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On Points at Infinity of Real Spectra of Polynomial Rings

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1. Introduction

Let R be a real closed field and let $z_0, \dots, z_n$ be independent variables. A basic fact in mathematics is the way the n-dimensional projective space $\operatorname{Proj} R[z_0, \dots, z_n]$ and other rational projective schemes such as $(\mathbb{P}^1_R)^n$ are glued together from affine charts of the form $\operatorname{Spec} R[x_1, \dots, x_n]$. Given two such coordinate charts $\operatorname{Spec} R[x_1, \dots, x_n]$ and $\operatorname{Spec} R[y_1, \dots, y_n]$, it is often easy to write down formulas describing the coordinate transformation from the x to the y coordinates. The subject of this paper is a part of the analogous story for real spectra (see Definition 1.1), which is more interesting because the real spectrum $\operatorname{Sper} R[x_1, \dots, x_n]$ already contains much information "at infinity".

To explain this in more detail, we first recall the definition of real spectrum and other related objects studied in this paper.

Notation and Conventions. All the rings we consider will be commutative with 1. "Order" will always mean "total order". Throughout this paper, A will stand for $R[x_1, \dots, x_n]$.

Let B be a ring. A point α in the real spectrum of B is, by definition, the data of a real prime ideal $\mathfrak{p}_\alpha$ of B (i.e., such that the quotient ring $B/\mathfrak{p}_\alpha$ admits an ordering) and an ordering $\le_\alpha$ of the ring $B/\mathfrak{p}_\alpha$ or, equivalently, of the field of fractions of $B/\mathfrak{p}_\alpha$.

Another way of defining the point α is as a homomorphism from B to a real closed field, where two homomorphisms are identified if they have the same kernel $\mathfrak{p}$ and induce the same total ordering on $B/\mathfrak{p}$.

The ideal $\mathfrak{p}_\alpha$ is called the *support* of α. The ordered quotient ring $(B/\mathfrak{p}_\alpha, \le_\alpha)$ is denoted by $B[\alpha]$ and its ordered field of fractions by $B(\alpha)$. Sometimes we write $\alpha = (\mathfrak{p}_\alpha, \le_\alpha)$.

Definition 1.1. The *real spectrum* of B, denoted by $\operatorname{Sper} B$, is the collection of all pairs $\alpha = (\mathfrak{p}_\alpha, \le_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of B and $\le_\alpha$ is a total ordering of $B/\mathfrak{p}_\alpha$.

For an element $f \in B$, we use $f(\alpha)$ to denote the natural image of f in $B[\alpha]$. When an order-theoretic statement involving $f(\alpha)$ is made, the reference is to the

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order $\leq_\alpha$. For example, the inequality $f(\alpha) > 0$ really means $f(\alpha) >_\alpha 0$. The notation $|f(\alpha)|$ will mean $f(\alpha)$ if $f(\alpha) \geq_\alpha 0$ and $-f(\alpha)$ if $f(\alpha) \leq_\alpha 0$. If B is an R-algebra then $B[\alpha]$ contains an order-isomorphic copy of R.

Two kinds of points occur in Sper B: finite points and points at infinity.

Definition 1.2. Let B be an R-algebra and α a point of Sper B. We say that α is *finite* if for each $f \in B$ there exists an $N \in R$ such that $|f(\alpha)| < N$. Otherwise, we say that α is a *point at infinity.*

Notation. The subset of Sper B consisting of all the finite points will be denoted by $\operatorname{Sper}^* B$.

Given any ordered domain D, let $\bar{D}$ denote the convex hull of D in its field of fractions $D_{(0)}$:

$$\bar{D} := \{f \in D_{(0)} \mid d > |f| \text{ for some } d \in D\}.$$

The ring $\bar{D}$ is a valuation ring because, for any element $f \in D_{(0)}$, either $f \in \bar{D}$ or $f^{-1} \in \bar{D}$. We define $R_\alpha := \overline{B[\alpha]}$. In this way, to every point $\alpha \in \operatorname{Sper} B$ we can canonically associate a valuation ν_α of $B(\alpha)$ determined by the valuation ring R_α (see Section 2 for more details). In other words, we have a canonical map $\operatorname{Sper} B \to \bigcup_{\mathfrak{p} \in \operatorname{Spec} B} S_{\mathfrak{p}}$, where $S_{\mathfrak{p}}$ denotes the Zariski–Riemann surface of the residue field $\kappa(\mathfrak{p})$.

The real spectrum Sper B is endowed with the *spectral* (or *Harrison*) *topology.* By definition, this topology has basic open sets of the form

$$U(f_1, \dots, f_k) = \{\alpha \mid f_1(\alpha) > 0, \ \dots, \ f_k(\alpha) > 0\}$$

with $f_1, \dots, f_k \in B$.

The purpose of this paper is to study the analogue of projectivization of the affine space Spec A by adding a divisor at infinity in the framework of real spectra. Unlike the Zariski spectrum, Sper A intrinsically contains much information at infinity, embodied precisely in its points at infinity. For example, Sper $R[x]$ has two points at infinity, δ_+ and δ_-. Thus Sper $R[x]$ naturally contains the two-point compactification $R \cup \{\delta_+, \delta_-\}$ of R. The points δ_+ and δ_- can also be viewed as finite points of Sper $R[1/x]$. More generally, take a point $\delta \in \operatorname{Sper} A$ at infinity. Then there exists a (nonunique) set $T \subset \{1, \dots, n\}$ with the following property. Let

$$y_j = x_j \qquad \text{if } \ j \in \{1, \dots, n\} \setminus T, \tag{1}$$

$$y_j = 1/x_j \quad \text{if } \ j \in T. \tag{2}$$

We require that $|y_j(\delta)| < N$ for some $N \in R$. Let $B = R[y_1, \dots, y_n]$. In Section 3 we will associate to δ a point δ^* in $\operatorname{Sper}^*(B)$. Furthermore, we will define a combinatorial invariant t_δ of points $\delta \in \operatorname{Sper} A$—that is, a mapping from Sper A to a certain finite set. This defines a partition

