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# Plurisubharmonic Exhaustion Functions and Almost Complex Stein Structures

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## 1. Introduction

One of the fundamental results of classical complex analysis establishes the equivalence between the holomorphic disc convexity of a domain in an affine complex space, the Levi convexity (the positive semidefiniteness of the Levi form) of its boundary, and the existence of a strictly plurisubharmonic exhaustion function. On the other hand, in the works of Eliashberg and Gromov [7], McDuff [20], and other authors, the convexity properties of strictly pseudoconvex domains in almost complex manifolds are substantially used and allow one to obtain many interesting results concerning symplectic and contact structures. It turns out that the notion of pseudoconvexity playing a fundamental role in classical complex analysis admits deep analogues in the symplectic category. Furthermore, Eliashberg and Thurston [8] introduced and studied so-called confoliations, which can be viewed as contact structures with degeneracies. Their theory, developed in [8] mainly for manifolds of real dimension 3, links the geometry and topology of contact structures with the theory of foliations. One of the main examples of confoliations is given by the distribution of holomorphic tangent spaces on the boundary of a weakly pseudoconvex domain in an almost complex manifold (in general, with a nonintegrable almost complex structure). However, in contrast to the situation for the classical case of $\mathbb{C}^n$, only a few properties of such domains are known. The goal of our paper is to study the basic convexity properties of (weakly) pseudoconvex domains in almost complex manifolds.

The paper is organized as follows. Sections 2 and 3 are essentially preliminary and contain the properties of almost complex structures used in the proofs of the main results. In Section 4 we prove that if $(M, J)$ is an almost complex manifold admitting a strictly plurisubharmonic function and $\Omega$ is a relatively compact domain in $M$ with smooth boundary such that the Levi form of $b\Omega$ is positive semidefinite at every point, then $\Omega$ admits a bounded strictly plurisubharmonic exhaustion function (Theorem 4.4). This generalizes the known result of Diederich and Fornaess [5; 6] dealing with the case of domains in $\mathbb{C}^n$. In particular, this means that the domains satisfying the hypothesis of Theorem 4.4 are Stein manifolds in the sense of Eliashberg and Gromov [7] and admit the canonical symplectic structure defined by the Levi form of a strictly plurisubharmonic exhaustion function.

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As an application we obtain a characterization of pseudoconvex domains in almost complex manifolds similar to the classical results of complex analysis (Theorem 5.4). As another consequence of Theorem 4.4 we obtain that in its setting the domain $\Omega$ is taut; that is, every sequence $(D_k)$ of holomorphic discs in $\Omega$ either contains a subsequence, convergent in the compact open topology, or $(D_k)$ is compactly divergent (Theorem 6.1).

Next we prove a result on the approximation of confoliations (in the sense of Eliashberg and Thurston [8]) by contact structures (Theorem 7.1). Similar results are obtained in [8] only in the case of complex dimension 2. In Section 8 we study Bishop discs in pseudoconvex domains and generalize a recent result of [18] (this result of [18] gave an affirmative answer to the question raised by Ivashkovich and Rosay [11] on foliations of Levi-flat hypersurfaces by pseudoholomorphic discs). Finally, in the last section we discuss the problem of constructing almost complex Stein structures on fiber bundles.

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## 2. Almost Complex Manifolds

All manifolds and almost complex structures are supposed to be of class $C^\infty$, though the main results require only a lower regularity. Let $(\tilde{M}, \tilde{J})$ and $(M, J)$ be almost complex manifolds and let $f$ be a smooth map from $\tilde{M}$ to $M$. We say that $f$ is $(\tilde{J}, J)$-holomorphic if $df \circ \tilde{J} = J \circ df$. Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and let $J_{st}$ be the standard structure on $\mathbb{C}^n$ for every $n$. If $(M', J') = (\mathbb{D}, J_{st})$, we call $f$ a $J$-holomorphic disc in $M$.

Every almost complex manifold $(M, J)$ can be viewed locally as the unit ball $\mathbb{B}$ in $\mathbb{C}^n$ equipped with a small almost complex deformation of $J_{st}$. Indeed, we have the following frequently used statement.

LEMMA 2.1. *Let $(M, J)$ be an almost complex manifold. Then for every point $p \in M$, every real $\alpha \geq 0$, and $\lambda_0 > 0$ there exist a neighborhood $U$ of $p$ and a coordinate diffeomorphism $z\colon U \to \mathbb{B}$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$, and the direct image $z_*(J) := dz \circ J \circ dz^{-1}$ satisfies $\|z_*(J) - J_{st}\|_{\mathcal{C}^\alpha(\bar{\mathbb{B}})} \leq \lambda_0$.*

*Proof.* There exists a diffeomorphism $z$ from a neighborhood $U'$ of $p \in M$ onto $\mathbb{B}$ satisfying $z(p) = 0$ and $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$. For $\lambda > 0$ consider the dilation $d_\lambda\colon t \mapsto \lambda^{-1}t$ in $\mathbb{R}^{2n}$ and the composition $z_\lambda = d_\lambda \circ z$. Then $\lim_{\lambda\to 0}\|(z_\lambda)_*(J) - J_{st}\|_{\mathcal{C}^\alpha(\bar{\mathbb{B}})} = 0$ for every real $\alpha \geq 0$. Setting $U = z_\lambda^{-1}(\mathbb{B})$ for $\lambda > 0$ small enough, we obtain the desired statement. $\square$

In the sequel we often denote $z_*(J)$ just by $J$ when local coordinates are fixed. Let $(M, J)$ be an almost complex manifold. We denote by $TM$ the real tangent bundle of $M$ and by $T_{\mathbb{C}}M$ its complexification. Recall that $T_{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$, where $T^{(1,0)}M := \{V \in T_{\mathbb{C}}M : JV = iV\} = \{X - iJX, X \in TM\}$ and $T^{(0,1)}M := \{V \in T_{\mathbb{C}}M : JV = -iV\} = \{X + iJX, X \in TM\}$. Let $T^*M$ denote the cotangent bundle of $M$. Identifying $\mathbb{C} \otimes T^*M$ with $T^*_{\mathbb{C}} M := \mathrm{Hom}(T_{\mathbb{C}} M, \mathbb{C})$ we define the set of complex forms of type $(1,0)$ on $M$ by $T^*_{(1,0)} M = \{w \in T^*_{\mathbb{C}} M : w(V) = 0 \ \forall V \in T^{(0,1)} M\}$ and the set of complex forms of type $(0,1)$ on $M$ by $T^*_{(0,1)} M = \{w \in T^*_{\mathbb{C}} M : w(V) = 0 \ \forall V \in T^{(1,0)} M\}$. Then $T^*_{\mathbb{C}} M = T^*_{(1,0)} M \oplus T^*_{(0,1)} M$. This allows us to define the operators $\partial_J$ and $\bar\partial_J$ on the space of smooth functions defined on $M$: given a complex smooth function $u$ on $M$, we set $\partial_J u = du_{(1,0)} \in T^*_{(1,0)} M$ and $\bar\partial_J u = du_{(0,1)} \in T^*_{(0,1)} M$. As usual, differential forms of any bidegree $(p,q)$ on $(M,J)$ are defined by means of the exterior product.

In what follows we will often work in local coordinates. Fixing local coordinates on a connected open neighborhood of a point $p \in (M,J)$, we can view it as a neighborhood $U$ of the origin (corresponding to $p$) in $\mathbb{R}^{2n}$ with the standard coordinates $(x,y)$. After an additional linear transformation we may assume that $J(0) = J_{st}$; recall that

$$J_{st}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \qquad J_{st}\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}.$$

In these coordinates the structure $J$ can be viewed as a smooth real $(2n \times 2n)$-matrix function $J : U \to M_{2n}(\mathbb{R})$ satisfying $J^2 = -I$ (the $2n$ unit matrix). After the preceding complexification we can identify $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ with the standard complex coordinates $z = x + iy$ and so deal with $T^{(1,0)}(M)$ instead of $T(M)$.

Consider a $J$-holomorphic disc $z : \mathbb{D} \to U$. After a straightforward verification the conditions $J(z)^2 = -I$ and $J(0) = J_{st}$ imply that for every $z$ the endomorphism $u$ of $\mathbb{R}^2$ defined by $u(z) := -(J_{st} + J_\delta(Z))^{-1}(J_{st} - J_\delta(Z))$ is anti-$\mathbb{C}$-linear; that is, $u \circ J_{st} = -J_{st} \circ u$. Thus $u$ is a composition of the complex conjugation and a $\mathbb{C}$-linear operator. Denote by $Q_J(z)$ the complex $n \times n$ matrix such that $u(z)(v) = Q_J(z)\bar v$ for any $v \in \mathbb{C}^n$. The entries of the matrix $Q_J(z)$ are smooth functions of $z$ and $Q_J(0) = 0$. The $J$-holomorphy condition $J(z) \circ dz = dz \circ J_{st}$ can be written in the form

$$z_{\bar\zeta} + Q_J(z)\bar z_{\bar\zeta} = 0. \tag{2.1}$$

Similarly to the proof of Lemma 2.1, consider the isotropic dilations $d_\lambda$. Since the structures $J_\lambda := (d_\delta)_*(J)$ converge to $J_{st}$ in any $C^\alpha$ norm as $\lambda \to 0$, we have $Q_{J_\lambda} \to 0$ in any $C^\alpha$ norm. Thus, shrinking $U$ if necessary and using the isotropic dilations of coordinates as in the proof of Lemma 2.1, we can assume that for given $\alpha > 0$ we have $\|Q_J\|_{C^\alpha} \ll 1$ on the unit ball of $\mathbb{C}^n$. In particular, the system (2.1) is elliptic.

