Michigan Math. J. 56 (2008)

Functions of Vanishing Mean Oscillation Associated with Operators and Applications

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1. Introduction

Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbb{R}^n)$ with suitable upper bounds on its heat kernels, and suppose L has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$. In this paper, we introduce and develop a new function space VMO_L of vanishing mean oscillation associated with the operator L. Using the theory of tent spaces and the Littlewood–Paley theory, we prove that a Hardy space H^1_L of Auscher, Duong, and McIntosh introduced in [ADMc] is the dual of our new VMO_{L^*} in which L^* is the adjoint operator of L. We also give an equivalent characterization of the space VMO_L in the context of the theory of tent spaces.

A locally integrable function f on $\mathbb{R}^n$ is said to be in $\mathrm{BMO}(\mathbb{R}^n)$, the space of bounded mean oscillation, if

$$\|f\|_{\mathrm{BMO}} = \sup_B |B|^{-1} \int_B |f(x) - f_B|\,dx < \infty, \tag{1.1}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and where f_B stands for the mean of f over B; that is,

$$f_B = |B|^{-1} \int_B f(x)\,dx.$$

The quotient space of $\mathrm{BMO}(\mathbb{R}^n)$ with this seminorm over the constant functions is a Banach space. The space of BMO functions was introduced by John and Nirenberg [JN].

According to Sarason [Sa], a function f of $\mathrm{BMO}(\mathbb{R}^n)$ that satisfies the limiting condition

Received August 1, 2007. Revision received March 12, 2008.
The research of Donggao Deng was supported by NNSF of China (Grant no. 10571182). The research of Xuan Thinh Duong was supported by a grant from Australia Research Council. The research of Liang Song was supported by the Priming Scientific Research Foundation for the Junior Teachers in Sun Yat-Sen University. The research of Lixin Yan was supported by NCET of Ministry of Education of China and NNSF of China (Grant no. 10571182/10771221).

$$\lim_{a\to 0}\left(\sup_{B: r_B\le a}|B|^{-1}\int_B |f(x)-f_B|\,dx\right)=0 \tag{1.2}$$

is said to be of *vanishing mean oscillation* on $\mathbb{R}^n$. The subspace of $\mathrm{BMO}(\mathbb{R}^n)$ consisting of the functions of vanishing mean oscillation is denoted by $\mathrm{VMO}(\mathbb{R}^n)$, and we endow $\mathrm{VMO}(\mathbb{R}^n)$ with the norm of $\mathrm{BMO}(\mathbb{R}^n)$. See [Sa] for several alternative characterizations of functions in $\mathrm{VMO}(\mathbb{R}^n)$.

The famous result of Fefferman and Stein [FS] identified $\mathrm{BMO}(\mathbb{R}^n)$ with the dual of the Hardy space $H^1(\mathbb{R}^n)$. In [CW, Sec. 4], Coifman and Weiss introduced a modified version of $\mathrm{VMO}(\mathbb{R}^n)$, denoted by $\mathrm{CMO}(\mathbb{R}^n)$, the space of functions of the closure in the BMO norm of the space $C_0(\mathbb{R}^n)$ of continuous functions with compact support. They then proved that the space $H^1(\mathbb{R}^n)$ is the dual of $\mathrm{CMO}(\mathbb{R}^n)$. See [B; BCrSi; U] for characterizations of functions in $\mathrm{CMO}(\mathbb{R}^n)$ and relations among $\mathrm{BMO}(\mathbb{R}^n)$, $\mathrm{VMO}(\mathbb{R}^n)$, $\mathrm{CMO}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$, and local spaces.

Recently, a $\mathrm{BMO}_L(\mathbb{R}^n)$ space associated with an operator L was introduced and studied in [DY1]. Roughly speaking, if L is the infinitesimal generator of an analytic semigroup $\{e^{-tL}\}_{t\ge 0}$ on L^2 with kernel $p_t(x,y)$ (which decays fast enough), then we can view $P_t f = e^{-tL}f$ as an average version of f (at the scale t) and use the quantity

$$P_{t_B}f(x)=\int_{\mathbb{R}^n} p_{t_B}(x,y)f(y)\,dy \tag{1.3}$$

to replace the mean value f_B in our definition (1.1) of the classical BMO space, where t_B is scaled to the radius of the ball B. We then say that a function f (with suitable bounds on growth) is in $\mathrm{BMO}_L(\mathbb{R}^n)$ if

$$\sup_B |B|^{-1}\int_B |f(x)-P_{t_B}f(x)|\,dx<\infty.$$

In [DY2], Duong and Yan characterized the space of $\mathrm{BMO}_L(\mathbb{R}^n)$ functions as the dual of a new Hardy space $H^1_{L^*}(\mathbb{R}^n)$ of Auscher, Duong, and McIntosh [ADMc] associated with the adjoint operator L^* of L. This gives a generalization of the duality of $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ of Fefferman and Stein [FS]. Indeed, a valid choice of e^{-tL} is the Poisson integral of f defined by

$$e^{-t\sqrt{\triangle}}f(x)=\int_{\mathbb{R}^n} p_t(x-y)f(y)\,dy,\quad t>0,$$

where $p_t(x-y)=c_n t/(t^2+|x-y|^2)^{(n+1)/2}$. For this choice of e^{-tL}, the spaces $H^1_{\sqrt{\triangle}}(\mathbb{R}^n)$ and $\mathrm{BMO}_{\sqrt{\triangle}}(\mathbb{R}^n)$ coincide with the classical Hardy space and BMO space, respectively.

This paper continues a line of study in [ADMc; DY1; DY2] to introduce and develop a new function space $\mathrm{VMO}_L(\mathbb{R}^n)$, of vanishing mean oscillation associated with operators, that generalizes the classical VMO space. We will say that a function f of $\mathrm{BMO}_L(\mathbb{R}^n)$ is in $\mathrm{VMO}_L(\mathbb{R}^n)$ if it satisfies the limiting conditions $\gamma_1(f)=\gamma_2(f)=\gamma_3(f)=0$, where:

$$\gamma_1(f) = \lim_{a\to 0}\left[\sup_{B:r_B\le a}\left(|B|^{-1}\int_B |f(x) - P_{t_B}f(x)|^2\,dx\right)^{1/2}\right];$$

$$\gamma_2(f) = \lim_{a\to \infty}\left[\sup_{B:r_B\ge a}\left(|B|^{-1}\int_B |f(x) - P_{t_B}f(x)|^2\,dx\right)^{1/2}\right];$$

$$\gamma_3(f) = \lim_{a\to \infty}\left[\sup_{B\subset B(0,a)^c}\left(|B|^{-1}\int_B |f(x) - P_{t_B}f(x)|^2\,dx\right)^{1/2}\right].$$

See Section 3.2. With the choice $P_t f = p_t * f$, where p_t is the Poisson kernel, the classical space $\mathrm{CMO}(\mathbb{R}^n)$ (of Coifman and Weiss) coincides with our $\mathrm{VMO}_{\sqrt{\Delta}}(\mathbb{R}^n))$ space. We also give an equivalent characterization of VMO_L space in the context of the theory of tent spaces initiated by Coifman, Meyer, and Stein in [CMS1; CMS2]; see Propositions 3.3 and 3.6.

The main purpose of Section 4 is to prove our main result, Theorem 4.1, which gives a generalization of Coifman and Weiss's [CW] result on the duality of $H^1(\mathbb{R}^n)$ and $\mathrm{CMO}(\mathbb{R}^n)$ spaces. We will show that if L has a bounded holomorphic functional calculus on L^2 and if the kernel $p_t(x,y)$ of the operator P_t in (1.3) satisfies an upper bound of Poisson type, then the dual of our new space $\mathrm{VMO}_{L^*}(\mathbb{R}^n)$ is the Hardy space $H^1_L(\mathbb{R}^n)$, where L^* denotes the adjoint operator of L. We then give applications to large classes of differential operators such as the Schrödinger operators and second-order elliptic operators of divergence form.

Throughout this paper, c will denote (possibly different) constants that are independent of the essential variables.

2. Preliminaries

2.1. Holomorphic Functional Calculi of Operators

We first give some preliminary definitions of holomorphic functional calculi as introduced by McIntosh [Mc]. Let $0 \le \omega < \nu < \pi$. We define the closed sector in the complex plane $\mathbb{C}$ as

$$S_\omega = \{z \in \mathbb{C} : |\arg z| \le \omega\} \cup \{0\}$$

and denote the interior of S_ω by S^0_ω.