$$\operatorname{Sper}(A) = \coprod_t U_t \tag{3}$$

of Sper A into a finite disjoint union of sets U_t, where each U_t is defined to be the set of points δ of Sper A on which t_δ has constant value t. The first result, Proposition 3.1, describes a homeomorphism between each U_t and a certain subspace $U_t^* \subset \mathrm{Sper}^*(B)$. Let δ be a point in U_t and δ^* its image in U_t^*. The main theorems, Theorems 3.1 and 3.2, describe in detail the relation between the associated valuations ν_δ and ν_{δ^*}. Geometrically, a finite point in Sper A can be interpreted as a *semicurvette* (see Definition 2.2) centered at a point of R^n. The homeomorphism $U_t \cong U_t^*$ allows us to interpret points at infinity as semicurvettes centered at infinity.

This paper originally grew out of the authors' joint work with Madden [7] on the Pierce–Birkhoff conjecture. Certain definitions and constructions worked only for finite points of Sper A, so a need naturally arose to cover Sper A by subspaces, each of which is homeomorphic to a subspace of $\mathrm{Sper}^* B$ for some other polynomial ring B. Eventually, we found another way of getting around this difficulty and were able to deal in a uniform way with all points of Sper A whether finite or infinite. However, we hope that the decomposition (3) will later prove useful to someone who is faced with finiteness problems similar to ours.

2. The Valuation Associated to a Point in the Real Spectrum

CONVENTION. Given a valuation ν of a field K, we adopt the usual convention that $\nu(0) = \infty$, which is taken to be greater than any element of the value group of ν.

For a point α in Sper B, we have defined the valuation ν_α of $B(\alpha)$ to be the one whose valuation ring is

$$R_\alpha = \{x \in B(\alpha) \mid \exists z \in B[\alpha],\ |x| \leq_\alpha z\}.$$

The maximal ideal of R_α is $M_\alpha = \{x \in B(\alpha) \mid |x| < 1/|z|\ \forall z \in B[\alpha] \setminus \{0\}\}$; its residue field k_α comes equipped with a total ordering induced by $\leq_\alpha$. We will denote by Γ_α the value group of ν_α.

By definition, we have a natural ring homomorphism

$$B \to R_\alpha \tag{4}$$

whose kernel is $\mathfrak{p}_\alpha$. The valuation ν_α has the following properties:

- $\nu_\alpha(B[\alpha]) \geq 0$; and
- if B is an R-algebra then, for any positive elements $y, z \in B(\alpha)$,

$$\nu_\alpha(y) < \nu_\alpha(z) \implies y > Nz \quad \forall N \in R. \tag{5}$$

(An example at the end of the paper shows that the converse of (5) is not true in general.)

When B is an R-algebra, we have another valuation ring $\mathcal{N}_\alpha$ that is naturally associated to α—namely, the convex hull of R in $B(\alpha)$:

$$\mathcal{N}_\alpha = \{x \in B(\alpha) \mid \exists N \in R,\ |x| \leq_\alpha N\}. \tag{6}$$

We will call the corresponding valuation the *natural valuation associated to* α. If $\mathfrak{n}_\alpha$ is the maximal ideal of $\mathcal{N}_\alpha$, then

$$M_\alpha \subset \mathfrak{n}_\alpha \subset \mathcal{N}_\alpha \subset R_\alpha \subset B(\alpha). \tag{7}$$

REMARK 2.1. Let B be an R-algebra and take a point $\alpha \in \operatorname{Sper}^* B$ (see Definition 1.2). Then $R_\alpha = \mathcal{N}_\alpha$. Thus, for points $\alpha \in \operatorname{Sper}^* B$, the valuation ν_α of $B(\alpha)$ depends on the ordering $\leq_\alpha$ but not on the ring $B[\alpha]$.

Points of Sper B admit the following geometric interpretation. (We refer the reader to [3; 4; 8, p. 89; 9] for the construction and properties of generalized power series rings and fields.)

DEFINITION 2.1. Let k be a field and let Γ be an ordered abelian group. The *generalized formal power series ring* $k[[u^\Gamma]]$ is the ring formed by elements of the form $\sum_\gamma a_\gamma u^\gamma$ $(a_\gamma \in k)$ such that the set $\{\gamma \mid a_\gamma \neq 0\}$ is well-ordered.

The ring $k[[u^\Gamma]]$ is equipped with the natural u-adic valuation v with values in Γ defined by $v(f) = \inf\{\gamma \mid a_\gamma \neq 0\}$ for $f = \sum_\gamma a_\gamma u^\gamma \in k[[u^\Gamma]]$. Specifying both a total ordering on k and $\dim_{\mathbb{F}_2}(\Gamma/2\Gamma)$ sign conditions defines a total ordering on $k[[u^\Gamma]]$. In this ordering, $|u|$ is smaller than any positive element of k. For example, if $u^\gamma > 0$ for all $\gamma \in \Gamma$ then $f > 0$ if and only if $a_{v(f)} > 0$.

DEFINITION 2.2. Let k be an ordered field. A *k-curvette* on $\operatorname{Sper}(B)$ is a morphism of the form

$$\alpha\colon B \to k[[u^\Gamma]],$$

where Γ is an ordered group. A *k-semicurvette* is a k-curvette α together with a choice of the sign data $\operatorname{sgn} x_1, \dots, \operatorname{sgn} x_r$, where $x_1, \dots, x_r$ are elements of B whose t-adic values induce an $\mathbb{F}_2$-basis of $\Gamma/2\Gamma$.

Given an ordered field k, a k-semicurvette α determines a prime ideal $\mathfrak{p}_\alpha$ (the ideal of all the elements of B that vanish identically on α) and a total ordering on $B/\mathfrak{p}_\alpha$ induced by the ordering of the ring $k[[u^\Gamma]]$ of formal power series.