According to classsical results [31], the Cauchy–Green transform

$$T_{\mathrm{CG}}(f) = \frac{1}{2\pi i} \iint_{\mathbb{D}} \frac{f(\tau)}{\zeta - \tau}\, d\tau \wedge d\bar\tau$$

is a continuous linear operator from $C^\alpha(\bar{\mathbb{D}})$ into $C^{\alpha+1}(\bar{\mathbb{D}})$ for any nonintegral $\alpha > 0$. Hence the operator

$$\Psi_J : z \to w = z + T_{\mathrm{CG}} Q_J(z)\bar z_{\bar\zeta}$$

takes the space $C^{\alpha}(\bar{\mathbb{D}})$ into itself and we can write equation (2.1) in the form $[\Psi_J(z)]_{\bar{\zeta}} = 0$. Thus, the disc $z$ is $J$-holomorphic if and only if the map $\Psi_J(z)\colon \mathbb{D} \to \mathbb{C}^n$ is $J_{st}$-holomorphic. If the norm of $Q_J$ (that is, the initial neighborhood $U$) is small enough, then by the implicit function theorem the operator $\Psi_J$ is invertible and we obtain a bijective correspondence between *small enough* $J$-holomorphic discs and usual holomorphic discs. This easily implies (a) the existence of a $J$-holomorphic disc through a given point in a given tangent direction and (b) the smooth dependence of such a disc on the central point, the given tangent vector, and the almost complex structure. It also implies the interior elliptic regularity of $J$-holomorphic discs (the Nijenhuis–Woolf theorem [22]; see [26] for a short complete proof).

## 3. Levi Form and Plurisubharmonic Functions

We recall some standard definitions playing a substantial role in the sequel. Let $r$ be a $C^2$ function on $(M, J)$. We denote by $J^*dr$ the differential form acting on a vector field $X$ by $J^*dr(X) := dr(JX)$. For example, if $J = J_{st}$ on $\mathbb{R}^2$, then $J^*dr = r_y dx - r_x dy$. The value of *the Levi form of* $r$ at a point $p \in M$ and a vector $t \in T_p(M)$ is defined by

$$L_r^J(p; t) := -d(J^*dr)(X, JX)$$

where $X$ is an arbitrary smooth vector field in a neighborhood of $p$ satisfying $X(p) = t$. This definition is independent of the choice of vector fields. For instance, if $J = J_{st}$ in $\mathbb{R}^2$, then $-d(J^*dr) = \Delta r dx \wedge dy$ ($\Delta$ denotes the Laplacian). In particular, $L_r^{J_{st}}\left(0, \frac{\partial}{\partial x}\right) = \Delta r(0)$.

The following properties of the Levi form are fundamental.

PROPOSITION 3.1. *Let $r$ be a real function of class $C^2$ in a neighborhood of a point $p \in M$.*

(i) *If $F\colon (M, J) \to (M', J')$ is a $(J, J')$-holomorphic map and $\varphi$ is a real function of class $C^2$ in a neighborhood of $F(p)$, then for any $t \in T_p(M)$ we have $L^J_{\varphi \circ F}(p; t) = L^{J'}_{\varphi}(F(p), dF(p)(t))$.*

(ii) *If $z\colon \mathbb{D} \to M$ is a $J$-holomorphic disc satisfying $z(0) = p$ and if $dz(0)(e_1) = t$ (here $e_1$ denotes the vector $\frac{\partial}{\partial \Re \zeta}$ in $\mathbb{R}^2$), then $L_r^J(p; t) = \Delta(r \circ z)(0)$.*

Property (i) expresses the invariance of the Levi form with respect to biholomorphic maps. Property (ii) is often useful in order to compute the Levi form if a vector $t$ is given.

*Proof of Proposition 3.1.* (i) Since the map $F$ is $(J, J')$-holomorphic, we have

$$J'^*dr(dF(X)) = dr(J'dF(X)) = dr(dF(JX)) = d(r \circ F)(JX);$$

that is, $F^*(J'^*dr) = J^*d(r \circ F)$. By the invariance of the exterior derivative we obtain that $F^*(dJ'^*dr) = dJ^*d(r \circ F)$. Again using the holomorphy of $F$, we get $dJ'^*dr(dF(X), J'dF(X))F^*(dJ'^*dr)(X, JX) = dJ^*d(r \circ F)(X, JX)$, which implies (i).

(ii) Since $z$ is a $(J_{st}, J)$-holomorphic map, (i) implies that

$$L^J_r(p,t) = L^{J_{st}}_{r\circ z}(0, e_1) = \Delta(r \circ z)(0).$$

This proves the proposition. □

As usual, we say that an upper semicontinuous function $r$ on $(M, J)$ is *plurisubharmonic* if its composition with any $J$-holomorphic disc is subharmonic on $\mathbb{D}$. For a $C^2$ function this is, in view of Proposition 3.1, equivalent to the positive semidefiniteness of the Levi form:

$$L^J_r(p,t) \geq 0 \quad \text{for any} \;\; p \in M \;\; \text{and} \;\; t \in T_p(M).$$

We use the standard notation $\mathrm{PSH}(M)$ for the class of plurisubharmonic functions in $M$. We say that a $C^2$ function $r$ is *strictly plurisubharmonic* on $M$ if $L^J_r(p,t) > 0$ for any $p \in M$ and $t \in T_p(M)\backslash\{0\}$.

Since the system (2.1) is written in complex coordinates, it is convenient to define the Levi form on the complexified bundle $T^{1,0}(M)$. Recall that the bundles $T(M)$ and $T^{1,0}(M)$ are canonically isomorphic by $X \mapsto V = X - iJX \in T^{1,0}(M)$ for any section $X$ of $T(M)$ or equivalently $X = 2\Re V \in T(M)$ for any section $V$ of $T^{1,0}(M)$. So we simply define the Levi form of $r$ on $T^{1,0}(M)$ by $L^J_r(p; V) = -d(J^*dr)(2\Re V, 2J\Re V)(p)$ for any smooth vector field $V$ on $T^{1,0}(M)$.

Our approach is based on the observation that the Levi form of a function $r$ at a point $p$ in an almost complex manifold $(M, J)$ coincides with the Levi form with respect to the standard structure $J_{st}$ of $\mathbb{R}^{2n}$ if *suitable* local coordinates near $p$ are chosen. Let us explain how to construct these adapted coordinate systems.

As before, by choosing local coordinates near $p$ we may identify a neighborhood of $p$ with a neighborhood of the origin and assume that $J$-holomorphic discs are solutions of (2.1).

LEMMA 3.2. *There exists a second-order polynomial local diffeomorphism $\Phi$ fixing the origin and with linear part equal to the identity such that in the new coordinates the matrix function $Q$ (we drop the index $J$) from* (2.1) *satisfies*

$$Q(0) = 0, Q_z(0) = 0.$$

Thus, by a suitable local change of coordinates one can remove the linear terms in $z$ in the matrix $Q$. We stress that in general it is impossible to get rid of first-order terms containing $\bar{z}$ since this would impose a restriction on the Nijenhuis tensor $J$ at the origin.

The analogue of this statement is well known in the classical theory of elliptic systems on the complex plane [31]. In the form just given the statement appeared first in Chirka's notes [3]. In [30] it is shown that, in an almost complex manifold of (complex) dimension 2, such a normalization is possible along a given embedded $J$-holomorphic disc; the proof requires a solution of some $\bar\partial$-type problems. Since the assertion claims the normalization only at a given point, the proof is much simpler and works in any dimension. For convenience of the reader we include the proof following [30].

*Proof of Lemma 3.2.* Set $z' = \Phi(z)$ and $J' := \Phi_*(z)$. The $J'$-holomorphy equations for the disc $z'$ are similar to (2.1) with the matrix $Q'$ instead of $Q$. We need to establish a relation between the matrices $Q$ from (2.1) and $Q'$. We have

$$z'_\zeta = (-\Phi_z Q + \Phi_{\bar{z}})\bar{z}_\zeta, \qquad \bar{z}'_{\bar{\zeta}} = (\bar{\Phi}_{\bar{z}} - \bar{\Phi}_z Q)\bar{z}_{\bar{\zeta}}.$$

Substituting these expressions into the $J'$-holomorphy equation for $z'$, we obtain the condition $N(z)\bar{z}_{\bar{\zeta}} = 0$ with

$$N(z) = (-\Phi_z Q + \Phi_{\bar{z}}) + Q'(\bar{\Phi}_{\bar{z}} - \bar{\Phi}_z Q).$$

Since by the Nijenhuis–Woolf theorem for every point $q$ and every vector $v \in T_q^{1,0}(M)$ there exists a solution $z$ of (2.1) satisfying $z(0) = q$, $dz(0)\left(\frac{\partial}{\partial\zeta}\right) = v$, we obtain $N = 0$; that is,

$$Q' = (\Phi_z Q - \Phi_{\bar{z}})(\bar{\Phi}_{\bar{z}} - \bar{\Phi}_z Q)^{-1}. \tag{3.1}$$

Set $\Phi(z) = z + \sum_{k,j=1}^n \phi_{kj} z_k \bar{z}_j$, where $\phi_{kj}$ are vectors in $\mathbb{C}^n$ with the entries $(\phi_{kj}^s)_{s=1}^n$. Then $\Phi_z = \sum_{k=1}^n \Phi_k z_k$ and the matrices $\Phi_k \in M_n(\mathbb{C})$ have the entries $(\phi_{kj}^s)_{j,s=1}^n$. Furthermore, $\Phi_z = I + O(|z|)$, where $I$ is the unit $(n \times n)$-matrix.

