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Weakly 1-Complete Surfaces with Singularities and Applications

G. Tomassini & V. Vâjâitu

1. Introduction

Throughout this paper, complex spaces are assumed to be reduced and with countable topology. A curve, surface, et cetera will be a complex space of the appropriate pure dimension.

Let X be a complex space. We say that X is *weakly 1-complete* if there exists a continuous plurisubharmonic (psh) function $\varphi: X \to \mathbb{R}$ such that φ is exhaustive—that is, if for every $c \in \mathbb{R}$ the sublevel set $\{x \in X : \varphi(x) < c\}$ is relatively compact in X. If we may choose φ strictly plurisubharmonic (spsh) outside a compact subset of X, then X is called *1-convex*.

For 1-convexity of a space X, one has mainly two equivalent characterizations [9]:

- X is *cohomologically 1-convex*—that is, for every coherent analytic sheaf $\mathcal{F}$ on X, the cohomology groups $H^q(X, \mathcal{F})$, $q = 1, 2, \dots$, have finite dimension (as complex vector spaces).
- The space X is a *proper modification of a Stein space at a finite number of points.* In other words, there is a Stein space Y, a proper holomorphic map $\pi : X \to Y$ with $\pi_\star(\mathcal{O}_X) \simeq \mathcal{O}_Y$ (in particular, π is surjective and has connected fibers), and a finite set $B \subset Y$ such that π induces a biholomorphism between $X \setminus \pi^{-1}(B)$ and $Y \setminus B$.

Thus each 1-convex space is holomorphically convex so that it admits "many holomorphic functions". However, there are weakly 1-complete spaces whose global holomorphic functions are only the constants. A class of examples is furnished by "toroidal groups", which are connected complex Lie groups G with $\mathcal{O}(G) = \mathbb{C}$. (By [8], every complex n-dimensional toroidal group is isomorphic to $\mathbb{C}^n/\Gamma$ for some discrete subgroup Γ of $\mathbb{C}^n$; moreover, Γ is weakly 1-complete with a real-analytic defining function [5].)

Perhaps the simplest example is $X = \mathbb{C}^2/\Gamma$, where Γ is the lattice generated by $\{(0,1), (1,0), (i, i\lambda)\}$ and λ is an irrational number in the unit interval. As a real Lie group, X is real-analytically equivalent to the product of a 3-dimensional real

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torus and the real line. Using the absolute value exhaustion function of $\mathbb{R}$, one finds a smooth, proper exhaustion φ of X. Clearly X is weakly 1-complete because the exhaustion function is essentially linear and thus the Levi form vanishes identically. Let $f \in \mathcal{O}(X)$ and let $\tilde{f}$ be its lift to $\mathbb{C}^2$. Since $\tilde{f}$ must be periodic with an irrational period, a look at its Fourier series will show that it, and therefore f, is identically constant. On the other hand, if λ is rational then X is holomorphically convex (in fact, it is the product of $\mathbb{C}^\star$ with an elliptic curve).

Another source of examples of weakly 1-complete manifolds is the bundle spaces of certain topologically trivial vector bundles over compact complex manifolds. We restrict our remark here to the case of a complex line bundle where the bundle space is weakly 1-complete—for instance, if F is a holomorphic line bundle over a compact complex manifold M such that, with respect to some hermitian metric on the fibres, the Chern form $c(F)$ vanishes identically. If $\pi : F \to M$ is the bundle projection, if $\|\cdot\|_x$ is the norm on the fiber F_x, and if $\varphi(\xi) := \log\|\xi\|_x$ where $\pi(\xi) = x$, then φ yields an exhaustion of the bundle space F. A simple calculation shows that φ is psh (in fact it is Levi flat). A particular instance of a bundle satisfying the condition stated is a topologically trivial line bundle over a compact Kähler manifold. Using Hodge theory, one can always find a metric so that the associated Chern form vanishes identically. See [15] for details.

In this circle of ideas one would like to know the answer to the following question, which might be seen as a reformulation of the classical Levi problem.

($\star$) *Describe weakly 1-complete spaces that are holomorphically convex.*

In the sequel we focus on ($\star$) for singular complex surfaces. It is important to note that Ohsawa [11] states that a smooth, connected, weakly 1-complete surface is holomorphically convex provided that it admits a nonconstant holomorphic function. However, his proof contains a gap, which is corrected in Remark 2 (see Section 3).