REMARK 2.2. Conversely, by using [9, Satz 21, p. 62] one can show that every finite point of $\operatorname{Sper}(A)$ can be represented by a semicurvette in this way.

We will sometimes describe points in the real spectrum by specifying the corresponding semicurvettes.

For a certain number $p \in \{0, 1, \dots, n-1\}$ and two points δ, δ^* living in different spaces, we shall need to compare $(n-p)$-tuples of elements such as $(\nu_\delta(x_{p+1}(\delta)), \dots, \nu_\delta(x_n(\delta))) \in \Gamma_\delta^{n-p}$ and $(\nu_{\delta^*}(y_{p+1}(\delta^*)), \dots, \nu_{\delta^*}(y_n(\delta^*))) \in \Gamma_{\delta^*}^{n-p}$ and also be able to say that they are in some sense "equivalent". To do this, we must embed Γ_δ in some "universal" ordered group.

NOTATION AND CONVENTION. Denote by Γ the ordered group $\mathbb{R}^n_{\text{lex}}$. This means that elements of Γ are compared as words in a dictionary: we say that $(a_1, \dots, a_n) < (a'_1, \dots, a'_n)$ if and only if there exists a $j \in \{1, \dots, n\}$ such that $a_q = a'_q$ for all $q < j$ and $a_j < a'_j$.

The reason for introducing Γ is that, by Abhyankar's inequality, rank $\nu_\delta \leq \dim A = n$ for all $\delta \in \operatorname{Sper} A$ and so the value group Γ_δ of ν_δ can be embedded into Γ as an ordered subgroup (of course, this embedding is far from being unique). Let Γ_+ be the semigroup of nonnegative elements of Γ.

Fix a strictly positive integer ℓ. In order to deal rigorously with ℓ-tuples of elements of Γ_δ despite the nonuniqueness of the embedding $\Gamma_\delta \subset \Gamma$, we introduce the category $\mathcal{OGM}(\ell)$ as follows. An object in $\mathcal{OGM}(\ell)$ is an ordered abelian group G together with a fixed *ℓ-tuple* of generators $(a_1, \dots, a_\ell)$; such an object will be denoted by $(G, (a_1, \dots, a_\ell))$. A morphism from $(G, (a_1, \dots, a_\ell))$ to $(G', (a'_1, \dots, a'_\ell))$ is a homomorphism $G \to G'$ of ordered groups that maps a_j to a'_j for each j.

Given $(G, (a_1, \dots, a_\ell)), (G', (a'_1, \dots, a'_\ell)) \in Ob(\mathcal{OGM}(\ell))$, the notation

$$(a_1, \dots, a_\ell) \underset{\circ}{\sim} (a'_1, \dots, a'_\ell) \tag{8}$$

will mean that $(G, (a_1, \dots, a_\ell))$ and $(G', (a'_1, \dots, a'_\ell))$ are isomorphic in $\mathcal{OGM}(\ell)$.

Take an ℓ-tuple

$$a = (a_1, \dots, a_\ell) \in \Gamma_+^\ell.$$

Let $G \subset \Gamma$ be the ordered group generated by $a_1, \dots, a_\ell$. Then $(G, (a_1, \dots, a_\ell)) \in Ob(\mathcal{OGM}(\ell))$. For each $\delta \in \operatorname{Sper}(A)$, let Γ_δ denote the value group of the associated valuation ν_δ and Γ^*_δ the subgroup of Γ_δ generated by $\nu_\delta(x_1(\delta)), \dots, \nu_\delta(x_n(\delta))$. In this way, we associate to δ an object $(\Gamma^*_\delta, \nu_\delta(x_1(\delta)), \dots, \nu_\delta(x_n(\delta))) \in Ob(\mathcal{OGM}(n))$.

Notation. Let Γ be an ordered group. Consider an ℓ-tuple $a = (a_1, \dots, a_\ell) \in \Gamma^\ell$. We denote by $\operatorname{Rel}(a)$ the set

$$\operatorname{Rel}(a) = \left\{ (m_1, \dots, m_\ell, m_{\ell+1}, \dots, m_{2\ell}) \in \mathbb{Z}^{2\ell} \;\middle|\; \sum_{j=1}^{\ell} m_j a_j > 0 \text{ and } \sum_{j=\ell+1}^{2\ell} m_j a_{j-\ell} = 0 \right\}.$$

Remark 2.3. Let a be as above and let G be the group generated by $a_1, \dots, a_\ell$, so that $(G, a) \in Ob(\mathcal{OGM}(\ell))$. There is a surjective group morphism $\pi_a \colon \mathbb{Z}^\ell \to G$, $\varepsilon_i \mapsto a_i$. If $m \in \mathbb{Z}^\ell$, then $\pi_a(m) = m \cdot a$. The data $\operatorname{Rel}(a) = \{z(a), p(a)\}$, with $z(a) = \{m \in \mathbb{Z}^\ell \mid m \cdot a = 0\}$ and $p(a) = \{m \in \mathbb{Z}^\ell \mid m \cdot a > 0\}$, determine the kernel of π_a and tell us what elements of $\mathbb{Z}^\ell$ have positive images in G. Thus $\operatorname{Rel}(a)$ determines the isomorphism class of (G, a) in $\mathcal{OGM}(\ell)$ and vice versa; the set $\operatorname{Rel}(a)$ and the isomorphism class of $(G, (a_1, \dots, a_\ell))$ are equivalent sets of data.

3. Points at Infinity of Sper (A)

In this section, we study the structure of the set of points at infinity in $\operatorname{Sper}(A)$. We express $\operatorname{Sper} A$ as a finite disjoint union of subsets, each of which is homeomorphic to a subset of the set $\operatorname{Sper}^* B$ of bounded points of another polynomial ring B. For a point $\delta \in \operatorname{Sper} A$, let δ^* denote the image of δ in $\operatorname{Sper}^* B$; by construction, we will then have $A(\delta) = B(\delta^*)$. We shall study in detail the relation between the valuations ν_{δ^*} and ν_δ of $A(\delta)$. Among other things, we show that R_δ is the localization of R_{δ^*} at a suitable prime ideal.