On the other hand, the Taylor expansion of $Q$ has the form

$$Q(z) = A(z) + B(\bar{z}) + O(|z|^2)$$

where $A(z) = \sum_{k=1}^n A_k z_k$, $A_k \in M_n(\mathbb{C})$, and $B(\bar{z}) = \sum_{k=1}^n B_k \bar{z}_k$, $B_k \in M_n(\mathbb{C})$. Substituting this into the expression for $Q'$, we obtain that

$$Q' = \sum_{k=1}^n (A_k - \Phi_k) z_k + B(\bar{z}) + O(|z|^2). \tag{3.2}$$

Now if we set $\Phi_k = A_k$ (this condition uniquely determines the quadratic part of $\Phi$), we obtain that $Q'(z'(z))_z(0) = 0$. Since $z' = z + O(|z|^2)$, this implies that $Q'_{z'}(0) = 0$, which proves the lemma. $\square$

REMARK 3.3. It follows from the relation (3.2) (which does not change if we add terms of type $O(|z|^2)$ to the expansion of $\Phi$) that the condition $Q'_{\bar{z}'}(0) = 0$ cannot be achieved by a change of variables. One can show that this would require restrictions on the Nijenhuis tensor of $J$ at 0. This is a reflection of the asymmetry between the $\partial$- and $\bar{\partial}$-operators in the Cauchy–Riemann equations (2.1).

LEMMA 3.4. *Assume that the local coordinates in a neighborhood of a point $p \in M$ are chosen according to Lemma 3.2; that is, assume $Q(0) = 0$ and $Q_z(0) = 0$ in* (2.1). *Then any $J$-holomorphic disc $z\colon \zeta \to z(\zeta)$ satisfying $z(0) = 0$ has the Taylor expansion*

$$z(\zeta) = t\zeta + a\zeta^2 + b\bar{\zeta}^2 + o(|\zeta|^2)$$

*where $t, a, b \in \mathbb{C}^n$.*

*Proof.* Indeed, we have $z(\zeta) = t\zeta + a\zeta^2 + b\bar{\zeta}^2 + c\zeta\bar{\zeta} + o(|\zeta|^2)$, where $t, a, b, c \in \mathbb{C}^n$. The condition $Q_z(0) = 0$ implies that $Q(z) = B(\bar{z}) + o(|z|)$, with $B$ as in the proof of Lemma 3.2. Substituting these expressions into (2.1), we obtain

$$2b\bar{\zeta} + c\zeta + B(\bar{t})t\bar{\zeta} + o(|\zeta|) = 0$$

so that $c = 0$. □

The following useful statement allows us to extend to the almost complex case all invariance properties of the Levi form without additional computations. Essentially it follows from the results of Chirka [3].

PROPOSITION 3.5. *Assume that the local coordinates in a neighborhood of a point* $p \in M$ *are chosen as in Lemmas 3.2 and 3.4* (*i.e.,* $Q(0) = 0$ *and* $Q_z(0) = 0$ *in* (2.1)). *Then for any function r of class* $C^2$ *in a neighborhood of the origin we have*

$$L_r^J(0; t) = L_r^{J_{st}}(0; t)$$

*for every* $t \in T_0(M)$.

*Proof.* According to the Nijenhuis–Woolf theorem and Lemma 3.4, there exists a solution $z$ of (2.1) such that $z(\zeta) = t\zeta + a\zeta^2 + b\bar{\zeta}^2 + o(|\zeta|^2)$. Consider the Taylor expansion of $r$ at the origin:

$$r(z) = r(0) + 2\Re L(z) + 2\Re K(z) + H(z, \bar{z}) + o(|z|^2)$$

where $L$ is a $\mathbb{C}$-linear form, $K$ is a complex quadratic form, and $H$ is the complex Hessian of $r$. Then the terms of degree $\leq 2$ in $\Re L \circ z(\zeta)$ and $\Re K \circ z(\zeta)$ are harmonic and thus the Laplacian $\Delta(r \circ z)(0)$ coincides with $L_r^{J_{st}}(0; t)$. On the other hand, $\Delta(r \circ z)(0)$ is equal to $L_r^J(0; t)$ by (ii) of Proposition 3.1. This completes the proof. □

As a consequence of Proposition 3.5 we obtain the following statement.

PROPOSITION 3.6. *For every point p of an almost complex manifold* $(M, J)$ *there exists a neighborhood U and local coordinates* $z \colon U \to \mathbb{C}^n$ *such that for any function r of class* $C^2$ *in a neighborhood of p we have* $L_r^J(p; t) = L_{r\circ z^{-1}}^{J_{st}}(0; dz(p)(t))$ *for any* $t \in T_p(M)$.

## 4. Bounded Strictly Plurisubharmonic Exhaustion Functions

We begin with the fundamental definition of Levi convexity. Let $p$ be a boundary point of a domain $\Omega$ in an almost complex manifold $(M, J)$; assume that $b\Omega$ is of class $C^2$ in a neighborhood $U$ of $p$. Then $\Omega \cap U = \{q \in U : r(q) < 0\}$ where $r$ is a real function of class $C^2$ on $U$, $dr(p) \neq 0$.

DEFINITION 4.1. $\Omega$ is called *Levi convex* at $p \in b\Omega$ if $L_r^J(p; t) \geq 0$ for any $t \in T_p(b\Omega) \cap J(T_p(b\Omega))$ and *strictly Levi convex* at $p$ if $L_r^J(p; t) > 0$ for any nonzero

$t \in T_p(b\Omega) \cap J(T_p(b\Omega))$. If $\Omega$ is a relatively compact domain with $C^2$ boundary in an almost complex manifold $(M, J)$, then $\Omega$ is called Levi convex if it is Levi convex at every boundary point.

It is easy to show that this definition does not depend on the choice of defining functions. In [7; 8; 20], strictly Levi convex domains are called "$J$-convex". We prefer the terminology closer to traditional complex analysis.

In order to justify the terminology we give here a simple geometric proof of the following natural statement obtained (together with other results) by Barraud and Mazzilli [1] and Ivashkovich and Rosay [11].

PROPOSITION 4.2. *Let $\Omega$ be a domain with $C^2$ boundary in an almost complex manifold $(M, J)$. Suppose that for a point $p \in b\Omega$ there exists a vector $t \in T_p(b\Omega) \cap J(T_p(b\Omega))$ such that $L^J_r(p; t) > 0$. Then there exists a $J$-holomorphic disc $f$ such that $f(0) = p$ and $f(\mathbb{D}\backslash\{0\})$ is contained in $M\backslash\bar{\Omega}$.*

*Proof.* We fix local coordinates near $p$ such that $p = 0$ and $J(0) = J_{st}$. Denote by $e_j$, $j = 1, \dots, n$, the vectors of the standard basis of $\mathbb{C}^n$. By an additional change of coordinates we may achieve that the map $f : \zeta \mapsto \zeta e_1$ is $J$-holomorphic on $\mathbb{D}$. We can assume that $L^J_r(0, e_1) = 1$ so that

$$r(z) = 2\Re z_n + 2\Re \sum a_{jk} z_j z_k + \sum h_{jk} z_j \bar{z}_k + o(|z|^2)$$

with

$$h_{11} = \Delta(r \circ f)(0) = 1.$$

Now for every $\delta > 0$ consider the nonisotropic dilation

$$\Lambda_\delta : (z_1, z_2, \dots, z_n) \mapsto (\delta^{-1/2} z_1, \delta^{-1} z_2, \dots, \delta^{-1} z_n).$$

The $J$-holomorphicity of the map $f$ implies that the direct images $J_\delta := (\Lambda_\delta)_*(J)$ converge to $J_{st}$ as $\delta \to 0$ in $C^\alpha$ norm for every positive real $\alpha$ on any compact subset of $\mathbb{C}^n$. Similarly, the functions $r_\delta := \delta^{-1} r \circ \Lambda^{-1}$ converge to the function $r_0 := 2\Re z_n + |z_1|^2 + 2\Re\beta z_1^2$ (for some $\beta \in \mathbb{C}$).

Consider a $J_{st}$-holomorphic disc $\hat{z} : \zeta \mapsto \zeta e_1 - \beta\zeta^2 e_n$. According to the Nijenhuis–Woolf theorem (see e.g. [26, Thm. 3.1.1]), for every $\delta \geq 0$ small enough there exists a $J_\delta$-holomorphic disc $z^\delta$ such that the family $(z^\delta)_{\delta \geq 0}$ depends smoothly on the parameter $\delta$ and for every $\delta \geq 0$ we have $z^\delta(\zeta) = \zeta e_1 + o(|\zeta|)$ and $z^0 = \hat{z}$. Since $(r_0 \circ z^0)(\zeta) = |\zeta|^2$, we obtain that for $\delta > 0$ small enough $(r_\delta \circ z^\delta)(\zeta) = A_\delta(\zeta) + o(|\zeta|^2)$ where $A_\delta$ is a positive defined quadratic form on $\mathbb{R}^2$. Since the structures $J_\delta$ and $J$ are biholomorphic, the desired conclusion follows. □

REMARK 4.3. It follows by the Nijenhuis–Woolf theorem that in the hypothesis of Proposition 4.2 there exists a smooth 1-parameter family of $J$-holomorphic discs $f_t, t \in ]-1, 1[$ forming a foliation of a neighborhood $U$ of $p$ in $M$ and $f_0$ coincides with the disc $f$ given by Proposition 4.2. The intersections of these discs with $\Omega$ give a family of $J$-holomorphic discs with boundaries attached to $b\Omega$ and filling $U \cap \Omega$.

Recall also that a continuous map $u\colon \Omega \to [a, 0[ \subset \mathbb{R}$ is called a *bounded exhaustion function* for $\Omega$ if for every $a \leq b < 0$ the pull-back $u^{-1}([a, b])$ is compact in $\Omega$.

Our first main result is the following theorem.

THEOREM 4.4. *Let $(M, J)$ be an almost complex manifold and let $\Omega \subset M$ be a relatively compact Levi convex domain with $C^3$ boundary such that there exists a $C^2$ strictly plurisubharmonic function $\psi$ in a neighborhood $U$ of $b\Omega$. Let $r$ be any $C^3$ defining function for $\Omega \cap U$. Then there exist a neighborhood $U'$ of $b\Omega$ and constants $A > 0$, $0 < \eta_0 < 1$, such that for any $0 < \eta \leq \eta_0$ the function $\rho = -(-re^{-A\psi})^\eta$ is strictly plurisubharmonic on $\Omega \cap U'$. If $U$ is a neighborhood of $\bar{\Omega}$, then $\rho$ is strictly plurisubharmonic on $\Omega$.*

Thus, $\rho$ is a bounded strictly plurisubharmonic exhaustion function for $\Omega$. The proof is based on the method of Diederich and Fornaess [5] (slightly modified by Range in [23]). In its first step the following expression for the Levi form of $\rho$ is determined.

