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Generalized Test Ideals and Symbolic Powers

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Dedicated to Professor Mel Hochster
on the occasion of his sixty-fifth birthday

Introduction

Ein, Lazarsfeld, and Smith proved in [ELS] the following uniform behavior of symbolic powers of ideals in affine regular rings of equal characteristic 0: If h is the largest height of any associate prime of an ideal $I \subseteq R$, then $I^{(hn+kn)} \subseteq (I^{(k+1)})^n$ for all integers $n \geq 1$ and $k \geq 0$. Here, if W is the complement of the union of the associate primes of I, then the mth symbolic power $I^{(m)}$ of I is defined to be the contraction of $I^m R_W$ to R, where R_W is the localization of R at the multiplicative system W. To prove this, the authors introduced the notion of asymptotic multiplier ideals, which is a variant of multiplier ideals associated to filtrations of ideals and is formulated in terms of resolution of singularities. The uniform behavior of symbolic powers follows immediately from a combination of properties (of asymptotic multiplier ideals) whose proofs require deep vanishing theorems.

In [HHu5], Hochster and Huneke generalized the result in [ELS] to the case of arbitrary regular rings of equal characteristic (i.e., both of equal characteristic 0 and of positive prime characteristic) in a completely different way. Furthermore, they used in [HHu6] similar ideas to prove more subtle behaviors of symbolic powers of ideals in a regular ring of equal characteristic. Their methods depend on the theory of tight closure and reduction to positive characteristic and thus require neither resolution of singularities nor vanishing theorems, which are proved only in characteristic 0. In this paper, by combining the ideas of Ein–Lazarsfeld–Smith and Hochster–Huneke, we give a slight generalization of Hochster and Huneke's results in [HHu6].

Tight closure is an operation defined on ideals or modules in positive characteristic; it was introduced by Hochster and Huneke [HHu2] in the 1980s. The test ideal $\tau(R)$ of a Noetherian ring R of prime characteristic p is the annihilator ideal of all tight closure relations in R, and it plays a central role in the theory of tight closure. In [HaY] and [Ha], Hara and Yoshida introduced a generalization of the test ideal $\tau(R)$, the ideal $\tau(\mathfrak{a}_\bullet)$ associated to a filtration of ideals $\mathfrak{a}_\bullet$, and

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showed that their generalized test ideal $\tau(\mathfrak{a}_\bullet)$ is a characteristic-p analogue of the asymptotic multiplier ideal $\mathcal{J}(\mathfrak{a}_\bullet)$. In particular, in fixed prime characteristic, the generalized test ideal $\tau(\mathfrak{a}_\bullet)$ satisfies several nice properties similar to those of the asymptotic multiplier ideal $\mathcal{J}(\mathfrak{a}_\bullet)$ that are needed to prove Ein–Lazarsfeld–Smith's result—for example, an analogue of Skoda's theorem (see [HaT, Thms. 4.1, 4.2]) and the subadditivity theorem [HaY, Thm. 4.5; Ha, Prop. 2.10]. In this paper, we first prove that the formation of generalized test ideals commutes with localization without the assumption of F-finiteness (cf. [HaT, Prop. 3.1]). This result makes it easier to study further properties of generalized test ideals in non-F-finite rings. Then, employing the strategy of Ein–Lazarsfeld–Smith, we use generalized test ideals instead of asymptotic multiplier ideals to prove the following behavior of symbolic powers.

MAIN THEOREM. *Let R be an excellent regular ring of characteristic $p > 0$ (resp., a regular algebra essentially of finite type over a field of characteristic 0), and let $I \subsetneq R$ be an ideal of positive height. Let h denote the largest analytic spread of IR_P as P runs through the associated primes of I.*

(1) *If $(R,\mathfrak{m})$ is local then, for all integers $n \geq 1$ and $k \geq 0$,*

$$I^{(hn+kn+1)} \subseteq \mathfrak{m}(I^{(k+1)})^n.$$

(2) *If R/I is F-pure (resp., of dense F-pure type) then, for all integers $n \geq 1$ and $k \geq 0$,*

$$I^{(hn+kn-1)} \subseteq (I^{(k+1)})^n.$$

These results are a slight generalization of those in [HHu6] and are closely related to a conjecture concerning the existence of evolutions: Eisenbud and Mazur [EiM] asked whether $P^{(2)} \subseteq \mathfrak{m}P$ for a prime ideal P in a regular local ring $(R,\mathfrak{m})$ of equal characteristic 0. Their conjecture fails in positive characteristic (see [EiM]), whereas our results hold true even in positive characteristic.

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1. Generalized Test Ideals

In [Ha], Hara generalized the notion of tight closure in order to define a positive-characteristic analogue of asymptotic multiplier ideals, which are "multiplier ideals associated to graded families of ideals" (definition to follow). In this section, we quickly review the definition and basic properties of this analogue.

Throughout this paper, all rings are excellent Noetherian reduced commutative rings with unity. For a ring R, we denote by R° the set of elements of R that are not in any minimal prime ideal. A *graded family of ideals* $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\geq 1}$ on R means a collection of ideals $\mathfrak{a}_m \subseteq R$ satisfying $\mathfrak{a}_1 \cap R^\circ \neq \emptyset$ and $\mathfrak{a}_k \cdot \mathfrak{a}_l \subseteq \mathfrak{a}_{k+l}$

for all $k, l \geq 1$. Just for convenience, we decree that $\mathfrak{a}_0 = R$. One of the most important examples of graded families of ideals is a collection of symbolic powers $\mathfrak{a}^{(\bullet)} = \{\mathfrak{a}^{(m)}\}_{m \geq 1}$.

Let R be a ring of characteristic $p > 0$, and let $F\colon R \to R$ be the Frobenius map that sends $x \in R$ to $x^p \in R$. For each integer $e > 0$, the ring R when viewed as an R-module via the e-times iterated Frobenius map $F^e\colon R \to R$ is denoted by ${}^eR$. Since R is assumed to be reduced, we can identify $F^e\colon R \to {}^eR$ with the natural inclusion map $R \hookrightarrow R^{1/p^e}$. We say that R is *F-pure* if $R^{1/p}$ is a pure extension of R and that R is *F-finite* if $R^{1/p}$ is a finitely generated R-module. For example, complete local rings with perfect residue fields are F-finite. An F-finite ring R is said to be *strongly F-regular* if, for any $c \in R^\circ$, there exists $q = p^e$ such that $c^{1/q}R \hookrightarrow R^{1/q}$ splits as an R-linear map.