We are interested here in the case of a singular space X, but we cannot reduce this to the case of nonsingular X owing to Markoe's example [6] of a nonholomorphically convex locally irreducible surface Y whose normalization $Y^\star$ is holomorphically convex. (It is worth remarking that in this example $Y^\star$ is homeomorphic to Y through the normalization map!)

Our main result is Theorem 1.

THEOREM 1. *Let X be an irreducible complex surface that is weakly 1-complete. Then X is* holomorphically convex *provided that there exists a nonconstant holomorphic function f on X.*

In Section 4 we give two applications of Theorem 1:

- a variant of Simha's theorem [13] concerning the "Restraumproblem" for holomorphically convex surfaces; and
- a criterion for holomorphic convexity of pseudoconvex domains in complex 2-dimensional tori (see Corollary 1 in Section 4 for the precise statement).

We note that the present proof of Theorem 1 works for low dimension of X because a holomorphically convex curve Γ can be written as an increasing union of 1-convex open subsets and Γ is Stein if it is irreducible and noncompact.

2. Preliminaries

Here we recall a few notions and lemmas that we need to prove our main theorem.

Let X be a complex space. A function $\varphi\colon X \to \mathbb{R}$ is said to be *plurisubharmonic* (*psh*) if it is upper semicontinuous and, for any holomorphic map $h\colon \Delta \to X$ (Δ is the unit disk in $\mathbb{C}$), $\varphi \circ h$ is subharmonic in Δ (possibly identically $-\infty$). We call φ *strictly psh* if, for any $\theta \in C_0^\infty(X, \mathbb{R})$, there exists an $\varepsilon > 0$ such that $\varphi + \varepsilon\theta$ is psh.

It is known [3, Thm. 5.3.1] that a (strictly) psh function is locally the restriction of a (strictly) psh function in an open set in some $\mathbb{C}^N$ in which X is locally embedded; that is, our definition coincides with the usual one as given in [9].

We shall use the following well-known criterion (due to Narasimhan) of holomorphic convexity.

LEMMA 1. *Let X be a complex space and $\varphi\colon X \to \mathbb{R}$ a continuous psh function. Suppose that there exists a sequence $\{c_\nu\}_\nu$ of real numbers tending to infinity such that every $\{\varphi < c_\nu\}$ is holomorphically convex. Then X is holomorphically convex.*

From [2] we quote the following statement.

LEMMA 2. *Let D be an open set in a Stein space X such that, for any positive integer j, $H^j(D, \mathcal{O}) = 0$. Then D is Stein.*

LEMMA 3. *Let $\pi : X \to Y$ be a finite surjective holomorphic map of complex spaces. Then X is 1-convex if and only if Y is.*

Proof. By [9] we know that 1-convexity is equivalent to cohomological 1-convexity. Moreover, it has been proved in [16] that cohomological q-convexity, *a fortiori* cohomological 1-convexity, is invariant under finite holomorphic surjections. The proof of the lemma follows. □

A key fact in our proof of Theorem 1 is the following particular case of [17, Prop. 4].

LEMMA 4. *Let X be an irreducible surface on which there is a nonconstant holomorphic function f. Assume that X has isolated singularities at worst.*

Let K be a compact set in X and let $Z_1, \dots, Z_m$ be the irreducible components of $\{f = 0\}$ that meet K. Let Ω be an open set in X that intersects every Z_j, $j = 1, \dots, m$.

Then there exist a compact set L in X and an $\varepsilon > 0$ such that, if g is a holomorphic function on X with $\sup_{x \in L} |g(x) - f(x)| < \varepsilon$, then the irreducible components of $\{g = 0\}$ that meet K also meet Ω.