We employ the following subspaces of the space $H(S^0_\nu)$ of all holomorphic functions on S^0_ν:

$$H_\infty(S^0_\nu) = \{b \in H(S^0_\nu) : \|b\|_\infty < \infty\},$$

where $\|b\|_\infty = \sup\{|b(z)| : z \in S^0_\nu\}$, and

$$\Psi(S^0_\nu) = \{\psi \in H(S^0_\nu) : \exists s > 0,\ |\psi(z)| \le c|z|^s(1 + |z|^{2s})^{-1}\}.$$

Let $0 \le \omega < \pi$. A closed operator L in $L^2(\mathbb{R}^n)$ is said to be of type ω if $\sigma(L) \subset S_\omega$ and if, for each $\nu > \omega$, there exists a constant c_ν such that

$$\|(L - \lambda\mathcal{I})^{-1}\| \le c_\nu|\lambda|^{-1}, \quad \lambda \notin S_\nu.$$

If L is of type ω and $\psi \in \Psi(S_\nu^0)$, then we define $\psi(L) \in \mathcal{L}(L^2, L^2)$ by

$$\psi(L) = \frac{1}{2\pi i}\int_\Gamma (L - \lambda\mathcal{I})^{-1}\psi(\lambda)\,d\lambda, \tag{2.1}$$

where Γ is the contour $\{\xi = re^{\pm i\theta} : r \geq 0\}$ parameterized clockwise around S_ω and $\omega < \theta < \nu$. Clearly, this integral is absolutely convergent in $\mathcal{L}(L^2, L^2)$, and it is straightforward to show, using Cauchy's theorem, that the definition is independent of the choice of $\theta \in (\omega, \nu)$. If, in addition, L is one-to-one and has dense range and if $b \in H_\infty(S_\nu^0)$, then $b(L)$ can be defined by

$$b(L) = [\psi(L)]^{-1}(b\psi)(L),$$

where $\psi(z) = z(1+z)^{-2}$. It can be shown that $b(L)$ is a well-defined linear operator in $L^2(\mathbb{R}^n)$. We say that L has a bounded H_∞ calculus in L^2 if there exists $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$ and, for $b \in H_\infty(S_\nu^0)$,

$$\|b(L)\| \leq c_{\nu,2}\|b\|_\infty.$$

For a detailed study of operators that have holomorphic functional calculi, see [Mc].

2.2. Assumptions and Notation

Assume that the operator L, acting on $L^2(\mathbb{R}^n)$, is one-to-one. Suppose L is a linear operator of type ω on $L^2(\mathbb{R}^n)$ with $\omega < \pi/2$; then L generates a holomorphic semigroup e^{-zL}, $0 \leq |\mathrm{Arg}(z)| < \pi/2 - \omega$. Assume the following two conditions.

ASSUMPTION A. The holomorphic semigroup e^{-zL}, $|\mathrm{Arg}(z)| < \pi/2 - \omega$, is represented by a kernel $p_z(x, y)$ that satisfies an upper bound

$$|p_z(x, y)| \leq c_\theta h_{|z|}(x, y)$$

for $x, y \in \mathbb{R}^n$; $|\mathrm{Arg}(z)| < \pi/2 - \theta$ for $\theta > \omega$, and h_t is given by

$$h_t(x, y) = t^{-n/m}s\left(\frac{|x - y|}{t^{1/m}}\right), \tag{2.2}$$

in which m is a fixed positive constant and s is a positive, bounded, decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\varepsilon}s(r) = 0 \tag{2.3}$$

for some $\varepsilon > 0$.

ASSUMPTION B. The operator L has a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$. That is, there exists a $c_{\nu,2} > 0$ such that $b(L) \in \mathcal{L}(L^2, L^2)$ and, for $b \in H_\infty(S_\nu^0)$,

$$\|b(L)\|_{2,2} \leq c_{\nu,2}\|b\|_\infty.$$

We now give some consequences of Assumptions A and B that will be useful in the sequel.

First, if $\{e^{-tL}\}_{t\geq 0}$ is a bounded analytic semigroup on $L^2(\mathbb{R}^n)$ whose kernel $p_t(x, y)$ satisfies the estimate (2.2) then, for all $k \in \mathbb{N}$, the time derivatives of p_t satisfy

$$\left|\frac{\partial^k p_t}{\partial t^k}(x,y)\right| \le ct^{-(n+km)/m} s\left(\frac{|x-y|}{t^{1/m}}\right) \tag{2.4}$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^n$. For each $k \in \mathbb{N}$, the function s might depend on k but always satisfies (2.3). See [O, Thm. 6.17].

Second, L has a bounded H_∞-calculus in $L^2(\mathbb{R}^n)$ if and only if, for any nonzero function $\psi \in \Psi(S_\nu^0)$, L satisfies the square function estimate and its reverse,

$$c_1\|f\|_2 \le \left(\int_0^\infty \|\psi_t(L)f\|_2^2 \frac{1}{t}\,dt\right)^{1/2} \le c_2\|f\|_2, \tag{2.5}$$

for some $0 < c_1 \le c_2 < \infty$, where $\psi_t(\xi) = \psi(t\xi)$. Note that different choices of $\nu > \omega$ and $\psi \in \Psi(S_\nu^0)$ lead to equivalent quadratic norms of f. See [Mc].

As noted in [Mc], positive self-adjoint operators satisfy the quadratic estimate (2.5). So do normal operators with spectra in a sector as well as maximal accretive operators. For definitions of these classes of operators, we refer the reader to [Yo].

We now define the class of functions upon which the operators e^{-tL} act. For any $\beta > 0$, a function $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be *a function of β-type* if f satisfies

$$\left(\int_{\mathbb{R}^n} \frac{|f(x)|^2}{1+|x|^{n+\beta}}\,dx\right)^{1/2} \le c < \infty. \tag{2.6}$$

We denote by $\mathcal{M}_\beta$ the collection of all functions of β-type. If $f \in \mathcal{M}_\beta$, then the norm of f in $\mathcal{M}_\beta$ is denoted by

$$\|f\|_{\mathcal{M}_\beta} = \inf\{c \ge 0 : (2.6) \text{ holds}\}.$$

It is easy to see that $\mathcal{M}_\beta$ is a Banach space under the norm $\|f\|_{\mathcal{M}_\beta}$. For any given operator L, we let $\Theta(L) = \sup\{\varepsilon > 0 : (2.3) \text{ holds}\}$ and define

$$\mathcal{M} = \begin{cases} \mathcal{M}_{\Theta(L)} & \text{if } \Theta(L) < \infty, \\ \bigcup_{\beta:0<\beta<\infty} \mathcal{M}_\beta & \text{if } \Theta(L) = \infty. \end{cases}$$

Note that if L is the Laplacian $\triangle$ on $\mathbb{R}^n$, then $\Theta(\triangle) = \infty$. When $L = \sqrt{\triangle}$, we have $\Theta\big(\sqrt{\triangle}\big) = 1$.

For any $(x,t) \in \mathbb{R}^n \times (0,+\infty)$ and $f \in \mathcal{M}$, we define

$$P_t f(x) = e^{-tL} f(x) = \int_{\mathbb{R}^n} p_t(x,y) f(y)\,dy \tag{2.7}$$

and

$$Q_t f(x) = tLe^{-tL} f(x) = \int_{\mathbb{R}^n} -t\left(\frac{d}{dt} p_t(x,y)\right) f(y)\,dy. \tag{2.8}$$

It follows from the estimate (2.4) that the operators $P_t f$ and $Q_t f$ are well-defined. Moreover, the operator Q_t has the following properties.

(i) For any $t_1, t_2 > 0$ and almost all $x \in \mathbb{R}^n$,

$$Q_{t_1} Q_{t_2} f(x) = t_1 t_2 \left(\frac{d^2 P_t}{dt^2}\bigg|_{t=t_1+t_2} f\right)(x).$$

(ii) The kernel $q_{t^m}(x,y)$ of Q_{t^m} satisfies

$$|q_{t^m}(x,y)| \le ct^{-n} s\left(\frac{|x-y|}{t}\right), \tag{2.9}$$

where the function s satisfies condition (2.3).

3. The Spaces VMO_L Associated with Operators

In this section, we assume that L is an operator satisfying assumptions A and B of Section 2.2. The aim of this section is to introduce and study a new function space VMO_L of vanishing mean oscillation, associated with an operator L, that generalizes the classical VMO spaces.

3.1. The Function Space $\mathrm{BMO}_L(\mathbb{R}^n)$

Following [DY1], we say that $f \in \mathcal{M}$ is *of bounded mean oscillation associated with an operator L* (abbreviated as BMO_L) if

$$\sup_B |B|^{-1} \int_B |f(x) - P_{t_B} f(x)|\, dx = \|f\|_{\mathrm{BMO}_L} < \infty, \tag{3.1}$$

where the supremum is taken over all balls in $\mathbb{R}^n$ and where $t_B = r_B^m$ for r_B the radius of the ball B of $\mathbb{R}^n$. The class of functions of $\mathrm{BMO}_L(\mathbb{R}^n)$ modulo $\mathcal{K}_L$, where

$$\mathcal{K}_L = \{f \in \mathcal{M} : P_t f(x) = f(x) \text{ for almost all } x \in \mathbb{R}^n \text{ and all } t > 0\}, \tag{3.2}$$

is a Banach space with the norm $\|f\|_{\mathrm{BMO}_L}$ defined as in (3.1). We refer to [DY2, Sec. 6] for a discussion of the dimensions of $\mathcal{K}_L$ when L is a second-order elliptic operator of divergence form or a Schrödinger operator.

We now list two important properties of the spaces $\mathrm{BMO}_L(\mathbb{R}^n)$. For the proofs, we refer the reader to Sections 2 and 3 of [DY1].

First, under the extra condition that L satisfies a conservation property of the semigroup $P_t(1) = 1$ for every $t > 0$, it can be verified that $\mathrm{BMO}(\mathbb{R}^n)$ is a subspace of $\mathrm{BMO}_L(\mathbb{R}^n)$. Moreover, the spaces $\mathrm{BMO}(\mathbb{R}^n)$, $\mathrm{BMO}_\triangle(\mathbb{R}^n)$, and $\mathrm{BMO}_{\sqrt{\triangle}}(\mathbb{R}^n)$ coincide, and their norms are equivalent.