Let R be a ring of characteristic $p > 0$ and let M be an R-module. For each integer $e > 0$, we denote $\mathbb{F}^e(M) = \mathbb{F}^e_R(M) := M \otimes_R {}^eR$ and regard it as an R-module by the action of R on ${}^eR$ from the right. Then we have the induced e-times iterated Frobenius map $F^e_M\colon M \to \mathbb{F}^e(M)$. The image of $z \in M$ via this map is denoted by $z^q := F^e_M(z) \in \mathbb{F}^e(M)$, where $q = p^e$. For an R-submodule N of M, we denote by $N^{[q]}_M$ the image of the induced map $\mathbb{F}^e(N) \to \mathbb{F}^e(M)$.

Now we recall the definition of $\mathfrak{a}^k_\bullet$-tight closure, which is a variant of tight closure associated to a graded family of ideals $\mathfrak{a}_\bullet$ with exponent k.

Definition 1.1 [Ha, Def. 2.7]. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}$ be a graded family of ideals on a ring R of characteristic $p > 0$, and let $k \geq 1$ be an integer. Let $N \subseteq M$ be (not necessarily finitely generated) R-modules. The $\mathfrak{a}^k_\bullet$-*tight closure* of N in M, denoted by $N^{*\mathfrak{a}^k_\bullet}_M$, is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists a $c \in R^\circ$ such that

$$c\mathfrak{a}_{kq} z^q \subseteq N^{[q]}_M$$

for all large $q = p^e$. The $\mathfrak{a}^k_\bullet$-tight closure $I^{*\mathfrak{a}^k_\bullet}$ of an ideal $I \subseteq R$ is defined by $I^{*\mathfrak{a}^k_\bullet} = I^{*\mathfrak{a}^k_\bullet}_R$.

Remark 1.2. When $\mathfrak{a}_m = R$ for all $m \geq 1$, $\mathfrak{a}^k_\bullet$-tight closure is nothing but classical tight closure. That is, the *tight closure* I^* of an ideal $I \subseteq R$ is defined to be the ideal consisting of all elements $x \in R$ for which there exists a $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all large $q = p^e$. We say that R is *weakly F-regular* if all ideals in R are tightly closed (i.e., if $I^* = I$ for all ideals $I \subseteq R$) and that R is *F-regular* if all of its local rings are weakly F-regular. The reader is referred to [HHu2] for classical tight closure theory.

The test ideal $\tau(R)$ plays a central role in classical tight closure theory. We use the $\mathfrak{a}^k_\bullet$-tight closure of a zero submodule to define the ideal $\tau(\mathfrak{a}^k_\bullet)$, which is a generalization of the test ideal $\tau(R)$.

Proposition-Definition 1.3 (cf. [Ha, Prop.-Def. 2.9], [HaY, Def.-Thm. 6.5]). *Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}$ be a graded family of ideals on an excellent reduced ring R of characteristic $p > 0$, and let $k \geq 1$ be an integer. Let $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct* *sum, taken over all maximal ideals* $\mathfrak{m}$ *of* R, *of the injective hulls of the residue fields* $R/\mathfrak{m}$.

(1) *The following ideals are equal to each other*:

- $\bigcap_M \operatorname{Ann}_R(0_M^{*\mathfrak{a}_\bullet^k})$, *where* M *runs through all finitely generated* R*-modules*;
- $\bigcap_{M\subset E} \operatorname{Ann}_R(0_M^{*\mathfrak{a}_\bullet^k})$, *where* M *runs through all finitely generated* R*-submodules of* E;
- $\bigcap_{J\subseteq R}(J : J^{*\mathfrak{a}_\bullet^k})$, *where* J *runs through all ideals of* R.

These ideals are denoted by $\tau(\mathfrak{a}_\bullet^k)$ (*or* $\tau(k\cdot\mathfrak{a}_\bullet)$) *and are called the* generalized test ideals *associated to* $\mathfrak{a}_\bullet$ *with exponent* k. *Given a real number* $t>0$ *and an ideal* $\mathfrak{a}\subseteq R$ *such that* $\mathfrak{a}\cap R^\circ \neq \emptyset$, *if* $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}$ *is defined by* $\mathfrak{a}_m = \mathfrak{a}^{\lceil tm\rceil}$ *then we simply denote this ideal by* $\tau(\mathfrak{a}^t)$. *Also, for convenience we decree that* $\tau(\mathfrak{a}_\bullet^0)=\tau(R)$.

(2) *If* R *is a* $\mathbb{Q}$*-Gorenstein normal local ring, then*

$$\tau(\mathfrak{a}_\bullet^k) = \operatorname{Ann}_R(0_E^{*\mathfrak{a}_\bullet^k}).$$

(3) *If* $(R,\mathfrak{m})$ *is a* d*-dimensional Gorenstein local ring and if* $x_1,\dots,x_d$ *is a system of parameters for* R, *then*

$$\tau(\mathfrak{a}_\bullet^k) = \bigcap_{l\geq 0}((x_1^l,\dots,x_d^l) : (x_1^l,\dots,x_d^l)^{*\mathfrak{a}_\bullet^k}) = \operatorname{Ann}_R(0^{*\mathfrak{a}_\bullet^k}_{H^d_\mathfrak{m}(R)}).$$

REMARK 1.4 (cf. [Ha, Obs. 2.8]). With notation as in Proposition-Definition 1.3, the generalized test ideal $\tau(\mathfrak{a}_\bullet^k)$ is equal to the unique maximal element among the set of ideals $\{\tau(\mathfrak{a}_{p^e}^{k/p^e})\}_{e\geq 0}$ with respect to inclusion. The existence of a maximal element follows from the ascending chain condition on ideals (because R is Noetherian), and the uniqueness follows from the inclusion $\tau(\mathfrak{a}_q^{k/q})\subseteq\tau(\mathfrak{a}_{qq'}^{k/qq'})$ for any powers q,q' of p. If $\mathfrak{a}_\bullet$ is a descending filtration, then the ideal $\tau(\mathfrak{a}_\bullet^k)$ is equal to the unique maximal element among the set of ideals $\{\tau(\mathfrak{a}_m^{k/m})\}_{m\geq 1}$.