LEMMA 5. *Let X be a complex space and $\mathcal{F}$ an analytic sheaf on X. Assume that there exists a positive integer q such that $H^q(X,\mathcal{F})$ has finite dimension (as a complex vector space). Then, for any holomorphic function h on X, there is a nonconstant holomorphic polynomial P in one complex variable such that $P(h)H^q(X,\mathcal{F}) = 0$.*

Proof. If $H^q(X,\mathcal{F}) = 0$, the assertion is evident. So assume that $H^q(X,\mathcal{F}) \neq 0$. Let $\{\xi_1,\dots,\xi_m\}$ be a basis (of cohomology classes) of $H^q(X,\mathcal{F})$ over $\mathbb{C}$. Fix an index j, $1 \leq j \leq m$. Because $H^q(X,\mathcal{F})$ is also naturally an $\mathcal{O}(X)$-module, it makes sense to consider the cohomology classes $h^l\xi_j$, $l \in \mathbb{N}$. Of course $\xi_j, h\xi_j,\dots,h^m\xi_j$ are dependent over $\mathbb{C}$; thus there is a nonconstant holomorphic polynomial P_j in one complex variable such that $P_j(h)\xi_j$ is the zero cohomology class. Setting $P = P_1\cdots P_m$, it follows that P is a nonconstant holomorphic polynomial in one complex variable such that $P(h)\xi_j = 0$ for all j. Thus $P(h)H^q(X,\mathcal{F}) = 0$. □

Finally, we introduce the singular set of a holomorphic function and give an important property that is used in the proof of Theorem 1.

Let Y be a complex space of pure dimension n. Let $\mathrm{Sing}(Y)$ and $\mathrm{Reg}(Y)$ denote the sets of (respectively) singular and regular points of Y. Let g be a holomorphic function defined on Y. We define the singular set $\mathrm{Sing}(g)$ of g to be the union of $\mathrm{Sing}(Y)$ with the set of critical points of $g|_{\mathrm{Reg}(Y)}$.

Observe that $\mathrm{Sing}(g)$ is an analytic subset of Y. As a matter of fact, since $\mathrm{Sing}(g)$ is obviously closed in Y, its analyticity is a local question and so we may assume (i) that Y is an analytic subset of a Stein open set D in some complex Euclidean space $\mathbb{C}^N$ and (ii) that the ideal sheaf of Y in D is generated by holomorphic functions $h_1,\dots,h_m$ on D. If $\tilde{g}$ is an extension of g to D, then one checks easily that $\mathrm{Sing}(g) = \mathrm{Sing}(Y) \cup (Y \cap \Sigma)$, where

$$\Sigma := \{z \in D : \mathrm{rank}_z\, J(h_1,\dots,h_m,\tilde{g}) \leq N - n\}.$$

Here $J(\cdot)$ is the Jacobian of the corresponding holomorphic mapping; whence the analyticity of $\mathrm{Sing}(g)$.

Moreover, if Γ is an irreducible component of $\mathrm{Sing}(g)$ of positive dimension k and if Γ does not lie entirely in $\mathrm{Sing}(Y)$ (this holds, e.g., when Y has isolated singularities at worst), then $g|_\Gamma$ is constant. To see this, observe that if $y_0 \in W := \mathrm{Reg}(\Gamma) \setminus \mathrm{Sing}(Y)$ then around y_0 we regard Γ as a locally closed submanifold of $\mathbb{C}^n$. We parameterize Γ locally at y_0, which may be chosen as the origin of $\mathbb{C}^n$, so that $\Gamma = \{0\} \times \mathbb{C}^k$ (as germs at 0 in $\mathbb{C}^n$). Therefore,

$$\frac{\partial g}{\partial z_j}(0,\cdot) = 0, \quad j = n-k+1,\dots,n.$$

Thus $g(0,\cdot)$ is constant on a neighborhood of y_0 in W and hence on W. Then the continuity of g implies that it is constant on Γ.

It is worth noting that the foregoing property of g does not hold if Y has nonisolated singularities. For instance, take $Y = \mathbb{C} \times \{y^2 = z^3\} \subset \mathbb{C}^3$ and g induced by the first projection of Y onto $\mathbb{C}$.

3. Proof of Theorem 1

Recall that X is a weakly 1-complete irreducible surface on which there is a nonconstant holomorphic function f. Let φ be the function that displays the weak 1-completeness of X.

We divide the proof into two steps. In Step 1 we deal with the particular case when X has isolated singularities; the general case is then considered in Step 2.