Second, we note that a variant of the John–Nirenberg inequality holds for functions in $\mathrm{BMO}_L(\mathbb{R}^n)$. That is, there exist positive constants c_1 and c_2 such that, for every ball B and $\alpha > 0$,

$$|\{x \in B : |f(x) - P_{r_B^m} f(x)| > \alpha\}| \le c_1 |B| \exp\left\{-\frac{c_2 \alpha}{\|f\|_{\mathrm{BMO}_L}}\right\}.$$

This and (3.1) imply that, for any $f \in \mathrm{BMO}_L(\mathbb{R}^n)$ and $1 \le p < \infty$, the norms

$$\|f\|_{p,\mathrm{BMO}_L} = \sup_B \left(|B|^{-1} \int_B |f(x) - P_{r_B^m} f(x)|^p\, dx\right)^{1/p} \tag{3.3}$$

with different choices of p are all equivalent.

3.2. The Spaces VMO_L *Associated with Operators*

Let us introduce a new function space $\mathrm{VMO}_L(\mathbb{R}^n)$ associated with the semigroup $\{e^{-tL}\}_{t>0}$.

DEFINITION 3.1. We say that a function $f \in \mathrm{BMO}_L(\mathbb{R}^n)$ is in VMO_L, the space of functions of vanishing mean oscillation associated with the semigroup $\{e^{-tL}\}_{t>0}$, if it satisfies the limiting conditions $\gamma_1(f) = \gamma_2(f) = \gamma_3(f) = 0$, where

$$\gamma_1(f) = \lim_{a\to 0}\left[\sup_{B:r_B\le a}\left(|B|^{-1}\int_B |f(x) - P^m_{r_B} f(x)|^2\,dx\right)^{1/2}\right],$$

$$\gamma_2(f) = \lim_{a\to \infty}\left[\sup_{B:r_B\ge a}\left(|B|^{-1}\int_B |f(x) - P^m_{r_B} f(x)|^2\,dx\right)^{1/2}\right],$$

$$\gamma_3(f) = \lim_{a\to \infty}\left[\sup_{B\subset B(0,a)^c}\left(|B|^{-1}\int_B |f(x) - P^m_{r_B} f(x)|^2\,dx\right)^{1/2}\right];$$

we endow $\mathrm{VMO}_L(\mathbb{R}^n)$ with the norm of $\mathrm{BMO}_L(\mathbb{R}^n)$.

Note that if L is the Laplacian $\triangle$ on $\mathbb{R}^n$, then it follows that the space $\mathrm{VMO}_\triangle(\mathbb{R}^n)$ (or $\mathrm{VMO}_{\sqrt{\triangle}}(\mathbb{R}^n)$) is equivalent to the space $\mathrm{CMO}(\mathbb{R}^n)$ of Coifman and Weiss (i.e., the space of functions of the closure in the BMO norm of the space $C_0(\mathbb{R}^n)$ of continuous functions with compact support), and their norms are equivalent. See Proposition 3.6.

3.3. Properties of Functions in $\mathrm{VMO}_L(\mathbb{R}^n)$

In [CMS1] and [CMS2], the authors introduced and studied a new family of function spaces called *tent spaces*. These spaces are useful for the study of a variety of problems in harmonic analysis. See also [De]. In this paper, we will adopt the approach of tent spaces to study our new VMO spaces.

3.3.1. Tent Spaces and Applications

We will use $\mathbb{R}^{n+1}_+$ to denote the usual upper half-space in $\mathbb{R}^{n+1}$. The notation $\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$ denotes the standard cone (of aperture 1) with vertex $x \in \mathbb{R}^n$. For any closed subset $F \subset \mathbb{R}^n$, $\mathcal{R}(F)$ will be the union of all cones with vertices in F; that is, $\mathcal{R}(F) = \bigcup_{x\in F}\Gamma(x)$. If O is an open subset of $\mathbb{R}^n$, then the "tent" over O, denoted by $\hat{O}$, is given as $\hat{O} = [\mathcal{R}(O^c)]^c$.

For any function $f(y,t)$ defined on $\mathbb{R}^{n+1}_+$, we will denote

$$\mathcal{A}(f)(x) = \left(\int_{\Gamma(x)} |f(y,t)|^2 \frac{1}{t^{n+1}}\,dy\,dt\right)^{1/2}$$

and

$$\mathcal{C}(f)(x) = \sup_{x\in B}\left(|B|^{-1}\int_{\hat{B}} |f(y,t)|^2 \frac{1}{t}\,dy\,dt\right)^{1/2}.$$

As in [CMS1], the tent space T^p_2 is defined as the space of functions f such that $\mathcal{A}(f) \in L^p(\mathbb{R}^n)$ when $p < \infty$. The resulting equivalence classes are then equipped with the norm $\|f\|_{T_2^p} = \|\mathcal{A}(f)\|_p$. When $p = \infty$, the space T_2^∞ is the class of functions f for which $\mathcal{C}(f) \in L^\infty(\mathbb{R}^n)$ and the norm $\|f\|_{T_2^\infty} = \|\mathcal{C}(f)\|_\infty$.

In what follows, let $T_{2,c}^p$ be the set of all $f \in T_2^p$ with compact support in $\mathbb{R}_+^{n+1}$. We denote by $T_{2,0}^\infty$ the linear subspace of T_2^∞ consisting of those functions f that satisfy the condition

$$\eta_1(f) = \lim_{a\to 0}\left[\sup_{B:r_B\le a}\left(|B|^{-1}\int_{\hat{B}}|f(y,t)|^2\frac{1}{t}\,dy\,dt\right)^{1/2}\right] = 0,$$

and we endow $T_{2,0}^\infty$ with norm of T_2^∞. Finally, we denote by $T_{2,V}^\infty$ the closure of the set $T_{2,c}^2$ in $T_{2,0}^\infty$, and we endow $T_{2,V}^\infty$ with the norm of T_2^∞.

Let $\mathcal{H}$ be the set of all $f \in T_2^\infty$ satisfying the following three conditions:

(i) $f \in T_{2,0}^\infty$;

(ii) $\eta_2(f) = \lim_{a\to+\infty}\left[\sup_{B:r_B\ge a}\left(|B|^{-1}\int_{\hat{B}}|f(y,t)|^2\frac{1}{t}\,dy\,dt\right)^{1/2}\right] = 0$;

(iii) $\eta_3(f) = \lim_{a\to+\infty}\left[\sup_{B:B\subset(B(0,a))^c}\left(|B|^{-1}\int_{\hat{B}}|f(y,t)|^2\frac{1}{t}\,dy\,dt\right)^{1/2}\right] = 0$.

It can be verified that $\mathcal{H}$ is a closed linear subspace of T_2^∞. Note that conditions (ii) and (iii) are not consequences of (i). To see this, set

$$f(x,t) = \begin{cases} 1 & \text{if } (x,t) \in \bigcup_{k=1}^\infty R_k, \\ 0 & \text{otherwise,} \end{cases}$$

where $R_k = [7\cdot 2^{k-3}, 9\cdot 2^{k-3}] \times [1,2]$. It follows from the fact $\{R_k\}_{k=1}^\infty$ are pairwise disjoint, together with $\int_{R_k}|f(x,t)|^2(1/t)\,dx\,dt = 2^{k-2}\ln 2$, that the function $f(x,t)$ satisfies condition (i). However, $f(x,t)$ does not satisfy condition (ii) or (iii).

LEMMA 3.2. *Let $T_{2,V}^\infty$ be defined as before. Then we have*

(a) $(T_{2,V}^\infty)^* = T_2^1$ *and*

(b) $f \in T_{2,V}^\infty$ *if and only if* $f \in \mathcal{H}$.

Proof. For the proof of (a), we refer to [Wa, Thm. 1.7, p. 542]. Let us prove (b). Since $T_{2,c}^2 \subset \mathcal{H}$ and since $\mathcal{H}$ is a closed linear subspace of T_2^∞, we have that $T_{2,V}^\infty = \overline{T_{2,c}^2} \subset \mathcal{H}$. We now assume that $f \in \mathcal{H}$; we want to prove $f \in T_{2,V}^\infty$. It follows from the definition of $\mathcal{H}$ that for any $\eta > 0$, there exist $a_0 > 0$, $b_0 > 0$, and $c_0 > 0$ such that

$$\sup_{B:r_B\le a_0}|B|^{-1}\int_{\hat{B}}|f(y,t)|^2\frac{1}{t}\,dy\,dt \le \eta, \qquad \sup_{B:r_B\ge b_0}|B|^{-1}\int_{\hat{B}}|f(y,t)|^2\frac{1}{t}\,dy\,dt \le \eta, \tag{3.4}$$

and

$$\sup_{B:B\subset(B(0,c_0))^c}|B|^{-1}\int_{\hat{B}}|f(y,t)|^2\frac{1}{t}\,dy\,dt \le \eta. \tag{3.5}$$

Let $K_0 = \max(a_0^{-1}, b_0, c_0)$ and define

$$g(y,t) = f(y,t)\chi_{\{y\in B(0,2K_0),\,t\in(2K_0^{-1},2K_0)\}}(y,t).$$

Obviously, $g \in T_{2,c}^2$. We now prove

$$\|f-g\|^2_{T_2^\infty} < c\eta. \tag{3.6}$$

Let us verify the estimate (3.6) by examining the balls B of $\mathbb{R}^n$ in three cases.