LEMMA 4.5. *Under the hypothesis of Theorem 4.4 there exists a neighborhood $U'$ of $b\Omega$ such that for every $p \in \Omega \cap U'$ and $v \in T_p(M)$ we have*

$$L^J_\rho(p; v) = \eta(-r)^{\eta-2} e^{-\eta A\psi} D(v)$$

*where*

$$\begin{aligned} D(v) = {} & Ar^2(p)[L^J_\psi(p; v) - \eta A|\partial_J\psi(p)(v)|^2] \\ & + (-r(p))[L^J_r(p; v) - 2\eta A \Re \partial_J r(p)(v)\overline{\partial_J\psi(p)(v)}] \\ & + (1-\eta)|\partial_J r(p)(v)|^2. \end{aligned}$$

In the case of the standard structure of $\mathbb{C}^n$ this formula is already contained in [5]. Its proof, however, becomes possible only by using Proposition 3.5. Any attempt to show it by direct computations gets stuck in painfully long calculations since, in principle, the given almost complex structure $J$ depends on the point and, hence, also has to be differentiated.

*Proof of Lemma 4.5.* Fix a point $p \in \Omega$ close enough to $b\Omega$ and choose coordinates $z\colon W \to \mathbb{C}^n$ in a neighborhood $W$ of $p$ according to Lemma 3.2. In these coordinates the neighborhood $V$ can be identified with a neighborhood $z(V)$ of the origin in $\mathbb{C}^n$. For simplicity of notation, we denote again by $J$ the direct image $z_*(J)$ and by $\rho$ (resp. $r$) the composition $\rho \circ z^{-1}$ (resp. $r \circ z^{-1}$). In particular, we have $J(0) = J_{st}$. Consider a vector $t \in T^{1,0}_0(\mathbb{C}^n)$. According to [5] we have the following expression for the Levi form of the function $\rho = -(-re^{-A\psi})^\eta$ with respect to the standard structure $J_{st}$:

$$L^{J_{st}}_\rho(0; t) = \eta(-r)^{\eta-2} e^{-\eta A\psi} D_{st}(t)$$

where

$$\begin{aligned}D_{st}(t) = {} & Ar^2(p)[L_\psi^{J_{st}}(0;t) - \eta A|\partial_{J_{st}}\psi(0)(t)|^2] \\ & + (-r(p))[L_r^{J_{st}}(0;t) - 2\eta A \Re \partial_{J_{st}} r(0)(t)\overline{\partial_{J_{st}}\psi(0)(t)}] \\ & + (1-\eta)|\partial_{J_{st}} r(0)(t)|^2.\end{aligned}$$

On the other hand, by Proposition 3.5, for every real function $\varphi$ of class $C^2$ in a neighborhood of the origin, its Levi form at 0 with respect to $J$ coincides with the Levi form with respect to $J_{st}$. Furthermore, the condition $J(0) = J_{st}$ implies $\partial_{J_{st}}\varphi(0) = \partial_J\varphi(0)$. Therefore, for any vector $v \in T_p^{1,0}(M)$ we have a similar expression for the Levi form $L_\rho^J(p; v)$ with respect to $J$ without any additional calculations. This proves the lemma. □

Now the proof of Theorem 4.4 is quite similar to [5; 23]. For the sake of completeness we include the details.

*Proof of Theorem 4.4.* Recall that any almost complex manifold admits a Hermitian metric (see e.g. [13]). Fix such a metric $d\mu^2$ on $M$ and denote by $\|\cdot\|_p$ the norm induced by $d\mu^2$ on $T_p^{1,0}(M)$. Then we have the orthogonal decomposition $T_p^{1,0}(M) = T_p^t \oplus T_p^n$ where $T_p^t = \{v \in T_p^{1,0}(M) : \partial r_J(p)(v) = 0\}$ is the "tangent" space and its orthogonal complement $T_p^n$ is the "normal" space. If $p \in b\Omega$, then $T_p^t$ is canonically isomorphic to the holomorphic tangent space $T_p(b\Omega) \cap J(T_p(b\Omega))$ by the canonical identification of $T_p(M)$ and $T_p^{1,0}(M)$. So for any vector $v \in T_p^{1,0}(M)$ we have the decomposition $v = v^t + v^n \in T_p^t \oplus T_p^n$ into the "tangent" and "normal" components. Fix a neighborhood $U$ of a point $q \in b\Omega$ of the form $U = b\Omega \times ]-\varepsilon, \varepsilon[$ and denote by $\pi : W \to b\Omega$ the natural projection.

*Step 1: Estimate of the Levi form of r from below.* Since $r$ is of class $C^3$, given a smooth vector field $V : p \to V(p) \in T_p^{1,0}(M)$ the function $p \mapsto L_r^J(p, V^t(p))$ is of class $C^1$ on $W$. Furthermore, the function

$$p \mapsto L_r^J(p; V^t(p)) - L_r^J(\pi(p); (d\pi(p)(V(p)))^t)$$

vanishes on $b\Omega$. In local coordinates the last expression is just a quadratic form in $V$ with coefficients of class $C^1$ in $p$ vanishing on $b\Omega$. Therefore, there exists a constant $C_1 > 0$ such that

$$|L_r^J(p; V^t(p)) - L_r^J(\pi(p); (d\pi(p)(V(p)))^t)| \le C_1|r(p)|\|V(p)\|_p^2.$$

Since $\Omega$ is Levi convex, we have $L_r^J(p; (d\pi(p)(V(p)))^t) \ge 0$ for every $V$. This implies

$$L_r^J(p; V^t(p)) \ge -C_1|r(p)|\|V(p)\|_p^2. \tag{4.1}$$

By a compactness argument we can assume that there exists a neighborhood $W$ of $b\Omega$ such that the estimate (4.1) holds for every $p \in \Omega \cap W$ and any $(1,0)$ vector field $V$ on $W$.

If $V_1$ and $V_2$ are $(1,0)$ vector fields, then we set

$$L_r^J(p; V_1, V_2) = -d(J^*dr)(2\Re V_1, 2J\Re V_2).$$

We have

$$\begin{aligned}&|L_r^J(p; V(p)) - L_r^J(p; V^t(p))|\\&\quad= |L_r^J(p; V^t(p), V^n(p)) + L_r^J(p; V^n(p), V^t(p)) + L_r^J(p; V^n(p), V^n(p))|\\&\quad\le C_2 \|V(p)\|_p \|V^n(p)\|_p\end{aligned}$$

and

$$\|V^n(p)\|_p \le C_3 |\partial_J r(V(p))|.$$

Together with (4.1) these estimates imply that

$$L_r^J(p; V(p)) \ge -C_4 |r(p)| \|V(p)\|_p^2 - C_4 \|V(p)\|_p |\partial_J r(V(p))| \tag{4.2}$$

for any $p \in \Omega \cap W$ and any vector field $V$.

*Step 2: Estimate of the expression $D(v)$ from above.* Fix $C_5 > 0$ such that $L_\psi^J(q; v) \ge C_5 \|v\|_p^2$ for any $q \in W \cap \Omega$ and any $v \in T_q^{1,0}(M)$. Then there exists $\eta_0(A)$ such that for any $0 < \eta < \eta_0(A)$

$$\begin{aligned}D(v) &\ge A r^2 (C_5 - C_5/2) \|v\|^2\\&\qquad + (-r)(r C_4 \|v\|^2 - C_6 |\partial_J r(p)(v)| \|v\|) + (1/2) |\partial_J r(p)(v)|^2.\end{aligned}$$

Since

$$C_6 |r(p)| |\partial_J r(p)(v)| \|v\| \le (1/4) |\partial_J r(p)(v)|^2 + C_7 r^2 \|v\|^2$$

we obtain

$$D(v) \ge r^2 (A C_5/2 - C_8) \|v\|^2$$

where all constants $C$ are positive and independent of $A$, $\eta$, and $v$. Now this is enough to fix $A > 2C_8/C_5$ and then $\eta_0 = \eta(A)$.

Finally, let $\psi$ be strictly plurisubharmonic on $\bar{\Omega}$. Then $L_\psi^J(p; v) \ge C_9 \|v\|^2$ for any $p \in \bar{\Omega}$ and $v \in T_p^{1,0}(M)$ with $C_9 > 0$; moreover, $K := \Omega \backslash W$ is a compact subset in $\Omega$ and there exists a $\delta > 0$ such that $r(p) \ge \delta$ for every $p \in K$. The expression for $D(v)$ given by Lemma 4.5 now holds in $\Omega$. Then for every $0 < \eta < \eta_1(A)$ in $\Omega$ we have $D(v) \ge (A\delta^2 C_9/2 - C_{10}) \|v\|^2$ for all $p \in K$ and $v \in T_p^{1,0}(M)$. So we take $A > \max(2C_8/C_5, 2C_{10}/C_9\delta^2)$ and then fix $\eta$. This proves the theorem. □

REMARKS 4.6. (1) As in the case of $\mathbb{C}^n$, for a fixed point $p \in b\Omega$ and any given $0 < \eta < 1$ there is a neighborhood $U$ of $p$, a strictly plurisubharmonic function $\psi$ in $U$, and $A > 0$ such that $\rho$ is strictly plurisubharmonic in $\Omega \cap U$. Indeed, in local coordinates given by Lemma 3.2 consider the function $\psi(z) = \sum_{j=1}^n |z_j|^2$. Then $\psi$ is strictly plurisubharmonic in $U$ and $d\psi(p) = 0$ so the result follows from the previous estimate of $D(v)$.

