**Michigan Math. J. 56 (2008)**

# Arithmetic of a Singular K3 Surface

Matthias Schütt

## 1. Introduction

This paper investigates the arithmetic of a particular singular K3 surface $X$ over $\mathbb{Q}$, the extremal elliptic fibration with configuration $[1,1,1,12,3^*]$. First of all, we determine the corresponding weight-3 form (cf. [Li, Ex. 1.6]) explicitly. For this, we calculate the action of Frobenius on the transcendental lattice by counting points and applying the Lefschetz fixed point formula. The proof is based on our previous classification of complex multiplication (CM) forms with rational coefficients in [S1]. In fact, we only have to compute one trace.

Then we compute the zeta-function of the surface. This is used to study the reductions of $X$ modulo some primes $p$. We emphasize that we are able to find a model with good reduction at 2. We subsequently verify conjectures of Tate and Shioda. The conjectures will be recalled in Section 4 and verified in Sections 5–7.

The final section is devoted to the twists of $X$. We show that these produce all newforms of weight 3 that have rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$.

## 2. The Extremal Elliptic K3 Fibration

There is a unique elliptic K3 surface $X$ with a section and singular fibres $I_1$, $I_1$, $I_1$, $I_{12}$, and $I_3^*$. The configuration is listed as $[1,1,1,12,3^*]$ under No. 166 in [ShiZ] and [S3, Tab. 2]. Since the fibration arises as cubic base change of the extremal rational elliptic surface $Y$ with singular fibres $I_1^*$, $I_4$, and $I_1$, we shall start by studying this surface.

In [MP], an affine Weierstrass equation of this fibration was given as

$$Y'\colon\ y^2 = x^3 - 3(s-2)^2(s^2-3)x + s(s-2)^3(2s^2-9). \tag{1}$$

It has discriminant

$$\Delta = 16\cdot 27(s-2)^7(s+2),$$

so the singular fibres are $I_1^*$ above $2$, $I_1$ above $-2$, and $I_4$ above $\infty$.

We shall look for a model of $Y$ over $\mathbb{Q}$ that has everywhere good reduction. The fibre of $Y'$ at $\infty$ has nonsplit multiplicative reduction, so $\mathrm{H}^2_{\text{ét}}(Y', \mathbb{Q}_\ell)$ is ramified. Therefore we twist equation (1) over the splitting field $\mathbb{Q}(\sqrt{-3})$. Performing some elementary transformations (cf. [S2, IV.1]), we obtain the equation

Received July 6, 2007. Revision received July 30, 2007.

$$Y\colon\ y^2 + sxy = x^3 + 2sx^2 + s^2x. \tag{2}$$

This elliptic surface has discriminant $\Delta = s^7(s+16)$ and everywhere good reduction. In particular, all components of reducible fibres are defined over $\mathbb{Q}$. (For $I_1^*$, this can be derived from Tate's algorithm [Si, IV, Sec. 9].) Upon reducing modulo 2, we obtain the equation from [Ito]. Then $Y$ inherits only the two singular fibres of types $I_4$ and $I_1^*$ with wild ramification at the latter fibre.

We now come to the singular K3 surface $X$. Consider the cubic base change

$$\pi : s \mapsto s^3.$$

Via pull-back from $Y$, this gives rise to an extremal elliptic K3 surface. The resulting Weierstrass equation reads

$$y^2 + s^2xy = x^3 + 2sx^2 + s^2x. \tag{3}$$

This Weierstrass model has a $D_7$ (resp. $A_{11}$) singularity in the fibre above 0 (resp. $\infty$). By $X$, we denote the minimal desingularization. This elliptic fibration has configuration $[1, 1, 1, 12, 3^*]$. The reducible singular fibres $I_3^*$ and $I_{12}$ sit at 0 and $\infty$. The three fibres of type $I_1$ can be found at the cube roots of $-16$.

By construction, the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts trivially on the trivial lattice $V$ of $X$ generated by the 0-section and the fibre components. Since $X$ is extremal, tensoring $V$ with $\mathbb{Q}$ gives the corresponding statement for $\mathrm{NS}(X)$.

The elliptic surface $Y$ has everywhere good reduction. Furthermore, the base change $\pi$ is nowhere degenerate upon reducing. Hence, the pull-back $X$ can have bad reduction only at the prime divisors of the degree of $\pi$ (i.e., at 3). Modulo 3, the Weierstrass model (3) obtains an additional $A_2$ singularity, so the reduction is in fact bad (cf. Section 5).

In terms of $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_\ell)$, the ramification is reflected in the contribution of the transcendental lattice $T_X$, the orthogonal complement of $\mathrm{NS}(X)$ in $\mathrm{H}^2(X, \mathbb{Z})$. Here we consider it as a two-dimensional $\ell$-adic Galois representation $\rho$. The reduction properties of $X$ imply that $\rho$ is ramified only at 3 and at the respective prime $\ell$. This agrees with the discriminant $d = d_{T_X}$ of $X$, which is 3. To see this, recall that

$$d_{T_X} = -d_{\mathrm{NS}(X)},$$

since $\mathrm{H}^2(X, \mathbb{Z})$ is unimodular. Consider the trivial lattice $V$ of $X$. Let $U$ denote the hyperbolic plane—that is, $\mathbb{Z}^2$ with intersection form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then

$$V = A_{11} \oplus D_7 \oplus U$$

has discriminant $d_V = -12 \cdot 4 = -48$. The Néron–Severi group $\mathrm{NS}(X)$ is obtained from $V$ by adding the sections. Here, $X$ is extremal, so the Mordell–Weil group $\mathrm{MW}(X)$ is finite. Hence

$$d_{\mathrm{NS}(X)} = \frac{d_V}{|\mathrm{MW}(X)|^2}.$$

Explicitly, we have $\mathrm{MW}(X) \cong \mathrm{MW}(Y)$, consisting of four elements (cf. Section 6). We obtain $d_{\mathrm{NS}(X)} = -3$ and $d_{T_X} = 3$. Then the (reduced) intersection form on $T_X$ can only be