DEFINITION 3.1. Let $B = R[y_1, \dots, y_n]$. For $T \in \{0,1\}^n$, let

$$A_T = R[x_i \mid T(i) = 0][x_i^{\pm 1} \mid T(i) = 1].$$

Let $b_T : B \to A_T$ be defined by $b_T(y_i) = x_i^{-1}$ if $T(i) = 1$ and by $b_T(y_i) = x_i$ if $T(i) = 0$.

We have localization morphisms $B \xrightarrow{b_T} A_T \xleftarrow{a_T} A$ (where a_T is the natural embedding), and these morphisms induce injective maps of the real spectrum of A_T into the real spectra of A and of B in the following way: a point of the real spectrum of A_T is a homomorphism of A_T to an ordered field and, by composition, this induces points of the real spectra of A and B.

DEFINITION 3.2. Let $a_T^\# : \operatorname{Sper} A_T \to \operatorname{Sper} A$ denote the embedding induced by a_T. Its image is denoted by

$$(\operatorname{Sper} A)_T = \{\alpha \in \operatorname{Sper} A \mid x_i(\alpha) \neq 0 \text{ whenever } T(i) = 1\}. \tag{9}$$

Similarly, $b_T^\# : \operatorname{Sper} A_T \to \operatorname{Sper} B$ will denote the embedding induced by b_T. Its image is denoted by

$$(\operatorname{Sper} B)_T = \{\beta \in \operatorname{Sper} B \mid y_i(\beta) \neq 0 \text{ whenever } T(i) = 1\}. \tag{10}$$

Let $\gamma = b_T^\# \circ a_T^\# : (\operatorname{Sper} A)_T \to (\operatorname{Sper} B)_T$ be the induced homeomorphism.

Suppose R is a real closed field and S is any R-algebra. For any $\delta \in \operatorname{Sper} S$, let M_δ be the maximal ideal of R_δ. Let $\mathcal{N}_\delta$ be the natural valuation ring of $S(\delta)$ and let $\mathfrak{n}_\delta$ be the maximal ideal of $\mathcal{N}_\delta$. Recall the inclusions (7). Consider the four-element set consisting of formal symbols $\{\varepsilon, 1/\infty, 1, \infty\}$.

DEFINITION 3.3. For $a \in S$, define

$$\tau_\delta(a) = \begin{cases} \varepsilon & \text{if } a(\delta) \in M_\delta, \\ 1/\infty & \text{if } a(\delta) \in \mathfrak{n}_\delta \setminus M_\delta, \\ 1 & \text{if } a(\delta) \in \mathcal{N}_\delta \setminus \mathfrak{n}_\delta, \\ \infty & \text{if } a(\delta) \in R_\delta \setminus \mathcal{N}_\delta. \end{cases}$$

In other words, $\tau_\delta(a)$ is ε, $1/\infty$, 1, or ∞ according as $a(\delta)$ is a nonunit, an infinitely small unit, a unit comparable with an element of R, or an infinitely large unit in R_δ, respectively. Note that

$$\delta \text{ is finite} \iff R_\delta = \mathcal{N}_\delta \iff M_\delta = \mathfrak{n}_\delta \iff \{\tau_\delta(a) \mid a \in A\} \subseteq \{\varepsilon, 1\}.$$

In what follows, we will work with elements $t \in \{\varepsilon, 1/\infty, 1, \infty\}^n$, which we consider as maps $t : \{1, \dots, n\} \to \{\varepsilon, 1/\infty, 1, \infty\}^n$.

DEFINITION 3.4. For a point $\delta \in \operatorname{Sper} A$, let $t_\delta : \{1, \dots, n\} \to \{\varepsilon, 1/\infty, 1, \infty\}^n$ be the map defined by $t_\delta(i) = \tau_\delta(x_i)$. If $t \in \{\varepsilon, 1/\infty, 1, \infty\}^n$, we let $U_t = \{\delta \in \operatorname{Sper} A \mid t_\delta(i) = t(i)\}$.

Let $V \subset \{\varepsilon, 1/\infty, 1, \infty\}^n$ be given by

$$V = \{t \in \{\varepsilon, 1/\infty, 1, \infty\}^n \mid t^{-1}(1/\infty) \neq \emptyset \Rightarrow t^{-1}(\infty) \neq \emptyset\}.$$

REMARK 3.1. In Theorem 3.1(3) we will see that $U_t \neq \emptyset$ if and only if $t \in V$. Therefore,

$$\operatorname{Sper} A = \coprod_{t \in V} U_t. \tag{11}$$

Take an element $t \in V$. We shall consider subsets $T \in \{0,1\}^n$ satisfying the following conditions:

$$T(i) = 1 \quad \text{if } t(i) = \infty, \tag{12}$$

$$T(i) = 0 \quad \text{if } t(i) \in \{\varepsilon, 1/\infty\}. \tag{13}$$

PROPOSITION 3.1. *Assume that T satisfies* (12) *and* (13). *Then*:

(i) $U_t \subseteq (\operatorname{Sper} A)_T$;
(ii) $\gamma(U_t) \subset \operatorname{Sper}^* B$.

Proof. (i) Take a point $\delta \in U_t$. By (13), whenever $T(i) = 1$ we have $\tau_\delta(x_i) = t(i) \neq \varepsilon$. Hence $x_i(\delta) \neq 0$ and so $\delta \in (\operatorname{Sper} A)_T$ by (9).

(ii) This is equivalent to saying that there exists an $N \in R$ such that $|y_i(\delta)| < N$ for all $i \in \{1, \dots, n\}$—in other words, that $y_i \in \mathcal{N}_\delta$.