Step 1: Case of X with Isolated Singularities

Let X have isolated singularities and let f be a nonconstant holomorphic function on X. Granting the discussion at the end of Section 2, it follows that $\mathrm{Sing}(f)$ is an analytic subset of X and, for each connected component Γ of $\mathrm{Sing}(f)$, $f|_\Gamma$ is constant. Thus, if K is a compact subset of X then $f(K \cap \mathrm{Sing}(f))$ is a finite subset of $\mathbb{C}$. Therefore, by Lemma 1—and since φ is continuous, so that every sublevel set $\{\varphi < c\}$, $c \in \mathbb{R}$, is weakly 1-complete—there is no loss in generality in assuming that

$$\Lambda := f(\mathrm{Sing}(f))$$

is a finite set of points in $\mathbb{C}$. It is also important to notice that every fiber of f is holomorphically convex (being 1-dimensional and weakly 1-complete, a fiber cannot contain an "infinite necklace"—i.e., a connected analytic curve each of whose infinitely many irreducible components is compact) and so $f^{-1}(\Lambda)$ is holomorphically convex, too.

Following an idea due to Ohsawa [11] we now define, for each $x \in X$, $N_x(f) :=$ the connected component of $f^{-1}(f(x))$ passing through x. Then put

$$B := \{x \in X : N_x(f) \text{ is compact}\}.$$

We start an analysis by cases according to whether B is the empty set or not.

Case I. In this case we assume that B equals the empty set.

Because $f^{-1}(\Lambda)$ is holomorphically convex, we know that if $\{A_i\}_{i \in I}$ denotes the collection of its compact irreducible components (I is an at most countable set of indices) then $\varphi|_{A_i}$ is a constant, say $t_i \in \mathbb{R}$; moreover, the set $\{t_i : i \in I\}$ is discrete in $\mathbb{R}$. Then, since φ is continuous and exhaustive, we infer readily that there are arbitrarily large real numbers c and correspondingly $\varepsilon = \varepsilon(c) > 0$ (small enough) such that on the level sets $\{\varphi = c'\}$, $c - \varepsilon \leq c' \leq c + \varepsilon$, there is no compact irreducible component of $f^{-1}(\Lambda)$.

Fix such c and ε. We claim that, for any $\delta \in (-\varepsilon, \varepsilon)$, the set $\{\varphi < c + \delta\}$ is 1-convex. Then, by Lemma 1 and the preceding discussion, the holomorphic convexity of X will follow.

It is important to observe that, for any $c' \in [c - \varepsilon, c + \varepsilon]$ and $x \in \{\varphi = c'\}$, the compact set $f^{-1}(f(x)) \cap \{\varphi = c'\}$ is contained in a Stein space, namely, the union of the noncompact irreducible components of $f^{-1}(f(x))$. (If A is a positive-dimensional compact analytic subset of some fiber of f, then A must meet $f^{-1}(\Lambda)$.)

In order to settle the claim, now observe also that, as a straightforward consequence of Siu's theorem [14] on the existence of Stein neighborhoods, the following condition is satisfied. There are Stein open sets V_j in X and D_j in $\mathbb{C}$, $j = 1, \dots, m$, such that setting $L := \{c - \delta \le \varphi \le c + \delta\}$ yields:

(1) for each index j, $f^{-1}(D_j) \cap L \subset V_j$;
(2) the $\{f^{-1}(D_j)\}_j$ cover L.

Select smooth functions ρ_j on $\mathbb{C}$ with compact support, $0 \le \rho_j \le 1$, and such that $S_j := \operatorname{supp} \rho_j \subset D_j$ and $\{f^{-1}(S_j)\}_j$ still cover L. Let ψ_j be a smooth strictly psh function on V_j, $j = 1, \dots, m$. Now, for every constant $M > 0$ we define a smooth function Ψ on $\Omega := \{c - \delta < \varphi < c - \delta\}$ (the interior set of L) by setting

$$\Psi(x) = \sum \psi_j(x)\rho_j(f(x)) + M|f(x)|^2, \quad x \in \Omega.$$

Straightforward computations show that, for M sufficiently large, this Ψ becomes strictly psh on Ω. This easily implies the claim, whence the holomorphic convexity of X.

REMARK 1. As a matter of fact, we stress that in this case we have proved that X is a proper modification of a Stein space in a discrete set of points. (We may also say that X is a nondegenerate holomorphically convex space.)

CASE II. Here we assume that B is not the empty set. We shall prove that, in fact, $B = X$.

