natural partial order on $R(G)$. If π_1 and π_2 are smooth finite-length representations of G, we write $\pi_1 \leq \pi_2$ if $\mathrm{s.s.}(\pi_1) \leq \mathrm{s.s.}(\pi_2)$. In particular, if π_1 is irreducible then $\pi_1 \leq \pi_2$ means that π_1 is a subquotient of π_2.

We denote by $i_{G,M}$ the functor of normalized parabolic induction and by $r_{M,G}$ the normalized Jacquet functor. The Aubert duality operator D_G is defined on $R(G)$ by

$$D_G = \sum_{M \leq G} (-1)^{|\Pi_M|} i_{G,M} \circ r_{M,G},$$

where the sum runs over the set of all standard Levi subgroups of G. Let π be an irreducible smooth representation of G. We define $\hat{\pi} = \pm D_G(\pi)$, taking the sign $+$ or $-$ so that $\hat{\pi}$ is a positive element in $R(G)$.

LEMMA 2.1. *Let π be an irreducible smooth representation of G and let χ be a character of G. Then $\widehat{\chi \otimes \pi} = \chi \otimes \hat{\pi}$ (where $\chi \otimes \pi$ denotes the representation of G defined by $(\chi \otimes \pi)(g) = \chi(g)\pi(g)$).*

Proof. If $P = MU$ is a standard parabolic subgroup of G and if σ is a smooth representation of M, then [BeZ, Prop. 1.9] implies

$$i_{G,M}(\chi \otimes \sigma) = \chi \otimes i_{G,M}(\sigma), \qquad r_{M,G}(\chi \otimes \pi) = \chi \otimes r_{M,G}(\pi).$$

The lemma now follows from the definition of D_G. □

Let $\Pi(P, A_M) = \{r_M(\alpha) \mid \alpha \in \Pi - \Pi_M\}$ denote the set of simple roots for the pair (P, A_M). Choose a W-invariant inner product $\langle \cdot, \cdot \rangle : \mathfrak{a}^* \times \mathfrak{a}^* \to \mathbb{R}$. As in [Si], identifying $\mathfrak{a}_M^*$ with the subspace $\iota(\mathfrak{a}_M^*) \subset \mathfrak{a}^*$, we set

$$(\mathfrak{a}_M)_+^* = \{x \in \mathfrak{a}_M^* \mid \langle x, \alpha \rangle > 0 \ \forall \alpha \in \Pi(P, A_M)\},$$

$$+\bar{\mathfrak{a}}_M^* = \left\{ x \in \mathfrak{a}_M^* \;\middle|\; x = \sum_{\alpha \in \Pi(P, A_M)} c_\alpha \alpha,\ c_\alpha \geq 0 \right\},$$

and

$$(\mathfrak{a}_M)_-^* = -(\mathfrak{a}_M)_+^*, \qquad -\bar{\mathfrak{a}}_M^* = -(+\bar{\mathfrak{a}}_M^*).$$

A set of Langlands data for G is a triple (P, ν, τ) with the following properties:

(1) $P = MU$ is a standard parabolic subgroup of G;
(2) $\nu \in (\mathfrak{a}_M)_-^*$; and
(3) τ is (the equivalence class of) an irreducible tempered representation of M.

We now state the Langlands classification (cf. [BoW; Ko; Si]).

THEOREM 2.2 (The Langlands classification). *Suppose (P, ν, τ) is a set of Langlands data for G. Then the induced representation $i_{G,M}(\exp \nu \otimes \tau)$ has a unique irreducible subrepresentation, which we denote by $L(P, \nu, \tau)$. Conversely, if π is an irreducible admissible representation of G, then there exists a unique (P, ν, τ) as before such that $\pi \cong L(P, \nu, \tau)$.*

This theorem describes the Langlands classification in the subrepresentation setting. It can also be formulated in the quotient setting, in which case $\nu \in (\mathfrak{a}_M)_+^*$. We work in the subrepresentation setting for technical reasons: if $\pi \cong L(P, \nu, \tau)$, then $\exp \nu \otimes \tau \leq r_{M,G}(\pi)$.

REMARK 2.3. This is a correction to our paper [BJ2], where the order $\geq$ and strict order $>$ on $\mathfrak{a}^*$ are defined incorrectly. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots and let $\beta_1, \ldots, \beta_n$ be the elements of $\sum \mathbb{R}\alpha_i$ satisfying $\langle \beta_i, \alpha_j \rangle = \delta_{ij}$. As in [BoW, Sec. XI.2], define $\mu \geq \nu$ if $\langle \mu - \nu, \beta_i \rangle \geq 0$ for all β_i. Define $\mu > \nu$ if $\mu \geq \nu$ and $\langle \mu - \nu, \beta_i \rangle > 0$ for some β_i.

3. A Combinatorial Lemma

In this section, we prove a technical lemma that plays a key role in the proof of Proposition 5.3.

Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (the set of simple roots). As in [BoW, Chap. IX], set $\mathcal{F} = \sum \mathbb{R}\alpha_i$. Then $\mathfrak{a}^* = \mathfrak{z}^* \oplus \mathcal{F}$, where $\mathfrak{z}^* = \{x \in \mathfrak{a}^* \mid \langle x, \alpha\rangle = 0 \ \forall \alpha \in \Pi\}$. Define $\beta_1, \ldots, \beta_n \in \mathcal{F}$ by $\langle \beta_i, \alpha_j\rangle = \delta_{ij}$. Then $\mathcal{F} = \sum \mathbb{R}\beta_i$. More generally, if $I \subset \{1, \ldots, n\}$, then $\mathfrak{a}^* = \mathfrak{z}^* + \sum_{i \notin I} \mathbb{R}\beta_i + \sum_{i \in I} \mathbb{R}\alpha_i$ (see [BoW, Chap. IV.6.6]). Note that if M is the standard Levi factor with $\Pi_M = \{\alpha_i \mid i \in I\}$, then $i_M(\mathfrak{a}_M^*) = \mathfrak{z}^* + \sum_{i \notin I} \mathbb{R}\beta_i$. The set of simple roots Π is a basis of an abstract root system in $\mathcal{F}$. Let

$$\bar{\mathcal{F}}_+ = \bar{\mathfrak{a}}_+^* \cap \mathcal{F} = \{x \in \mathcal{F} \mid \langle x, \alpha\rangle \geq 0 \ \forall \alpha \in \Pi\}.$$

LEMMA 3.1. *Let $x, y \in \bar{\mathcal{F}}_+$ and $w \in W$ with $w \neq 1$. Then $\langle wx, y\rangle \leq \langle x, y\rangle$.*

Proof. That $\langle wx - x, y\rangle \leq 0$ is an immediate consequence of Proposition 18 in [Bou, Chap. 5, Sec. 1.6]. □

We take a moment to recall a geometric result of Langlands (see e.g. [BoW, Chap. IV.4.6]), reformulating it slightly for the situation at hand. For $F \subset \{1, \ldots, n\}$, set

$$T_F = \left\{ x \in \mathcal{F} \;\middle|\; x = \sum_{i \notin F} c_i\beta_i + \sum_{j \in F} c_j\alpha_j \right.$$
$$\left. \text{with } c_i < 0 \text{ for } i \notin F \text{ and } c_j \geq 0 \text{ for } j \in F \right\},$$

noting that $T_F = -S_F$ in the notation of [BoW].