(2) We want to remind the reader of the fact, known already for the classical situation, that, in general, a bounded strictly plurisubharmonic exhaustion function for $\Omega$ cannot be chosen to be more than just Hölder continuous up to $b\Omega$ (see [5]). In fact, the supremum of all possible $0 < \eta < 1$ in Theorem 4.4 is an interesting invariant of $\Omega$ linked to the $\bar{\partial}$-Neumann problem in classical complex analysis (see [14]).

As an obvious consequence we obtain the following result.

COROLLARY 4.7. *J-holomorphic discs in $\bar{\Omega}$ must either lie completely in $b\Omega$ or cannot touch $b\Omega$ at all.*

In the case where $\Omega$ is strictly Levi convex, this statement is well known (see e.g. [20, Lemma 2.4]).

## 5. Characterization of Stein Structures: The Continuity Principle

If the various definitions of complex convexity are chosen in the correct way, then the classical results for the characterization of Stein domains with smooth boundary carry over to the almost holomorphic category. In order to make these facts available once and for all, we treat them here briefly. As we will see in doing so, Theorem 4.4 plays an important role.

Let $\Omega$ be a relatively compact domain in an almost complex manifold $(M, J)$. By a *Hartogs family* we mean a continuous map $f : \bar{\mathbb{D}} \times [0,1] \to M$ such that for every $t \in [0,1]$ the map $f_t := f(\cdot, t)$ is $J$-holomorphic on $\mathbb{D}$ and for every $t \in ]0,1]$ we have $f_t(\bar{\mathbb{D}}) \subset \Omega$ and $f_0(b\mathbb{D}) \subset \Omega$.

DEFINITION 5.1. We say that $\Omega$ is *disc-convex* if for any Hartogs family of discs we have $f_0(\mathbb{D}) \subset \Omega$.

The classical definition of pseudoconvexity also can be extended to the almost complex case without changes.

DEFINITION 5.2. A domain $\Omega$ in an almost complex manifold $(M, J)$ is *pseudoconvex* if for any compact subset $K$ in $\Omega$ its plurisubharmonically convex hull $\hat{K}_\Omega := \{p \in \Omega : u(p) \le \sup_{q \in K} u(q)\ \forall u \in \mathrm{PSH}(\Omega) \cap C(\Omega)\}$ is compact.

Finally, recall that a continuous proper map $u : \Omega \to [0, +\infty[$ is called an *exhaustion function* for a domain $\Omega$.

DEFINITION 5.3. A domain $\Omega$ in an almost complex manifold $(M, J)$ is a *Stein domain* if there exists a strictly plurisubharmonic exhaustion function on $\Omega$.

The following consequence of Theorem 4.4 is the almost complex analogue of results that are valid in the classical situation.

THEOREM 5.4. *Let $\Omega$ be a relatively compact domain with boundary of class $C^3$ in an almost complex manifold $(M, J)$. Suppose that $M$ admits a strictly plurisubharmonic function. Then the following conditions are equivalent:*

(i) *$\Omega$ is Levi convex;*
(ii) *$\Omega$ is disc-convex;*
(iii) *$\Omega$ admits a bounded strictly plurisubharmonic exhaustion function;*
(iv) *$\Omega$ is pseudoconvex;*
(v) *$\Omega$ is a Stein domain.*

*Proof.* (i) $\Rightarrow$ (iii) by Theorem 4.4. The proof of (iii) $\Rightarrow$ (ii) is quite similar to the case of $\mathbb{C}^n$. Next, we prove that (ii) $\Rightarrow$ (i). Suppose that (ii) holds and $\Omega$ is not Levi convex. Then there exists a boundary point $p \in b\Omega$ and a vector $v \in T_p(b\Omega) \cap J(T_p(b\Omega))$ such that $L_r^J(p, v) < 0$, where $r$ is a local defining function of $\Omega$ near $p$. This condition is open and stable with respect to small translations of $b\Omega$ along the normal direction at $p$, so the same holds for the hypersurfaces $b\Omega_t = \{r = -t\}$ if $t \geq 0$ is small enough. It follows from Proposition 4.2 that for every $t$ there exists a $J$-holomorphic disc $f_t$ such that $(\rho \circ f_t)(0) = -t$ and $(\rho \circ f_t)(\zeta) < -t$ if $\zeta \in \bar{\mathbb{D}} \setminus \{0\}$. This family depends continuously on $t$ and so this is a Hartogs family such that $f_0(\mathbb{D})$ is not contained in $\Omega$: a contradiction. So (i), (ii), and (iii) are equivalent. The proofs of (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv) are quite similar to the classical arguments for the standard $\mathbb{C}^n$. This completes the proof. □

Eliashberg and Gromov [7] proved that every Stein domain admits the canonical symplectic structure defined by the symplectic form $\omega_u = -dJ^*du$, where $u$ is a strictly plurisubharmonic exhaustion function. Clearly, $\omega_u(v, Jv) > 0$ for any nonzero tangent vector $v$ that is tamed by the symplectic form $\omega$ in the sense of Gromov. Furthermore, if $\tilde{u}$ is another strictly plurisubharmonic exhaustion function for $\Omega$, then the symplectic manifolds $(\Omega, \omega_u)$ and $(\Omega, \omega_{\tilde{u}})$ are symplectomorphic [7]. In this sense the foregoing symplectic structure defined by a strictly plurisubharmonic exhaustion function is canonical. Thus, we have the following statement.

COROLLARY 5.5. *Let $(M, J)$ be an almost complex manifold admitting a strictly plurisubharmonic function. Then a relatively compact domain $\Omega$ with $C^3$ boundary in $M$ admits a canonical symplectic structure if and only if $\Omega$ is pseudoconvex.*

## 6. Normal Families and Taut Manifolds

Let $(M, J)$ be an almost complex manifold. Recall that it is called *taut* if any sequence $(D_k)$ of $J$-holomorphic discs in $M$ either contains a compactly convergent subsequence or is compactly divergent. It is well known and easily follows from standard elliptic estimates for the Cauchy–Green kernel (see e.g. [21]) that the limit in the compact open topology of a sequence of $J$-holomorphic discs also is a $J$-holomorphic disc. Since Gromov's compactness theorems for $J$-holomorphic discs can be viewed as normal family-type theorems, the class of taut almost complex manifolds is important. It is well known that any complete hyperbolic manifold is taut; the inverse in general is not true.

THEOREM 6.1. *Suppose that an almost complex manifold $(M, J)$ admits a strictly plurisubharmonic function. Then any relatively compact pseudoconvex domain with $C^3$ boundary in $M$ is taut.*

In the case of $\mathbb{C}^n$ this is an immediate corollary of the Diederich–Fornaess result [5; 6] on the existence of a bounded strictly plurisubharmonic exhaustion function and the Montel theorem. In the case of an arbitrary complex manifold (with an integrable structure) the result is due to Sibony [25]. We also point out that if in the hypothesis of Theorem 6.1 the boundary of the domain is strictly Levi convex, then the domain is complete hyperbolic [9; 11].

*Proof of Theorem 6.1.* Since $M$ admits a strictly plurisubharmonic function, it follows from the results of [9; 11] that $M$ is hyperbolic at every point (in the sense of Royden [24]) and so is hyperbolic for the Kobayashi distance by the classical Royden theorem [24] (see e.g. [11; 16] for the almost complex version of this theorem). In particular, a domain $\Omega$ satisfying the hypothesis of Theorem 6.1 is a hyperbolic manifold and the Kobayashi distance determines the usual topology on $\Omega$ (see e.g. [13]). By Theorem 4.4, $\Omega$ admits a bounded strictly plurisubharmonic exhaustion function $\rho$. Now given a family of $J$-holomorphic discs $(D_k)$ in $\Omega$ it suffices to apply the argument of the proof of [25, Cor. 5] since it only uses the subharmonicity of $\rho \circ D_k$ and so does not require any modifications. □

As in the classical case, this result has many important applications (e.g., the Lindelöf principle for pseudoholomorphic maps). Since they follow likewise for standard complex structures, we do not go into any detail.

## 7. Approximation of Confoliations by Contact Structures

According to Eliashberg and Thurston [8], a tangent hyperplane field $\xi = \{\alpha = 0\}$ on a $(2n+1)$-dimensional manifold $\Gamma$ is called a *positive confoliation* if there exists an almost complex structure $J$ on the bundle $\xi$ such that

$$d\alpha(X, JX) \geq 0$$

for any vector $X \in \xi$. The 1-form $\alpha$ is defined up to multiplication by a nonvanishing function. Thus, the confoliation condition for $\xi$ is equivalent to the existence of a compatible Levi convex CR-structure (in general, nonintegrable). In other words, if $\Gamma = \{r = 0\}$ is a smooth Levi convex hypersurface in an almost complex manifold $(M, J)$, then the distribution of its holomorphic tangent spaces $\xi = T\Gamma \cap J(T\Gamma)$ is a confoliation: we can set $\alpha = J^*dr$. In particular, if $\Gamma$ is a strictly Levi convex hypersurface, then $\xi = \{J^*dr = 0\}$ is a contact structure. Recall that a tangent hyperplane field $\xi = \{\alpha = 0\}$ on $\Gamma$ is called a *contact structure* if $\alpha \wedge (d\alpha)^n \neq 0$ on $\Gamma$. One of the main questions considered by Eliashberg and Thurston concerns the possibility of deforming a given confoliation to a contact structure or approximating it by contact structures. Combining the contact topology techniques with the geometric foliation theory they obtained several results of this type in the case where $\Gamma$ is of real dimension 3. Our next result works in any dimension.