Case 1: $r_B < a_0$ or $r_B > b_0$. From the estimate (3.4), we have

$$\int_{\hat{B}} |f(y,t)-g(y,t)|^2 \frac{1}{t}\, dy\, dt \le 2\int_{\hat{B}} |f(y,t)|^2 \frac{1}{t}\, dy\, dt \le 2\eta|B|.$$

Case 2: $a_0 \le r_B \le b_0$ and $B \subset B(0,c_0)^c$. By the estimate (3.5), we obtain

$$\int_{\hat{B}} |f(y,t)-g(y,t)|^2 \frac{1}{t}\, dy\, dt \le 2\int_{\hat{B}} |f(y,t)|^2 \frac{1}{t}\, dy\, dt \le 2\eta|B|.$$

Case 3: $a_0 \le r_B \le M_0$ and $B \cap B(0,c_0) \ne \emptyset$. In this case, it follows from the definition of g that

$$|f(y,t)-g(y,t)| = \begin{cases} |f(y,t)| & \text{if } y \in B \text{ and } t \in (0, 2K_0^{-1}), \\ 0 & \text{otherwise.} \end{cases}$$

This gives

$$\int_{\hat{B}} |f(y,t)-g(y,t)|^2 \frac{1}{t}\, dy\, dt \le \int_0^{(2K_0)^{-1}} \int_B |f(y,t)|^2 \frac{1}{t}\, dy\, dt. \tag{3.7}$$

We use $B(x_B, r_B)$ to denote B centered with x_B and of radius r_B. Then there exists a $k \in \mathbb{N}$ such that $2^{k-1}a_0 \le r_B \le 2^k a_0$. Consider the ball $B(x_B, 2^k a_0)$. This ball is contained in the cube $Q[x_B, 2^{k+1}a_0]$ centered at x and of side length $2^{k+1}a_0$. We then divide this cube $Q[x_B, 2^{k+1}a_0]$ into $[2^{k+1}([\sqrt{n}\,]+1)]^n$ small cubes $\{Q_{x_{k_i}}\}_{i=1}^{N_k}$ centered at x_{k_i} and of equal side length $([\sqrt{n}\,]+1)^{-1}a_0$, where $N_k = [2^{k+1}([\sqrt{n}\,]+1)]^n$. For any $i = 1, 2, \ldots, N_k$, each of these small cubes $Q_{x_{k_i}}$ is then contained in the corresponding ball B_{k_i} with the same center x_{k_i} and radius $r = a_0$. Consequently, for the ball $B(x_B, 2^k a_0)$, there exists a corresponding collection of balls $B_{k_1}, B_{k_2}, \ldots, B_{k_{N_k}}$ such that:

(i) each ball B_{k_i} is of radius a_0;
(ii) $B(x_B, 2^k a_0) \subset \bigcup_{i=1}^{N_k} B_{k_i}$;
(iii) there exists a constant $c > 0$ independent of k such that $N_k \le c2^{kn}$;
(iv) each point of $B(x_B, 2^k a_0)$ is contained in at most a finite number M of the balls B_{k_i}, where M is independent of k.

These properties (i)–(iv), together with the estimate (3.4), show that

$$\begin{aligned}
\int_{\hat{B}} |f(y,t)-g(y,t)|^2 \frac{1}{t}\, dy\, dt &\le \int_0^{(2K_0)^{-1}} \int_{\bigcup_{i=1}^{N_k} B_{k_i}} |f(y,t)|^2 \frac{1}{t}\, dy\, dt \\
&\le \sum_{i=1}^{N_k} \int_{\widehat{B_{k_i}}} |f(y,t)|^2 \frac{1}{t}\, dy\, dt \\
&\le c\eta \sum_{i=1}^{N_k} |B_{k_i}| \\
&\le c\eta|B|.
\end{aligned}$$

Estimate (3.6) follows readily. This proves that $f \in T^{\infty}_{2,V}$, whence the proof of Lemma 3.2 is complete. □

3.3.2. A Characterization of $\mathrm{VMO}_L(\mathbb{R}^n)$
Using Lemma 3.2, we can prove the following proposition.

PROPOSITION 3.3. *Assume that the operator L satisfies Assumptions A and B in Section 2.2. Then the following conditions are equivalent*:

(a) f *is in a function in* $\mathrm{VMO}_L(\mathbb{R}^n)$;
(b) $f \in \mathcal{M}$ *and* $Q_{t^m}(I - P_{t^m})f \in T^{\infty}_{2,V}$, *with* $\|f\|_{\mathrm{VMO}_L} \sim \|Q_{t^m}(I - P_{t^m})f\|_{T^{\infty}_2}$.

Proof. We first prove the implication (a) ⇒ (b). Suppose $f \in \mathrm{VMO}_L(\mathbb{R}^n)$. In order to prove $Q_{t^m}(I - P_{t^m})f \in T^{\infty}_{2,V}$, we will prove that there exists a positive constant $c > 0$ such that, for any ball $B = B(x_B, r_B)$,

$$|B|^{-1} \int_{\hat{B}} |Q_{t^m}(\mathcal{I} - P_{t^m})f(x)|^2 \frac{1}{t}\, dx\, dt \le c \sum_{k=1}^{\infty} 2^{-k\varepsilon} \delta_k(f, B), \tag{3.8}$$

where

$$\delta_k(f, B) = \sup_{B' \subset 2^{k+1}B : r_{B'} \in [2^{-1}r_B, 2r_B]} \left(\frac{1}{|B'|} \int_{B'} |f(y) - P_{r^m_{B'}} f(y)|^2\, dy \right). \tag{3.9}$$

Once the estimate (3.8) is proved, $Q_{t^m}(I - P_{t^m})f \in T^{\infty}_{2,V}$ follows readily. Indeed, by the condition $f \in \mathrm{VMO}_L(\mathbb{R}^n)$, we have that $f \in \mathrm{BMO}_L(\mathbb{R}^n)$ and then $\delta_k(f, B) \le c\|f\|^2_{\mathrm{BMO}_L}$ for some constant $c > 0$. Moreover, for any $k \in \mathbb{N}$ we have that

$$\begin{aligned} \lim_{a \to 0} \sup_{B : r_B \le a} \delta_k(f, B) &= \lim_{a \to \infty} \sup_{B : r_B \ge a} \delta_k(f, B) \\ &= \lim_{a \to \infty} \sup_{B : B \subset B(0,a)^c} \delta_k(f, B) \\ &= 0. \end{aligned} \tag{3.10}$$

By estimate (3.8) we have that

$$\begin{aligned} &|B|^{-1} \int_{\hat{B}} |Q_{t^m}(\mathcal{I} - P_{t^m})f(x)|^2 \frac{1}{t}\, dx\, dt \\ &\quad \le c \sum_{k=1}^{k_0} 2^{-k\varepsilon} \delta_k(f, B) + c \sum_{k=k_0}^{\infty} 2^{-k\varepsilon} \|f\|^2_{\mathrm{BMO}_L} \\ &\quad \le c \sum_{k=1}^{k_0} 2^{-k\varepsilon} \delta_k(f, B) + c 2^{-k_0\varepsilon} \|f\|^2_{\mathrm{BMO}_L}. \end{aligned}$$

Note that if k_0 is large enough then the quantity $2^{-k_0\varepsilon}\|f\|^2_{\mathrm{BMO}_L}$ is sufficiently small. Fix a k_0. We then use the property (3.10) to obtain $\eta_1(f) = \eta_2(f) = \eta_3(f) = 0$, where $\{\eta_i(f)\}_{i=1}^3$ of $\mathcal{H}$ are defined in Section 3.3.1. This gives $Q_{t^m}(I - P_{t^m})f \in T^{\infty}_{2,V}$, from which the proof of (b) follows.

We now prove estimate (3.8). Note that

$$Q_{t^m}(\mathcal{I} - P_{t^m}) = Q_{t^m}(\mathcal{I} - P_{t^m})(\mathcal{I} - P_{r^m_{2B}}) + Q_{t^m}(I - P_{t^m})P_{r^m_{2B}}.$$

Hence, (3.8) follows from the estimates (3.11) and (3.12):

$$|B|^{-1}\int_{\hat{B}}|Q_{t^m}(\mathcal{I}-P_{t^m})(\mathcal{I}-P_{r_{2B}^m})f(x)|^2\frac{1}{t}\,dx\,dt \le c\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k(f,B); \tag{3.11}$$

$$|B|^{-1}\int_{\hat{B}}|P_{r_{2B}^m}Q_{t^m}(\mathcal{I}-P_{t^m})f(x)|^2\frac{1}{t}\,dx\,dt \le c\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k(f,B). \tag{3.12}$$

We will prove these two estimates by adapting the argument in [DY2, Lemma 4.6]. Let $b_1=(\mathcal{I}-P_{r_{2B}^m})f\chi_{2B}$ and $b_2=(\mathcal{I}-P_{r_{2B}^m})f\chi_{(2B)^c}$. From (2.5), we obtain

$$\begin{aligned}\int_{\hat{B}}|Q_{t^m}(\mathcal{I}-P_{t^m})b_1(x)|^2\frac{1}{t}\,dx\,dt &\le \iint_{\mathbb{R}^{n+1}_+}|Q_{t^m}(\mathcal{I}-P_{t^m})b_1(x)|^2\frac{1}{t}\,dx\,dt\\ &\le c\|b_1\|^2_{L^2(\mathbb{R}^n)}\\ &= c\int_{2B}|(\mathcal{I}-P_{r_{2B}^m})f(x)|^2\,dx\\ &\le c|B|\delta_2(f,B).\end{aligned} \tag{3.13}$$

On the other hand, for any $x\in B$ and $y\in(2^kB)^c$, one has $|x-y|\ge c2^kr_B$. By (2.4), we obtain

$$\begin{aligned}|Q_{t^m}(\mathcal{I}-P_{t^m})b_2(x)| &\le c\int_{\mathbb{R}^n\setminus 2B}\frac{t^\varepsilon}{(t+|x-y|)^{n+\varepsilon}}|(\mathcal{I}-P_{r_{2B}^m})f(y)|\,dy\\ &\le c\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\frac{t^\varepsilon}{(t+|x-y|)^{n+\varepsilon}}|(\mathcal{I}-P_{r_{2B}^m})f(y)|\,dy\\ &\le c\left(\frac{t}{r_B}\right)^{\varepsilon}\sum_{k=1}^{\infty}2^{-k(n+\varepsilon)}r_B^{-n}\int_{2^{k+1}B}|(\mathcal{I}-P_{r_{2B}^m})f(y)|\,dy.\end{aligned} \tag{3.14}$$

For a fixed positive integer k, the same argument as in Case 3 of Lemma 3.2 shows that for any ball $B(x_B,2^kr_B)$, $k=1,2,\ldots$, there exists a corresponding collection of balls $B_{k_1},B_{k_2},\ldots,B_{k_{N_k}}$ such that:

(i) each ball B_{k_i} is of radius r_{2B};
(ii) $B(x_B,2^kr_B)\subset\bigcup_{i=1}^{N_k}B_{k_i}$;
(iii) there exists a constant $c>0$ independent of k such that $N_k\le c2^{kn}$;
(iv) each point of $B(x_B,2^kr_B)$ is contained in at most a finite number M of the balls B_{k_i}, where M is independent of k.