THEOREM 3.2. *The sets T_F partition $\mathcal{F}$ into 2^n disjoint subsets. In particular, if $F \neq F'$, then $T_F \cap T_{F'} = \emptyset$. Further, if $x \in \mathcal{F}$, then there exists a unique $F \subset \Pi$ such that $x \in T_F$.*

Proof. This follows immediately from Lemmas IV.6.9–11 in [BoW]. □

Let $x \in \mathcal{F}$ and let $F \subset \{1, \ldots, n\}$ be the corresponding subset. Then $x = \sum_{i \notin F} c_i\beta_i + \sum_{j \in F} c_j\alpha_j$ for unique $c_i < 0$ $(i \notin F)$ and $c_j \geq 0$ $(j \in F)$. We then let x_0 denote the projection onto $\text{span}_{i \notin F}\{\beta_i\}$; that is,

$$x_0 = \sum_{i \notin F} c_i\beta_i.$$

Set $W^{M,A} = \{w \in W \mid w\alpha > 0 \text{ for all } \alpha \in \Pi_M\}$. We then have the following statement.

LEMMA 3.3. *Let $P = MU$ be a standard parabolic subgroup of G. Let $F \subset \{1, \ldots, n\}$ be such that $\Pi_M = \{\alpha_i \mid i \in F\}$. If $x \in T_F$ and $w \in W^{M,A}$ with $w \neq 1$, then $(wx)_0 \neq x_0$.*

Proof. Suppose the simple roots of G are labeled so that $\Pi_M = \{\alpha_{k+1}, \dots, \alpha_n\}$. Let $x = c_1\beta_1 + \cdots + c_k\beta_k + c_{k+1}\alpha_{k+1} + \cdots + c_n\alpha_n \in \mathcal{F}$, with $c_1, \dots, c_k < 0$ and $c_{k+1}, \dots, c_n \geq 0$. It suffices to show that, if $wx = d_1\beta_1 + \cdots + d_k\beta_k + d_{k+1}\alpha_{k+1} + \cdots + d_n\alpha_n$, then $d_i \neq c_i$ for some $i \in \{1, \dots, k\}$.

Let

$$w = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

be the matrix for the action of w with respect to the basis $\beta_1, \dots, \beta_k, \alpha_{k+1}, \dots, \alpha_n$. Observe that, for $i, j \in \{1, \dots, k\}$, we have

$$\begin{aligned} \langle w\beta_i, \beta_j\rangle &= \langle a_{1i}\beta_1 + \cdots + a_{ki}\beta_k + a_{(k+1)i}\alpha_{k+1} + \cdots + a_{ni}\alpha_n, \beta_j\rangle \\ &= a_{1i}\langle\beta_1, \beta_j\rangle + \cdots + a_{ki}\langle\beta_k, \beta_j\rangle. \end{aligned}$$

Also, for $j \in \{1, \dots, k\}$ and $l \in \{k+1, \dots, n\}$ we have

$$\begin{aligned} \langle w\alpha_l, \beta_j\rangle &= \langle a_{1l}\beta_1 + \cdots + a_{kl}\beta_k + a_{(k+1)l}\alpha_{k+1} + \cdots + a_{ni}\alpha_n, \beta_j\rangle \\ &= a_{1l}\langle\beta_1, \beta_j\rangle + \cdots + a_{kl}\langle\beta_k, \beta_j\rangle. \end{aligned}$$

Let

$$Q = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \end{pmatrix}, \qquad R = \begin{pmatrix} \langle\beta_1, \beta_1\rangle & \dots & \langle\beta_1, \beta_k\rangle \\ \vdots & & \vdots \\ \langle\beta_k, \beta_1\rangle & \dots & \langle\beta_k, \beta_k\rangle \end{pmatrix},$$

and

$$S = \begin{pmatrix} \langle w\beta_1, \beta_1\rangle & \dots & \langle w\beta_k, \beta_1\rangle & \langle w\alpha_{k+1}, \beta_1\rangle & \dots & \langle w\alpha_n, \beta_1\rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ \langle w\beta_1, \beta_k\rangle & \dots & \langle w\beta_k, \beta_k\rangle & \langle w\alpha_{k+1}, \beta_k\rangle & \dots & \langle w\alpha_n, \beta_k\rangle \end{pmatrix}.$$

Then, noting $R = R^T$, we have $RQ = S$. Now consider wx:

$$wx = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kn} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} Q \\ * \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_k \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} Q\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \\ * \end{pmatrix};$$

here the entries for $*$ are left unspecified because they do not play a role in what follows. That is, if $wx = d_1\beta_1 + \cdots + d_k\beta_k + d_{k+1}\alpha_{k+1} + \cdots + d_n\alpha_n$, then

$$\begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix} = Q\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = R^{-1}S\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

(noting that R is invertible because $\beta_1, \dots, \beta_k$ are linearly independent and $\langle\cdot,\cdot\rangle$ is nondegenerate).