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

## 3. The Associated Newform

By a result of Livné [Li, Ex. 1.6], $X$ (or $T_X$ resp. $\rho$) has an associated newform of weight 3 with CM by $\mathbb{Q}(\sqrt{-3})$. Our aim is to determine this newform explicitly. Let $p$ be a prime of good reduction; that is, let $p \neq 3$. Then

$$\det \rho(\mathrm{Frob}_p) = \left(\frac{p}{3}\right) p^2. \tag{4}$$

To find the trace of $\rho$, we use the Lefschetz fixed point formula. We have already seen that $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ operates trivially on $\mathrm{NS}(X)$. Hence, the Lefschetz fixed point formula gives

$$\#X(\mathbb{F}_p) = 1 + 20p + \operatorname{tr} \rho(\mathrm{Frob}_p) + p^2.$$

Using a computer program, we calculated the following traces at the first good primes.

| $p$ | 2 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\operatorname{tr}\rho$ | 0 | 0 | $-13$ | 0 | $-1$ | 0 | 11 | 0 | 0 | $-46$ | 47 |

By inspection, these traces coincide with the Fourier coefficients of the newform $f = \sum_n a_n q^n$ of level 27 and weight 3 from [S1, Tab. 1]. We shall now prove that this holds at every prime.

PROPOSITION 3.1.

$$L(T_X, s) = L(f, s).$$

*Proof.* The proof makes use of Livné's modularity result for $X$. Let $g$ denote the associated newform. Since $g$ has CM by $\mathbb{Q}(\sqrt{-3})$, it is a twist of $f$ by [S1, Thm. 3.4]. In our special situation, $g$ is unramified outside 3 (i.e., it has level $3^r$), since $X$ has good reduction elsewhere. Hence we need only compare the few possible newforms with such a level.

Let $\psi_f$ denote the Größencharakter of conductor 3 that corresponds to $f$, and analogously for $\psi_g$. Then [S1, Prop. II.11.1]—and, in particular, the reasoning of [S1, Sec. II.11.4]—show that there are three possibilities in total:

$$\psi_g = \psi_f \quad \text{or} \quad \psi_g = \psi_f \otimes \left(\frac{3}{\cdot}\right)_3 \quad \text{or} \quad \psi_g = \psi_f \otimes \left(\frac{3}{\cdot}\right)_3^2. \tag{5}$$

Since the third residue symbol evaluates nontrivially at the factors of $p = 7$, it suffices to compare the Fourier coefficients (or traces) at 7. Thus $f = g$. A priori, this guarantees only that all but finitely many traces coincide. But here, the associated Galois representations $\rho$ and $\rho_f$ are simple, since their traces are not even. Hence Proposition 3.1 follows. □

COROLLARY 3.2. *The zeta-function of X is*

$$\zeta(X,s) = \zeta(s)\zeta(s-1)^{20}L(f,s)\zeta(s-2).$$

*Proof.* We verify the corollary at every local Euler factor. On the one hand, at all primes $p \neq 3$, this follows from Proposition 3.1 and $\rho(X/\mathbb{Q}) = 20$. On the other hand, the reduction $X_3$ at 3 is singular. Hence, the local Euler factor can be defined only via the number of points of this singular variety over the fields $\mathbb{F}_{3^r}$, since the Lefschetz fixed point formula is not available.

The idea is that, with respect to the number of points, $X_3$ looks like $\mathbb{P}^2$ blown up in 19 rational points. Let us explain what we mean by this. We compare $X_3$ to $Y_3$, the reduction of $Y$ modulo 3. The base change $\pi$ is purely inseparable modulo 3. Hence, over any finite field $\mathbb{F}_{3^r}$, the smooth fibres of $X_3$ have the same number of points as the corresponding fibres of $Y_3$. On the other hand, $X_3$ has ten additional $\mathbb{P}^1$s in the singular fibres at 0 and $\infty$. These are all defined over $\mathbb{F}_3$. Thus

$$\#X_3(\mathbb{F}_{3^r}) = \#Y_3(\mathbb{F}_{3^r}) + 10 \cdot 3^r.$$

Recall that $Y_3$ is $\mathbb{P}^2$ blown up in nine points. These points are all rational over $\mathbb{F}_3$, since the absolute Galois group operates trivially. We deduce that

$$\#X_3(\mathbb{F}_{3^r}) = \#(\mathbb{P}^2(19))(\mathbb{F}_{3^r}).$$

This gives the local Euler factor

$$\zeta_3(X,T) = \frac{1}{(1-T)(1-3T)^{20}(1-3^2T)}.$$

The level 27 implies $L_3(f,s) = 1$ by classical theory. This can also be read off from [S1, Tab. 2] and the nebentypus $\chi_{-3} = (\cdot/3)$ of $f$. This completes the proof of Corollary 3.2. □

## 4. The Conjectures for the Reductions

In this section, we shall discuss conjectures of Tate, Shioda, and Artin for smooth projective surfaces over finite fields. In particular, these conjectures apply to (supersingular) reductions of varieties defined over number fields.

Corollary 3.2 will be very useful: If $X$ has good reduction at $p$, then the corollary gives the local $\zeta$-function of the smooth variety $X/\mathbb{F}_p$. Explicitly, let $p \neq 3$. We obtain

$$\begin{aligned} P_2(X/\mathbb{F}_p, T) &= \det(1 - \mathrm{Frob}_p\, T; \mathrm{H}^2_{\text{ét}}(X/\bar{\mathbb{F}}_p, \mathbb{Q}_\ell)) \\ &= (1-pT)^{20}(1 - a_pT + \chi_{-3}(p)p^2T^2). \end{aligned}$$

This is exactly where the Tate conjecture enters. To formulate it, consider a finite field $k$ and a smooth projective variety $Z/k$. Define the Picard number of $Z$ over $k$:

$$\rho(Z/k) = \operatorname{rk} \mathrm{NS}(Z)^{\mathrm{Gal}(\bar{k}/k)}.$$

We employ the convention $\rho(Z) = \rho(Z/\bar{k})$ when the field of definition of $Z$ is understood.