Applying these properties (i)–(iv), one obtains

$$\begin{aligned}&|Q_{t^m}(\mathcal{I}-P_{t^m})b_2(x)|\\ &\quad\le c\left(\frac{t}{r_B}\right)^{\varepsilon}\sum_{k=1}^{\infty}2^{-k(n+\varepsilon)}r_{2B}^{-n}\int_{\bigcup_{i=1}^{N_{k+1}}B_{k_i}}|(\mathcal{I}-P_{r_{2B}^m})f(y)|\,dy\\ &\quad\le c\left(\frac{t}{r_B}\right)^{\varepsilon}\sum_{k=1}^{\infty}2^{-k(n+\varepsilon)}\sum_{i=1}^{N_{k+1}}|B_{k_i}|^{-1}\int_{B_{k_i}}|(\mathcal{I}-P_{r_{2B}^m})f(y)|\,dy\le\end{aligned}$$

$$\leq c\left(\frac{t}{r_B}\right)^{\varepsilon}\sum_{k=1}^{\infty}2^{-k\varepsilon}\sup_{i:1\leq i\leq N_{k+1}}\left(|B_{k_i}|^{-1}\int_{B_{k_i}}|(\mathcal{I}-P_{r_{2B}^m})f(y)|^2\,dy\right)^{1/2}$$

$$\leq c\left(\frac{t}{r_B}\right)^{\varepsilon}\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k^{1/2}(f,B). \tag{3.15}$$

Therefore,

$$\iint_{\hat{B}}|Q_{t^m}(\mathcal{I}-P_{t^m})b_2(x)|^2\frac{1}{t}\,dx\,dt \leq \frac{c}{r_B^{2\varepsilon}}\iint_{\hat{B}}t^{2\varepsilon-1}\,dx\,dt\left(\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k^{1/2}(f,B)\right)^2$$
$$\leq c|B|\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k(f,B).$$

This, together with (3.13), yields estimate (3.11).

Let us prove (3.12). One writes

$$P_{r_B^m}Q_{t^m}(I-P_{t^m})f(x) = P_{r_B^m}Q_{t^m}(I-P_{r_{2B}^m})f(x)-P_{r_B^m+t^m}Q_{t^m}(I-P_{r_{2B}^m-t^m})f(x).$$

Note that, by (2.4), for $0<t<r_B$ the kernel $k_{t,r_B}(x,y)$ of the operator $P_{r_B^m}Q_{t^m} = (t^m/(t^m+r_B^m))Q_{(t^m+r_B^m)}$ satisfies

$$|k_{t,r_B}(x,y)| \leq c\left(\frac{t}{r_B}\right)^m\frac{r_B^{\varepsilon}}{(r_B+|x-y|)^{n+\varepsilon}}. \tag{3.16}$$

Estimate (3.16) holds for the kernel of the operator $P_{r_B^m+t^m}Q_{t^m}$. We then use the argument as in (3.14) and (3.16) to show that, for any $x\in B$ and $0<t<r_B$,

$$|P_{r_B^m}Q_{t^m}(I-P_{r_{2B}^m})f(x)| \leq c\left(\frac{t}{r_B}\right)^m\int_{\mathbb{R}^n}\frac{r_B^{\varepsilon}}{(r_B+|x-y|)^{n+\varepsilon}}|(I-P_{r_{2B}^m})f(x)|\,dx$$
$$\leq c\left(\frac{t}{r_B}\right)^m\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k^{1/2}(f,B)$$

and

$$P_{r_B^m+t^m}Q_{t^m}(I-P_{r_{2B}^m-t^m})f(x) \leq c\left(\frac{t}{r_B}\right)^m\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k^{1/2}(f,B).$$

From this it follows that

$$\int_{\hat{B}}|P_{r_B^m}Q_{t^m}(I-P_{t^m})f(x)|^2\frac{1}{t}\,dx\,dt$$
$$\leq cr_B^{-2m}\int_{\hat{B}}t^{2m-1}\,dx\,dt\left(\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k^{1/2}(f,B)\right)^2$$
$$\leq c|B|\sum_{k=1}^{\infty}2^{-k\varepsilon}\delta_k(f,B).$$

Estimate (3.12) is then obtained. We have proved estimate (3.8), so the implication (a) $\Rightarrow$ (b) follows.

In order to prove the implication (b) $\Rightarrow$ (a), we need the following lemma.

LEMMA 3.4. *For any $f \in \mathrm{BMO}_L$ and $g \in H^1_{L^*} \cap L^2$, we have the following identity with constant $b_m = \frac{36}{5}m$:*

$$\int_{\mathbb{R}^n} f(x)g(x)\,dx = b_m \int_{\mathbb{R}^{n+1}_+} F(x,t)G(x,t)\frac{1}{t}\,dx\,dt, \tag{3.17}$$

where $F(x,t) = Q_{t^m}(\mathcal{I} - P_{t^m})f(x)$ and $G(x,t) = Q^*_{t^m}g(x)$.

Proof. See [DY2, Prop. 5.1]. □

Proof of the implication (b) $\Rightarrow$ (a) *of Proposition 3.3* (cont.). First, it follows from condition (b) together with [DY2, Thm. 3.2] that $f \in \mathrm{BMO}_L(\mathbb{R}^n)$. On the other hand, for any $g \in L^2$ we have $(I - P^*_{r_B^m})g \in L^2$. Also, it follows from [DY2, Lemma 4.5] that $(I - P^*_{r_B^m})g \in H^1_L$, and then $(I - P^*_{r_B^m})g \in H^1_L \cap L^2$. The duality argument for the L^2-space shows that, for any ball B of $\mathbb{R}^n$,

$$\begin{aligned}
&\left(|B|^{-1}\int_B |f(x) - P_{r_B^m}f(x)|^2\,dx\right)^{1/2}\\
&\quad= \sup_{\|g\|_{L^2(B)\le 1}} |B|^{-1/2}\left|\int_B (I - P_{r_B^m})f(x)g(x)\,dx\right|\\
&\quad= \sup_{\|g\|_{L^2(B)\le 1}} |B|^{-1/2}\left|\int_{\mathbb{R}^n} f(x)(I - P^*_{r_B^m})g(x)\,dx\right|.
\end{aligned} \tag{3.18}$$

We apply Lemma 3.4 to obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n} f(x)(I - P^*_{r_B^m})g(x)\,dx\right|\\
&\quad= b_m\left|\int_{\mathbb{R}^{n+1}_+} Q_{t^m}(I - P_{t^m})f(x)Q^*_{t^m}(I - P^*_{r_B^m})g(x)\frac{1}{t}\,dx\,dt\right|\\
&\quad\le c\int_{\mathbb{R}^{n+1}_+} |Q_{t^m}(I - P_{t^m})f(x)||Q^*_{t^m}(I - P^*_{r_B^m})g(x)|\frac{1}{t}\,dx\,dt\\
&\quad\le c\int_{\widehat{4B}} |Q_{t^m}(I - P_{t^m})f(x)||Q^*_{t^m}(I - P^*_{r_B^m})g(x)|\frac{1}{t}\,dx\,dt\\
&\qquad+ c\sum_{k=2}^{\infty}\int_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}} |Q_{t^m}(I - P_{t^m})f(x)||Q^*_{t^m}(I - P^*_{r_B^m})g(x)|\frac{1}{t}\,dx\,dt\\
&\quad= \mathrm{A}_1 + \sum_{k=2}^{\infty}\mathrm{A}_k.
\end{aligned}$$

Using Hölder's inequality and the square function estimate (2.5), we have

$$\begin{aligned}
\mathrm{A}_1 &\le \left(\int_{\widehat{4B}} |Q_{t^m}(I-P_{t^m})f(x)|^2 \frac{1}{t}\,dx\,dt\right)^{1/2} \left(\int_0^\infty \|Q^*_{t^m}(I-P^*_{r_B^m})g\|_2^2 \frac{1}{t}\,dt\right)^{1/2} \\
&\le c\|g\|_2 \left(\int_{\widehat{4B}} |Q_{t^m}(I-P_{t^m})f(x)|^2 \frac{1}{t}\,dx\,dt\right)^{1/2} \\
&\le c|B|^{1/2} \left(|4B|^{-1} \int_{\widehat{4B}} |Q_{t^m}(I-P_{t^m})f(x)|^2 \frac{1}{t}\,dx\,dt\right)^{1/2}
\end{aligned}$$

since $\|g\|_2 \le 1$.