We would like to show that $R^{-1}S\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \neq \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$ or, equivalently, that $S\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \neq R\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$. We have

$$S\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$
$$= \begin{pmatrix} c_1\langle w\beta_1, \beta_1\rangle + \cdots + c_k\langle w\beta_k, \beta_1\rangle + c_{k+1}\langle w\alpha_{k+1}, \beta_1\rangle + \cdots + c_n\langle w\alpha_n, \beta_1\rangle \\ \vdots \\ c_1\langle w\beta_1, \beta_k\rangle + \cdots + c_k\langle w\beta_k, \beta_k\rangle + c_{k+1}\langle w\alpha_{k+1}, \beta_k\rangle + \cdots + c_n\langle w\alpha_n, \beta_k\rangle \end{pmatrix}$$

and (using $R = R^T$)

$$R\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = \begin{pmatrix} c_1\langle \beta_1, \beta_1\rangle + \cdots + c_k\langle \beta_k, \beta_1\rangle \\ \vdots \\ c_1\langle \beta_1, \beta_k\rangle + \cdots + c_k\langle \beta_k, \beta_k\rangle \end{pmatrix}.$$

Therefore, the ith entry of $R\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} - S\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is equal to

$$\begin{aligned} &c_1\langle \beta_1, \beta_i\rangle + \cdots + c_k\langle \beta_k, \beta_i\rangle \\ &\quad - [c_1\langle w\beta_1, \beta_i\rangle + \cdots + c_k\langle w\beta_k, \beta_i\rangle + c_{k+1}\langle w\alpha_{k+1}, \beta_i\rangle + \cdots + c_n\langle w\alpha_n, \beta_i\rangle] \\ &= c_1\langle \beta_1 - w\beta_1, \beta_i\rangle + \cdots + c_k\langle \beta_k - w\beta_k, \beta_i\rangle \\ &\quad - c_{k+1}\langle w\alpha_{k+1}, \beta_i\rangle - \cdots - c_n\langle w\alpha_n, \beta_i\rangle. \end{aligned}$$

Now, $w \in W^{M,A}$ implies that for $j = k+1, \ldots, n$ we have $w\alpha_j > 0$ and hence $\langle w\alpha_j, \beta_i\rangle \geq 0$. Lemma 3.1 tells us that, for $j = 1, \ldots, k$, $\langle \beta_j - w\beta_j, \beta_i\rangle \geq 0$. By assumption, $c_1, \ldots, c_k < 0$ and $-c_{k+1}, \ldots, -c_n \leq 0$, so the ith entry is ≤ 0. Now fix $i \in \{1, \ldots, k\}$ such that $\beta_i - w\beta_i \neq 0$. Since the inner product is symmetric and W-invariant, we have

$$0 < \langle \beta_i - w\beta_i, \beta_i - w\beta_i\rangle = \langle \beta_i, \beta_i\rangle - 2\langle w\beta_i, \beta_i\rangle + \langle \beta_i, \beta_i\rangle = 2\langle \beta_i - w\beta_i, \beta_i\rangle.$$

Therefore, the ith entry is less than zero, from which the lemma follows. □

4. Criterion for Temperedness

In this section, we give a variation of the Casselman criterion for temperedness (cf. [Ca; Wa]). The arguments later in this paper use exponents in $\mathfrak{a}^*$ (rather than in the different $\mathfrak{a}_M^*$ that arise) to facilitate comparison. Thus, in this section, we reformulate the Casselman criterion in terms of exponents in $\mathfrak{a}^*$ (Corollary 4.4) to set up these later arguments. Our starting point is the Cassleman criterion as formulated in [Wa, Prop. III.2.2.].

Let π be an irreducible admissible representation of G. Let

$$\mathcal{M}_{\min}(\pi) = \{L \text{ standard Levi} \mid r_{L,G}(\pi) \neq 0 \text{ but } r_{N,G}(\pi) = 0\ \forall N < L\}.$$

We define

$$Exp(\pi) = \{\iota(\mu) \mid \exp \mu \otimes \rho \le r_{L,G}(\pi)$$
$$\text{for some irreducible unitary } \rho \text{ and some } L \in \mathcal{M}_{\min}(\pi)\}$$

(noting that such a ρ is necessarily supercuspidal). If π is reducible, we say $\nu \in Exp(\pi)$ when $\nu \in Exp(\pi')$ for some irreducible $\pi' \le \pi$.

We use $\mathcal{E}xp_M(\pi)$ for the exponents of $r_{M,G}(\pi)$ defined by Waldspurger and $\mathcal{E}xp_M^{\mathbb{R}}(\pi)$ for their real parts (see [Wa, Secs. I.3 and I.1] resp.). In particular, if π is a representation of G and χ is a character of A_M, let

$$V_\chi = \{v \in V_U \mid \exists n \in \mathbb{N} \text{ such that } [(r_{M,G}\pi)(a) - \chi(a)]^n v = 0 \ \forall a \in A_M\},$$

where V_U is the space for $r_{M,G}\pi$. Then $\chi \in \mathcal{E}xp_M(\pi)$ if $V_\chi \ne 0$. We have that $\nu \in \mathfrak{a}_M^*$ is in $\mathcal{E}xp_M^{\mathbb{R}}(\pi)$ if there is a character χ of A_M such that $\chi = \exp \nu$.

LEMMA 4.1. *Let π be an irreducible representation and $L \in \mathcal{M}_{\min}(\pi)$. If $M > L$ is a standard Levi factor and $\xi \in \mathcal{E}xp_L^{\mathbb{R}}(\pi)$, then $r_M^L(\xi) \in \mathcal{E}xp_M^{\mathbb{R}}(\pi)$, and every $\mu \in \mathcal{E}xp_M^{\mathbb{R}}(\pi)$ has this form (for some $L \in \mathcal{M}_{\min}$ and $\xi \in \mathcal{E}xp_L^{\mathbb{R}}(\pi)$).*

Proof. This follows from [BeZ, Prop. 1.9(f)] and (Jacquet) restriction in stages. □

LEMMA 4.2. *Let $A = (\langle \alpha_i, \alpha_j \rangle)_{i,j \in \{1,\dots,n\}}$. Then $A^{-1} = (\langle \beta_i, \beta_j \rangle)_{i,j \in \{1,\dots,n\}}$ and the entries of A^{-1} are nonnegative.*

Proof. The characterization of A^{-1} is an immediate consequence of $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. For the nonnegativity of its entries, see [K, Lemma 8.57]. □

LEMMA 4.3. *Consider the following condition (cf. Condition (ii) in Proposition III.2.2 of* [W]*): For every standard parabolic subgroup $P = MU$ of G and every $\xi \in \mathcal{E}xp_M^{\mathbb{R}}(\pi)$, we have $\xi \in +(\bar{\mathfrak{a}}_M)^*$. This condition holds if and only if every exponent $\nu \in Exp(\pi)$ satisfies $\nu \in +\bar{\mathfrak{a}}^*$.*

Proof. We check both directions. Observe that both conditions require that the $\mathfrak{z}^*$ component be zero.

($\Leftarrow$). Let $P = LU$ be a standard parabolic subgroup, with $\Pi_L = \{\alpha_i \mid i \in I_L\}$. If we do not have $L \ge M$ for some $M \in \mathcal{M}_{\min}(\pi)$, then $r_{L,G}\pi = 0$ and there is nothing to prove. Thus, we assume $L \ge M$ for some $M \in \mathcal{M}_{\min}(\pi)$.