The notion of $\mathfrak{a}_\bullet^k$-test elements is useful for studying the behavior of the generalized test ideal $\tau(\mathfrak{a}_\bullet^k)$.

DEFINITION 1.5 (cf. [HaY, Def. 6.3]). Let $\mathfrak{a}_\bullet$ be a graded family of ideals on a ring R of characteristic $p>0$, and let $k\geq 1$ be an integer. An element $d\in R^\circ$ is called an $\mathfrak{a}_\bullet^k$*-test element* if, for every ideal $I\subseteq R$ and every $x\in R$, the following holds: $x\in I^{*\mathfrak{a}_\bullet^k}$ if and only if $dx^q\mathfrak{a}_{kq}\subseteq I^{[q]}$ for all powers $q=p^e$ of p.

An $\mathfrak{a}_\bullet^k$-test element exists in nearly every ring of interest.

PROPOSITION 1.6 (cf. [HaY, Thm. 1.7]). *Let* R *be a reduced ring of characteristic* $p>0$ *and let* $c\in R^\circ$. *Assume that one of the following conditions holds*:

(1) R *is* F*-finite and the localized ring* R_c *is strongly* F*-regular*;
(2) R *is an algebra of finite type over an excellent local ring* B, *and the localized ring* R_c *is Gorenstein and* F*-regular.*

Then some power c^n *of* c *is an* $\mathfrak{a}_\bullet^k$*-test element for all graded families of ideals* $\mathfrak{a}_\bullet$ *on* R *and for all integers* $k\geq 1$.

Proof. One can prove this proposition by an argument similar to that in the proof of [HaY, Thm. 1.7], but we will sketch the proof here for the reader's convenience.

(1) Take a power c^n that satisfies [HHu1, Rem. 3.2]. Let I be an ideal of R, and fix any $z \in I^{*\mathfrak{a}_\bullet^k}$ and any power q of p. Since $z \in I^{*\mathfrak{a}_\bullet^k}$, there exists a $d \in R^\circ$ such that $dz^Q\mathfrak{a}_{kQ} \subseteq I^{[Q]}$ for every power Q of p. By the choice of c^n, there exists a power q' of p and a $\phi \in \operatorname{Hom}_R(R^{1/q'}, R)$ such that $\phi(d^{1/q'}) = c^n$. Since $dz^{qq'}(\mathfrak{a}_{kq})^{[q']} \subseteq dz^{qq'}\mathfrak{a}_{kqq'} \subseteq I^{[qq']}$, it follows that $d^{1/q'}z^q\mathfrak{a}_{kq}R^{1/q'} \subseteq I^{[q]}R^{1/q'}$. Applying ϕ to both sides gives $c^nz^q\mathfrak{a}_{kq} \subseteq I^{[q]}$. Hence c^n is an $\mathfrak{a}_\bullet^k$-test element.

(2) Put $R' = R \otimes_B \hat{B}$, where $\hat{B}$ denotes the completion of B. Since B is excellent, $R \to R'$ is faithfully flat with regular fibers. In particular, R' is a reduced algebra of finite type over $\hat{B}$, and R'_c is Gorenstein and F-regular by [HHu3, Thm. 7.3(c)]. Thus, one can assume that B is complete. By using a Γ-construction argument (see [HHu3, Secs. 6, 7]), one can then reduce the problem to the case (1). The reader is referred to [HHu3, Thm. 6.1, Lemma 6.13, Lemma 6.19] for details. □

When the ring is the quotient of a regular local ring, we have a criterion for the triviality of the generalized test ideal $\tau(\mathfrak{a}_\bullet^k)$.

Proposition 1.7 (cf. [F, Thm. 1.12]). *Let $(S, \mathfrak{m})$ be a complete regular local ring of characteristic $p > 0$, and let $I \subsetneq S$ be a radical ideal. Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m\ge1}$ be a graded family of ideals on S and let $k \ge 1$ be an integer. Denote $R = S/I$ and $\mathfrak{a}_{R,\bullet} = \{\mathfrak{a}_mR\}_{m\ge1}$. Fix any $c \in S \setminus I$ whose image in R is an $\mathfrak{a}_{R,\bullet}^k$-test element. Then the generalized test ideal $\tau(\mathfrak{a}_{R,\bullet}^k)$ is trivial if and only if there exists $q = p^e$ such that $c(I^{[q]} : I)\mathfrak{a}_{kq} \not\subseteq \mathfrak{m}^{[q]}$.*

Proof. The proof is similar to those in [F, Thm. 1.12] and [HaW, Prop. 2.6]. □

In fixed prime characteristic, the generalized test ideal $\tau(\mathfrak{a}_\bullet^k)$ satisfies properties analogous to those of the asymptotic multiplier ideal $\mathcal{J}(\mathfrak{a}_\bullet^k)$.

Lemma 1.8 [T2, Lemma 4.5]. *Let $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}$ be a graded family of ideals on a ring R of characteristic $p > 0$. Then, for all integers $k, l \ge 0$,*

$$\mathfrak{a}_l\tau(\mathfrak{a}_\bullet^k) \subseteq \tau(\mathfrak{a}_\bullet^{k+l}).$$

Proof. It is enough to show that $(0_M^{*\mathfrak{a}_\bullet^k} : \mathfrak{a}_l) \supseteq 0_M^{*\mathfrak{a}_\bullet^{k+l}}$ for all finitely generated R-modules M, but this is immediate because $\mathfrak{a}_l^{[q]}\mathfrak{a}_{kq} \subseteq \mathfrak{a}_{(k+l)q}$ for every $q = p^e$. □

Given graded families of ideals $\mathfrak{a}_\bullet$, $\mathfrak{b}_\bullet$ on a ring R of characteristic $p > 0$ and integers $k, l \ge 1$, we can define $\mathfrak{a}_\bullet^k\mathfrak{b}_\bullet^l$-tight closure as follows: If N is a submodule of an R-module M, then an element $z \in M$ is in the $\mathfrak{a}_\bullet^k\mathfrak{b}_\bullet^l$-tight closure $N_M^{*\mathfrak{a}_\bullet^k\mathfrak{b}_\bullet^l}$ of N in M if and only if there exists a $c \in R^\circ$ such that $cz^q\mathfrak{a}_{kq}\mathfrak{b}_{lq} \subseteq N_M^{[q]}$ for all large $q = p^e$. The *generalized test ideal* $\tau(\mathfrak{a}_\bullet^k\mathfrak{b}_\bullet^l)$ is defined by $\tau(\mathfrak{a}_\bullet^k\mathfrak{b}_\bullet^l) = \bigcap_M \operatorname{Ann}_R(0_M^{*\mathfrak{a}_\bullet^k\mathfrak{b}_\bullet^l})$, where M runs through all finitely generated R-modules.