CONJECTURE 4.1 [T1, (C); T2]. *Let $q = p^r$ and let $Z/\mathbb{F}_q$ be a smooth projective variety. Denote the order of the zero of $P_2(Z/\mathbb{F}_q, T)$ at $T = 1/q$ by $u$. Then $u = \rho(Z/\mathbb{F}_q)$.*

The Tate conjecture is known for elliptic K3 surfaces with a section in characteristic $p > 3$ [T2, Thm. (5.6)].

We can consider the Weierstrass model (3) over any $\mathbb{F}_q$, $q = p^r$. Denote the minimal resolution of the $A_n$ and $D_m$ singularities by $X/\mathbb{F}_q$. This surface coincides with the reduction $X_p$ of $X$ if and only if $p$ is a good prime (i.e., iff $p \neq 3$). On the other hand, $X_3$ contains an $A_2$ singularity, so it is not smooth. The desingularization $X/\mathbb{F}_3$ will be sketched in the next section.

PROPOSITION 4.2. *Let $p$ be a prime. Consider $X/\mathbb{F}_p$. Then*

$$\rho(X/\mathbb{F}_p) = \begin{cases} 20 & \text{if } p \equiv 1 \bmod 3, \\ 21 & \text{if } p \equiv 0, 2 \bmod 3. \end{cases}$$

If $p > 3$, Proposition 4.2 follows from the (known) Tate conjecture. The proofs for $p = 2$ and 3 will be given in the next three sections. We will also verify the following conjecture of Shioda.

Let $L$ be a number field and $Z$ a singular K3 surface over $L$. If $\mathfrak{p}$ is a prime of $L$, denote the residue field of $L$ at $\mathfrak{p}$ by $L_\mathfrak{p}$. We call $\mathfrak{p}$ supersingular if and only if $Z/L_\mathfrak{p}$ is supersingular (i.e., iff $\rho(Z/\bar{L}_\mathfrak{p}) = 22$).

Shioda's conjecture concerns the surface $Z/L_\mathfrak{p}$ at a supersingular prime $\mathfrak{p}$. We can compare two lattices of rank 2: On the one hand, we have the transcendental lattice $T_Z$ of $Z/L$; on the other hand, we can use the natural embedding

$$\mathrm{NS}(Z/\bar{L}) \subseteq \mathrm{NS}(Z/\bar{L}_\mathfrak{p})$$

to define the orthogonal complement

$$T_\mathfrak{p} = \mathrm{NS}(Z/\bar{L})^\perp \subset \mathrm{NS}(Z/\bar{L}_\mathfrak{p}).$$

CONJECTURE 4.3 [Sh2, Conj. 4.1]. *Let $Z$ be a singular K3 surface over a number field $L$ and let $\mathfrak{p}$ be a supersingular prime. Then the two lattices $T_Z$ and $T_\mathfrak{p}$ are similar.*

In other words, the claim is that $T_\mathfrak{p}$ is isomorphic to $T_Z(-m)$ for some $m \in \mathbb{Q}_{>0}$. We will verify this conjecture for the extremal elliptic K3 fibration $X$ at the primes 2 and 3 in the next three sections. This will be achieved by finding explicit generators of the respective $T_\mathfrak{p}$. As a by-product, we will thus verify the following theorem.

THEOREM 4.4 [A, (6.8)]. *Let $Z$ be a supersingular K3 surface over a finite field $k$ of $p^r$ elements. If $r$ is odd, then* $\mathrm{Gal}(\bar{k}/k)$ *operates nontrivially on* $\mathrm{NS}(Z)$.

At the time of this paper, the Brauer group $\mathrm{Br}(Z)$, if finite, was only known to have cardinality a square or twice a square. Hence, Artin was able to prove this theorem only for odd $p$. Recently, the second alternative has been ruled out in [LLoR]. Hence, Artin's argumentation in [A, Sec. 6] also applies to $p = 2$.

Note that Theorem 4.4 agrees perfectly with Proposition 4.2 for our surface $X$. In fact, the known part of Tate's Conjecture 4.1 shows that at a supersingular prime $p > 3$ (i.e., at $p \equiv -1 \bmod 3$),

$$\rho(X/\mathbb{F}_{p^2}) = 22.$$

## 5. $X/\mathbb{F}_3$

In this section, we shall consider $X/\mathbb{F}_3$. This will be special because $\pi$ is purely inseparable modulo 3. As a consequence, the base change $X/\mathbb{F}_3$ from $Y/\mathbb{F}_3$ via $\pi$ has only three singular fibres. They have types $I_3^*$, $I_{12}$, and $I_3$. In particular, the elliptic fibration $X/\mathbb{F}_3$ is extremal (cf. the classification of [Ito]) and supersingular. By construction, it also is unirational. We emphasize that the reduction $X_3$ is not smooth.

We shall now verify Tate's and Shioda's conjectures for $X/\mathbb{F}_3$. Consider the 4-torsion sections of $X/\mathbb{F}_3$ that come from $X/\mathbb{Q}$ upon reducing (see Section 6 for a detailed study). These sections meet the $O$-component of the $I_3$ fibre. Denote the other components of the $I_3$ fibre (not meeting $O$) by $\Theta_1$ and $\Theta_2$. By construction, they also do not meet the generators of the trivial lattice $V$ of $X/\mathbb{Q}$ (embedded into $\mathrm{NS}(X/\mathbb{F}_3)$). Hence, they are orthogonal to $\mathrm{NS}(X/\mathbb{Q}) \subset \mathrm{NS}(X/\bar{\mathbb{F}}_3)$. In the next lemma we claim that this is already all of $\mathrm{NS}(X/\bar{\mathbb{F}}_3)$.