THEOREM 7.1. *Let $\Omega$ be a relatively compact pseudoconvex domain with $C^\infty$ boundary in an almost complex manifold $(M, J)$. Assume that there exists a smooth strictly plurisubharmonic function $\psi$ in a neighborhood of $b\Omega$. Then the confoliation of holomorphic tangent spaces $T(b\Omega) \cap J(T(b\Omega))$ can be approximated in any $C^k$ norm by contact structures.*

*Proof.* Fix a small enough neighborhood $U$ of $b\Omega$ such that $r$ is a defining function of $\Omega \cap U$ and for some $0 < \eta < 1$ and $A > 0$ the function $\rho = -(-re^{-A\psi})^{\eta}$ is strictly plurisubharmonic on $\Omega \cap U$ by Theorem 4.4. Set also $\varphi = re^{-A\psi}$. For $\delta > 0$ small enough, consider the hypersurfaces $\Gamma_\delta = \{\rho = -\delta\} = \{\varphi = -\delta^{1/\eta}\}$. Since $\rho$ does not have critical points near $b\Omega$, we can fix a sequence $(\delta_j)_j$ of noncritical values of $\rho$ converging to 0. Then every $\Gamma_j := \Gamma_{\delta_j}$ is a strictly Levi convex hypersurface and admits the canonical contact structure $\xi_j$ defined by the form $J^*d\rho$. Since a contact form is defined up to multiplication by a nonvanishing function and $d\rho = \eta(-\varphi)^{\eta-1}d\varphi$, we have $\xi_j = \{J^*d\varphi = 0\}$. Fix a smooth Riemannian metric on $M$. Shrinking the neighborhood $U$ if necessary we can assume that it is foliated by the normals to $b\Omega$. For any $j$ consider a smooth diffeomorphism $\pi_j : b\Omega \to \Gamma_j$ that associates to every point of $p \in b\Omega$ the point of $\Gamma_j$ defined by the intersection of $\Gamma_j$ with the normal of $b\Omega$ at $p$. Then the pull-back $\alpha_j := \pi_j^*(J^*d\varphi)$ defines a contact structure $\xi_j^* = \{\alpha_j = 0\}$ on $b\Omega$ and the sequence of forms $(\alpha_j)_j$ converges to the form $J^*d\varphi$ on $b\Omega$ in any $C^k$ norm. This proves the theorem. □

REMARK 7.2. In a suitable neighborhood of a fixed point $p \in b\Omega$ there always exists a smooth strictly plurisubharmonic function. So every confoliation can be approximated locally by contact structures.

A similar approach also can be useful for the filling of real 2-spheres contained in the boundary of a Levi convex domain. The results of this type are well known in the strictly Levi convex case. In the weakly Levi convex case a related result was obtained by Hind [10] and used in [8] for the study of confoliations. Here we just sketch our approach. Let $\Omega$ be a relatively compact pseudoconvex domain with $C^\infty$ boundary in an almost complex manifold $(M, J)$ of complex dimension 2. Assume again that there exists a smooth strictly plurisubharmonic function $\psi$ in a neighborhood of $b\Omega$ and that $J$ is tamed by a symplectic structure $\omega$ on $M$. Let $S^2$ be a real sphere of dimension 2 smoothly embedded into $b\Omega$. Then $b\Omega$ can be approximated by the strictly Levi convex hypersurfaces $\Gamma_\delta$ introduced in the proof of Theorem 7.1. Consider a general family of 2-spheres $S^2_\delta \subset \Gamma_\delta$ smoothly depending on a parameter $\delta \geq 0$ such that $S^2_0 = S^2$. Since every $\Gamma_\delta$ is strictly Levi convex and $J$ is tamed, according to known results (see [10] and the references there) for any $\delta > 0$ there exists a 1-parameter family of $J$-holomorphic discs $(D_{\delta,t})_{t\in[0,1]}$ with boundaries contained in $S^2_\delta$ and filling a real compact Levi-flat hypersurface $\Sigma_\delta$ whose boundary coincides with $S^2_\delta$. One can control these discs as $\delta$ tends to 0 and then obtain a filling of the initial sphere $S^2$ by pseudoholomorphic discs. For instance, it follows by a standard argument using the Stokes formula that the area of discs attached to $S^2_\delta$ are bounded by the area of the sphere $S^2_\delta$ (see more details in [10]) and so are uniformly bounded. This allows one essentially to reduce the considerations to the strictly pseudoconvex case.

## 8. Pseudoholomorphic Discs Attached to Levi Convex Hypersurfaces

We say that a real hypersurface $\Gamma = \{r = 0\}$ in an almost complex manifold $(M, J)$ is *Levi-flat* if $L^J_r(p,t) = 0$ for every $p \in \Gamma$ and every $t \in T_p(\Gamma) \cap J(T_p(\Gamma))$. In

[11], Ivashkovich and Rosay give an example of a Levi-flat hypersurface in an almost complex manifold of complex dimension 3 that does not contain complex hypersurfaces (in a deep contrast with the case of complex manifolds). In the same paper they ask whether through every point of such a hypersurface there passes a $J$-holomorphic disc in any prescribed direction. An affirmative answer was given by Kruzhilin and Sukhov [18]. The results of this paper allow us to simplify the most technical part of the proof of [18] and to generalize this result.

As usual, by a *Bishop disc* for $\Gamma$ we mean a $J$-holomorphic disc $f : \mathbb{D} \to M$ continuous on $\bar{\mathbb{D}}$ and such that $f(b\mathbb{D}) \subset \Gamma$. Let $\Gamma$ be a Levi convex hypersurface and let $U$ be a neighborhood of a point $p \in \Gamma$. If $r$ is a defining function of $\Gamma$ on $U$, we use the following standard notation: $U^+ = \{q \in U : r(q) > 0\}$ and $U^- = \{q \in U : r(q) < 0\}$.

PROPOSITION 8.1. *Let $\Gamma$ be a smooth Levi convex hypersurface. For any point $p \in \Gamma$ there exists a neighborhood $U$ of $p$ with the following property: If $f : \mathbb{D} \to U$ is a Bishop disc for $\Gamma$ then $f(\mathbb{D}) \subset \Gamma \cup U^-$.*

We point out that in the classical case of $\mathbb{C}^n$ this statement is obvious since the translations of $\Gamma$ in the normal direction are Levi convex (see [2]). Proposition 8.1 is obtained in [18] for the special case where $\Gamma$ is Levi-flat.

*Proof of Proposition 8.1.* Since the statement is local, we may work in suitable coordinates. So without loss of generality we assume that $\Gamma$ is defined in a neighborhood of the origin in $\mathbb{C}^n$ with the standard complex coordinates $z$ and $J(0) = J_{st}$. After an isotropic dilation of coordinates we may achieve that the unit ball $\mathbb{B}$ is contained in $U$ and $J$ is as close to $J_{st}$ as we need in the $C^3$ norm on $U$. Furthermore, we denote by $\|\cdot\|_p$ the norm on $T_p\mathbb{C}^n$ defined by a Hermitian metric compatible with $J$. We can assume that this norm is arbitrarily close to the Euclidean norm $|\cdot|$ and that the function $z \mapsto |z|^2$ is strictly plurisubharmonic on $U$.

Similarly to [18], the key statement is the following lemma.

LEMMA 8.2. *For every small enough neighborhood $U$ of the origin there exist a neighborhood $U' \subset U$ of the origin and an integral $N > 1$ such that for any $\varepsilon > 0$ the hypersurface $\Gamma_\varepsilon = \{z \in U' : r_\varepsilon(z) := r(z) + \varepsilon|z|^2 - \varepsilon/N = 0\}$ is strictly Levi convex in $U'$.*

*Proof.* Consider a smooth vector field $V$ on $\Gamma_\varepsilon$ such that $V(z)$ is in $T_z(\Gamma_\varepsilon) \cap J(T_z(\Gamma_\varepsilon))$ for every $z \in \Gamma_\varepsilon$. It is sufficient to consider the case where $\|V(z)\|_z = 1$ for any $z$. Then the estimate (4.2) implies

$$L_r^J(z, V) \geq -C_0(|r(z)| + |\partial_J r(V)|) \tag{8.1}$$

(here and below, $C_j$ denote positive constants independent of $\varepsilon$ and $N$). On the other hand, the condition $V(z) \in T_z(\Gamma_\varepsilon) \cap J(T_z(\Gamma_\varepsilon))$ means that $\partial_J r(V) = -\varepsilon\partial_J|z|^2(V)$ and so

$$|\partial_J r(V)| \leq C_1\varepsilon|z|. \tag{8.2}$$

For $N > 1$ the relation $z \in \Gamma_\varepsilon \cap (1/N)\mathbb{B}$ implies $r(z) = \varepsilon(1/N - |z|^2) > 0$ and so the inequalities (8.1) and (8.2) give

$$L^J_r(z,V) \geq -C_2\varepsilon(1/N - |z|^2 + |z|) \geq -(C_2/2N)\varepsilon.$$

On the other hand, there exists $\alpha > 0$ such that $L^J_{|z|^2}(z,V) \geq \alpha$. Thus

$$L^J_{r_\varepsilon}(z,V) \geq \varepsilon(\alpha - C_2/2N).$$

Therefore any $N > C_2/\alpha$ satisfies the assertion of the lemma. □

Fix now a neighborhood $U$ (we drop the prime) of $p$ small enough satisfying the hypothesis of lemma. Assume that $f(\mathbb{D})$ intersects $U^+$. Then a connected component $\Sigma$ of this intersection is a Riemann surface in $U^+$ with the boundary contained in $\Gamma$. So there exists an $\varepsilon > 0$ such that $\Sigma$ touches $\Gamma^+_\varepsilon$ from inside at some point, which contradicts Corollary 4.7.

Since a Levi-flat hypersurface is Levi convex "from the both sides", as a consequence we obtain the following result of [18].

COROLLARY 8.3. *Let $\Gamma$ be a smooth Levi-flat hypersurface. For any point $p \in \Gamma$ there exists a neighborhood $U$ of $p$ with the following property: If $f : \mathbb{D} \to U$ is a Bishop disc for $\Gamma$ then $f(\mathbb{D}) \subset \Gamma$.*

Recall that this proposition easily implies the following result due to [18].