If $T(i) = 0$, then $t(i) \neq \infty$ by (12); hence $y_i(\delta) = x_i(\delta) \in \mathcal{N}_\delta$. If $T(i) = 1$ then, by (13), $t(i) \in \{1, \infty\}$ and so $y_i(\delta) = 1/x_i(\delta) \in \mathcal{N}_\delta$. This proves that $y_i \in \mathcal{N}_\delta$, as desired. □

Take a point $\delta \in U_t$. Take $T \in \{0,1\}^n$ satisfying (12) and (13), so that $\delta \in (\operatorname{Sper} A)_T$ by Proposition 3.1(i). Let B be as in Definition 3.1 and let $\delta^* = \gamma(\delta)$. Then the localization morphisms a_T and b_T induce a canonical isomorphism

$$\phi \colon A(\delta) \cong B(\delta^*) \tag{14}$$

of ordered fields.

Renumbering the coordinates if necessary, we may assume the existence of a p $(0 \le p \le n)$ such that

$$\begin{aligned} \nu_\delta(x_j(\delta)) &= 0 \quad \text{for } 1 \le i \le p \quad \text{and} \\ \nu_\delta(x_j(\delta)) &> 0 \quad \text{for } j > p. \end{aligned} \tag{15}$$

In other words,

$$\{1, \dots, p\} = t_\delta^{-1}(\{1/\infty, 1, \infty\}) \quad \text{and} \tag{16}$$

$$\{p+1, \dots, n\} = t_\delta^{-1}(\varepsilon). \tag{17}$$

We use $R_{>0}$ to denote the set of strictly positive elements of R.

THEOREM 3.1. *The valuation ν_{δ^*} of $B(\delta^*)$ associated to δ^* has the following properties.*

(1) $\nu_{\delta^*}(y_j(\delta^*)) = 0$ *for* $j \in t_\delta^{-1}(1)$.
(2) $\nu_{\delta^*}(y_j(\delta^*)) > 0$ *for* $j \in t_\delta^{-1}(\{1/\infty, \infty\})$.

(3) *Assume that $t_\delta^{-1}(1/\infty) \neq \emptyset$. Then there exist a $q \in t_\delta^{-1}(\infty)$ and a strictly positive integer N such that, for all $j \in t_\delta^{-1}(1/\infty)$,*

$$N\nu_{\delta^*}(y_q(\delta^*)) > \nu_{\delta^*}(y_j(\delta^*)). \tag{18}$$

In particular, if $t_\delta^{-1}(1/\infty) \neq \emptyset$ then $t_\delta^{-1}(\infty) \neq \emptyset$.

(4) *The valuation ring R_δ is the localization of R_{δ^*} at a prime ideal; this gives rise to a surjective order-preserving homomorphism $\tilde{\phi}\colon \Gamma_{\delta^*} \to \Gamma_\delta$ of value groups whose kernel is an isolated subgroup.*

(5) *For all $j \in \{1, \dots, n\}$, $\tilde{\phi}(\nu_{\delta^*}(y_j(\delta^*))) = \nu_\delta(x_j(\delta))$.*

(6) *For $j \in \{1, \dots, p\}$, $\nu_{\delta^*}(y_j(\delta^*)) \in \ker(\tilde{\phi})$. In particular: given any $j \in \{1, \dots, p\}$, $t \in \{p+1, \dots, n\}$, and $N' \in \mathbb{N}$, we have $N'\nu_{\delta^*}(y_j(\delta^*)) < \nu_{\delta^*}(y_t(\delta^*))$.*

(7) *Assume that $\nu_\delta(x_{p+1}(\delta)), \dots, \nu_\delta(x_n(\delta))$ are $\mathbb{Q}$-linearly independent. Then*

$$(\nu_{\delta^*}(y_{p+1}(\delta^*)), \dots, \nu_{\delta^*}(y_n(\delta^*))) \underset{\circ}{\sim} (\nu_\delta(x_{p+1}(\delta)), \dots, \nu_\delta(x_n(\delta))) \tag{19}$$

in $\mathcal{OGM}(n-p)$.

Proof. (1) Take $j \in t_\delta^{-1}(1)$. By definition this means that $\tau_\delta(x_j) \in \mathcal{N}_\delta \setminus \mathfrak{n}_\delta$, so $1/|y_j(\delta^*)| < c$ for some $c \in R$. Hence $1/y_j(\delta^*) \in R_{\delta^*}$ and the result follows.

(2) Take $j \in t_\delta^{-1}(\{1/\infty, \infty\})$. First, we show that

$$|y_j(\delta^*)| < c \quad \forall c \in R_{>0}. \tag{20}$$

Indeed, if $t_\delta(j) = 1/\infty$ then $T(j) = 0$ by (13), so $y_j(\delta^*) = x_j(\delta) \in \mathfrak{n}_\delta$ and (20) follows. If $t_\delta(j) = \infty$ then $T(j) = 1$ by (12) and so $y_j(\delta^*) = 1/x_j(\delta) \in \mathfrak{n}_\delta$; again, (20) follows. By (20), $1/|y_j(\delta^*)| > N$ for every $N \in R$. By the boundedness of δ^*, for each $f \in B(\delta^*)$ we have $|f| < N'$ for some $N' \in R$. Hence $1/|y_j(\delta^*)| > f$ for each $f \in B(\delta^*)$, so $1/y_j(\delta^*) \notin R_{\delta^*}$. This proves that $\nu_{\delta^*}(y_j(\delta^*)) > 0$.

(3) Take $j \in t_\delta^{-1}(1/\infty)$. Since $x_j(\delta) \in \mathfrak{n}_\delta \setminus M_\delta$, it follows that $1/x_j(\delta) \in R_\delta$. Since $R_\delta = \{w \in A(\delta) \mid \exists z \in A[\delta], |w| \leq_\delta z\}$ and since each $z \in A[\delta]$ is a sum of monomials in the $x_i(\delta)$, we can write R_δ as a union of intervals in $A(\delta)$:

$$R_\delta = \bigcup_{r \in R_{>0},\, m \in \mathbb{N},\, i=1,\dots,n} [-r|x_i(\delta)|^m, r|x_i(\delta)|^m]. \tag{21}$$

Hence there exists a $q \in \{1, \dots, n\}$ such that

$$\left|\frac{1}{x_j(\delta)}\right| < r|x_q(\delta)|^N \quad \text{for all } \ j \in t_\delta^{-1}(1/\infty). \tag{22}$$

Because $j \in t_\delta^{-1}(1/\infty)$, we have $|1/x_j(\delta)| > c$ for all $c \in R$. For $l \in t_\delta^{-1}(\{\varepsilon, 1/\infty, 1)\}$, we have $|x_l(\delta)| < C$ for some $C \in R$. Thus $q \in t_\delta^{-1}(\infty)$ in (22).