The set B is open. Indeed, let $x_0 \in B$ and let U be a relatively compact open neighborhood of $N_{x_0}(f)$ such that $f^{-1}(f(x_0)) \cap \partial U = \emptyset$; this is a standard topological fact and can be found, for instance, in [10] (see [10, Chap. 5, Sec. 3, Prop. 2]). In particular, $f(x_0)$ does not belong to $f(\partial U)$. Take W to be an open neighborhood of x_0 in U such that $f(\bar{W}) \cap f(\partial U) = \emptyset$. It follows that, for each $x \in W$, $f^{-1}(f(x)) \cap \partial U = \emptyset$. Hence for such x, $N_x(f)$ lies in U so that $N_x(f)$ is compact; as a result, $W \subset B$.

The set B contains $Y := X \setminus f^{-1}(\Lambda)$. Indeed, since B is open and nonempty and since Y is connected and dense in X, it suffices to verify that $B \cap Y$ is closed in Y. So consider $x_0 \in Y$, a point of adherence of $B \cap Y$, such that there exists a sequence of points $\{x_\nu\}_\nu$ in $B \cap Y$ converging to x_0. Assume, in order to reach a contradiction, that $x_0 \notin B$; hence $N_{x_0}(f)$ is not compact.

Now, since $f^{-1}(f(x_0))$ is smooth, it follows that $N_{x_0}(f)$ is the (unique) connected noncompact component of $f^{-1}(f(x_0))$ through x_0. On the other hand, $\bigcup_\nu N_{x_\nu}(f)$ is relatively compact in X (at this point we use that X is weakly 1-complete). Hence there is an open set Ω in X that meets $N_{x_0}(f)$ and is disjoint from the closure of $\bigcup_\nu N_{x_\nu}(f)$.

Applying Lemma 4 with K a small compact neighborhood of x_0, it follows that for ν sufficiently large, every irreducible component of $\{f = f(x_\nu)\}$ that meets K should meet Ω, too. In particular, there is such an irreducible component in $N_{x_\nu}(f)$, which contradicts the choice of Ω.

The set B contains $f^{-1}(\Lambda)$. Let $t_0 \in \Lambda$ and suppose, in order to reach a contradiction, that there exists an irreducible component Γ of $f^{-1}(t_0)$ that is noncompact. Consider x_0 a regular point of Γ and choose a sequence $\{x_\nu\}_\nu$ of points in X converging to x_0 so that $f(x_\nu) \notin \Lambda$. Thus $x_\nu \in B$. The desired contradiction follows as before, applying again Lemma 4. Therefore, $f^{-1}(t_0)$ has no noncompact irreducible component and, since it is holomorphically convex, it follows that $N_x(f)$ is compact for all $x \in f^{-1}(t_0)$. Thus B contains $f^{-1}(\Lambda)$. Hence $B = X$ as desired, completing Step 1.

We note before proceeding to Step 2 that, because f has compact level sets, the Stein factorization theorem gives a commutative diagram of holomorphic maps:

$$\begin{array}{ccc} X & \xrightarrow{\ \sigma\ } & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & \mathbb{C}, & \end{array}$$

where σ is proper and f' has discrete fibers (in particular, X' is at most of dimension 1 and contains no compact analytic curve; hence X' is Stein). Thus X is holomorphically convex.

Remark 2. The gap in Ohsawa's proof is the following assertion (see [11, p. 155, ll. 25–30]). Let M be a complex manifold and let L be a compact subset of M. Let there be a sequence $\{F_k\}_k$ of compact connected complex hypersurfaces contained in L and a sequence of points $x_k \in F_k$ converging to a point a contained in a connected hypersurface F. Suppose that, for an open neighborhood U of x_0, the sequence $\{U \cap F_k\}_k$ converges in the Hausdorff distance to $U \cap F$. Then $\{F_k\}_k$ converges uniformly to F. Notice that [12] corrects a different gap in [11] from the gap addressed here.

Nevertheless the proof of [11] can be settled as follows. First, using the singular set of f and Lemma 1, one has: For every $x_0 \in X$, there is an $r > 0$ such that $f^{-1}(t)$ is smooth for all $t \in \mathbb{C}$ with $0 < |t - f(x_0)| < r$.