THEOREM 8.4. *Let $\Gamma$ be a Levi-flat hypersurface in an almost complex manifold $(M, J)$. Then for every point $p \in \Gamma$ and every vector $t \in T_p(\Gamma) \cap J(T_p(\Gamma))$ there exists a $J$-holomorphic disc $f$ such that $f(0) = p$, $df(0)\left(\frac{\partial}{\partial \Re\zeta}\right) = t$, and $f(\mathbb{D}) \subset \Gamma$.*

This is a consequence of the analysis of the Bishop equation given in [18] (essentially based on the implicit function theorem) to show that the family of Bishop discs of $\Gamma$ is "rich enough".

Combining Proposition 8.1 with Remark 4.6(1) and using the Hopf lemma in the standard way, we get the following result.

PROPOSITION 8.5. *In the hypothesis of Theorem 4.4, for any point $p \in b\Omega$ there exists a neighborhood $U$ of $p$ with the following property: If $f : \mathbb{D} \to U$ is a Bishop disc for $b\Omega$ then either $f(\mathbb{D}) \subset b\Omega$ or the disc $f(\mathbb{D})$ is contained in $\Omega$ and is transversal to $b\Omega$ at every boundary point.*

In particular, if $b\Omega$ contains no $J$-holomorphic discs then there exists a Bishop disc transversal to $b\Omega$ at $p$. The standard arguments then imply that Bishop discs fill $U \cap \Omega$ for a suitably chosen neighborhood $U$ of $p$ (further results in this direction are contained in [18; 30]).

## 9. Examples of Almost Complex Stein Structures on Fibered Spaces

Let $(X, B, Y)$ be complex spaces and let $(X, \pi, B)$ be a locally trivial holomorphic fiber bundle with typical fiber $Y$ and projection $\pi : X \to B$. In 1953, J.-P. Serre asked whether the total space $X$ is Stein if $B$ and $Y$ are Stein. The conjecture was for quite some time that the answer always is in the affirmative, until at first Skoda [27] and then Demailly [4] gave their famous counterexamples. In each of them, the main obstructions are features of the typical fibers. Therefore, the question arose: For what kind of fibers might the answer to the conjecture of Serre still be positive? In this situation, Stehlé [28; 29] proved that the conjecture is true if the typical fiber $Y$ admits a bounded plurisubharmonic exhaustion function. This was one of the motivations for constructing such exhaustion functions on smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ (see [5; 6; 12]).

In this section we want to study analogous situations in the almost complex realm. In particular, we want to ask whether Theorem 4.4 again can be used to prove that an analogue of the Serre conjecture in the almost complex category holds for locally trivial fiber bundles with Stein basis and a domain as considered in Theorem 4.4. As it will turn out, this holds (almost) true. However, as we also will see, it is not clear how much sense this analogue of the question of Serre makes for almost complex manifolds, since natural examples for such a situation like tangent bundles with their natural almost complex structure are no longer locally trivial over the base manifold, as we also will show.

### *9.1. Locally Trivial Bundles*

We point out that the Cartesian product of Stein almost complex manifolds $(M_j, J_j)$, $j = 1, 2$, is, of course, Stein for the structure $J_1 \otimes J_2$ since the canonical projections $\pi_j : M_1 \times M_2 \to M_j$ are holomorphic and, if $\rho_j$ are strictly plurisubharmonic exhaustion functions on $M_j$, $j = 1, 2$, then $\rho_1 \circ \pi_1 + \rho_2 \circ \pi_2$ is a strictly plurisubharmonic exhaustion function for $M_1 \times M_2$.

This construction can be generalized in the following way. Let $((X, J_X), (B, J_B), (Y, J_Y))$ be almost complex manifolds with the respective almost complex structures. Let, furthermore, $\pi : X \to B$ be a surjective map (with maximal rank everywhere) that is holomorphic with respect to the structures $(J_X, J_B)$.

DEFINITION 9.1. The triple $(X, \pi, B)$ is called a *locally trivial holomorphic fiber bundle* with *typical fiber* $(Y, J_Y)$ if there is for each point $p \in B$ a neighborhood $U = U(p)$ and a diffeomorphism

$$\Phi : X|_U \to U \times Y$$

that is biholomorphic with respect to the almost complex structures $J_X$ on the restriction $X|_U = \pi^{-1}(U)$ and the canonical product structure $(J_B, J_Y)$ on $U \times Y$.

This definition is the exact analogue of the definition in the classical case. With it, the Serre conjecture still holds with the following two additional hypotheses.

(1) $(B, J_B)$ admits a strictly plurisubharmonic exhaustion function with a unique critical point (we must be able to use an analogue of the “lemme de recouvrement” of [28; 29]).

(2) $(Y, J_Y)$ admits a bounded plurisubharmonic exhaustion function—for instance, if (as in Theorem 4.4) $Y$ is a smoothly bounded Levi convex domain in an almost complex manifold admitting a strictly plurisubharmonic function.

*Proof of Hypotheses (1) and (2).* The approach of [28; 29] is based on the following "lemme de recouvrement".

LEMMA 9.2. *Let $B$ be a Stein space and let $U = (U_\alpha)$ be an open covering of $B$. Then there exists a sequence $(B_j)$ of relatively compact Stein subsets in $U_\alpha$ such that the $B_j$ form a locally finite covering of $B$ and satisfy the following properties.*

(i) *$C_j = \bigcup_{i \le j} B_j$ is Stein.*

(ii) *There exists on $C_j$ a function $h_j$ that is continuous, strictly plurisubharmonic, bounded, and negative (resp. $h_j \ge 1$) in a neighborhood of the closure in $C_j$ of $C_j - B_j$ (resp. $C_j - C_{j-1}$).*

The proof of this statement in [28; 29] uses an embedding of $B$ into $\mathbb{C}^n$, an exhaustion by a sequence of Euclidean balls, and (affine) convex bumpings of these balls. In our situation we can replace the balls by the sublevel sets of a strictly plurisubharmonic exhaustion function and convex bumpings by strictly pseudoconvex bumpings. The other arguments of [28; 29] can be carried over to our situation without modifications. $\square$

### *9.2. Almost Complex Fiber Bundles of Riemann Surfaces*

An obstruction for almost complex manifolds to be holomorphically fibered lies in the fact, that, in general, almost complex structures do not admit almost complex subvarieties of complex dimension $> 1$. This makes it natural to study at least the case of Riemann surfaces as fibers. To be precise, we give the following definition.

DEFINITION 9.3. Let $(X, I)$ and $(M, J)$ be almost complex manifolds of complex dimensions $n+1$ and $n$, respectively, and such that there exists a holomorphic submersion $\pi : X \to M$. Assume that $M$ admits an open covering $\{U_\alpha\}$ with the following properties.

(i) For every $\alpha$ there exists a diffeomorphism $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times R$ where $R$ is a Riemann surface.

(ii) The map $\Phi_\alpha$ is fiber preserving; that is, $\pi' \circ \Phi_\alpha = \pi$ where $\pi' \colon U_\alpha \times R \to U_\alpha$ is the natural projection.

Then $(X, \pi, M)$ is called an *almost complex fiber bundle* of Riemann surfaces.

REMARK 9.4. Notice that it is not assumed that $\Phi_\alpha$ is biholomorphic for $U_\alpha \times R$ carrying the almost complex product structure of the restriction of $J$ to $U_\alpha$ and a given complex structure on $R$.

This definition—including the fact underlying the warning in Remark 9.4—makes sense, since any almost complex structure on a real surface is integrable by the classical results on the solvability of the Beltrami equation (see e.g. [21]). For instance, if $R$ is the unit disc $\mathbb{D}$, by solving the Beltrami equation we always can achieve that the restriction of the structure $(\Phi_\alpha)_*(I)$ on $\{a\} \times R$ is $J_{st}$ for every $a \in U_\alpha$. This justifies the choice of a Riemann surface as a generic fiber. We point out that, given Definition 9.3, the projection of $U_\alpha \times R$ to $R$, in general, is not holomorphic with respect to the structure $(\Phi_\alpha)_*(I)$ and its restriction to $R$.

PROPOSITION 9.5. *Let $(X, \pi, M)$ be an almost complex fiber bundle with a hyperbolic Riemann surface as a generic fiber. Suppose that at least one of the following conditions holds.*

(i) *The base $M$ admits a plurisubharmonic function and an almost complex structure $J$ on $M$ is integrable (for instance, $M$ is a hyperbolic Riemann surface).*
(ii) *$M$ admits a strictly plurisubharmonic exhaustion function $\rho$ with a unique critical point.*

*Then $X$ admits a strictly plurisubharmonic function. Consequently, every smoothly bounded Levi convex domain in $X$ is Stein.*

*Proof.* In case (i) the method of Stehlé [28; 29] can be applied by using Proposition 9.9 in order to construct plurisubharmonic functions in trivializing charts. The same concerns (ii): If $J_M$ is not integrable, then "le lemme de recouvrement" of [28; 29] can be established if we use the level sets of the function $\rho$ instead of real spheres and use strictly pseudoconvex bumping instead of the affine convex bumping of [28; 29]. We leave the details to the reader. □

### *9.3. The Tangent and Cotangent Bundle of an Almost Complex Manifold*

Among the most natural examples of fibered almost complex manifolds, the various jet bundles over an almost complex manifold $M$ should be considered. However, they have surprising properties, as we will see.

Namely, any almost complex structure $J$ on $M$ can be lifted to any of its jet bundles (in general, however, such a lift is not unique). In particular, this can be done to obtain an almost complex structure $\hat{J}$ on its tangent bundle $TM$ and an almost complex structure $\hat{J}^*$ on its cotangent bundle $T^*M$; see [32]. These lifts can be chosen in such a way that the following properties hold.