Let $\mu \in \mathfrak{a}_L^* \in \mathcal{E}xp_L^{\mathbb{R}}(\pi)$. By Lemma 4.1, $\mu = r_L^M(\xi)$ for some $\xi \in \mathcal{E}xp_M^{\mathbb{R}}(\pi)$. Then $\nu = \iota_M(\xi) \in \mathfrak{a}^* \in Exp(\pi)$. Note that

$$r_L(\nu) = r_L \circ \iota_M(\xi) = r_L^M \circ r_M \circ \iota_M(\xi) = r_L^M(\xi) = \mu.$$

Write $\nu = z + \sum_{i=1}^n c_i \alpha_i$. Then

$$\mu = r_L(\nu) = r_L(z) + \sum_{i=1}^n c_i r_L(\alpha_i) = r_L(z) + \sum_{i \notin I_L} c_i \alpha_i^L,$$

where $\alpha_i^L = r_L(\alpha_i)$ (a simple root in $\Pi(P, A_L)$ when $i \notin I_L$). Of course, $\mu \in +\bar{\mathfrak{a}}_L^*$ is then equivalent to

$$c_i \geq 0 \quad \text{for all } \ i \notin I_L.$$

On the other hand, the assumption $\nu \in +\bar{\mathfrak{a}}^*$ is equivalent to $c_i \geq 0$ for all i, from which it immediately follows that $\mu \in +\bar{\mathfrak{a}}_L^*$.

($\Rightarrow$). Consider $\nu \in Exp(\pi)$. Then $\nu \in \iota_M(\mathfrak{a}_M^*)$ for some $M \in \mathcal{M}_{\min}$. In particular, $\nu \in \operatorname{span}_{i \notin I_M}\{\beta_i\}$. Write

$$\nu = \sum_{i=1}^{n} c_i \alpha_i = \sum_{i \notin I_M} d_i \beta_i.$$

Our goal is to show $c_i \geq 0$ for all i. If $M_{\Pi - \{\alpha_i\}} \geq L$ for some $L \in \mathcal{M}_{\min}(\pi)$, then one can use the same basic argument as before to show that $c_i \geq 0$. However, this need not hold for all i. In particular, such an argument will tell us that $c_i \geq 0$ for all $i \notin I_M$; we need to extend this to show that $c_i \geq 0$ for all i.

If we let

$$A = \begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle & \dots & \langle \alpha_1, \alpha_n \rangle \\ \vdots & & \vdots \\ \langle \alpha_n, \alpha_1 \rangle & \dots & \langle \alpha_n, \alpha_n \rangle \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \langle \beta_1, \beta_1 \rangle & \dots & \langle \beta_1, \beta_n \rangle \\ \vdots & & \vdots \\ \langle \beta_n, \beta_1 \rangle & \dots & \langle \beta_n, \beta_n \rangle \end{pmatrix}$$

then, by Lemma 4.2,

$$A \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} \quad \text{and} \quad B \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

noting that $d_i = 0$ for $i \in I_M$. For convenience and without loss of generality, suppose the roots are ordered so that $I_M = \{m+1, \dots, n\}$. Let

$$C_1 = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}, \quad C_2 = \begin{pmatrix} c_{m+1} \\ \vdots \\ c_n \end{pmatrix}, \quad \text{and} \quad D_1 = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}.$$

We may then write (block matrices)

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \begin{pmatrix} D_1 \\ 0 \end{pmatrix}$$

so that $C_2 = B_{2,1} D_1$. Now

$$\begin{pmatrix} D_1 \\ 0 \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

implies $D_1 = A_{1,1} C_1 + A_{1,2} C_2$ and therefore

$$C_2 = B_{2,1} D_1 = B_{2,1} A_{1,1} C_1 + B_{2,1} A_{1,2} C_2.$$

Thus, $(I - B_{2,1} A_{1,2}) C_2 = B_{2,1} A_{1,1} C_1$. It follows from $BA = I$ that $I - B_{2,1} A_{1,2} = B_{2,2} A_{2,2}$. Because $A_{2,2}$ and $B_{2,2}$ are invertible (from e.g. linear independence considerations), this gives

$$C_2 = A_{2,2}^{-1} B_{2,2}^{-1} B_{2,1} A_{1,1} C_1.$$

Again, it follows from $BA = I$ that $B_{2,1}A_{1,1} = -B_{2,2}A_{2,1}$. Therefore,

$$C_2 = -A_{2,2}^{-1}A_{2,1}C_1.$$

Since $A_{2,2}$ is the matrix from Lemma 4.2 for a subroot system, $A_{2,2}^{-1}$ has nonnegative entries. Also, since $A_{2,1}$ contains no diagonal entries, $-A_{2,1}$ also has nonnegative entries. It now follows from $c_1, \dots, c_m \geq 0$ that $c_{m+1}, \dots, c_n \geq 0$, as needed. □

The following is now an immediate consequence of [Wa, Prop. III.2.2].

COROLLARY 4.4. *Let π be an irreducible admissible representation of G. Suppose that the central character of π is unitary. Then π is tempered if and only if every exponent $\nu \in Exp(\pi)$ satisfies $\nu \in +\bar{\mathfrak{a}}^*$.*

We now give a lemma (Lemma 4.6) for future use (see Section 6). Let M be a standard Levi subgroup of G and let w_0 be the longest element in $W^{M,A}$. Then [Au, p. 2180] $w_0(\Pi_M) \subset \Pi$ and $w_0(\Pi - \Pi_M) \subset -\Phi^+$. Let L be the standard Levi subgroup corresponding to the set of simple roots $w_0(\Pi_M)$. Then $L = w_0(M)$. Further, $w_0(A_M) = A_L$ and $w_0(\mathfrak{a}_M^*) = \mathfrak{a}_L^*$.