LEMMA 5.1. *Let $A_2$ denote the root lattice generated by $\Theta_1$ and $\Theta_2$. Then*

$$\mathrm{NS}(X/\bar{\mathbb{F}}_3) \cong \mathrm{NS}(X/\mathbb{Q}) \oplus A_2.$$

The proof of this lemma will be directly derived from the following classical result.

THEOREM 5.2 (Artin; Rudakov–Šafarevič). *Let $X$ be a supersingular K3 surface over a field of characteristic $p$. Then*

$$\operatorname{discr} \mathrm{NS}(X) = -p^{2\sigma_0} \quad \textit{for some} \quad \sigma_0 \in \{1, \dots, 10\}.$$

*Here, $\sigma_0$ is called the* Artin invariant.

As a consequence, in our situation we have $\operatorname{discr} \mathrm{NS}(X/\bar{\mathbb{F}}_3) = -3^{2\sigma_0}$ for some $\sigma_0 \in \{1, \dots, 10\}$. But then the previous sublattice $\mathrm{NS}(X/\mathbb{Q}) \oplus A_2$ of $\mathrm{NS}(X/\bar{\mathbb{F}}_3)$ has rank 22 and discriminant $-9$. Since this is the maximal possible discriminant, we deduce the equality of the two lattices. This proves Lemma 5.1.

COROLLARY 5.3. *The supersingular K3 surface $X/\mathbb{F}_3$ has Artin invariant $\sigma_0 = 1$.*

We shall now prove Proposition 4.2 for $p = 3$. Since $\rho(X/\mathbb{Q}) = 20$, the reduction of $\mathrm{NS}(X/\mathbb{Q})$ is clearly generated by divisors over $\mathbb{F}_3$. Using Lemma 5.1, we only have to study the field of definition of the two further generators $\Theta_1, \Theta_2$ of $\mathrm{NS}(X/\mathbb{F}_3)$.

For this, consider the elliptic curve over $\mathbb{F}_3(s)$ given by equation (3). It has non-split multiplicative reduction at $s = -1$. More precisely, the components $\Theta_1$ and $\Theta_2$ are conjugate in $\mathbb{F}_3\big(\sqrt{-1}\big)$. In particular, their sum is defined over $\mathbb{F}_3$, while the difference has eigenvalue $-1$ with respect to the conjugation. Hence, we deduce the claim of Proposition 4.2 that $\rho(X/\mathbb{F}_3) = 21$. This agrees with the Tate conjecture and Theorem 4.4. We also see that $\rho(X/\mathbb{F}_9) = 22$.

Finally, we come to Shioda's conjecture. By Lemma 5.1, we have

$$T_3 = \mathrm{NS}(X/\mathbb{Q})^{\perp} = A_2 \subset \mathrm{NS}(X/\bar{\mathbb{F}}_3).$$

Since $A_2$ has intersection matrix $\left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$, we deduce the validity of Conjecture 4.3 with $m = 1$.

## 6. NS($X$)

To verify the conjectures for the reduction $X_2$, we need a better knowledge of $\mathrm{NS}(X) = \mathrm{NS}(X/\bar{\mathbb{Q}})$. To be precise, we want to express the sections of the original fibration $X$ over $\mathbb{Q}$ in terms of $V \otimes_{\mathbb{Z}} \mathbb{Q}$. This is possible because the fibration is extremal such that

$$\mathrm{MW}(X) \cong \mathrm{NS}(X)/V$$

is only torsion. In terms of equation (3), the sections of the elliptic fibration $X$ are given by

$$\mathrm{MW}(X) \cong \mathbb{Z}/4 = \langle P \rangle = \langle (-s, 0) \rangle = \{(-s, 0), (0, 0), (-s, s^3), O\}.$$

They can be derived from the sections of $Y'$, as given in [MP], by following them through the base and variable changes. We will later see that $\mathrm{MW}(X)$ gives all torsion sections of $X/\bar{\mathbb{F}}_2$ upon reducing.

We want to express the section $P$ as a $\mathbb{Q}$-divisor in $V \otimes_{\mathbb{Z}} \mathbb{Q}$. This can be achieved by determining the precise components of the singular fibres that $P$ intersects.

In the $I_{12}$ fibre above $\infty$, we number the components cyclically $\Theta_0, \ldots, \Theta_{11}$ such that $\Theta_0$ meets $O$. The fibre of type $I_3^*$ above 0 is sketched in Figure 1. As usual, $C_0$ denotes the component meeting $O$. The components $D_i$ have multiplicity 2. The freedom of renumbering components is killed by the following elementary observation.

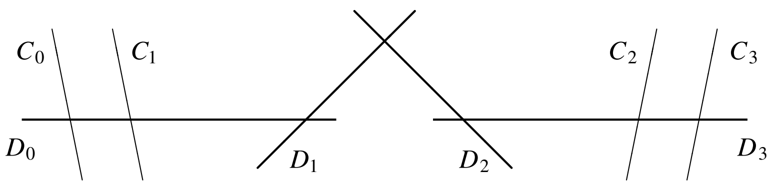


**Figure 1** The fibre of type $I_3^*$ at 0

LEMMA 6.1. *Up to renumbering, the section $P$ meets the components $C_3$ and $\Theta_9$.*

Lemma 6.1 enables us to describe $P$ in terms of the generators of the trivial lattice $V$. Since $P$ is given by polynomials (of low degree), it does not meet $O$. We will use the $\mathbb{Q}$-divisors

$$A = \tfrac{1}{4}(\Theta_1 + 2\Theta_2 + \cdots + 9\Theta_9 + 6\Theta_{10} + 3\Theta_{11}) \quad \text{and}$$
$$B = \tfrac{1}{4}(2C_1 + 4D_0 + 6D_1 + 8D_2 + 10D_3 + 5C_2 + 7C_3).$$