Theorem 1.9 ([Ha, Prop. 2.10]; cf. [T2, Prop. 4.4]). *Let R be a complete regular local ring of characteristic $p > 0$ or an F-finite regular ring of characteristic*

$p > 0$. *Let* $\mathfrak{a}_\bullet, \mathfrak{b}_\bullet$ *be graded families of ideals on* R *and fix any integers* $k, l \ge 0$. *Then*

$$\tau(\mathfrak{a}_\bullet^k \mathfrak{b}_\bullet^l) \subseteq \tau(\mathfrak{a}_\bullet^k)\tau(\mathfrak{b}_\bullet^l).$$

In particular,

$$\tau(\mathfrak{a}_\bullet^{kl}) \subseteq \tau(\mathfrak{a}_\bullet^k)^l.$$

2. Localization of Generalized Test Ideals

In [HaT], Hara and Takagi proved that the formation of generalized test ideals commutes with localization under the assumption of F-finiteness. In this section, we study the behavior of $\mathfrak{a}_\bullet^k$-tight closure and the ideal $\tau(\mathfrak{a}_\bullet^k)$ under localization without the assumption of F-finiteness.

PROPOSITION 2.1. *Let* $(R, \mathfrak{m})$ *be an excellent equidimensional reduced local ring of characteristic* $p > 0$, *let* $\mathfrak{a}_\bullet$ *be a graded family of ideals on* R, *and let* $k \ge 1$ *be an integer. Let* W *denote a multiplicatively closed subset of* R *and let* $\mathfrak{a}_{W,\bullet}$ *denote the extension of* $\mathfrak{a}_\bullet$ *on* R_W. *If* I *is an ideal of* R *that is generated by a subsystem of parameters for* R, *then*

$$(IR_W)^{*\mathfrak{a}_{W,\bullet}^k} = I^{*\mathfrak{a}_\bullet^k} R_W.$$

Proof. Since the ring R is excellent and reduced, by Proposition 1.6 it admits a $\mathfrak{b}_\bullet^l$-test element (say, $d \in R^\circ$) for any graded family of ideals $\mathfrak{b}_\bullet$ on R and for any integer $l \ge 1$. We first prove the following claim using an argument similar to that in the proof of [HHu4, Prop. 2.6].

Claim 1. For any given $c \in R^\circ$, *there exists an* $e_0 \in \mathbb{N}$ *such that* $\xi \in 0^{*\mathfrak{a}_\bullet^k}_{H^d_\mathfrak{m}(R)}$ *holds whenever* $\xi \in H^d_\mathfrak{m}(R)$ *and* $c\xi^q \mathfrak{a}_{kq} = 0$ *in* $H^d_\mathfrak{m}(R)$ *for some* $q = p^e \ge p^{e_0}$.

Proof of Claim 1. Let $0^F_{H^d_\mathfrak{m}(R)}$ denote the Frobenius closure of zero in $H^d_\mathfrak{m}(R)$; that is, $\xi \in H^d_\mathfrak{m}(R)$ is in $0^F_{H^d_\mathfrak{m}(R)}$ if and only if $\xi^q = 0$ in $H^d_\mathfrak{m}(R)$ for some power $q = p^e$. For each $e \in \mathbb{N}$, we denote

$$N_e = \{\xi \in H^d_\mathfrak{m}(R) \mid c\xi^{p^e} \mathfrak{a}_{kp^e} \subseteq 0^F_{H^d_\mathfrak{m}(R)}\}.$$

Now let $\xi \in N_{e+1}$ and put $q = p^e$. Then $c\xi^{pq}\mathfrak{a}_{kpq} \subseteq 0^F_{H^d_\mathfrak{m}(R)}$ by definition. In particular, there exists $q' = p^{e'}$ such that $c^{pq'}\xi^{pqq'}\mathfrak{a}_{kq}^{[pq']} \subseteq c^{q'}\xi^{pqq'}\mathfrak{a}_{kpq}^{[q']} = 0$ in $H^d_\mathfrak{m}(R)$. This implies that $c\xi^q\mathfrak{a}_{kq} \subseteq 0^F_{H^d_\mathfrak{m}(R)}$; that is, $\xi \in N_e$. Hence $\{N_e\}$ is a decreasing sequence on e. One can therefore choose $e_0 \in \mathbb{N}$ such that $N_e = N_{e_0}$ for all $e \ge e_0$, because $H^d_\mathfrak{m}(R)$ is Artinian. Then we can easily see that $N_{e_0} \subseteq 0^{*\mathfrak{a}_\bullet^k}_{H^d_\mathfrak{m}(R)}$, which implies the claim. □

Given a system of parameters $x_1, \dots, x_d$ for R, we may assume that I is generated by a subsystem of parameters $x_1, \dots, x_h$ $(h \le d)$.

Claim 2. Let $c \in R^\circ$ *and fix* e_0 *as given by Claim 1. If* $cz^q\mathfrak{a}_{kq} \subseteq I^{[q]}$ *for some* $q = p^e \ge p^{e_0}$, *then* $z \in I^{*\mathfrak{a}_\bullet^k}$.