Let us estimate A_k for $k = 2, 3, \ldots$. Using Hölder's inequality again, we have that

$$\mathrm{A}_k \le \mathrm{D}_k \cdot \mathrm{E}_k,$$

where

$$\mathrm{D}_k = \left(\int_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}} |Q^*_{t^m}(I-P^*_{r_B^m})g(x)||Q_{t^m}(I-P_{t^m})f(x)|^2 \frac{1}{t}\,dx\,dt\right)^{1/2}$$

and

$$\mathrm{E}_k = \left(\int_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}} |Q^*_{t^m}(I-P^*_{r_B^m})g(x)| \frac{1}{t}\,dx\,dt\right)^{1/2}.$$

Since $(\mathcal{I} - P^*_{r_B^m}) = m\int_0^{r_B} Q^*_{s^m}\frac{1}{s}\,ds$, we obtain

$$Q^*_{t^m}(\mathcal{I} - P^*_{r_B^m}) = m\int_0^{r_B} Q^*_{t^m}Q^*_{s^m}\frac{1}{s}\,ds = \int_0^{r_B} h\left(\frac{s}{t}\right)\Psi_{t,s}(L^*)\frac{1}{s}\,ds,$$

where $h(x) = mx^m(1+x^m)^{-2}$ and

$$\Psi_{t,s}(L^*)f(x) = (t^m+s^m)^2\left(\frac{d^2P^*_r}{dr^2}\bigg|_{r=t^m+s^m} f\right)(x).$$

It follows from estimate (2.4) that the kernel $\Psi_{t,s}(L^*)(x,y)$ of the operator $\Psi_{t,s}(L^*)$ satisfies

$$|\Psi_{t,s}(L^*)(x,y)| \le c\frac{(t+s)^\varepsilon}{(t+s+|x-y|)^{n+\varepsilon}},$$

where ε is the positive constant in (2.3). Also, it can be verified that

$$h\left(\frac{s}{t}\right)(t+s)^\varepsilon = \frac{t^{2m}s^m(t+s)^\varepsilon}{(t^m+s^m)^3} \le c\min(s^\varepsilon, (st)^{\varepsilon/2}).$$

Note that for any $(x,t) \in \widehat{2^{k+1}B} \setminus \widehat{2^kB}$ and $(y,s) \in \hat{B}$, we have $t+s+|x-y| \ge c2^kr_B$. By estimate (2.4), for $(x,t) \in \widehat{2^{k+1}B} \setminus \widehat{2^kB}$ we have

$$\begin{aligned}
|Q^*_{t^m}(I-P^*_{r_B^m})g(x)| &\le c\left|\int_0^{r_B} h\left(\frac{s}{t}\right)\Psi_{t,s}(L^*)(g)(x)\frac{1}{s}\,ds\right| \\
&\le c\int_0^{r_B}\int_B \frac{(st)^m}{(s+t)^{2m}} \cdot \frac{(s+t)^\varepsilon}{(s+t+|x-y|)^{n+\varepsilon}}|g(y)|\frac{1}{s}\,dy\,ds \\
&\le c(2^kr_B)^{-(n+\varepsilon)}\int_0^{r_B} s^{\varepsilon-1}\,ds\|g\|_{L^1(B)} \\
&\le c2^{-k\varepsilon}|2^{k+1}B|^{-1}|B|^{1/2},
\end{aligned}$$

where we used the estimate $\|g\|_1 \le |B|^{1/2}\|g\|_2 \le |B|^{1/2}$. This gives

$$\mathrm{D}_k \le c2^{-k\varepsilon/2}|B|^{1/4}\bigg(|2^{k+1}B|^{-1}\int_{\widehat{2^{k+1}B}}|Q_{t^m}(I-P_{t^m})f(x)|^2\frac{1}{t}\,dx\,dt\bigg)^{1/2}.$$

On the other hand, the same argument as before shows that

$$\begin{aligned}
\mathrm{E}_k^2 &= \int_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}}|Q_{t^m}^*(I-P_{r_B^m}^*)g(x)|\frac{1}{t}\,dx\,dt\\
&\le c\int_{\widehat{2^{k+1}B}}\int_0^{r_B}\int_B\frac{(st)^m}{(s+t)^{2m}}\cdot\frac{(s+t)^\varepsilon}{(s+t+|x-y|)^{n+\varepsilon}}|g(y)|\frac{1}{s}\,dy\,ds\frac{1}{t}\,dx\,dt\\
&\le c|2^kB|(2^kr_B)^{-(n+\varepsilon)}\int_0^{2^{k+1}r_B}\int_0^{r_B}(st)^{\varepsilon/2-1}\,ds\,dt\|g\|_{L^1(B)}\\
&\le c2^{-k\varepsilon/2}|B|^{1/2}.
\end{aligned}$$

Therefore,

$$\mathrm{A}_k \le c2^{-k\varepsilon/2}|B|^{1/2}\bigg(|2^{k+1}B|^{-1}\int_{\widehat{2^{k+1}B}}|Q_{t^m}(I-P_{t^m})f(x)|^2\frac{1}{t}\,dx\,dt\bigg)^{1/2}.$$

By (3.18), for any ball B of $\mathbb{R}^n$ we have

$$\bigg(|B|^{-1}\int_B|f(x)-P_{r_B^m}f(x)|^2\,dx\bigg)^{1/2} \le c\sum_{k=1}^{\infty}2^{-k\varepsilon/4}\sigma_k(f,B),$$

where

$$\sigma_k(f,B) = \bigg(|2^kB|^{-1}\int_{\widehat{2^kB}}|Q_{t^m}(I-P_{t^m})f(x)|^2\frac{1}{t}\,dx\,dt\bigg)^{1/2}.$$

We then follow the argument as in the proof of the implication (a) $\Rightarrow$ (b) of this proposition to show that $\gamma_1(f)=\gamma_2(f)=\gamma_3(f)=0$, where the $\{\gamma_i(f)\}_{i=1}^3$ are as defined in Section 3.2 (we omit the details). This proves $f\in \mathrm{VMO}_L(\mathbb{R}^n)$ and thus the implication (b) $\Rightarrow$ (a) of Proposition 3.3. □

3.3.3. Equivalence of Classical $\mathrm{CMO}(\mathbb{R}^n)$ *and* $\mathrm{VMO}_{\sqrt{\triangle}}(\mathbb{R}^n)$

We note that the space $\mathrm{VMO}(\mathbb{R}^n)$ is different from $\mathrm{CMO}(\mathbb{R}^n)$ of Coifman and Weiss, the space of functions of the closure in the BMO norm of the space $C_0(\mathbb{R}^n)$ of continuous functions with compact support (cf. [CW, p. 638]). See also [U]. As is well known, $\mathrm{VMO}(\mathbb{R}^n)\subsetneqq \mathrm{BMO}(\mathbb{R}^n)$. For example, the function $\log|x|$ belongs to $\mathrm{BMO}(\mathbb{R}^n)$ but not to $\mathrm{VMO}(\mathbb{R}^n)$. See [B] and [BCrSi] for relations among $\mathrm{BMO}(\mathbb{R}^n)$, $\mathrm{VMO}(\mathbb{R}^n)$, $\mathrm{CMO}(\mathbb{R}^n)$, $L^\infty(\mathbb{R}^n)$ and local spaces.

The aim of this section is to show that if L is the Laplacian $\triangle$ on $\mathbb{R}^n$, then the space $\mathrm{VMO}_{\sqrt{\triangle}}(\mathbb{R}^n)$ (or $\mathrm{VMO}_{\triangle}(\mathbb{R}^n)$) is equivalent to the space $\mathrm{CMO}(\mathbb{R}^n)$. First, we have the following proposition.

PROPOSITION 3.5. *The following statements are equivalent.*

(a) $f\in\mathrm{CMO}(\mathbb{R}^n)$.

(b) $f\in\mathcal{B}$, *where* $\mathcal{B}$ *is the subspace of* $\mathrm{BMO}(\mathbb{R}^n)$ *satisfying the following conditions*:

(b$_1$) $\lim_{a\to 0}\sup_{B:r_B\le a}\big(|B|^{-1}\int_B |f(x)-f_B|^2\,dx\big)^{1/2}=0$;
(b$_2$) $\lim_{a\to\infty}\sup_{B:r_B\ge a}\big(|B|^{-1}\int_B |f(x)-f_B|^2\,dx\big)^{1/2}=0$;
(b$_3$) $\lim_{a\to\infty}\sup_{B:B\subset B(0,a)^c}\big(|B|^{-1}\int_B |f(x)-f_B|^2\,dx\big)^{1/2}=0$.
Here $f_B=|B|^{-1}\int_B f(x)\,dx$.

Proof. The proof of the implication (a) $\Rightarrow$ (b) follows from the facts $C_0(\mathbb{R}^n)\subset\mathcal{B}$ and $\mathcal{B}$ is a closed subspace of BMO; thus $\mathrm{CMO}(\mathbb{R}^n)=\overline{C_0}(\mathbb{R}^n)\subset\mathcal{B}$.