COROLLARY 6.2. *In $V \otimes_{\mathbb{Z}} \mathbb{Q}$, the section $P$ is given as*

$$P = O + 2F - A - B.$$

*Proof.* Since $4P \equiv 0$ modulo $V$, it suffices to check the following intersection numbers:

$$(P.O) = 0, \quad (P.F) = 1, \quad (P.\Theta_i) = \delta_{i,9},$$
$$(P.D_j) = 0, \quad (P.C_j) = \delta_{j,3}, \quad (P^2) = -2.$$

## 7. $X/\mathbb{F}_2$

In this section, we shall consider the elliptic K3 surface $X/\mathbb{F}_2$. This coincides with the reduction $X_2$ of $X$ at 2. Reducing equation (3) modulo 2, we obtain the affine equation

$$X/\mathbb{F}_2\colon\ y^2 + s^2xy = x^3 + s^2x. \tag{6}$$

Recall that this has only two singular fibres. They sit at 0 and $\infty$ and have types $I_3^*$ and $I_{12}$. Since 2 remains inert in $\mathbb{Q}(\sqrt{-3})$, the local $L$-factor is given by

$$P_2(X/\mathbb{F}_2, T) = (1 - 2T)^{21}(1 + 2T)$$

because of Corollary 3.2. In accordance with the Tate conjecture, we shall prove that $X$ is a supersingular K3 surface. To be precise, we claim the following.

PROPOSITION 7.1. *$X/\mathbb{F}_2$ is a supersingular K3 surface with Picard numbers*

$$\rho(X/\mathbb{F}_2) = 21 \quad \textit{and} \quad \rho(X/\mathbb{F}_4) = 22.$$

This proposition completes the proof of Proposition 4.2. It will be established by finding explicit generators for $\mathrm{NS}(X/\bar{\mathbb{F}}_2)$. In addition to the reduction of $\mathrm{NS}(X/\mathbb{Q})$, we need two generators. By the formula of Shioda–Tate, these can be given as sections of the elliptic fibration $X/\bar{\mathbb{F}}_2$.

In detail, we compute some additional sections of $X/\bar{\mathbb{F}}_2$ that are not derived from $\mathrm{MW}(X/\mathbb{Q})$ by way of reduction. Then we determine two among them that supplement $\mathrm{NS}(X/\mathbb{Q})$ to generate $\mathrm{NS}(X/\bar{\mathbb{F}}_2)$. Let $\alpha$ be a generator of $\mathbb{F}_4$ over $\mathbb{F}_2$; that is, $\alpha^2 + \alpha + 1 = 0$. Among others, we found the following sections. Here, the inverse refers to the group law on the generic fibre.

| Section | Inverse |
|---|---|
| $Q = (1, 1)$ | $(1, 1 + s^2)$ |
| $(s^2, s^2)$ | $(s^2, s^2 + s^4)$ |
| $(s + s^3, s^3 + s^4)$ | $(s + s^3, s^4 + s^5)$ |
| $R = (s + \alpha s^3, \alpha^2 s^4 + \alpha s^5)$ | $(s + \alpha s^3, s^3 + \alpha^2 s^4)$ |
| $(1 + s^4, 1 + \alpha s^2 + \alpha^2 s^6)$ | $(1 + s^4, 1 + \alpha^2 s^2 + \alpha s^6)$ |

We shall now prove that $\mathrm{MW}(X/\bar{\mathbb{F}}_2)$ has rank 2 and can be generated by the sections $Q$ and $R$ together with the torsion section $P$. This will be achieved with the help of the *height pairing* on the Mordell–Weil group, as introduced by Shioda in [Sh1]. Let $V$ denote the trivial lattice of $X/\mathbb{F}_2$. This is exactly the reduction of the trivial lattice of $X/\mathbb{Q}$. The height pairing is defined via the orthogonal projection

$$\varphi\colon \mathrm{MW}(X/\bar{\mathbb{F}}_2) \to V^{\perp} \otimes_{\mathbb{Z}} \mathbb{Q} \subset \mathrm{NS}(X/\bar{\mathbb{F}}_2) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

This projection uses only the information concerning which (simple) component of a reducible fibre a section meets.

Let $(\cdot.\cdot)$ denote the intersection form on $\mathrm{NS}(X/\bar{\mathbb{F}}_2) \otimes_{\mathbb{Z}} \mathbb{Q}$. Shioda's height pairing is defined by

$$\begin{array}{rccccl} \langle \cdot, \cdot \rangle\colon & \mathrm{MW}(X/\bar{\mathbb{F}}_2) & \times & \mathrm{MW}(X/\bar{\mathbb{F}}_2) & \to & \mathbb{Q} \\ & \mathcal{P} & & \mathcal{Q} & \mapsto & -(\varphi(\mathcal{P}).\varphi(\mathcal{Q})). \end{array}$$

The height pairing is symmetric and bilinear. It induces the structure of a positive definite lattice on $\mathrm{MW}(X/\bar{\mathbb{F}}_2)/\mathrm{MW}(X/\bar{\mathbb{F}}_2)_{\mathrm{tor}}$ (the Mordell–Weil lattice). For $\mathcal{P}, \mathcal{Q} \in \mathrm{MW}(X/\bar{\mathbb{F}}_2)$, Shioda shows that

$$\langle \mathcal{P}, \mathcal{Q} \rangle = \chi(\mathcal{O}_X) - (\mathcal{P}.\mathcal{Q}) + (\mathcal{P}.O) + (\mathcal{Q}.O) - \sum_v \mathrm{contr}_v(\mathcal{P}, \mathcal{Q}).$$

Here the sum runs over the cusps, and $\mathrm{contr}_v$ can be computed strictly in terms of the components of the singular fibres that $\mathcal{P}$ and $\mathcal{Q}$ meet [Sh1, Thm. 8.6] (cf. [Sh1, (8.16)]). The following lemma is easily checked.