Proof of Claim 2. By a standard argument, we can reduce to the case where $h = d$ (i.e., $I = (x_1, \dots, x_d)$). If we put $\xi = [z + (x_1, \dots, x_d)] \in H^d_{\mathfrak{m}}(R)$, then by Claim 1 we have $\xi \in 0^{*\mathfrak{a}^k_\bullet}_{H^d_{\mathfrak{m}}(R)}$ because $c\xi^q \mathfrak{a}_{kq} = 0$ in $H^d_{\mathfrak{m}}(R)$. Hence $d\xi^Q \mathfrak{a}_{kQ} = 0$ in $H^d_{\mathfrak{m}}(R)$ for all powers Q of p. That is, there exists an integer $l = l(Q) \geq 0$ depending on Q such that $dz^Q \mathfrak{a}_{kQ}(x_1, \dots, x_d)^l \subseteq (x_1^{l+Q}, \dots, x_d^{l+Q})$. By the colon-capturing property for (classical) tight closure, we obtain $dz^Q \mathfrak{a}_{kQ} \subseteq (I^{[Q]})^*$ and $d^2 z^Q \mathfrak{a}_{kQ} \subseteq I^{[Q]}$ for all powers Q. This means that $z \in I^{*\mathfrak{a}^k_\bullet}$, as required. □

Finally, we prove the assertion of Proposition 2.1. It is enough to show that $(IR_W)^{*\mathfrak{a}^k_{W,\bullet}} \subseteq I^{*\mathfrak{a}^k_\bullet} R_W$. Suppose that $\alpha \in (IR_W)^{*\mathfrak{a}^k_{W,\bullet}}$. By definition, there exist $c \in (R_W)^\circ \cap R$ and an integer e_1 such that $c\alpha^q \mathfrak{a}_{kq} R_W \subseteq I^{[q]} R_W$ for all $q = p^e \geq p^{e_1}$. We may assume that $c \in R^\circ$ by prime avoidance. By Claim 2, we can take a positive integer e_2 such that $z \in I^{*\mathfrak{a}^k_\bullet}$ holds whenever $cz^q \mathfrak{a}_{kq} \subseteq I^{[q]}$ for some $q = p^e \geq p^{e_2}$. Fix any $e \geq e_3 := \max\{e_1, e_2\}$ and put $q = p^e$. Choose $u \in W$ such that $uc\alpha^q \mathfrak{a}_{kq} \subseteq I^{[q]}$. Then $c(u\alpha)^q \mathfrak{a}_{kq} \subseteq I^{[q]}$ and thus $u\alpha \in I^{*\mathfrak{a}^k_\bullet}$. That is, $\alpha \in I^{*\mathfrak{a}^k_\bullet} R_W$ as required. □

Corollary 2.2 (cf. [HaT, Prop. 3.1]). *Let $(R, \mathfrak{m})$ be a complete Gorenstein reduced local ring of characteristic $p > 0$, let $\mathfrak{a}_\bullet$ be a graded family of ideals on R, and let $k \geq 1$ be an integer. Let W denote a multiplicatively closed subset of R and let $\mathfrak{a}_{W,\bullet}$ denote the extension of $\mathfrak{a}_\bullet$ on R_W. Then*

$$\tau(\mathfrak{a}^k_{W,\bullet}) = \tau(\mathfrak{a}^k_\bullet) R_W.$$

Proof. The proof is based on arguments similar to those in [S2]. First, we will show that $\tau(\mathfrak{a}^k_\bullet) R_W \subseteq \tau(\mathfrak{a}^k_{W,\bullet})$. To see this, let $c \in \tau(\mathfrak{a}^k_\bullet)$ and $z \in (IR_W)^{*\mathfrak{a}^k_{W,\bullet}} \cap R$, where I is any given ideal of R such that $IR_W \neq R_W$. Let P be a prime ideal of R such that $P \cap W = \emptyset$, and let $\mathfrak{a}_{P,\bullet}$ denote the extension of $\mathfrak{a}_\bullet$ on R_P. Take an R-sequence $x_1, \dots, x_h$ in P whose images form a system of parameters for R_P. By Proposition 2.1,

$$\tau(\mathfrak{a}^k_\bullet)((x_1^l, \dots, x_h^l) R_P)^{*\mathfrak{a}^k_{P,\bullet}} = \tau(\mathfrak{a}^k_\bullet)(x_1^l, \dots, x_h^l)^{*\mathfrak{a}^k_\bullet} R_P \subseteq (x_1^l, \dots, x_h^l) R_P$$

for every $l \in \mathbb{N}$. Thus $\tau(\mathfrak{a}^k_\bullet) R_P \subseteq \tau(\mathfrak{a}^k_{P,\bullet})$ because R is Gorenstein. Since the image $\frac{z}{1}$ lies in $(IR_P)^{*\mathfrak{a}^k_{P,\bullet}}$, we have that $\frac{cz}{1} = \frac{c}{1} \cdot \frac{z}{1} \in IR_P$ for all such primes P. It then follows that $\frac{cz}{1} \in IR_W$; that is, $\frac{c}{1} \in \tau(\mathfrak{a}^k_{W,\bullet})$.

Next we prove the converse, starting with the case where $W = R \setminus P$ and $P \subset R$ is a prime ideal of height h. Then

$$\tau(\mathfrak{a}^k_{P,\bullet}) = \operatorname{Ann}_{R_P} 0^{*\mathfrak{a}^k_{P,\bullet}}_{H^h_{PR_P}(R_P)}$$

by definition. On the other hand, since R is complete and $H^d_{\mathfrak{m}}(R) = E$ is Artinian,

$$\tau(\mathfrak{a}^k_\bullet) R_P = (\operatorname{Ann}_R N) R_P = \operatorname{Ann}_{R_P} N^{\vee_{\mathfrak{m}} \vee_P},$$

where $N = 0^{*\mathfrak{a}^k_\bullet}_{H^d_{\mathfrak{m}}(R)}$ and

$$N^{\vee_{\mathfrak{m}}\vee_P} = \mathrm{Hom}_R(\mathrm{Hom}_R(N, H^d_{\mathfrak{m}}(R)), H^h_{PR_P}(R_P)) \subseteq H^h_{PR_P}(R_P)$$

by [S1, Lemma 3.1(iii)]. It is therefore enough to show the following claim in order to prove $\tau(\mathfrak{a}^k_\bullet)R_P \supseteq \tau(\mathfrak{a}^k_{P,\bullet})$.

Claim. $N^{\vee_{\mathfrak{m}}\vee_P} \subseteq 0^{*\mathfrak{a}^k_{P,\bullet}}_{H^h_{PR_P}(R_P)}$ *in* $H^h_{PR_P}(R_P)$.