For the proof of the implication (b) $\Rightarrow$ (a), we refer to [B, Thm. 7]. See also [U, Sec. 3, p. 166]. □

REMARK. It is well known that, for any $f\in\mathrm{BMO}(\mathbb{R}^n)$ and a constant $K>1$,

$$|f_B-f_{KB}|\le c(1+\log K)\|f\|_{\mathrm{BMO}},$$

where f_B is the mean of f on the ball B. This, together with the properties (b$_1$)–(b$_3$), shows that the condition (b$_3$) can be replaced by the following weak limiting condition:

(b$'_3$) $\lim_{|a|\to\infty}\big(|B+a|^{-1}\int_{B+a}|f(x)-f_{B+a}|^2\,dx\big)^{1/2}=0$

for any ball B of $\mathbb{R}^n$, where $B+a=\{x\in\mathbb{R}^n : x=a+y,\ y\in B\}$. We omit the proof.

PROPOSITION 3.6. *The spaces* $\mathrm{VMO}_\Delta(\mathbb{R}^n)$, $\mathrm{VMO}_{\sqrt{\Delta}}(\mathbb{R}^n)$, *and* $\mathrm{CMO}(\mathbb{R}^n)$ *coincide, and their norms are equivalent.*

Proof. Recall that $\mathcal{B}$ is the space in Proposition 3.5. We now assume that $\psi\in C_0^\infty$ satisfies the conditions

$$\int\psi(x)\,dx=0\quad\text{and}\quad|\psi(x)|+|x\nabla\psi(x)|\le\frac{c}{(1+|x|)^{n+\varepsilon}}$$

for some $\varepsilon>0$. We can argue as in Proposition 3.3 to show that $f\in\mathcal{B}$ if and only if $\psi_t*f\in T^\infty_{2,V}$; we omit the details here. This gives that the spaces $\mathrm{VMO}_\Delta(\mathbb{R}^n)$, $\mathrm{VMO}_{\sqrt{\Delta}}(\mathbb{R}^n)$, and $\mathrm{CMO}(\mathbb{R}^n)$ coincide and that their norms are equivalent. □

4. Duality between H^1_L and $\mathrm{VMO}_{L^*}(\mathbb{R}^n)$ Spaces

4.1. Hardy Space $H^1_L(\mathbb{R}^n)$

We continue with the assumption that L is an operator that satisfies Assumptions A and B of Section 2.2. Given a function $f\in L^1(\mathbb{R}^n)$, the area integral function $\mathcal{S}_L(f)$ associated with an operator L is defined by

$$\mathcal{S}_L(f)(x)=\bigg(\int_{\Gamma(x)}|Q_{t^m}f(y)|^2\frac{1}{t^{n+1}}\,dy\,dt\bigg)^{1/2}. \tag{4.1}$$

It follows from Assumption B that the area integral function $\mathcal{S}_L(f)$ is bounded on $L^2(\mathbb{R}^n)$ [Mc]. It then follows from Assumption A that $\mathcal{S}_L(f)$ is bounded on L^p, $1<p<\infty$ (see [ADMc, Thm. 6]). More specifically, there exist constants c_1,c_2 such that $0<c_1\le c_2<\infty$ and

$$c_1\|f\|_p \le \|\mathcal{S}_L(f)\|_p \le c_2\|f\|_p \tag{4.2}$$

for all $f \in L^p, 1 < p < \infty$. By duality, the operator $S_{L^*}(f)$ also satisfies estimate (4.2), where L^* is the adjoint operator of L.

The following definition was introduced in [ADMc]. We say that $f \in L^1$ belongs to a Hardy space associated with an operator L, denoted by H_L^1, if $S_L(f) \in L^1$. We define its H_L^1 norm by

$$\|f\|_{H_L^1} = \|\mathcal{S}_L(f)\|_{L^1}.$$

REMARKS. 1. If L is the Laplacian $\triangle$ on $\mathbb{R}^n$, then it follows from area integral characterization of Hardy space using convolution that the classical space $H^1(\mathbb{R}^n)$ and the spaces $H_\triangle^1(\mathbb{R}^n)$ and $H_{\sqrt{\triangle}}^1(\mathbb{R}^n)$ coincide and that their norms are equivalent. See [FS].

2. Recently, Duong and Yan proved in [DY2] that the dual of a Hardy space $H_L^1(\mathbb{R}^n)$ is the space $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$ of Section 3.1, where L^* is the adjoint operator of L. This gives a generalization of the duality of $H^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ described by Fefferman and Stein [FS].

4.2. Main Theorem and Its Proof

The aim of this section is to prove the following theorem.

THEOREM 4.1. *Assume that the operator L satisfies Assumptions A and B of Section 2.2. Denote by L^* the adjoint operator of L. Then the dual space of* $\mathrm{VMO}_L(\mathbb{R}^n)$ *is the space $H_{L^*}^1(\mathbb{R}^n)$ in the following sense.*

(i) *Suppose $f \in H_{L^*}^1(\mathbb{R}^n)$. Then the linear functional ℓ given by*

$$\ell(g) = \int_{\mathbb{R}^n} f(x)g(x)\,dx,$$

initially defined on the dense subspace $\mathrm{VMO}_L \cap L^2$, *has a unique extension to* $\mathrm{VMO}_L(\mathbb{R}^n)$.

(ii) *Conversely, every continuous linear functional ℓ on the* $\mathrm{VMO}_L(\mathbb{R}^n)$ *space can be realized as just described, with $f \in H_{L^*}^1(\mathbb{R}^n)$ and*

$$\|f\|_{H_{L^*}^1} \le c\|\ell\|.$$

In order to prove Theorem 4.1, we need to establish the following two lemmas. Consider the operator π_L initially defined on $T_{2,c}^p$ by

$$\pi_L(f)(x) = \int_0^\infty Q_{t^m}(f(\cdot,t))(x)\frac{1}{t}\,dt. \tag{4.3}$$

Note that, for any compact set K in $\mathbb{R}_+^{n+1}$,

$$\int_K |f(x,t)|^2\,dx\,dt \le C(K,p)\|\mathcal{A}(f)\|_p^2.$$

This and estimate (2.5) imply that the integral (4.3) is well-defined and that $\pi_L \in L^2$ for $f \in T_{2,c}^p$.

LEMMA 4.2. *The operator π_L, initially defined on $T^p_{2,c}$, extends to a bounded linear operator*:

(a) *from T^p_2 to L^p if $1 < p < \infty$*;
(b) *from T^1_2 to H^1_L*;
(c) *from T^∞_2 to* BMO_L;
(d) *from $T^\infty_{2,V}$ to* VMO_L.

Proof. For the proofs of (a), (b), and (c), we refer to [DY2, Lemma 4.3]. We now prove (d). Suppose $f \in T^\infty_{2,V}$. Let us prove that $\pi_L(f) \in \mathrm{VMO}_L(\mathbb{R}^n)$; then, by Proposition 3.3, we need only prove $Q_{t^m}(I - P_{t^m})\pi_L(f) \in T^\infty_{2,V}$. We will prove that there exists a positive constant $c > 0$ such that, for any ball $B = B(x_B, r_B)$,

$$|B|^{-1}\int_{\hat{B}}|Q_{t^m}(I - P_{t^m})\pi_L(f)(x)|^2\frac{1}{t}\,dx\,dt \le c\sum_{k=2}^{\infty}2^{-k\varepsilon/2}\omega_k(f, B), \tag{4.4}$$

where ε is the constant in (2.3) and

$$\omega_k(f, B) = \left(|2^kB|^{-1}\int_{\widehat{2^kB}}|f(x,t)|^2\frac{1}{t}\,dx\,dt\right). \tag{4.5}$$

Once estimate (4.4) is established, we can argue as in the proof of Proposition 3.3 to show that $Q_{t^m}(I - P_{t^m})\pi_L(f) \in T^\infty_{2,V}$. We omit the details here.

Let us verify estimate (4.4). Denote $\Psi_t(L) = Q_{t^m}(I - P_{t^m})$. Let $f_1 = f\chi_{\widehat{4B}}$ and $f_2 = f\chi_{(\widehat{4B})^c}$. One writes

$$\begin{aligned}
|B|^{-1}&\int_{\hat{B}}|Q_{t^m}(I - P_{t^m})\pi_L(f)(x)|^2\frac{1}{t}\,dx\,dt\\
&= \sum_{i=1}^{2}|B|^{-1}\int_{\hat{B}}|\Psi_t(L)\pi_L(f_i)(x)|^2\frac{1}{t}\,dx\,dt\\
&= \mathrm{I} + \mathrm{II}.
\end{aligned}$$

For the term I, using estimate (2.5) and property (a) of this lemma yields

$$\begin{aligned}
\mathrm{I} &\le c|B|^{-1}\int_0^\infty \|\Psi_t(L)\pi_L(f_1)\|^2_{L^2}\frac{1}{t}\,dt\\
&\le c|B|^{-1}\|\pi_L(f_1)\|^2_2\\
&\le c|B|^{-1}\|f\chi_{\widehat{4B}}\|^2_{T^2_2}\\
&\le c|4B|^{-1}\int_{\widehat{4B}}|f(x,t)|^2\frac{1}{t}\,dx\,dt\\
&= c\omega_2(f, B).
\end{aligned}$$