By (12) and (13), $T(j) = 0$ and $T(q) = 1$. Thus (22) becomes $|y_j(\delta^*)| > (1/r)|y_q(\delta^*)|^N$. Then $N\nu_{\delta^*}(y_q(\delta^*)) \geq \nu_{\delta^*}(y_j(\delta^*))$ by (5). Replacing N by $N+1$, we can make the inequality (18) strict.

(4) Since δ_* is finite, we have $R_{\delta^*} = \mathcal{N}_{\delta^*} = \mathcal{N}_\delta \subset R_\delta$. It is well known that every homomorphism between two valuation rings having the same field of fractions is a localization at a prime ideal.

The last statement of (4) follows from the general theory of composition of valuations [10, Chap. VI, Sec. 10, p. 43]. Recall that $\Gamma_\delta \cong (A(\delta) \setminus \{0\})/U(R_\delta)$ and that the valuation ν_δ can be identified with the natural homomorphism

$$A(\delta)\setminus\{0\} \to \frac{A(\delta)\setminus\{0\}}{U(R_\delta)}.$$

Similarly, ν_{δ^*} can be thought of as

$$B(\delta^*)\setminus\{0\} \to \frac{B(\delta^*)\setminus\{0\}}{U(R_{\delta^*})} \cong \Gamma_{\delta^*}.$$

From the isomorphism ϕ and the inclusion $R_{\delta^*} \hookrightarrow R_\delta$, we obtain a natural surjective homomorphism of ordered groups,

$$\tilde{\phi}\colon \frac{B(\delta^*)\setminus\{0\}}{U(R_{\delta^*})} \to \frac{A(\delta)\setminus\{0\}}{U(R_\delta)}. \tag{23}$$

(5) If $T(j) = 0$, then $\phi(x_j(\delta)) = y_j(\delta^*)$ implies that

$$\tilde{\phi}(y_j(\delta^*) \bmod U(R_{\delta^*})) = x_j(\delta) \bmod U(R_\delta).$$

If $T(j) = 1$ then $\phi(x_j(\delta)) = 1/y_j(\delta^*)$; hence

$$\tilde{\phi}(\nu_{\delta^*}(y_j(\delta^*))) = \nu_\delta(1/x_j(\delta)) = 0 = \nu_\delta(x_j(\delta)).$$

(6) This is an immediate consequence of (5) and the fact that $\nu_\delta(x_1(\delta)) = \cdots = \nu_\delta(x_p(\delta)) = 0$.

(7) By Remark 2.3, it suffices to prove that

$$\mathrm{Rel}(\nu_{\delta^*}(y_{p+1}(\delta^*)),\ldots,\nu_{\delta^*}(y_n(\delta^*))) = \mathrm{Rel}(\nu_\delta(x_{p+1}(\delta)),\ldots,\nu_\delta(x_n(\delta))). \tag{24}$$

Part (5) and the fact that $\nu_\delta(x_{p+1}(\delta)),\ldots,\nu_\delta(x_n(\delta))$ are $\mathbb{Q}$-linearly independent imply that so are $\nu_{\delta^*}(y_{p+1}(\delta^*)),\ldots,\nu_{\delta^*}(y_n(\delta^*))$. Hence, using Theorem 3.1(5) again, for any $(n-p)$-tuple $(m_{p+1},\ldots,m_n) \in \mathbb{Z}^{n-p}$ we have

$$\sum_{i=p+1}^{n} m_j\nu_\delta(x_j(\delta)) > 0 \iff \sum_{j=p+1}^{n} m_j\nu_{\delta^*}(y_j(\delta^*)) > 0.$$

This, together with the linear independence of $\nu_\delta(x_{p+1}(\delta)),\ldots,\nu_\delta(x_n(\delta))$ and of $\nu_{\delta^*}(y_{p+1}(\delta^*)),\ldots,\nu_{\delta^*}(y_n(\delta^*))$, proves the desired equality (24). □

Let G be an ordered group of rank r and let ℓ be a positive integer. Take ℓ elements $a_1,\ldots,a_\ell \in G$. Let $(0) = \Delta_r \subsetneqq \Delta_{r-1} \subsetneqq \cdots \subsetneqq \Delta_0 = G$ be the isolated subgroups of G.

Definition 3.5. We say that $a_1,\ldots,a_\ell$ are *scalewise $\mathbb{Q}$-linearly independent* if, for each $q \in \{0,\ldots,r-1\}$, the images in Δ_q/Δ_{q+1} of those a_i lying in $\Delta_q \setminus \Delta_{q+1}$ are $\mathbb{Q}$-linearly independent.

Remark 3.2. With notation as before, assume that $a_1,\ldots,a_\ell$ are scalewise $\mathbb{Q}$-linearly independent. Let $\lambda\colon G \to G'$ be a homomorphism of ordered groups. Then $\lambda(a_1),\ldots,\lambda(a_\ell)$ are scalewise $\mathbb{Q}$-linearly independent if and only if they are $\mathbb{Q}$-linearly independent if and only if all of them are nonzero. This is precisely the form in which we will use scalewise $\mathbb{Q}$-linear independence hereafter.