Lemma 7.2. *$Q$ meets $C_0$ and $\Theta_8$ while $R$ intersects $C_3$ and $\Theta_1$.*

We are now in a position to compute the projections $\varphi(Q)$ and $\varphi(R)$. However, we will postpone their explicit calculation, since we will need this only for the explicit verification of Shioda's Conjecture 4.3. Here, we can use Shioda's results from [Sh1] to prove that $\mathrm{NS}(X/\bar{\mathbb{F}}_3)$ is generated by the trivial lattice $V$ and the sections $P$, $Q$, and $R$.

Lemma 7.3. *Let $Q$ and $R$ be the sections of $X/\mathbb{F}_4$ as specified before. Then*

$$\begin{pmatrix} \langle Q, Q \rangle & \langle Q, R \rangle \\ \langle Q, R \rangle & \langle R, R \rangle \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

*Proof.* Since they are given by polynomials (of low degree), $Q$ and $R$ do not meet $O$. For the self-intersection numbers, we thus miss only the contributions from the singular fibres. These are derived from [Sh1, (8.16)] with the help of Lemma 7.2:

$$\langle Q, Q\rangle = 4 - \frac{4\cdot 8}{12} = \frac{4}{3},$$
$$\langle R, R\rangle = 4 - \frac{7}{4} - \frac{11}{12} = \frac{4}{3}.$$

For the remaining entry, we must furthermore analyze the intersection of $Q$ and $R$. We need to find the common zeroes of

$$\alpha s^3 + s + 1 = \alpha(s+\alpha^2)(s^2+\alpha^2 s+1) \quad \text{and}$$
$$\alpha s^5 + \alpha^2 s^4 + 1 = \alpha(s+1)(s+\alpha^2)(s^3+\alpha^2 s+1).$$

The elements of this factorization are irreducible over $\mathbb{F}_4$. Hence, the only common zero is $s=\alpha^2$. Since this occurs with multiplicity 1, the intersection is transversal. Hence, we deduce

$$\langle Q, R\rangle = 2 - 1 - \frac{1\cdot 4}{12} = \frac{2}{3}.$$

This finishes the proof of Lemma 7.3. □

PROPOSITION 7.4. $\mathrm{NS}(X/\bar{\mathbb{F}}_2)$ *has discriminant* $-4$. *It is generated by the trivial lattice* $V$ *and the sections* $P$, $Q$, *and* $R$, *so* $\mathrm{MW}(X/\bar{\mathbb{F}}_2) = \mathrm{MW}(X/\mathbb{F}_4) = \langle P, Q, R\rangle$.

*Proof.* Consider the lattice $N$ generated by the trivial lattice $V$ and the sections $P$, $Q$, and $R$. Clearly, this is a sublattice of $\mathrm{NS}(X/\bar{\mathbb{F}}_2)$. We can identify $N' = \langle V, P\rangle \cong \mathrm{NS}(X/\mathbb{Q}) \subset N$. Recall that this sublattice has discriminant $-3$.

Next we use the orthogonal projection $\varphi$ from $\mathrm{NS}(X/\bar{\mathbb{F}}_2)$ to $V^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $P \in V \otimes_{\mathbb{Z}} \mathbb{Q}$, it follows that

$$\operatorname{discr} N = (\operatorname{discr} N')\det\begin{pmatrix} (\varphi(Q).\varphi(Q)) & (\varphi(Q).\varphi(R)) \\ (\varphi(Q).\varphi(R)) & (\varphi(R).\varphi(R)) \end{pmatrix}.$$

By Lemma 7.3, the matrix has determinant $\frac{4}{3}$. Hence, $N$ has discriminant $-4$ and in particular rank 22. Since both values are the maximal possible (cf. Theorem 5.2), we obtain $N = \mathrm{NS}(X/\bar{\mathbb{F}}_2)$. The claim $\mathrm{MW}(X/\bar{\mathbb{F}}_2) = \langle P, Q, R\rangle$ then follows from the formula of Shioda–Tate. This proves Proposition 7.4. □

Proposition 7.4 implies Proposition 7.1. It verifies the Tate conjecture for $X/\mathbb{F}_{2^r}$ with any $r \in \mathbb{N}$. We also deduce the validity of Theorem 4.4. Furthermore, we have seen that $X_2$ has Artin invariant $\sigma_0 = 1$.

We conclude this section by verifying Conjecture 4.3 for $X/\mathbb{Q}$ and its reduction at 2. Note that it suffices to consider $\mathrm{NS}(X/\mathbb{Q})$ and $\mathrm{NS}(X/\mathbb{F}_4)$. The dual of $\mathrm{NS}(X/\mathbb{Q})$ will always refer to the embedding into $\mathrm{NS}(X/\mathbb{F}_4)$.

To give the explicit form of $\varphi(Q)$ and $\varphi(R)$, recall the $\mathbb{Q}$-divisors $A$ and $B$ from Section 6. The projections are easily computed as

$$\varphi(Q) = Q - O - 2F + \tfrac{1}{3}(\Theta_1 + 2\Theta_2 + \cdots + 8\Theta_8 + 6\Theta_9 + 4\Theta_{10} + 2\Theta_{11}),$$
$$\varphi(R) = R - O - 2F + B + \tfrac{1}{12}(11\Theta_1 + 10\Theta_2 + \cdots + \Theta_{11}).$$

Since the denominators are divisible by 3, it follows that $\varphi(Q), \varphi(R) \notin \mathrm{NS}(X/\mathbb{F}_4)$. In other words,

$$\langle \varphi(Q), \varphi(R)\rangle \supsetneqq \mathrm{NS}(X/\mathbb{Q})^{\perp} \supseteq \langle 3\varphi(Q), 3\varphi(R)\rangle. \tag{7}$$