Then the desired contradiction (at the end of the proof of [11, Thm. 1.1]) is obtained as follows (we retain the author's notations). Define $T := F_0 \cap \overline{\bigcup_{k \ge 1} F_{x_k}}$. Clearly, T is a nonempty compact set. On the other hand, it can be seen from [11, Sublemma 1.2] that T is also open in F_0. Thus $T = F_0$!

Step 2: The General Case

Let $\pi : \hat{X} \to X$ be the normalization map of X. There is a natural commutative diagram of holomorphic maps:

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\ \pi\ } & X \\ & {\scriptstyle \hat{f}}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathbb{C}. & \end{array}$$

Observe that $\hat{X}$ is irreducible (for normal complex spaces, irreducibility is equivalent to connectedness) and weakly 1-complete ($\varphi \circ \pi$ displays the weak 1-completeness of $\hat{X}$). Also, $\hat{f}$ is not constant on $\hat{X}$. From the discussion in Step 1 and the irreducibility of X, either one of the following cases may occur:

(a) there is a sequence $\{c_\nu\}_\nu$ of real numbers increasing to infinity such that every sublevel set $\{\varphi \circ \pi < c_\nu\}$ is 1-convex; or

(b) $\hat{f}$ has compact level sets.

If (a) holds true, then each $\{\varphi < c_\nu\}$ is 1-convex. As a matter of fact, this is a straightforward consequence of Lemma 3. Then Step 2 follows, whence the proof of the theorem in this case.

If (b) is fulfilled, then we assert that f has compact level sets, too. Assume, in order to reach a contradiction, that there exists a noncompact irreducible component Γ of $f^{-1}(f(x))$ for some $x \in X$ (note that the fibers of f are holomorphically convex). Since π is finite, $\pi^{-1}(\Gamma)$ is Stein because Γ is Stein. But every irreducible component C of $\pi^{-1}(\Gamma)$ is contained in a level set of $\hat{f}$; thus C is compact and so C is a point. Therefore $\pi^{-1}(\Gamma)$ is a discrete set of points in $\hat{X}$, so that Γ is discrete—which is absurd. This establishes the truth of the assertion. Then, using Stein's factorization theorem again, it follows that X is holomorphically convex, completing the proof of Theorem 1.

4. Applications

An important situation that appears often in complex analysis is the following: A complex space X is given together with a certain complex analytic subvariety $A \subset X$, and one wants to study properties of the complement $U := X \setminus A$; this is known as "the remaining space problem" or "Restraumproblem". It can, then, be important to know how the convexity properties of X and the nature of A influence the convexity of U.

For instance, a well-known theorem due to Simha [13] states that, if X is a locally irreducible Stein surface and if A is a complex curve, then $U = X \setminus A$ remains Stein. More specific questions in this area can be found in [1].

We now give the application alluded to in the Introduction.

THEOREM 2. *Let X be a holomorphically convex surface that is irreducible and locally irreducible. Let A be a complex curve in X such that A has no compact connected component. Then $X \setminus A$ is holomorphically convex.*

We remark that the condition on A is necessary. This is shown by the simple example of the nonholomorphically convex complement of the exceptional divisor of the blowing-up of $\mathbb{C}^2$ at the origin.

REMARK 3. Here we give an example to show that the hypothesis on local irreducibility of X is necessary. Let X be the Whitney umbrella: $X = \{x^2 = yz^2\} \subset \mathbb{C}^3$. Then $\pi\colon \mathbb{C}^2 \to X$, $(u, v) \to (uv, v^2, u)$, is the normalization map of X.

Observe that for $p \in X$, $\#\pi^{-1}(p) > 1$ precisely when $p = (0,t,0)$ with $t \neq 0$. Take a curve $\tilde{A}$ in $\mathbb{C}^2$ with $(0,1) \in \tilde{A}$ but $(0,-1) \notin \tilde{A}$. Then $A := \pi(\tilde{A})$ is a complex curve in X; therefore $X \setminus A$ is not Stein and hence is not holomorphically convex.