Take elements $t \in V$ and $T \in \{0,1\}^n$ satisfying (12) and (13). Let the notation be as before. Renumbering the variables if necessary, we may assume that there exists a $p \in \{0, \dots, n\}$ such that

$$\{1, \dots, p\} = t^{-1}(\{1/\infty, 1, \infty\}) \quad \text{and} \tag{25}$$

$$\{p+1, \dots, n\} = t^{-1}(\{\varepsilon\}). \tag{26}$$

Next, we prove a partial converse to Theorem 3.1 as follows.

THEOREM 3.2. *Take a point $\delta^* \in \operatorname{Sper}^* B \cap (\operatorname{Sper} B)_T$. Assume that the following conditions hold.*

(1) *For each $j \in t^{-1}(1)$, there exists a $c \in R$ such that $|y_j(\delta^*)| > c$.*
(2) *$\nu_{\delta^*}(y_j(\delta^*)) > 0$ for all $j \in t^{-1}(\{1/\infty, \infty\})$.*
(3) *If $t^{-1}(1/\infty) \neq \emptyset$, there exist $q \in t^{-1}(\infty)$ and $N \in \mathbb{N}$ such that $N\nu_{\delta^*}(y_q(\delta^*)) > \nu_{\delta^*}(y_j(\delta^*))$ for all $j \in t^{-1}(1/\infty)$.*
(4) *For all $j \in \{1, \dots, p\}$, $l \in \{p+1, \dots, n\}$, and $N' \in \mathbb{N}$, we have $N'\nu_{\delta^*}(y_j(\delta^*)) < \nu_{\delta^*}(y_l(\delta^*))$.*

Then $\delta^* \in \gamma(U_t)$. If, in addition, $\nu_{\delta^*}(y_{p+1}(\delta^*)), \dots, \nu_{\delta^*}(y_n(\delta^*))$ are scalewise $\mathbb{Q}$-linearly independent, then we have the isomorphism (19).

Proof. Let $\delta = \gamma^{-1}(\delta^*)$. We must show that $\delta \in U_t$—in other words, that

$$\tau_\delta(x_i) = t(i) \quad \text{for all } \ i \in \{1, \dots, n\}. \tag{27}$$

We check (27) case by case for all possible values of $t(i)$.

First of all, assumption (2) implies that

$$|y_i(\delta^*)| < c \quad \text{for all } \ c \in R_{>0} \ \text{ and } \ i \in t^{-1}(\{1/\infty, \infty\}). \tag{28}$$

Now, if $t(i) = \infty$ then by (12) we have $x_j(\delta) = 1/y_j(\delta^*)$, so

$$|x_i(\delta)| > N \quad \text{for all } \ N \in R, \tag{29}$$

which proves (27) for $t(i) = \infty$.

If $t(i) = 1$, then $x_i(\delta) = y_i(\delta^*)$ or $x_i(\delta) = 1/y_i(\delta^*)$. In either case, assumption (1) together with the boundedness of δ^* implies the existence of $c_1, c_2 \in R_{>0}$ such that

$$c_1 < |x_i(\delta)| < c_2, \tag{30}$$

which proves (27) for $t(i) = 1$.

If $t(i) = 1/\infty$ then, by (13), we have $x_i(\delta) = y_i(\delta^*)$. Then (28) yields

$$|x_i(\delta)| < c \quad \text{for all } \ c \in R_{>0}. \tag{31}$$

It remains to prove that

$$\nu_\delta(x_i(\delta)) = 0 \quad \text{if } \ t(i) = 1/\infty \tag{32}$$

and

$$\nu_\delta(x_i(\delta)) > 0 \quad \text{if } \ t(i) = \varepsilon. \tag{33}$$

Equation (32) is equivalent to saying that

$$1/x_i(\delta) \in R_\delta \quad \text{if } \ t(i) = 1/\infty. \tag{34}$$

To see this, we use assumption (3). The existence of $q \in t^{-1}(\infty)$ and a positive $N \in \mathbb{N}$ such that $N\nu_{\delta^*}(y_q(\delta^*)) > \nu_{\delta^*}(y_i(\delta^*))$ implies that $|y_i(\delta^*)| > |y_q(\delta^*)|^N$ by the implication (5); in other words, $|x_i(\delta)| > 1/|x_q(\delta)|^N$ or, equivalently, $1/|x_i(\delta)| < |x_q(\delta)|^N$. This proves (34) and hence (27) for $i \in t^{-1}(1/\infty)$.

Next, assume $t(i) = \varepsilon$. To prove (33), it suffices to show that

$$1/x_i(\delta) \notin R_\delta. \tag{35}$$

By (21), this is equivalent to saying that $1/|x_i(\delta)|$ is not bounded above by any element of the form cx_j^N with $j \in \{1, \dots, n\}$, $N \in \mathbb{N}$, and $c \in R$. We prove this last statement by contradiction. Suppose there were an inequality of the form

$$1/|x_i(\delta)| < cx_j^N \tag{36}$$

with $N \in \mathbb{N}$, $c \in R$, and $j \in \{1, \dots, n\}$. Since $\nu_{\delta^*}(y_i(\delta^*)) > 0$ by conditions (2) and (4) of Theorem 3.2, we have $|y_i(\delta^*)| < c$ for all positive $c \in R$. Since $x_i(\delta) = y_i(\delta^*)$ by (13), it follows that $|x_i(\delta)| < c$ and $1/|x_i(\delta)| > 1/c$ for all positive $c \in R$. On the other hand, if $t(j) \in \{\varepsilon, 1/\infty\}$ we have $\nu_{\delta^*}(y_j(\delta^*)) > 0$. Hence $|x_j(\delta)| = |y_j(\delta^*)| < \theta$ for all positive $\theta \in R$ and, if $t(j) = 1$, then $|x_j(\delta)|$ is bounded above by a constant from R by (30). This proves that $t(j) = \infty$ in (36).

Assumption (4) now implies that, for any constant $d \in R$ and any $N' \in \mathbb{N}$, we have $d|y_j(\delta^*)|^{N'} > |y_i(\delta^*)|$ and so $d/|x_j(\delta)|^{N'} > |x_i(\delta)|$, which contradicts (36). This completes the proof of (35), (33), and (27).