LEMMA 4.5. *With notation as before,*

$$\nu \in (+\mathfrak{a}^*) \cap \iota(\mathfrak{a}_M^*) \iff w_0\nu \in (-\mathfrak{a}^*) \cap \iota(\mathfrak{a}_L^*).$$

Proof. Let $\nu \in (+\mathfrak{a}^*) \cap \iota(\mathfrak{a}_M^*)$. Since $\nu \in +\mathfrak{a}^*$, we may write $\nu = \sum_{\alpha \in \Pi} x_\alpha \alpha$ with $x_\alpha \geq 0$ for all α. Further, since $\nu \in \iota(\mathfrak{a}_M^*)$, we have $x_\alpha = 0$ for all $\alpha \in \Pi_M$; that is, $\nu = \sum_{\alpha \in \Pi - \Pi_M} x_\alpha \alpha$. Since $w_0\alpha < 0$ for all $\alpha \in \Pi - \Pi_M$, it follows that $w_0\nu \in -\mathfrak{a}^*$. Because $w_0(\mathfrak{a}_M^*) = \mathfrak{a}_L^*$, we have $w_0\nu \in (-\mathfrak{a}^*) \cap \iota(\mathfrak{a}_L^*)$. The converse is similar. □

LEMMA 4.6. *Let π be an irreducible admissible representation of G. Suppose that the central character of π is unitary. Then $\hat{\pi}$ is tempered if and only if every exponent $\mu \in Exp(\pi)$ satisfies $\mu \in -\bar{\mathfrak{a}}^*$.*

Proof. Suppose $\nu \in Exp(\pi)$. Then there is an $M \in \mathcal{M}_{\min}(\pi)$ and a $\nu' \in \mathfrak{a}_M^*$ such that $\exp \nu' \otimes \rho \leq r_{M,G}\pi$ (for some irreducible unitary ρ) and $\nu = \iota_M(\nu')$. Set $\mu' = w_0(\nu')$. Then Lemma 2.1 and [Au, Thm. 1.7] imply

$$\exp \mu \otimes w_0\hat{\rho} = w_0(\exp \nu \otimes \hat{\rho}) = w_0(\widehat{\exp \nu \otimes \rho}) \leq r_{L,G}(\hat{\pi});$$

in other words, $\mu = \iota_L(\mu') \in Exp(\hat{\pi})$. Since $w_0 \circ \iota_M = \iota_L \circ w_0$, we see that $\mu = w_0\nu$. The result now follows from Lemma 4.5 and Corollary 4.4. □

5. Multiplicity 1 in the Jacquet Module of a Standard Module

In this section, we prove the main technical result needed in this paper: If (P, ν, τ) is a set of Langlands data, then $\exp \nu \otimes \tau$ is the only irreducible subquotient $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ with its central character, and it occurs with multiplicity 1.

Let θ be an irreducible representation of M. Let us write $|\omega_\theta| = \exp \nu_\theta$ for $\nu_\theta \in \mathfrak{a}_M^*$, where ω_θ is the central character of θ. Then we may (uniquely) write θ as $\exp \nu_\theta \otimes \theta'$ with $\nu_\theta \in \mathfrak{a}_M^*$ and θ' having unitary central character. We call $\iota(\nu_\theta)$ the central exponent for θ (a slight abuse of terminology, as it would be a little more natural to call ν_θ the central exponent). Note that $\exp \nu \otimes \tau$ has central exponent $\iota(\nu)$.

Let Exp denote the set of exponents defined in Section 4.

LEMMA 5.1. *Let θ be a representation of M and let χ be a character of M. Then*

$$Exp(\chi \otimes \theta) = \iota(\nu_\chi) + Exp(\theta).$$

Proof. It follows from $r_{L,M}(\chi \otimes \theta) = \chi \otimes r_{L,M}(\theta)$ that

$$\exp \mu \otimes \rho \le r_{L,M}(\theta) \iff \chi \cdot \exp \mu \otimes \rho \le r_{L,M}(\chi \otimes \theta).$$

Write $\chi = \exp \nu_\chi \otimes \chi_u$, where χ_u is a unitary character. Then

$$\chi \cdot \exp \mu \otimes \rho = \exp \nu_\chi \cdot \exp \mu \otimes \chi_u \rho = \exp(\nu_\chi + \mu) \otimes \chi_u \rho$$

and the claim follows. □

LEMMA 5.2. *Let L and M be standard Levi subgroups, and suppose that $L < M$. Let θ be an irreducible representation of M and let $\mu \in \mathfrak{a}_L^*$ with $\iota(\mu) \in Exp(\theta)$. Write*

$$\mu = \mu_M + \mu_L^M, \quad \mu_M \in \iota_M^L(\mathfrak{a}_M^*),\ \mu_L^M \in (\mathfrak{a}_L^M)^*.$$

If ω_θ is unitary, then $\mu_M = 0$. In general, $\mu_M = \iota_M^L(\nu_\theta)$.

Proof. Suppose ω_θ is unitary. Then $|\omega_\theta| = 1$. According to [Ca, p. 45],

$$|\omega_\theta(a)| = \exp \mu(a) \quad \text{for all } a \in A_M.$$

Since $\mu_L^M(a) = 0$ for all $a \in A_M$, it follows that $\mu_M(a) = 0$ for all $a \in A_M$. Therefore, $\mu_M = 0$.

Now we consider the general case. Write $\theta = \exp \nu_\theta \otimes \theta'$, with θ' having unitary central character. Suppose $\mu \in \mathfrak{a}_L^*$ satisfies $\iota(\mu) \in Exp(\theta)$. Lemma 5.1 tells us that $\mu = \iota_M^L(\nu_\theta) + \mu'$ for some $\mu' \in \mathfrak{a}_L^*$ such that $\iota(\mu') \in Exp(\theta')$. Therefore,

$$\mu = \iota_M^L(\nu_\theta) + \mu'_M + (\mu')_L^M.$$

Since $\mu'_M = 0$ and $\nu_\theta \in \mathfrak{a}_M^*$, it follows that $\mu_M = \iota_M^L(\nu_\theta)$. □

PROPOSITION 5.3. *Let $\pi = L(P, \nu, \tau)$. Then $\exp \nu \otimes \tau$ is the unique irreducible subquotient of $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ having central exponent $\iota(\nu)$, and it occurs with multiplicity* 1.

Proof. Let $\mathcal{F} = \sum_{i=1}^n \mathbb{R}\alpha_i = \sum_{i=1}^n \mathbb{R}\beta_i$ be as in Section 3, so that $\mathfrak{a}^* = \mathfrak{z}^* \oplus \mathcal{F}$. If $\mu \in \mathfrak{a}^*$, we denote by μ^0 the orthogonal projection of μ onto $\mathcal{F}$. Let $I_M = \{i \mid \alpha_i \in \Pi_M\}$. If $\mu \in \mathfrak{a}^*$, then we can write

$$\mu = z + \sum_{i \notin I_M} c_i \beta_i + \sum_{i \in I_M} c_i \alpha_i,$$

where $z \in \mathfrak{z}^*$. In particular, if $\mu \in Exp(\exp \nu \otimes \tau)$ then

$$z + \sum_{i \notin I_M} c_i \beta_i = \iota(\nu), \qquad \sum_{i \in I_M} c_i \alpha_i \in Exp(\tau).$$

Since $\nu \in (\mathfrak{a}_M)^*_-$, we have $c_i < 0$ for $i \notin I_M$. Corollary 4.4 implies that $c_i \geq 0$ for $i \in I_M$. Therefore, μ^0 satisfies the conditions of Lemma 3.3.