Proof of Claim. Take a system of parameters $x_1, \dots, x_d$ for R such that $\frac{x_1}{1}, \dots, \frac{x_h}{1}$ forms a system of parameters for R_P. Suppose that

$$\eta = \left[\frac{z}{1} + \left(\frac{x_1^t}{1}, \dots, \frac{x_h^t}{1}\right)\right] \in H^h_{PR_P}(R_P)$$

belongs to $N^{\vee_{\mathfrak{m}}\vee_P}$. To see that $\mathfrak{a}_{kq}\eta^q \subseteq N^{\vee_{\mathfrak{m}}\vee_P}$ for all $q = p^e$, we may assume that $t = 1$ without loss of generality. Since $N^{\vee_{\mathfrak{m}}\vee_P} = \mathrm{Ann}_{H^h_{PR_P}(R_P)}\tau(\mathfrak{a}^k_\bullet)R_P$ by [S2, Lemma 2.1(iv)], it follows that

$$\frac{z}{1}\tau(\mathfrak{a}^k_\bullet)R_P \subseteq \left(\frac{x_1}{1}, \dots, \frac{x_h}{1}\right)R_P.$$

Take any $a \in R \setminus P$ such that $az\tau(\mathfrak{a}^k_\bullet) \subseteq (x_1, \dots, x_h)$ and put

$$\eta_l = [az + (x_1, \dots, x_h, x^l_{h+1}, \dots, x^l_d)] \in H^d_{\mathfrak{m}}(R)$$

for each integer $l \geq 1$. Then, since $\eta_l \in \mathrm{Ann}_{H^d_{\mathfrak{m}}(R)}\tau(\mathfrak{a}^k_\bullet) = 0^{*\mathfrak{a}^k_\bullet}_{H^d_{\mathfrak{m}}(R)}$ (by the Matlis dual), we have that $\mathfrak{a}_{kq}\eta_l^q \subseteq \mathrm{Ann}_{H^d_{\mathfrak{m}}(R)}\tau(\mathfrak{a}^k_\bullet)$ for all $q = p^e$ by an argument similar to that in the proof of [HaY, Prop. 1.15]. That is, for every $l \geq 1$, $(az)^q\mathfrak{a}_{kq}\tau(\mathfrak{a}^k_\bullet) \subseteq (x_1^q, \dots, x_h^q, x^{lq}_{h+1}, \dots, x^{lq}_d)$. In particular,

$$\left(\frac{z}{1}\right)^q \mathfrak{a}_{kq}\tau(\mathfrak{a}^k_\bullet)R_P \subseteq \left(\frac{x_1^q}{1}, \dots, \frac{x_h^q}{1}\right)R_P$$

and thus $\mathfrak{a}_{kq}\eta^q \subseteq N^{\vee_{\mathfrak{m}}\vee_P}$ for all $q = p^e$.

Note that there exists some $\frac{c}{1} \in \tau(\mathfrak{a}^k_\bullet)R_P \cap (R_P)^\circ$. Then $\frac{c}{1}$ kills every element of $N^{\vee_{\mathfrak{m}}\vee_P}$. This means that, for any $\eta \in N^{\vee_{\mathfrak{m}}\vee_P}$, one has $\frac{c}{1}\mathfrak{a}_{kq}\eta^q = 0$ in $H^h_{PR_P}(R_P)$ for all $q = p^e$. We conclude that $N^{\vee_{\mathfrak{m}}\vee_P} \subseteq 0^{*\mathfrak{a}^k_{P,\bullet}}_{H^h_{PR_P}(R_P)}$. □

Finally, we consider the general case. Suppose $c \in \tau(\mathfrak{a}^k_{W,\bullet}) \cap R$. To see that $c \in \tau(\mathfrak{a}^k_\bullet)R_W \cap R$, it suffices to show that $c \in \tau(\mathfrak{a}^k_\bullet)R_P \cap R$ for every prime ideal $P \subset R$ such that $P \cap W = \emptyset$. Take a system of parameters $x_1, \dots, x_d$ such that $\frac{x_1}{1}, \dots, \frac{x_h}{1}$ forms a system of parameters for R_P. Then, by Proposition 2.1 and the definition of c,

$$c((x_1^l, \dots, x_h^l)R_P)^{*\mathfrak{a}^k_{P,\bullet}} = c((x_1^l, \dots, x_h^l)R_W)^{*\mathfrak{a}^k_{W,\bullet}}R_P \subseteq (x_1^l, \dots, x_h^l)R_P$$

for every $l \geq 1$. Thus, $c \in \tau(\mathfrak{a}^k_{P,\bullet}) \cap R = \tau(\mathfrak{a}^k_\bullet)R_P \cap R$. □

We can use Corollary 2.2 to show a Skoda-type theorem for symbolic powers of ideals.

Proposition 2.3 (cf. [Ha, Thm. 2.12]). *Let R be an excellent Gorenstein reduced local ring of characteristic $p > 0$, and let I be an ideal of R such that $I \cap R^\circ \neq \emptyset$. Suppose that the residue field of each of the rings R_P is infinite when P is an associated prime of I. Let $I^{(\bullet)} = \{I^{(m)}\}$ denote the graded family of symbolic powers of I, and let h denote the largest analytic spread of IR_P as P runs through the associated primes of I. Then, for every integer $k \geq 0$,*

$$\tau((h+k) \cdot I^{(\bullet)}) \subseteq I^{(k+1)}.$$

Proof. Let $I_P^{(\bullet)}$ denote the extension of $I^{(\bullet)}$ on R_P. After localization at P, the symbolic and ordinary powers of I are the same and so $I_P^{(\bullet)} = \{(IR_P)^m\}_{m \geq 1}$. By assumption, IR_P has a reduction ideal generated by at most h elements for each associated prime P of I. Then, by [HaY, Thm. 2.1] and the first half of the proof of Corollary 2.2,

$$\begin{aligned} \tau((h+k) \cdot I^{(\bullet)})R_P &\subseteq \tau((h+k) \cdot I_P^{(\bullet)}) = \tau((IR_P)^{h+k}) \\ &\subseteq (IR_P)^{k+1} \end{aligned}$$

for every associated prime P of I. Thus $\tau((h+k) \cdot I^{(\bullet)}) \subseteq I^{(k+1)}$, as required. □

3. Symbolic Powers in Positive Characteristic

In [EiM], Eisenbud and Mazur asked a question concerning the behavior of symbolic squares of ideals in regular local rings of equal characteristic 0. When the ring is of positive characteristic or of mixed characteristic, counterexamples to their question are known (see [EiM] for the case of positive characteristic and [KR] for the case of mixed characteristic). Nevertheless, Hochster and Huneke [HHu6] proved analogous results to their question in positive characteristic, using the classical tight closure theory. In this section, we use generalized test ideals to give a slight generalization to the results of Hochster and Huneke.