Assume that $\nu_{\delta^*}(y_{p+1}(\delta^*)), \dots, \nu_{\delta^*}(y_n(\delta^*))$ are scalewise $\mathbb{Q}$-linearly independent. It remains to prove the isomorphism

$$(\nu_\delta(x_{p+1}(\delta)), \dots, \nu_\delta(x_n(\delta))) \underset{\circ}{\sim} (\nu_{\delta^*}(y_{p+1}(\delta^*)), \dots, \nu_{\delta^*}(y_n(\delta^*))). \tag{37}$$

By Theorem 3.1(5), inequality (33), the assumed scalewise $\mathbb{Q}$-linear independence of $\nu_{\delta^*}(y_{p+1}(\delta^*)), \dots, \nu_{\delta^*}(y_n(\delta^*))$, and Remark 3.2, it follows that $\nu_\delta(x_{p+1}(\delta)), \dots, \nu_\delta(x_n(\delta))$ are also scalewise $\mathbb{Q}$-linearly independent. Now (37) is a consequence of Theorem 3.1(7), and the theorem is proved. □

Remark 3.3. Although at first glance the assumption of (scalewise) $\mathbb{Q}$-linear independence seems rather restrictive, we remark that any point $\delta \in \operatorname{Sper} A$ can be transformed into one for which this assumption holds by a sequence of blow-ups. For details we refer the reader to [7, Cor. 6.2], which shows how to achieve the usual $\mathbb{Q}$-linear independence of $\nu_\delta(x_{p+1}(\delta)), \dots, \nu_\delta(x_{p+1}(\delta))$—but it also works (after some minor and obvious modifications) for scalewise $\mathbb{Q}$-linear independence.

Example. Let $n = 5$, and let $\delta \in \operatorname{Sper} A$ be the point given by the following semicurvette. We let $\Gamma = \mathbb{Z}^2_{\text{lex}}$ and $k_\delta = R(z, w)$, where z and w are independent variables. Let the order on k_δ be given by the following inequalities:

$$0 < w < c < z \quad \text{for all } \ c \in R_{>0};$$

$$1/w^N < z \quad \text{for all } \ N \in \mathbb{N}.$$

As usual, we define the order on $k_\delta((u^\Gamma))$ by declaring u to be positive. Define the map $\delta\colon A \to k_\delta((u^\Gamma))$ as follows:

$$\begin{aligned}\delta(x_1) &= w,\\ \delta(x_2) &= 1 + u^{(0,1)},\\ \delta(x_3) &= z,\\ \delta(x_4) &= u^{(1,0)},\\ \delta(x_5) &= zu^{(1,0)}.\end{aligned}$$

Then $\nu_\delta(x_1) = \nu_\delta(x_2) = \nu_\delta(x_3) = 0$ and

$$\nu_\delta(x_4) = \nu_\delta(x_5) = (1,0) > 0. \tag{38}$$

We have $t_\delta^{-1}(1/\infty) = \{1\}$, $t_\delta^{-1}(1) = \{2\}$, and $t_\delta^{-1}(\infty) = \{3\}$. Let $T(i) = 1$ whenever $i \in t_\delta^{-1}(\infty)$ and $T(i) = 0$ otherwise. Let $\delta^* = \psi(\delta) \in \operatorname{Sper}^* B$. Then $\Gamma_{\delta^*} = \mathbb{Z}^4_{\text{lex}}$ and $k_{\delta^*} = R$. The semicurvette δ^* can be defined by the following map:

$$\begin{aligned}\delta^*(y_1) &= u^{(0,0,0,1)},\\ \delta^*(y_2) &= 1 + u^{(0,1,0,0)},\\ \delta^*(y_3) &= u^{(0,0,1,0)},\\ \delta^*(y_4) &= u^{(1,0,0,0)},\\ \delta^*(y_5) &= u^{(1,0,1,0)}.\end{aligned}$$

In this example, $\nu_\delta(x_4(\delta))$ and $\nu_\delta(x_5(\delta))$ are not $\mathbb{Q}$-linearly independent (cf. (38)) and the conclusion of Theorem 3.1 does not hold: we do *not* have an isomorphism

$$(\nu_{\delta^*}(y_4(\delta^*)), \nu_{\delta^*}(y_5(\delta^*))) \underset{\circ}{\sim} (\nu_\delta(x_4(\delta)), \nu_\delta(x_5(\delta))).$$

In fact, every point $\delta \in \operatorname{Sper} A$ can be transformed—after a sequence $\operatorname{Sper} A' \to \operatorname{Sper} A$ of affine monomial blow-ups with respect to δ—into a point $\delta' \in \operatorname{Sper} A'$ such that the nonzero elements of the set $\{\nu_{\delta'}(x_1), \dots, \nu_{\delta'}(x_n)\}$ are (scalewise) $\mathbb{Q}$-linearly independent [7, Cor. 6.2].

Let $A' = R[x'_1, x'_2, x'_3, x'_4, x'_5]$, and consider the map $\pi : A \to A'$ defined by:

$$\begin{aligned}\pi(x_j) &= x'_j \quad \text{for } j \in \{1,2,3,4\};\\ \pi(x_5) &= x'_4 x'_5.\end{aligned}$$

Let δ' be the unique preimage of δ under the natural map $\pi^* : \operatorname{Sper} A' \to \operatorname{Sper} A$ of the real spectra induced by π. (In the terminology of [7], π is an affine monomial blow-up along the ideal (x_4, x_5) with respect to δ, and δ' is the transform of δ by π.) In particular: $\Gamma_{\delta'} = \mathbb{Z}^2_{\text{lex}}$; $k_{\delta'} = R(z,w)$, as before; and δ is given by the following semicurvette:

$$\begin{aligned}\delta(x_1) &= w,\\ \delta(x_2) &= 1 + u^{(0,1)},\\ \delta(x_3) &= z,\\ \delta(x_4) &= u^{(1,0)},\\ \delta(x_5) &= z.\end{aligned}$$

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