We claim that

$$\mathrm{NS}(X/\mathbb{Q})^{\perp} = \langle 3\varphi(Q), \varphi(Q) + \varphi(R)\rangle = \langle 3\varphi(R), \varphi(Q) + \varphi(R)\rangle. \tag{8}$$

Since the ranks are 2, the inequality of (7) implies that the claim is equivalent to

$$\varphi(Q) + \varphi(R) \in \mathrm{NS}(X/\mathbb{F}_4).$$

Since $P \equiv -A - B$ modulo $V$, it suffices to show that

$$\varphi(Q) + \varphi(R) \equiv A + B \bmod \langle V, Q, R\rangle.$$

Explicitly, we have

$$\begin{aligned}
\varphi(Q) + \varphi(R) &= Q + R - 2O - 4F + \frac{1}{12}(11\Theta_1 + 10\Theta_2 + \cdots + \Theta_{11}) \\
&\quad + B + \frac{1}{3}(\Theta_1 + 2\Theta_2 + \cdots + 8\Theta_8 + 6\Theta_9 + 4\Theta_{10} + 2\Theta_{11}) \\
&\equiv B + \frac{1}{12}\sum_j (12 - j)\Theta_j + \frac{1}{3}\sum_j j\Theta_j \\
&= B + \frac{1}{12}\sum_j (12 - j + 4j)\Theta_j \\
&\equiv B + \frac{1}{4}\sum_j j\Theta_j \equiv A + B \bmod \langle V, Q, R\rangle.
\end{aligned}$$

This proves claim (8). We shall now verify Conjecture 4.3 for $X/\mathbb{Q}$ and its reduction at 2. Here $T_2 = \langle 3\varphi(Q), \varphi(Q) + \varphi(R)\rangle$ with intersection form

$$\begin{pmatrix} (3\varphi(Q).3\varphi(Q)) & (3\varphi(Q).\varphi(Q) + \varphi(R)) \\ (3\varphi(Q).\varphi(Q) + \varphi(R)) & (\varphi(Q) + \varphi(R).\varphi(Q) + \varphi(R)) \end{pmatrix}$$
$$= -\begin{pmatrix} 12 & 6 \\ 6 & 4 \end{pmatrix} = -2\begin{pmatrix} 6 & 3 \\ 3 & 2 \end{pmatrix} \sim -2\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Hence, Conjecture 4.3 holds with $m = 2$.

REMARK 7.5. In [DK], Dolgachev and Kondō give several models of the supersingular K3 surface in characteristic 2 with Artin invariant $\sigma_0 = 1$. The most natural might be the quasi-elliptic fibration

$$Z\colon\ y^2 = x^3 + t^2 x + t^{11}.$$

The only exceptional fibre of this fibration has type $I_{16}^*$. This shows that

$$\mathrm{NS}(Z) = U \oplus D_{20}.$$

We realize a $\bar{\mathbb{F}}_2$-isomorphic fibration with the preceding exceptional fibre in terms of our model $X/\mathbb{F}_2$ by giving an effective divisor $L$ of type $I_{16}^*$. Denote by $\iota R$ the conjugate of the section $R$ in $\mathbb{F}_4$ and by $[3P]$ the section $(-s, s^3)$ (the inverse of $P$ with respect to the group law on the fibre). Recall that both $R$ and $\iota R$ meet $C_3$ and $\Theta_1$ and that $[3P]$ meets $C_2$ and $\Theta_3$ (see Figure 2 in Section 8). Thus the required divisor $L$ (over $\mathbb{F}_2$) is given by

$$L = R + \iota R + 2(\Theta_1 + \Theta_0 + \Theta_{11} + \Theta_{10} + \Theta_9 + \Theta_8 + \Theta_7 + \Theta_6 + \Theta_5 + \Theta_4 + \Theta_3 + [3P] + C_2 + D_3 + D_2 + D_1 + D_0) + C_0 + C_1.$$

## 8. Twisting

This section concludes the investigation of the extremal elliptic K3 surface $X$ by commenting on twisting. Our aim is to show that the twists of the associated newform $f$ with rational coefficients are in correspondence with twists of $X$. This is nontrivial, since $f$ admits cubic twists.

For the quadratic twisting, this is well known. For instance, it follows from point counting that the twist of $X$ over $\mathbb{Q}(\sqrt{d})$ corresponds to the quadratic twist $f \otimes (d/\cdot)$. Note that there is a model of this twist over $\mathbb{Q}$ with bad reduction exactly at 3 and the primes dividing the discriminant of $\mathbb{Q}(\sqrt{d})$.

We now come to the cubic twists. For $X$, they can be achieved by considering the base change

$$\pi_d : s \mapsto ds^3$$

instead of the original $\pi$. Here, we can restrict to positive cube-free $d$. Let $X^{(d)}$ denote the pull-back from the rational elliptic surface $Y$ by $\pi_d$. An affine model is given by

$$X^{(d)}:\ y^2 + d^2 s^2 xy = x^3 + 2sx^2 + s^2 x. \tag{9}$$

Of course, $X^{(d)}$ has the same configuration of singular fibres as $X$ and also inherits the trivial action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\mathrm{NS}(X^{(d)})$. By construction, it has bad reduction exactly at the prime divisors of $3d$.

For instance, consider the twist $X^{(3)}$. Since this has good reduction away from 3, the arguments of Section 3 apply to determine the associated newform. Recall the Größencharakter $\psi_f$ associated to the newform $f$. Counting points modulo 7, we find that $X^{(3)}$ corresponds to the twist $\psi_f \otimes (3/\cdot)_3$ (with trace 11 at 7). We now give the general statement.