For $\mu = z + \sum_{i \notin I_M} c_i \beta_i + \sum_{i \in I_M} c_i \alpha_i$, let $p_M(\mu) = \sum_{i \notin I_M} c_i \beta_i$. If θ is an irreducible representation of M and if $\mu \in Exp(\theta)$, then Lemma 5.2 tells us that $\iota(\nu_\theta)^0 = p_M(\mu)$. It follows from [BeZ, Lemma 2.12] and [Ca, Sec. 6] that

$$Exp(i_{G,M}(\exp \nu \otimes \tau)) \subseteq W^{M,A} \cdot Exp(\exp \nu \otimes \tau).$$

We now combine the preceding observations. Let $\theta \leq r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ be irreducible. Then

$$\begin{aligned} Exp(\theta) \subseteq Exp(i_{G,M}(\exp \nu \otimes \tau)) \subseteq W^{M,A} \cdot Exp(\exp \nu \otimes \tau) \\ \Longrightarrow CentExp(\theta)^0 \in \{p_M(W^{M,A} \cdot Exp(\exp \nu \otimes \tau))\}, \end{aligned}$$

where $CentExp$ denotes the (M-)central exponent. Thus, to show that $\exp \nu \otimes \tau$ is the unique irreducible subquotient of $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ having central exponent $\iota(\nu)$, it suffices to show that $p_M(w\mu) \neq \iota(\nu)^0$ for any $\mu \in Exp(\exp \nu \otimes \tau)$ and $w \in W^{M,A}$ having $w \neq 1$. This follows from Lemma 3.3. $\square$

COROLLARY 5.4. *Let* $\pi = L(P, \nu, \tau)$. *Then* $\exp \nu \otimes \tau$ *is the unique irreducible subquotient of* $r_{M,G} \circ i_{G,M}(\exp \nu \otimes \tau)$ *having central character* $\exp \nu \otimes \omega_\tau$.

COROLLARY 5.4. Proposition 5.3 and Corollary 5.4 also hold for $O(2n, F)$; this is essentially the same combinatorial statement as for $\mathrm{Sp}(2n, F)$ or $\mathrm{SO}(2n+1, F)$. In particular, all three have the same Weyl group, the same concrete realization of the Langlands classification (cf. [BJ1] and the Appendix to [BJ3]), and the same relevant double-coset representatives for the Weyl group (cf. [J3, Lemma 3.6]).

6. The Dual Langlands Classification

In this section we give a nice application of the main result: the dual Langlands classification (Theorem 6.3).

If θ is an irreducible representation of G with unitary central character, we say that θ is anti-tempered if every exponent $\nu \in Exp(\theta)$ satisfies $\nu \in -\bar{\mathfrak{a}}^*$. Note that this is equivalent to having $\hat{\theta}$ be tempered (Lemma 4.6).

Let $P = MU$ be a standard parabolic subgroup of G. If $w_0 \in W^{M,A}$ is the longest element, then $L = w_0(M)$ is also the Levi factor of a standard parabolic subgroup Q of G. Further, if τ is an irreducible tempered representation of M, then $w_0\tau$ is an irreducible tempered representation of L and $\theta = w_0\hat{\tau}$ is an irreducible anti-tempered representation of L (Lemma 4.6).

LEMMA 6.1. *If* $\nu \in (\mathfrak{a}_M)^*_-$, *then* $\mu = w_0\nu \in (\mathfrak{a}_L)^*_+$.

Proof. Let $\nu \in (\mathfrak{a}_M)^*_-$ and $\mu = w_0\nu$. If $\gamma \in \Pi(Q, A_L)$, then $\gamma = r_L(\alpha_j)$ for some $\alpha_j \in \Pi - \Pi_L$. Proposition 1.1.4 of [Ca] implies $w_0^{-1}(\alpha_j) < 0$. It follows that $w_0^{-1}(\alpha_j) = \sum_{i=1}^n c_i \alpha_i$, $c_i \leq 0$. Then

$$w_0^{-1}(\gamma) = r_M\left(\sum_{i=1}^{n} c_i\alpha_i\right) = \sum_{\alpha\in\Pi(P,A_M)} c_\alpha\alpha,$$

where $c_\alpha \le 0$ and not all c_α are equal to 0. By assumption, $\langle\nu,\alpha\rangle < 0$ for all $\alpha \in \Pi(P, A_M)$. It follows that

$$\langle\mu,\gamma\rangle = \langle w_0^{-1}\mu, w_0^{-1}\gamma\rangle = \left\langle\nu, \sum_{\alpha\in\Pi(P,A_M)} c_\alpha\alpha\right\rangle = \sum_{\alpha\in\Pi(P,A_M)} c_\alpha\langle\nu,\alpha\rangle > 0,$$

so $\mu \in (\mathfrak{a}_L)^*_+$. □

LEMMA 6.2. *Let $\pi = L(P,\nu,\tau)$. Then $\hat\pi$ is the unique irreducible subrepresentation of $i_{G,L}(\exp\mu\otimes\theta)$ with L,μ,θ as before.*

Proof. We have $\exp\nu\otimes\tau \le r_{M,G}(\pi)$. Corollary 5.4 tells us that $\exp\nu\otimes\tau$ is the unique irreducible subquotient of $r_{M,G}(\pi)$ having central character $\exp\nu\otimes\omega_\tau$. Let Z_M denote the center of M and $Z_L = w_0(Z_M)$. Combining Lemma 2.1 and [Au, Thm. 1.7], we have

$$\exp\mu\otimes\theta = w_0(\exp\nu\otimes\hat\tau) = w_0(\widehat{\exp\nu\otimes}\,\tau) \le r_{L,G}(\hat\pi),$$

and this is the unique irreducible subquotient of $r_{L,G}\circ i_{G,L}(\exp\mu\otimes\theta)$ having central character $\exp\mu\otimes\omega_\theta$.