(a) The canonical projection $\pi: TM \to M$ is $(\hat{J}, J)$-holomorphic (resp., the canonical projection $\pi: T^*M \to M$ is $(\hat{J}^*, J)$-holomorphic).
(b) The embedding $e: M \to TM$ (resp., $M \to T^*M$) as the zero section is $(J, \hat{J})$-holomorphic (resp., $(J, \hat{J}^*)$-holomorphic).
(c) If $(N, I)$ is another almost complex manifold and $f: N \to M$ is a holomorphic map, then the canonical lift of $f$ is a holomorphic map between $(TN, \hat{I})$ and $(TM, \hat{J})$ (resp., between $(T^*M, \hat{J}^*)$ and $(T^*N, \hat{I}^*)$).

More explicitly, the structures $\hat{J}$ and $\hat{J}^*$ can be described in local coordinates. (From now on we use tensor notation for summation in local coordinates, using upper and lower indices for coordinates of vectors and covectors, respectively.)

Let $x = (x^1, \dots, x^{2n})$ be local coordinates on $M$ and let $t = (t^1, \dots, t^{2n})$ be fiber coordinates on $TM$ (resp., $p = (p_1, \dots, p_{2n})$ be fiber coordinates on $T^*M$). Then the structure $\hat{J}$ is given by the matrix

$$\hat{J} = \begin{pmatrix} J_k^s & 0 \\ t_l \partial J_k^s / \partial x^l & J_k^s \end{pmatrix}_{\substack{k=1,\dots,2n \\ s=1,\dots,2n}}$$

with $J_k^s$ being the components of $J$.

In order to describe the structure $\hat{J}^*$, recall first some basic differential tensor calculus on the cotangent bundle. If $T$ is a (1, 2)-tensor field on $M$, define a (1, 1)-tensor field $\gamma T$ on $T^*M$ as follows:

$$\gamma T = \begin{pmatrix} 0 & 0 \\ p_a T_{kj}^a & 0 \end{pmatrix}.$$

(It can easily be verified that this definition is independent of local coordinates.) Furthermore, let $N_J$ be the Nijenhuis tensor associated to the almost complex structure $J$ on $M$ (a (1, 2)-tensor field on $M$). Then, given vector fields $X, Y$ on $M$, we have

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y].$$

(Of course, $J^2$ in this expression can be replaced by $-1$.)

If now $\tilde{J}$ denotes the complete lift of $J$ to the cotangent bundle,

$$\tilde{J} = \begin{pmatrix} J_j^k & 0 \\ p_a(\partial J_k^a / \partial x_j - \partial J_j^a / \partial x_k & J_j^k \end{pmatrix}$$

(in coordinates), then *the lift $\hat{J}^*$ of $J$ to $T^*M$* is defined by

$$\hat{J}^* = \tilde{J} + \gamma(N_J J). \tag{9.1}$$

It can be verified that this is, indeed, an almost complex structure satisfying properties (a)–(c). We point out that, by contrast, the complete lift $\tilde{J}$ defines an almost complex structure on $T^*M$ if and only if $J$ is integrable (see more details in [32]).

Let now $x$ be local trivializing coordinates on an open subset $U \subset M$. Then every tangent space $T_a(M)$, $a \in U$, can be identified with the linear vector space $\mathbb{R}^{2n}$ equipped with the complex structure $\hat{J}(a) = \hat{J}|_{\{a\}\times T_a(M)}$ (resp. $\hat{J}^t(a)$, the transposed structure, for the fiber $T_a^*(M)$ of the cotangent bundle). In the case where $J$ is an integrable structure, the local coordinates $x$ can be chosen so that $J = J_{st}$ in these coordinates; in particular, $\hat{J}(a) = J_{st}$ on $T_a(M)$ for all $a \in U$. However, for arbitrary $J$, it is, in general, impossible to choose open sets $U$ and local coordinates on them such that the structure $J(a)$ carried over to $\mathbb{R}^{2n}$ by the trivializing map would be independent of $a \in U$. In other words, in general the tangent bundles of an almost complex manifold are not holomorphically locally trivial with respect to the lifted almost complex structure as given by (9.1). But, since every (linear) complex structure on $\mathbb{R}^{2n}$ is conjugate to $J_{st}$, applying this conjugation for each $a \in M$ fixed to $T_a(M)$ allows us to achieve, at least for this $a$, $\hat{J}|_{\{a\}\times T_a(M)} = J_{st}$ (resp. $\hat{J}^*|_{\{a\}\times T_a^*(M)} = J_{st}$). Anyhow, a coordinate map on an open subset $U$ of $M$

always gives rise to a fiber-preserving map $\pi^{-1}(U) \to U \times \mathbb{R}^{2n}$, which is as such compatible with the topological structure of the fiber manifold. This leads us to the following definition.

DEFINITION 9.6. Let $TM$ (resp. $T^*M$) be the tangent (resp. cotangent) bundle of an almost complex manifold $(M, J)$ equipped with the almost complex structure $\hat{J}$ (resp. $\hat{J}^*$) described previously and the canonical projection $\pi$. An open connected subset $X$ in $TM$ (resp. in $T^*M$) is called a *tangent* (resp. *cotangent*) *almost complex fiber manifold* over $M$ if there exists a domain $Y \in \mathbb{R}^{2n}$ (called the generic fiber) such that for every point $a \in M$ one can find a neighborhood $U$ in $M$ and a diffeomorphism $\Phi\colon \pi^{-1}(U) \to U \times Y$ satisfying the following conditions.

(i) $\Phi$ is fiber preserving; that is, $\pi' \circ \Phi = \pi$ where $\pi'\colon U \times Y \to U$ is the canonical projection.

(ii) $\Phi_*(\hat{J})|_{\{a'\}\times Y} = J_{st}$ (resp. $\Phi_*(\hat{J}^*)|_{\{a'\}\times Y} = J_{st}$).

Thus, if we denote by $J'$ the direct image $\Phi_*(\hat{J})$ (resp. $\Phi_*(\hat{J}^*)$) then $\Phi$ is a biholomorphism with respect to $\hat{J}$ (resp. $\hat{J}^*$) and $J'$. In particular, the canonical projection $\pi' = \pi \circ \Phi^{-1}$ is holomorphic. Condition (ii) means that the restriction of $J'$ on every fiber $\{a'\} \times Y$ coincides with the standard complex structure of $\mathbb{C}^n$. Thus $Y$ is a "generic fiber" not just topologically but also with respect to a complex structure. In other words, $\pi : (X, \hat{J}) \to (M, J')$ is locally trivial. We stress that the structure $J'$ in general, of course, does not coincide with $J \otimes J_{st}$ if $J$ is not integrable.

REMARK 9.7. We do not impose any pseudoconvexity assumption on $Y$ in Definition 9.6.

REMARK 9.8. The restriction of the structure $\hat{J}$ to any fiber of the tangent bundle is a complex structure on the tangent space of $M$, so it is conjugate to the standard structure if a local trivializing chart is fixed. For this reason, in (ii) of Definition 9.6 we consider $J_{st}$.

Our goal is to construct plurisubharmonic functions on a tangent or cotangent almost complex fiber manifold over $M$ using plurisubharmonic functions on $M$ and on a generic fiber $Y$. Since $J'$ is not the tensor product of almost complex structures from the base and the fiber, even in the topologically trivial situation this requires some additional considerations.

PROPOSITION 9.9. *Let $U \times Y$ be equipped with an almost complex structure $J'$ as before. Suppose that $U$ admits a strictly plurisubharmonic function $\rho$ such that for some constant $c > 0$ we have $L_\rho^{J'}(a, v) \geq c\|v\|^2$ for any point $a \in \bar{U}$ and any vector $v \in T_aM$. Suppose also that $Y$ is a bounded domain in $\mathbb{R}^{2n}$. Then $U \times Y$ admits a strictly plurisubharmonic function.*

*Proof.* Since the projection $\pi'\colon U \times Y \to U$ is holomorphic, the function $\rho \circ \pi'$ is plurisubharmonic with respect to $J'$. Unfortunately, the projection of $U \times Y$

onto $Y$ in general is not holomorphic. This can be easily seen from the preceding definitions of the structures $\hat{J}$ and $\hat{J}^*$. Since $Y$ is a bounded domain in $\mathbb{C}^n$ it admits a strictly plurisubharmonic function $\phi$ (with respect to $J_{st}$) with the Levi form bounded away from 0. Then the Levi form of $\phi$ with respect to $J'$ in the vertical direction (i.e., on a vector tangent to the fiber $\{a\} \times Y$) coincides with its Levi form with respect to $J_{st}$. Other eigenvalues of the Levi form of $\phi$ can be negative but are uniformly bounded since the fiber $Y$ is bounded. Now, since the function $\rho$ is strictly plurisubharmonic on $U$ and its Levi form is bounded away from 0 on $U$, there exists a constant $c_1 > 0$ such that the function $c_1 \rho \circ \pi + \phi$ is strictly plurisubharmonic on $U \times Y$. □

As a consequence we obtain the following statement.

COROLLARY 9.10. *Suppose that an almost complex manifold* $(M, J)$ *admits a strictly plurisubharmonic exhaustion function* $\rho$. *Let* $D_c = \{\rho < c\}$ *be a sublevel set for* $\rho$ *such that* $\rho$ *has only one critical point* (*hence minimum*) *in* $D_c$. *Then any tangent or cotangent almost complex fiber bundle with a bounded generic fiber over* $D_c$ *admits a strictly plurisubharmonic function. A smoothly bounded Levi convex domain in* $TD_c$ *or* $T^*D_c$ *is Stein.*

*Proof.* For the proof it suffices to use "le lemme de recouvrement" due to [28; 29]. As we pointed out previously, since the level sets of $\rho$ are noncritical they can be used in the proof of this statement in [28; 29] instead of the Euclidean spheres. Furthermore, the convex affine bumpings of [28; 29] can be replaced by strictly pseudoconvex bumpings. This allows us to avoid an application of the embedding theorem for complex Stein spaces used in [28; 29]. □

REMARK 9.11. An interesting question with respect to the material presented in this section arises: Is there a small additional hypothesis (a kind of curvature condition weaker than the full integrability) that turns the tangent bundle into a locally trivial holomorphic tangent bundle as defined in Definition 9.1?

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