As a matter of fact, more generally, for every irreducible Stein surface X that is not locally irreducible there is a complex curve A in X such that $X \setminus A$ is not Stein. Indeed, let $\pi : \tilde{X} \to X$ be the normalization map and $x_0 \in X$ a point such that $\pi^{-1}(x_0) = \{\tilde{x}_1, \dots, \tilde{x}_m\}$ with $m \geq 2$. Let f be a holomorphic function on $\tilde{X}$ such that $f(\tilde{x}_1) \neq 0$ but $f(\tilde{x}_j) = 0$ for $j = 2, \dots, m$. Then $A := \pi(\{f = 0\})$ is as desired because if $X \setminus A$ were Stein then $\tilde{X} \setminus \pi^{-1}(A)$ would be Stein, too. But this is not possible since $\tilde{x}_1$ is isolated in $\pi^{-1}(A)$.

The following lemma will be used in the proof of Theorem 2.

LEMMA 6. *Let X be a locally irreducible weakly 1-complete surface and let A be a Stein curve in X. Then $X \setminus A$ is weakly 1-complete.*

Proof. Let U be a Stein open neighborhood of A in X; see [14]. Then, by [13] it follows that $U \setminus A$ is Stein. Hence there exists a strictly psh exhaustion function $\psi\colon U \setminus A \to \mathbb{R}$. Let $\varphi\colon X \to \mathbb{R}$ be psh and exhaustive (it exists because X is weakly 1-complete). Choose V an open neighborhood of A in X such that $\bar{V} \subset U$. Then select $\chi : [0,\infty) \to [0,\infty)$ rapidly increasing and convex such that $\chi \circ \varphi > \psi$ on ∂V. Define the function $\Phi\colon X \setminus A \to \mathbb{R}$ as follows:

$$\Phi = \begin{cases} \max(\chi \circ \varphi, \psi) & \text{on } V \setminus A; \\ \chi \circ \varphi & \text{on } X \setminus V. \end{cases}$$

Clearly Φ is continuous, exhaustive, and psh; hence $X \setminus A$ is weakly 1-complete. □

REMARK 4. In this circle of ideas we note that if X is a weakly 1-complete manifold and if $A \subset X$ is a Stein hypersurface (not necessarily smooth), then $X \setminus A$ is weakly 1-complete.

Conceptually speaking, the proof of this statement goes essentially along the same lines just described. Let us note a few details. The hypersurface A defines a canonical holomorphic line bundle L over X. Since A is Stein, there exists a Stein open neighborhood U of A and so $L|_U > 0$. Choose a holomorphic section $\sigma \in \Gamma(X, L)$ such that $A = \{\sigma = 0\}$; then let h be a smooth hermitian metric on L such that the function $\psi := -\log\|\sigma\|_h^2$, which is defined on $X \setminus A$, is strictly psh on $U \setminus A$. Then repeat the patching procedure used previously.

Proof of Theorem 2. First notice that, since A is holomorphically convex, the hypothesis implies readily that for each connected component A' of A there is a noncompact irreducible component Γ of A with $\Gamma \subset A'$.

We shall write A as an increasing union of analytic subsets $\{\Sigma_n\}_n$, $n = 0, 1, \dots$, such that Σ_0 and all the sets $\Sigma_{n+1} \setminus \Sigma_n$ are Stein curves. In order to do this, we proceed as follows. Let $\{A_i\}_{i\in I}$ be the decomposition of A into its irreducible components; I is an almost countable set of indices. We write I as an increasing union of subsets $\{I_n\}_n$ by setting $I_0 := \{i \in I : A_i \text{ is noncompact}\}$ and, if I_n is defined, we put

$$I_{n+1} := I_n \cup \{i \in I \setminus I_n : \exists j \in I_n \text{ such that } A_i \cap A_j \neq \emptyset\}.$$

It is obvious to see that the sets

$$\Sigma_n := \bigcup_{j\in I_n} A_j, \quad n = 0, 1, \dots,$$

fulfill the desired property.

Applying Lemma 6 we deduce that, for each n, $X \setminus \Sigma_n$ is weakly 1-complete and hence holomorphically convex by Theorem 1. Because X is holomorphically convex, to conclude the theorem we must show that, for any point $a \in A$ and any sequence $\{x_\nu\}_\nu \subset X \setminus A$ converging to a, there exists a holomorphic function f on $X \setminus A$ that is unbounded on this sequence. But this is obvious because, since $\{A_\lambda\}_\lambda$ is locally finite, $\{\Sigma_n\}_n$ is locally stationary; thus there is an $n_0 \in \mathbb{N}$ with $a \in \Sigma_{n_0}$ and an open neighborhood U of a such that $U \cap \Sigma_n = U \cap \Sigma_{n_0}$. The proof follows since $X \setminus \Sigma_{n_0}$ is holomorphically convex and contains $X \setminus A$. □

In this circle of ideas, a straightforward application of [11] and [6] yields the following result.