We now need the following standard result [Ca; G; Wa]: If (ρ, V) is an admissible representation of L and ω is a character of Z_L, write

$$V_\omega = \{v\in V \mid \exists n\in\mathbb{N} \text{ such that } [\rho(z)-\omega(z)]^n v = 0\ \forall z\in Z_L\}.$$

Then $V = \bigoplus_\omega V_\omega$ as a direct sum of L-modules. In particular, let $\rho = r_{L,G}(\hat\pi)$ and $\lambda = \exp\mu\otimes\omega_\theta$. Then V_λ is just the L-module $\exp\mu\otimes\theta$ (since it is the unique subquotient of $r_{L,G}(\hat\pi)$ having this central character), so it appears as a direct summand in $r_{L,G}(\hat\pi)$. The lemma now follows from Frobenius reciprocity. □

THEOREM 6.3 (The dual Langlands classification). *Let $Q = LU$ be a standard parabolic subgroup of G, let $\mu \in (\mathfrak{a}_L)^*_+$, and let θ be an* (*equivalence class of an*) *anti-tempered representation of L. Then the induced representation $i_{G,L}(\exp\mu\otimes\theta)$ has a unique irreducible subrepresentation, which we denote by $DL(Q,\mu,\theta)$. Conversely, if π is an irreducible admissible representation of G then there is a unique triple (Q,μ,θ), with Q a standard parabolic subgroup, $\mu \in (\mathfrak{a}_L)^*_+$, and θ an anti-tempered representation of L such that $\pi \cong DL(Q,\mu,\theta)$.*

Further, suppose that $\hat\pi = L(P,\nu,\tau)$ in the Langlands classification. If $P = MU$ and $w_0 \in W^{M,A}$ is the longest element, then $L = w_0(M)$, $\mu = w_0\nu$, and $\theta = w_0\hat\tau$.

Proof. If (P,ν,τ) is the Langlands data for $\hat\pi$, it follows immediately from Lemma 6.2 that (Q,μ,θ) is the dual Langlands data for π. This shows the existence of dual Langlands data. Conversely, if one starts with dual Langlands data (Q,μ,θ) for π, then Lemma 6.2 implies that (P,μ,τ) is Langlands data for $\hat\pi$. The uniqueness of dual Langlands data then follows from the uniqueness of Langlands data.

The relationship given between the dual Langlands data for π and the Langlands data for $\hat{\pi}$ is immediate from the preceding discussion. □

COROLLARY 6.4. *Let $\pi = DL(Q, \mu, \theta)$. Then the multiplicity of π in the induced representation $i_{G,L}(\exp \nu \otimes \theta)$ is* 1.

Proof. This follows from the corresponding result for the Langlands classification and the previous theorem. □

REMARK 6.5. By Remark 5.5, we know that Corollary 5.4 holds for $O(2n, F)$. Further, by [J3] we have a duality operator for $O(2n, F)$ with the properties from [Au, Thm. 1.7]. It is then a straightforward matter to check that Lemma 6.2 and Theorem 6.3 hold for $O(2n, F)$ as well.

7. Dual Langlands Classification for GL(n, F) and the Zelevinsky Classification

We close by considering the case of general linear groups. In this case, the dual Langlands classification, suitably interpreted, is the same as the Zelevinsky classification.

We start by reviewing some notation regarding general linear groups, most of which is taken from [Z]. If π_1 and π_2 are admissible representations of $\mathrm{GL}(k_1, F)$ and $\mathrm{GL}(k_2, F)$, respectively, then we define $\pi_1 \times \pi_2 = i_{G,M}(\pi_1 \otimes \pi_2)$, where $M \cong \mathrm{GL}(k_1, F) \times \mathrm{GL}(k_2, F)$ is the Levi factor of a standard parabolic subgroup of $G = \mathrm{GL}(k_1 + k_2, F)$. Let $\nu = |\det|$. Let ρ be an irreducible supercuspidal representation of $\mathrm{GL}(m, F)$ and let $k \geq 0$ be an integer. The set $\Delta = [\rho, \nu^k\rho] = \{\rho, \nu\rho, \dots, \nu^k\rho\}$ is called a *segment*. The induced representation $\rho \times \nu\rho \times \cdots \times \nu^k\rho$ has a unique irreducible subrepresentation, which we denote by $\langle\Delta\rangle$, and a unique irreducible quotient, which we denote by $\delta(\Delta)$. For $\mathrm{GL}(n, F)$, the Aubert involution coincides with the Zelevinsky involution (cf. [Au, Thm. 2.3]) and $\widehat{\delta(\Delta)} = \langle\Delta\rangle$. The representation $\delta(\Delta)$ is square integrable if the segment is balanced—that is, of the form $\Delta = [\nu^{-k}\rho, \nu^k\rho]$, where ρ is unitary and k is a half-integer. Any irreducible square-integrable representation of $\mathrm{GL}(n, F)$ is isomorphic to $\delta(\Delta)$ for some balanced segment Δ. In addition, if τ is a tempered representation of $\mathrm{GL}(n, F)$, then $\tau \cong \delta_1 \times \cdots \times \delta_s$ for some square integrable representations $\delta_1, \dots, \delta_s$; this follows from the irreducibility of induced-from-unitary representations of $\mathrm{GL}(n, F)$. All this implies the following description of the dual Langlands classification for $\mathrm{GL}(n, F)$.

PROPOSITION 7.1 (Dual Langlands classification for $\mathrm{GL}(n, F)$).

(a) *Suppose $\Delta_1, \dots, \Delta_k$ are balanced segments and $\alpha_1 \geq \cdots \geq \alpha_k$ are real numbers. Then the induced representation $\nu^{\alpha_1}\langle\Delta_1\rangle \times \cdots \times \nu^{\alpha_k}\langle\Delta_k\rangle$ has a unique irreducible subrepresentation; denote it by $\pi(\nu^{\alpha_1}\Delta_1, \dots, \nu^{\alpha_k}\Delta_k)$.*

(b) *The representations $\pi(\nu^{\alpha_1}\Delta_1, \dots, \nu^{\alpha_k}\Delta_k)$ and $\pi(\nu^{\beta_1}\Delta'_1, \dots, \nu^{\beta_\ell}\Delta'_\ell)$ are isomorphic if and only if $k = \ell$, $(\alpha_1, \dots, \alpha_k) = (\beta_1, \dots, \beta_k)$, and $(\Delta'_1, \dots, \Delta'_k) = (\Delta_{p(1)}, \dots, \Delta_{p(k)})$ for some permutation p such that $\alpha_i = \alpha_{p(i)}$ for all $i \in \{1, \dots, k\}$.*

(c) *Any irreducible admissible representation of* $\mathrm{GL}(n, F)$ *is isomorphic to some representation of the form* $\pi(\nu^{\alpha_1}\Delta_1, \ldots, \nu^{\alpha_k}\Delta_k)$.