THEOREM 8.1. *$X^{(d)}$ corresponds to the twist $\psi_f \otimes (d/\cdot)_3$; that is, $L(T_{X^{(d)}}, s) = L(\psi_f \otimes (d/\cdot)_3, s)$.*

The proof of this theorem is organized as follows. We exhibit another extremal elliptic fibration on the surfaces $X^{(d)}$. This will have fibres with CM by $\mathbb{Q}(\sqrt{-3})$ such that we can perform the twisting fibrewise. Then the claim will follow.

We want to give an elliptic fibration on $X^{(d)}$ with the component $D_0$ as a section. Therefore, we work with the affine equation of the first blow-up of the Weierstrass model (9) at the $D_3$ singularity:

$$y'^2 + d^2 s^2 x' y' = s x'(x' + 1)^2. \tag{10}$$

Then $D_0 = \{s = y' = 0\}$, so the fibration is the affine projection on the $x'$-coordinate. We employ the usual notation, replacing $x'$ by $t$ and likewise for $s$ and $y'$. Then equation (10) becomes

$$y^2 + d^2 t x^2 y = t(t + 1)^2 x.$$

In order to obtain a projective model for this, we first homogenize

$$s^3 y^2 z + d^2 s^2 t x^2 y = t(t + s)^2 x z^2.$$

Then the change of variable $y \mapsto \frac{t+s}{s} y$ gives

$$X^{(d)}\colon\ s(t + s) y^2 z + d^2 s t x^2 y = t(t + s) x z^2. \tag{11}$$

This has six singularities, two in each fibre above 0, $-1$, and $\infty$. Their resolution produces three fibres of type IV$^*$. These can be read off directly from the original fibration. We sketch this in Figure 2.

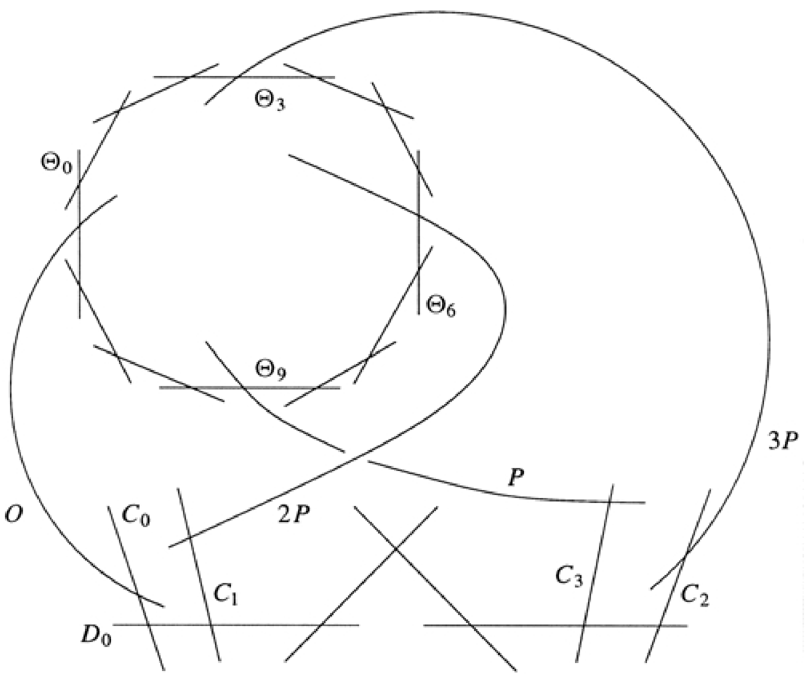


**Figure 2** The fibration on $X^{(d)}$ with three fibres of type IV$^*$

The sections of the new fibration are $D_0$, $\Theta_3$, and $\Theta_9$, and the remaining components form the three singular fibres of type IV*. In particular, they are all defined over $\mathbb{Q}$, as we have already seen.

The smooth fibres of this new fibration are elliptic curves with CM by $\mathbb{Q}(\sqrt{-3})$. However, the impact of twisting on the associated newforms is not visible so far. Therefore, we transform equation (11) into Weierstrass form. The procedure from [C, Sec. 8] gives

$$X^{(d)}:\ y^2 + d^2 s^2 t^2 (s+t)^2 y = x^3.$$

By [IRo, 18, Thm. 4], the Größencharaktere associated to the smooth fibres are twisted by $(d/\cdot)_3$ upon moving from $X^{(1)}$ to $X^{(d)}$. Using the Lefschetz fixed point formula, we can express the trace of Frobenius on $T_{X^{(d)}}$ as the sum of the traces on the smooth fibres. Hence Theorem 8.1 follows. □

REMARK 8.2. From the classification of [S1], it follows that any newform of weight 3 with rational coefficients and CM by $\mathbb{Q}(\sqrt{-3})$ can be realized geometrically by some twist of $X$.

For the only other CM field with nonquadratic twists, $\mathbb{Q}(\sqrt{-1})$, the corresponding statement can be established using the Fermat quartic in $\mathbb{P}^3$.

COROLLARY 8.3. *Up to the bad Euler factors, we have*

$$\zeta(X^{(d)}, s) \mathrel{\hat{=}} \zeta(s)\zeta(s-1)^{20} L\left(\psi_f \otimes \left(\frac{d}{\cdot}\right)_3, s\right)\zeta(s-2).$$

ACKNOWLEDGMENTS. This paper is condensed from the last chapter of my Ph.D. thesis. I would like to express my gratitude to my advisor K. Hulek for all his support and encouragement during this period. I thank B. van Geemen and T. Shioda for many useful comments and discussions.

Generous support from the DFG-Schwerpunkt 1094 “Globale Methoden in der komplexen Geometrie” and the network Arithmetic Algebraic Geometry, a Marie Curie Research Training Network, is gratefully acknowledged. The paper was finished while I enjoyed the hospitality of the Dipartimento di Matematica “Frederico Enriques” of Milano University. I am very grateful for the stimulating atmosphere.

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Department of Mathematical Sciences
University of Copenhagen
2100 Copenhagen Ø
Denmark

mschuett@math.ku.dk