COROLLARY 1. *Let $\mathbb{T}^2$ be a complex 2-dimensional torus and let $D \subset \mathbb{T}^2$ be a connected open set that is locally Stein. Then D is holomorphically convex if and only if $\mathcal{O}(D) \neq \mathbb{C}$.*

Proof. Consider the boundary distance function $\delta\colon D \to (0, \infty)$ from the boundary ∂D of D computed with respect to the flat Kähler metric on $\mathbb{T}^2$ that has vanishing holomorphic bisectional curvature. By [7] we deduce that $-\log\delta$ is psh. Obviously, $-\log\delta$ is exhaustive. Thus D is weakly 1-complete and so the corollary follows by [11]. □

A cohomological condition for local Steinness is provided by the following.

PROPOSITION 1. *Let X be an irreducible complex surface and let $D \subset X$ be an open set with $H^1(D, \mathcal{O})$ of finite dimension (as a complex vector space). Then D is locally Stein.*

Proof. Let $x_0 \in \partial D$. Let U be a connected Stein open neighborhood of x_0. We show that $V := U \cap D$ is a Stein open subset of U.

For this we use Coen's criterion [2] (see our Lemma 2). Now, in order to apply this, because $H^j(V, \mathcal{O}) = 0$ for all integers $j \geq 2$ it remains only to check that $H^1(V, \mathcal{O}) = 0$.

First we remark that $H^1(V, \mathcal{O})$ has finite dimension. Indeed, from the Mayer–Vietoris sequence (see [4]) one has an exact sequence

$$H^1(D, \mathcal{O}) \oplus H^1(U, \mathcal{O}) \to H^1(V, \mathcal{O}) \to H^2(D \cup U, \mathcal{O}).$$

Since $H^1(U, \mathcal{O}) = 0$ and since $H^1(D, \mathcal{O})$ and $H^2(D \cup U, \mathcal{O})$ have finite dimension, it follows that $H^1(V, \mathcal{O})$ has finite dimension, too.

Now, let h be a holomorphic function on V that is not constant on any 2-dimensional irreducible component of V; thus the sets $\{h = c\}$, $c \in \mathbb{C}$, are 1-dimensional Stein curves. (We can produce h as a restriction to V of a suitable holomorphic function on U.) Given Lemma 5, there is a nonconstant holomorphic polynomial P in one complex variable such that $P(h)H^1(V, \mathcal{O}) = 0$.

Let $\mathcal{I}$ be the ideal subsheaf of $\mathcal{O}$ generated by $P(h)$. Then, on the one hand, since the morphism $\mathcal{O} \to \mathcal{I}$ induced by $P(h)$ is an isomorphism it follows that the canonically induced map $\alpha\colon H^1(V, \mathcal{O}) \to H^1(V, \mathcal{I})$ is bijective; on the other hand, the short exact sequence $0 \to \mathcal{I} \to \mathcal{O} \to \mathcal{O}/\mathcal{I} \to 0$ induces in cohomology a surjection map $\beta\colon H^1(V, \mathcal{I}) \to H^1(V, \mathcal{O})$. Thus $\beta \circ \alpha\colon H^1(V, \mathcal{O}) \to H^1(V, \mathcal{O})$ is surjective. But the image of $\beta \circ \alpha$ is $P(h)H^1(V, \mathcal{O})$; hence $H^1(V, \mathcal{O}) = 0$ and this concludes the proof of the lemma. □

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G. Tomassini
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa
Italia

tomassini@sns.it

V. Vâjâitu
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O. Box 1-764
RO-014700 Bucharest
Romania

viorel.vajaitu@imar.ro

and

Université des Sciences et Technologies de Lille 1
Laboratoire Paul Painlevé
UFR de Mathèmatiques
Bât. M2
F-59655 Villeneuve d'Ascq Cedex
France

viorel.vajaitu@math.univ-lille1.fr