Proof. This follows from Theorem 6.3, with a modification in expressing inducing data. More precisely, let $Q = LU$ be a standard parabolic subgroup of $\mathrm{GL}(n, F)$, let $\mu \in (\mathfrak{a}_L)^*_+$, and let θ be an anti-tempered representation of L. The Levi factor L is of the form $L \cong \mathrm{GL}(n_1, F) \times \cdots \times \mathrm{GL}(n_r, F)$. Then $\exp \mu = \nu^{\gamma_1} \otimes \cdots \otimes \nu^{\gamma_r}$, where $\gamma_1 > \cdots > \gamma_r$, and $\theta = \theta_1 \otimes \cdots \otimes \theta_r$, where each θ_i is an anti-tempered representation of $\mathrm{GL}(n_i, F)$. We know from the previous discussion that θ_i can be expressed as $\theta_i \cong \langle \Delta_{i,1}\rangle \times \langle \Delta_{i,2}\rangle \times \cdots \times \langle \Delta_{i,k_i}\rangle$ for some balanced segments $\Delta_{i,1}, \ldots, \Delta_{i,k_i}$. This expression is not unique—we have $\theta_i \cong \langle \Delta_{i,p(1)}\rangle \times \langle \Delta_{i,p(2)}\rangle \times \cdots \times \langle \Delta_{i,p(k_i)}\rangle$ for any permutation p of $\{1, \ldots, k_i\}$. Now

$$i_{G,L}(\exp \mu \otimes \theta) \cong \nu^{\gamma_1}\langle \Delta_{1,1}\rangle \times \cdots \times \nu^{\gamma_1}\langle \Delta_{1,k_1}\rangle \times \cdots \times \nu^{\gamma_r}\langle \Delta_{r,1}\rangle \times \cdots \times \nu^{\gamma_r}\langle \Delta_{r,k_r}\rangle,$$

and the proposition follows from Theorem 6.3. □

Next we review the Zelevinsky classification [Z]. We say that the segments Δ_1 and Δ_2 are *linked* if $\Delta_1 \not\subset \Delta_2$ and $\Delta_2 \not\subset \Delta_1$ and if $\Delta_1 \cup \Delta_2$ is also a segment. Suppose that Δ_1 and Δ_2 are linked and that $\Delta_1 = [\rho_1, \nu^{k_1}\rho_1]$ and $\Delta_2 = [\rho_2, \nu^{k_2}\rho_2]$. If $\rho_2 = \nu^{\ell}\rho_1$ for some $\ell > 0$, we say that Δ_1 *precedes* Δ_2.

Let $(\Delta_1, \Delta_2, \ldots, \Delta_k)$ be a sequence of segments. We say that $\Delta_1, \Delta_2, \ldots, \Delta_k$ are *Zelevinsky data* if, for each pair of indices $i < j$, Δ_i does not precede Δ_j.

If $\Delta_1, \ldots, \Delta_k$ are Zelevinsky data, then the representation $\langle \Delta_1\rangle \times \cdots \times \langle \Delta_k\rangle$ has a unique irreducible subrepresentation that we denote by $\langle \Delta_1, \ldots, \Delta_k\rangle$. Any irreducible admissible representation of $\mathrm{GL}(n, F)$ is isomorphic to some representation of the form $\langle \Delta_1, \ldots, \Delta_k\rangle$, and the choice of $\Delta_1, \ldots, \Delta_k$ is unique up to a permutation [Z, Thm. 6.1].

In the next proposition, we give the connection between Zelevinsky classification and dual Langlands classification for $\mathrm{GL}(n, F)$.

Proposition 7.2. *Suppose* $\Delta_1, \ldots, \Delta_k$ *are balanced segments and* $\alpha_1 \geq \cdots \geq \alpha_k$ *are real numbers. Let* $\Delta'_i = \nu^{\alpha_i}\Delta_i$. *Then* $\Delta'_1, \ldots, \Delta'_k$ *are Zelevinsky data and* $\langle \Delta'_1, \ldots, \Delta'_k\rangle = \pi(\nu^{\alpha_1}\Delta_1, \ldots, \nu^{\alpha_k}\Delta_k)$.

Proof. Suppose $\Delta_1, \Delta_2, \ldots, \Delta_k$ are balanced segments and $\alpha_1 \geq \cdots \geq \alpha_k$ are real numbers. Set $\Delta'_i = \nu^{\alpha_i}\Delta_i$. We claim that $\Delta'_1, \ldots, \Delta'_k$ are Zelevinsky data. Suppose, on the contrary, that there exist indices $i < j$ such that Δ'_i precedes Δ'_j. Set $\alpha = \alpha_i$, $\beta = \alpha_j$, $\Delta_i = [\nu^{-r}\rho, \nu^{r}\rho]$, and $\Delta_j = [\nu^{-s}\rho, \nu^{s}\rho]$, where $r, s \geq 0$. Then $\Delta'_i = [\nu^{\alpha-r}\rho, \nu^{\alpha+r}\rho]$ and $\Delta'_j = [\nu^{\beta-s}\rho, \nu^{\beta+s}\rho]$. Suppose first that $\Delta'_i \cap \Delta'_j = \emptyset$. Then $\alpha + r + 1 = \beta - s$, which is impossible because $\alpha \geq \beta$ and $r, s \geq 0$. Therefore,

$$\alpha - r < \beta - s \leq \alpha + r < \beta + s.$$

If $r \geq s$ then $\alpha + r \geq \beta + s$, contradicting the above expression. Similarly, $r < s$ implies $\alpha - r > \beta - s$, again contradicting this expression. Therefore, if $i < j$ then Δ'_i does not precede Δ'_j. □

Conversely, if we start with Zelevinsky data then we can obtain corresponding dual Langlands data. Suppose $\Delta'_1, \ldots, \Delta'_k$ are Zelevinsky data. For $i \in \{1, \ldots, k\}$, write $\Delta'_i = \nu^{\beta_i}\Delta''_i$, where Δ''_i is balanced and β_i is a real number. Let $(\nu^{\alpha_1}\Delta_1, \ldots, \nu^{\alpha_k}\Delta_k)$ be a permutation of $(\nu^{\beta_1}\Delta''_1, \ldots, \nu^{\beta_k}\Delta''_k)$ such that $\alpha_1 \geq \cdots \geq \alpha_k$. According to Proposition 7.2, $\Delta'_{p(1)}, \ldots, \Delta'_{p(k)}$ are Zelevinsky data and $\langle \Delta'_{p(1)}, \ldots, \Delta'_{p(k)} \rangle = \pi(\nu^{\alpha_1}\Delta_1, \ldots, \nu^{\alpha_k}\Delta_k)$. Then from [Z, Thm. 6.1(b)], we have $\pi(\nu^{\alpha_1}\Delta_1, \ldots, \nu^{\alpha_k}\Delta_k) \cong \langle \Delta'_1, \ldots, \Delta'_k \rangle$.

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