Theorem 3.1 (cf. [HHu6, Thm. 3.5]). *Let $(R, \mathfrak{m})$ be an excellent regular local ring of characteristic $p > 0$ and let $I \subsetneq R$ be any nonzero ideal. Let h be the largest analytic spread of IR_P as P runs through the associated primes of I. Then, for all integers $n \geq 1$ and $k \geq 0$,*

$$I^{(hn+kn+1)} \subseteq \mathfrak{m}(I^{(k+1)})^n.$$

In particular, if $h = 2$ then

$$I^{(3)} \subseteq \mathfrak{m}I.$$

Proof. We may assume that R is a complete local ring. Suppose that we have the assertion in the complete regular case. Although $\hat{I} = I\hat{R}$ may have more associated primes, the biggest analytic spread as one localizes at these primes cannot increase (see [HHu5, Disc. 2.3(c)] for details). Also, $I^{(m)}\hat{R} = \hat{I}^{(m)}$ for every integer $m \geq 1$. Therefore

$$I^{(hn+kn+1)} = \hat{I}^{(hn+kn+1)} \cap R \subseteq \hat{\mathfrak{m}}(\hat{I}^{(k+1)})^n \cap R = \mathfrak{m}(I^{(k+1)})^n,$$

as required, by the faithful flatness of the completion.

Moreover, if $R/\mathfrak{m}$ is a finite field (in this case, R is F-finite), then we replace R by $R[t]_M$ and I by $IR[t]_M$ where t is indeterminate and M is the maximal ideal of $R[t]$ generated by $\mathfrak{m}$ and t. Observe that the issues are unaffected by this replacement because the associated primes of $IR[t]_M$ are simply those of the form $PR[t]_M$, where P is an associated prime of I (see [HHu5, Disc. 2.3(b)] for further explanation). By this trick, we can assume that R is a complete regular local ring with infinite residue field, or an F-finite regular local ring such that the residue field of each of the rings R_P is infinite when P is an associated prime of I.

Let $I^{(\bullet)} = \{I^{(m)}\}$ denote the graded family of symbolic powers of I, and let l denote the largest integer such that $\tau(l \cdot I^{(\bullet)}) = R$. Such an integer l always exists because $\tau(0 \cdot P^{(\bullet)}) = R$. Note that the ideal $\tau((l+1) \cdot P^{(\bullet)})$ is contained in $\mathfrak{m}$. By Lemma 1.8, Theorem 1.9 and Proposition 2.3, it follows that

$$\begin{aligned} I^{(hn+kn+1)} = I^{(hn+kn+1)}\tau(l \cdot I^{(\bullet)}) &\subseteq \tau((hn+kn+l+1) \cdot I^{(\bullet)}) \\ &\subseteq \tau((l+1) \cdot I^{(\bullet)})\tau((h+k) \cdot I^{(\bullet)})^n \\ &\subseteq \mathfrak{m}(I^{(k+1)})^n. \end{aligned}$$

□

LEMMA 3.2. *Let R be an excellent regular local ring of characteristic $p > 0$, and let $I \subsetneq R$ be an ideal of height at least 2 such that R/I is F-pure. Let $I^{(\bullet)} = \{I^{(m)}\}$ denote the graded family of symbolic powers of I, and let $\hat{I}^{(\bullet)} = \{\hat{I}^{(m)}\}$ denote the extension of $I^{(\bullet)}$ on the completion $\hat{R}$. Then the generalized test ideal $\tau(\hat{R}, \hat{I}^{(\bullet)})$ associated to $\hat{I}^{(\bullet)}$ is trivial. Moreover, if R is F-finite, then the generalized test ideal $\tau(R, I^{(\bullet)})$ associated to $I^{(\bullet)}$ is also trivial.*

Proof. By [F, Thm. 1.12], the ring R/I is F-pure if and only if $(I^{[q]} : I) \not\subseteq \mathfrak{m}^{[q]}$ for all $q = p^e$. Also, by Proposition 1.7, the generalized test ideal $\tau(\hat{R}, \hat{I}^{(\bullet)})$ is trivial if and only if $I^{(q)}\hat{R} = \hat{I}^{(q)} \not\subseteq (\mathfrak{m}\hat{R})^{[q]}$ for some $q = p^e$. Therefore, it is enough to show that $(I^{[q]} : I) \subseteq I^{(q)}$ for every $q = p^e$. Let $P_1, \dots, P_k$ be the minimal prime ideals of the radical ideal I. Then, by definition, $I^{(q)} = \bigcap_{i=1}^k P_i^{(q)}$. On the other hand, we have the following claim.

Claim. $(I^{[q]} : I) = \bigcap_{i=1}^k (P_i^{[q]} : P_i)$

Proof of Claim. It is enough to show that $(P_i^{[q]} : P_i) = (P_i^{[q]} : I)$ for all $i = 1, \dots, k$. Since the inclusion $(P_i^{[q]} : P_i) \subseteq (P_i^{[q]} : I)$ is clear, we will prove the reverse inclusion. Let $x \in (P_i^{[q]} : I)$. Then $xP_1 \cdots P_k \subseteq P_i^{[q]}$ and it follows that $xP_iR_{P_i} \subseteq P_i^{[q]}R_{P_i}$. Since the Frobenius map is flat, $P_i^{[q]}$ is a P_i-primary ideal. This implies that $xP_i \in P_i^{[q]}$. □

We may therefore assume that I is a prime ideal of height $h \geq 2$. Because the Frobenius map is flat, $I^{[q]}$ is I-primary and thus $(I^{[q]} : I)$ is also I-primary. Then

$$\begin{aligned}(I^{[q]} : I) &= (IR_I^{[q]} : IR_I) \cap R \\ &= (I^{h(q-1)}R_I + I^{[q]}R_I) \cap R \\ &\subseteq I^{(q)},\end{aligned}$$

since R_I is a regular local ring of dimension $h \geq 2$.