We now estimate term II. Denote by $\Psi_{s,t}(L) = Q_{t^m}(I - P_{t^m})Q_{s^m}$. It follows from estimate (2.4) that the kernel $k_{s,t}(x, y)$ of $\Psi_{s,t}(L)$ satisfies

$$\begin{aligned}
|k_{s,t}(x, y)| &\le c\frac{t^ms^m}{(t^m + s^m)^2}\frac{(t + s)^\varepsilon}{(t + s + |x - y|)^{n+\varepsilon}}\\
&\le c\min((ts)^{\varepsilon/2}, t^{-\varepsilon/2}s^{3\varepsilon/2})\frac{1}{(t + s + |x - y|)^{n+\varepsilon}},
\end{aligned} \tag{4.6}$$

where ε is the constant in (2.3). Observe that for any $(x,t) \in \hat{B}$ and $(y,s) \in \widehat{2^{k+1}B} \setminus \widehat{2^k B}$ we have $t + s + |x-y| \geq c2^k r_B$. From (4.6) it can be verified that $\int_{\mathbb{R}^{n+1}_+} |k_{s,t}(x,y)| \frac{1}{s}\, dy\, ds \leq c < \infty$. Using Hölder's inequality and elementary integration, we have that there exists a positive constant c such that

$$\begin{aligned}
\mathrm{II} &\leq c|B|^{-1} \int_{\hat{B}} \left| \int_{\mathbb{R}^{n+1}_+} k_{s,t}(x,y) f(y,s) \chi_{(\widehat{4B})^c} \frac{1}{s}\, dy\, ds \right|^2 \frac{1}{t}\, dx\, dt \\
&\leq c|B|^{-1} \int_{\hat{B}} \int_{(\widehat{4B})^c} |k_{s,t}(x,y)| |f(y,s)|^2 \frac{1}{s}\, dy\, ds \frac{1}{t}\, dx\, dt \\
&\leq c \sum_{k=2}^{\infty} |B|^{-1} \int_{\hat{B}} \int_{\widehat{2^{k+1}B} \setminus \widehat{2^k B}} \frac{(ts)^{\varepsilon/2}}{(t+s+|x-y|)^{n+\varepsilon}} |f(y,s)|^2 \frac{1}{s}\, dy\, ds \frac{1}{t}\, dx\, dt \\
&\leq c \sum_{k=2}^{\infty} (2^k r_B)^{-\varepsilon} |2^{k+1}B|^{-1} |B|^{-1} \int_{\hat{B}} \int_{\widehat{2^{k+1}B}} (ts)^{\varepsilon/2} |f(y,s)|^2 \frac{1}{s}\, dy\, ds \frac{1}{t}\, dx\, dt \\
&\leq c \sum_{k=2}^{\infty} 2^{-k\varepsilon/2} |2^{k+1}B|^{-1} \int_{\widehat{2^{k+1}B}} |f(y,s)|^2 \frac{1}{s}\, dy\, ds \\
&\leq c \sum_{k=2}^{\infty} 2^{-k\varepsilon/2} \omega_k(f,B).
\end{aligned}$$

Estimate (4.4) then follows readily. Hence, the proof of Lemma 4.2 is complete. □

As a consequence of Lemma 4.2, we have the following corollary.

LEMMA 4.3. $\mathrm{VMO}_L \cap L^2$ *is dense in* VMO_L.

Proof. For any $f \in \mathrm{VMO}_L$, we have $Q_{s^m}(\mathcal{I} - P_{s^m}) f \in T^\infty_{2,V}$. By the definition of $T^\infty_{2,V}$, there exists a family of functions $\{g_k(x,s)\}_k \in T^2_{2,c}$ such that

$$\| Q_{s^m}(\mathcal{I} - P_{s^m}) f - g_k(\cdot,s) \|_{T^\infty_2} \to 0.$$

Define $f_k = \frac{36m}{5} \int_0^\infty Q_{s^m} g_k(\cdot,s) \frac{1}{s}\, ds$. Then it can be verified that $f_k \in \mathrm{VMO}_L \cap L^2$. Moreover, by Lemma 4.2 we have that

$$\begin{aligned}
\|f - f_k\|_{\mathrm{VMO}_L} &\leq c\| Q_{s^m} f - g_k(\cdot,s)\|_{T^\infty_{2,V}} \\
&\leq c\| Q_{s^m} f - g_k(\cdot,s)\|_{T^\infty_2} \\
&\to 0
\end{aligned}$$

as $k \to \infty$. This proves Lemma 4.3. □

Proof of Theorem 4.1. The proof of (i) follows from Lemma 4.3 and the fact that the dual of a Hardy space $H^1_{L^*}(\mathbb{R}^n)$ is the space $\mathrm{BMO}_L(\mathbb{R}^n)$ of Section 3.1, where L^* is the adjoint operator of L.

We now prove (ii). Define

$$\Omega_L = \{h : h(x,t) = Q_{t^m}(I - P_{t^m}) g(x) \text{ for some } g \in \mathrm{VMO}_L\} \subset T^\infty_{2,V}.$$

Note that, for every $h(x,t) \in T^\infty_{2,V}$, by Lemma 4.2 we have

$$\mathcal{R}(h)(x) = b_m \int_0^\infty Q_{t^m}(h(\cdot,t))(x)\frac{1}{t}\,dt \in \mathrm{VMO}_L.$$

Therefore, for any $\ell \in (\mathrm{VMO}_L)'$ and $g \in \mathrm{VMO}_L$, we have

$$\ell(g) = \ell \circ \mathcal{R} \circ Q_{t^m}(\mathcal{I} - P_{t^m})g \tag{4.7}$$

for all $g \in \mathrm{VMO}_L \cap L^2$. Furthermore, it follows from Lemma 4.2 that $\ell \circ \mathcal{R}$ is a continuous linear functional on Ω_L that satisfies

$$\|\ell \circ \mathcal{R}\|_{T^\infty_{2,V}\to\mathbb{C}} \le \|\ell\|_{(\mathrm{VMO}_L)'}\|\mathcal{R}\|_{T^\infty_{2,V}\to\mathrm{VMO}_L} \le c\|\ell\| < \infty.$$

Applying the Hahn–Banach theorem, we can extend $\ell \circ \mathcal{R}$ to a continuous linear functional on $T^\infty_{2,V}$. Note that Lemma 3.2(a) implies that the dual of $T^\infty_{2,V}$ is equivalent to T^1_2. By restricting attention to Ω_L, we can conclude that if ℓ is a continuous linear functional ℓ on the space $\mathrm{VMO}_L(\mathbb{R}^n)$, then it follows from (4.7) that there exists a $w(x,t) \in T^1_2$ with $\|w\|_{T^1_2} \le C\|\ell \circ \mathcal{R}\|$ such that

$$\begin{aligned}
\ell(g)(x) &= \ell \circ \mathcal{R} \circ Q_{t^m}(\mathcal{I} - P_{t^m})g \\
&= \int_{\mathbb{R}^{n+1}_+} w(x,t)Q_{t^m}(\mathcal{I} - P_{t^m})g(x)\frac{1}{t}\,dx\,dt \\
&= \int_{\mathbb{R}^n} g(x)\int_0^\infty Q^*_{t^m}(\mathcal{I} - P^*_{t^m})w(\cdot,t)(x)\frac{1}{t}\,dt\,dx \\
&= \int_{\mathbb{R}^n} g(x)f(x)\,dx.
\end{aligned}$$

Using Lemma 4.2(b) for the adjoint operator L^* of L, we obtain $f \in H^1_{L^*}$ and $\|f\|_{H^1_{L^*}} \le c\|w\|_{T^1_2} \le c\|\ell \circ \mathcal{R}\| \le c\|\ell\|$. This proves (ii), which completes the proof of Theorem 4.1. □

4.3. Applications

Assumptions A and B of Section 2.2 are satisfied by large classes of differential operators. We will list some of them.

1. Let $A = A(x) = ((a_{i,j})(x))_{i,j}$ be an $n \times n$ matrix where the coefficients $a_{i,j}$ are complex-valued $L^\infty(\mathbb{R}^n)$ functions. Assume that this matrix satisfies the following elliptic (or "accretivity") condition:

$$\lambda|\xi|^2 \le \mathrm{Re}\,A\xi\cdot\bar{\xi} \equiv \mathrm{Re}\sum_{i,j} a_{i,j}(x)\xi_j\bar{\xi}_i, \quad \|A\|_\infty \le \Lambda,$$

for $\xi \in \mathbb{C}^n$ and for some λ, Λ such that $0 < \lambda \le \Lambda < \infty$. We define the second-order divergence form operator

$$Lf = -\mathrm{div}(A\nabla f),$$

which we interpret in the usual weak sense via a sesquilinear form.

Such a complex elliptic operator L has a bounded H_∞ calculus in $L^2(\mathbb{R}^n)$ [AT]. Note that when A has real entries, or when $n = 1, 2$ in the case of complex entries, the operator L generates an analytic semigroup e^{-tL} on $L^2(\mathbb{R}^n)$ with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound; that is,

$$|p_t(x,y)| \le \frac{C}{t^{n/2}} \exp\left\{-c\frac{|x-y|^2}{t}\right\} \tag{4.8}$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

2. Let $0 \le V \in L^1_{\text{loc}}(\mathbb{R}^n)$. The Schrödinger operator with potential V is defined by

$$L = -\Delta + V(x) \quad \text{on } \mathbb{R}^n \ (n \ge 3).$$

The operator L is a self-adjoint positive definite operator; hence it has a bounded H_∞ calculus in $L^2(\mathbb{R}^n)$ [Mc]. From the Feynman–Kac formula it is well known that the semigroup kernels $p_t(x, y)$ associated with e^{-tL} satisfy the estimates

$$0 \le p_t(x,y) \le \frac{1}{(4\pi t)^{n/2}} \exp\left\{-\frac{|x-y|^2}{4t}\right\}.$$

Note that unless V satisfies additional conditions, the heat kernel can be a discontinuous function of the space variables and the Hölder continuous estimates may fail to hold (see [Da]).

Acknowledgments. Liang Song and Chaoqiang Tan thank Y. S. Han for helpful discussions and suggestions. The authors thank the referee for helpful comments.

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