The latter assertion is immediate, because the formation of generalized test ideals commutes with completion if the ring is F-finite (this follows from an argument similar to that in the proof of [HaT, Prop. 3.2]). □

THEOREM 3.3 (cf. [HHu6, Thm. 3.6]). *Let R be an excellent regular ring of characteristic $p > 0$, and let $I \subsetneq R$ be an ideal of height at least 2 such that R/I is F-pure. Let h denote the largest height of any minimal prime of I. Then, for all integers $n \geq 1$ and $k \geq 0$,*

$$I^{(hn+kn-1)} \subseteq (I^{(k+1)})^n.$$

In particular, if $h = 2$ then

$$I^{(3)} \subseteq I^2.$$

Proof. The problem reduces to the local case. Then, by essentially the same argument as used in the proof of Theorem 3.1, we can assume that R is a complete regular local ring with infinite residue field or an F-finite regular local ring such that the residue field of each of the rings R_P is infinite when P is an associated prime of I. By virtue of Lemma 3.2, the generalized test ideal $\tau(1 \cdot I^{(\bullet)})$ is trivial. Then, for all integers $n \geq 1$ and $k \geq 0$, it follows from Lemma 1.8, Theorem 1.9, and Proposition 2.3 that

$$\begin{aligned}I^{(hn+kn-1)} = I^{(hn+kn-1)}\tau(1 \cdot I^{(\bullet)}) &\subseteq \tau((hn+kn) \cdot I^{(\bullet)}) \\ &\subseteq \tau((h+k) \cdot I^{(\bullet)})^n \\ &\subseteq (I^{(k+1)})^n.\end{aligned}$$

□

REMARK 3.4. The proof of Theorem 3.3 tells us the following. Given an ideal I of an excellent regular ring R of characteristic $p > 0$ such that $I \cap R^\circ \neq \emptyset$, if the generalized test ideal $\tau(l \cdot I^{(\bullet)})$ associated to $I^{(\bullet)}$ is trivial for some integer $l \geq 1$ then $I^{(hn+kn-l)} \subseteq (I^{(k+1)})^n$ for all integers $n \geq 1$ and $k \geq 0$, where h is the largest analytic spread of IR_P as P runs through the associated primes of I.

In general, the Eisenbud–Mazur conjecture fails in positive characteristic (see [EiM]), but there is no counterexample to the codimension 2 case. Theorem 3.3 suggests that symbolic powers of an ideal in a regular ring behave nicely when the quotient of the ring by the ideal is F-pure. We expect that, even in positive characteristic, their conjecture holds true for ideals of codimension 2 if the quotient of the regular ring by the ideal is F-pure.

CONJECTURE 3.5. *Let $(R, \mathfrak{m})$ be an excellent regular local ring of characteristic $p > 0$, and let $I \subsetneq R$ be an unmixed ideal of height 2 such that R/I is F-pure. Then $I^{(2)} \subseteq \mathfrak{m}I$.*

This conjecture is known to hold when R/I is Cohen–Macaulay (because such an I is a licci ideal and then $P^{(2)} \subseteq \mathfrak{m}P$ is established in [EiM]). Also, it is easy to check that the conjecture holds for squarefree monomial ideals.

4. Symbolic Powers in Characteristic 0

By using the standard descent theory of [HHu7, Chap. 2], we can generalize the positive characteristic results of Section 3 to the case of equal characteristic.

THEOREM 4.1. *Let $(R, \mathfrak{m})$ be a regular local ring, essentially of finite type over a field of characteristic 0, and let $I \subsetneq R$ be any nonzero ideal. Let h be the largest analytic spread of IR_P as P runs through the associated primes of I. Then, for all integers $n \geq 1$ and $k \geq 0$,*

$$I^{(hn+kn+1)} \subseteq \mathfrak{m}(I^{(k+1)})^n.$$

In particular, if $h = 2$ then

$$I^{(3)} \subseteq \mathfrak{m}I.$$

Proof. We employ the same strategy as in the proof of Theorem 3.1 but use asymptotic multiplier ideals instead of generalized test ideals. Let $I^{(\bullet)} = \{I^{(m)}\}$ denote the graded family of symbolic powers of I, and let l denote the largest integer such that $\mathcal{J}(l \cdot I^{(\bullet)}) = R$. Note that the ideal $\mathcal{J}((l+1) \cdot I^{(\bullet)})$ is contained in $\mathfrak{m}$. Applying [L, Thm. 11.1.19], the subadditivity formula [L, Thm. 11.2.3], and Skoda's theorem [L, Thm. 9.6.21], we obtain

$$\begin{aligned} I^{(hn+kn+1)} = I^{(hn+kn+1)}\mathcal{J}(l \cdot I^{(\bullet)}) &\subseteq \mathcal{J}((hn+kn+l+1) \cdot I^{(\bullet)}) \\ &\subseteq \mathcal{J}((l+1) \cdot I^{(\bullet)})\mathcal{J}((h+k) \cdot I^{(\bullet)})^n \\ &\subseteq \mathfrak{m}(I^{(k+1)})^n. \end{aligned}$$

□

DEFINITION 4.2. Let R be a ring that is finitely generated over a field k of characteristic 0. Then the ring R is said to be of *dense F-pure type* if there exist a finitely generated $\mathbb{Z}$-subalgebra $A \subseteq k$ and a finitely generated A-algebra R_A, free over A, such that $R \cong R_A \otimes_A k$ and such that, for all maximal ideals μ in a Zariski dense subset of Spec A with residue field $\kappa = A/\mu$, the fiber rings $R_A \otimes_A \kappa$ are F-pure.

THEOREM 4.3. *Let R be a regular algebra, essentially of finite type over a field of characteristic 0, and let $I \subsetneq R$ be an ideal of height at least 2 such that R/I is of dense F-pure type. Let h denote the largest height of any minimal prime of I. Then, for all integers $n \geq 1$ and $k \geq 0$,*

$$I^{(hn+kn-1)} \subseteq (I^{(k+1)})^n.$$

In particular, if $h = 2$ then

$$I^{(3)} \subseteq I^2.$$

Proof. We use the standard descent theory of [HHu7, Chap. 2] to reduce the problem to the case of positive characteristic. This reduction step is essentially the same as that used in the proof of [HHu5, Thm. 4.4] and [HHu6, Thm. 4.2]. After reduction to characteristic $p > 0$, the assertion immediately follows from Theorem 3.3. □

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