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A Vanishing Theorem for Finitely Supported Ideals in Regular Local Rings

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To Mel Hochster, on the occasion of his 65th birthday

Introduction

In [L5, p. 747, (b)] there is a vanishing conjecture for an ideal I in a d-dimensional regular local ring $(R,\mathfrak{m})$. (A stronger "CM" conjecture on that page was disproved by Hyry [Hy, p. 389, Ex. 3.6].) Suppose there is a map $f\colon X \to \operatorname{Spec}(R)$ that factors as a finite sequence of blowups with smooth centers and is such that $I\mathcal{O}_X$ is invertible. Let E be the closed fiber $f^{-1}\{\mathfrak{m}\}$. The conjecture is that

$$\mathrm{H}^i_E(X,(I\mathcal{O}_X)^{-1}) = 0 \quad \text{for all } i \neq d.$$

This statement implies, with $\ell(I)$ denoting the analytic spread of I and $\sim$ denoting "adjoint ideal of" (a.k.a. multiplier ideal with exponent 1), that

$$\widetilde{I^{n+1}} = I\widetilde{I^n} \quad \text{for all } n \geq \ell(I)-1,$$

which in turn implies a number of "Briançon-Skoda with coefficients" results; see [L5, pp. 745–746]. The conjectured statement holds true when $d=2$, and it was proved by Cutkosky [Cu] for R essentially of finite type over a field of characteristic 0 (in which case it is closely related to vanishing theorems in the theory of multiplier ideals; see [La]). In these two situations, the assumed principalization f is known to exist for any $I \neq (0)$.

In this paper we show that vanishing holds for those R-ideals that are *finitely supported*—in other words, those for which there is a sequence of blowups (as before) in which all the centers are closed points.

In addition, we deduce that the adjoint ideal of a finitely supported ideal I is itself finitely supported, with point basis obtained by subtracting $\min(d-1, r_\beta)$ componentwise from the point basis (r_β) of I. (The terminology is explained in Sec. 3.)

More consequences of vanishing are scattered throughout Sections 3–4. For example, for finitely supported I, Proposition 3.4 generalizes the $\widetilde{I^{n+1}} = I\widetilde{I^n}$ relation; furthermore, if I is the integral closure $\bar{J}$ of a d-generated ideal J—whence $J\widetilde{I^{d-1}} = \widetilde{I^d}$—then Proposition 4.2 gives that $J\widetilde{I^{d-2}} = \widetilde{I^{d-1}} \cap J \neq \widetilde{I^{d-1}}$ (unless

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$I = R$) and that $J : I = \widetilde{J^{d-1}} + J = \widetilde{I^{d-1}} + J$. Moreover, for $1 \le t \le d$ we have $J\widetilde{J^{t-1}} = \widetilde{J^t}$ if and only if $t > d(1 - 1/\mathrm{ord}_\alpha(J))$.

1. Reformulation of Vanishing

Let K be a field. We denote by Greek letters $\alpha, \beta, \gamma, \ldots$ regular local rings of dimension ≥ 2 and with fraction field K; we refer to such objects as "points".

From now on α will be a d-dimensional point with maximal ideal $\mathfrak{m}_\alpha$, and $f : X \to \mathrm{Spec}(\alpha)$ will be a proper birational map with X regular (i.e., the local ring $\mathcal{O}_{X,x}$ is regular for every $x \in X$).

Let $E_1, E_2, \ldots, E_r$ be the $(d-1)$-dimensional reduced irreducible components of the closed fiber $E := f^{-1}\{\mathfrak{m}_\alpha\}$. The local ring on X of the generic point of E_i is a discrete valuation ring R_i whose corresponding valuation we denote by v_i. Because the regular ring α is universally catenary [GD2, (5.6.4)], the residue field of R_i has transcendence degree $d-1$ over $\alpha/\mathfrak{m}_\alpha$. There is then a unique point β_i infinitely near to α such that v_i is the order valuation ord_{β_i} associated with β_i; see [L2, Sec. 1, pp. 204, 208]. (The *first neighborhood* of α consists of all points of the form $\mathcal{O}_{Z,z}$, where $\varphi : Z \to \mathrm{Spec}(\alpha)$ is the blowup of $\mathfrak{m}_\alpha$ and $z \in \varphi^{-1}\{\mathfrak{m}_\alpha\}$; a point β is *infinitely near to* α if there is a finite sequence of points beginning with α, ending with β, and such that each member other than α is in the first neighborhood of the preceding member.)

We say that a point β' is *proximate to* another point β'', and write $\beta' \succ \beta''$, when β' is infinitely near to β'' and the valuation ring of $\mathrm{ord}_{\beta''}$ is the localization of β' at a height-1 prime ideal. For each i, j such that $\beta_i \succ \beta_j$, let $\mathfrak{p}_{ij}$ be the height-1 prime ideal in β_i such that the localization $(\beta_i)_{\mathfrak{p}_{ij}}$ is the valuation ring R_j of v_j. Using induction on the length of the blowup sequence from β_j to β_i, one checks that $v_i(\mathfrak{p}_{ij}) = 1$.

LEMMA 1.1. *Let I be a nonzero α-ideal. Then, for each $i = 1, 2, \ldots, r$,*

$$v_i(I) \ge \sum_{\{j \mid \beta_j \prec \beta_i\}} v_j(I).$$

By convention, the sum of the empty family of integers is 0.

Proof. After reindexing, we may assume that $\beta_1, \beta_2, \ldots, \beta_s$ are all the β_j such that $\beta_j \prec \beta_i$. Then use that for some β_i-ideal I_i we have $I\beta_i = \mathfrak{p}_{i1}^{v_1(I)} \cdots \mathfrak{p}_{is}^{v_s(I)} I_i$. □

DEFINITION 1.2. A divisor $\sum_{i=1}^r n_i E_i$ is *full* if, for each i, it holds that $n_i \ge 0$ and that (with the preceding notation)

$$n_i \ge \sum_{\{j \mid \beta_j \prec \beta_i\}} n_j.$$

EXAMPLES 1.2.2. (a) For any nonzero α-ideal I, the divisor $\sum_{i=1}^r v_i(I) E_i$ is full.

(b) Any finite sum of full divisors is full.

(c) If $D = \sum_i n_i E_i$ is full and $0 \le c \in \mathbb{R}$, then $\lfloor cD \rfloor := \sum_i \lfloor cn_i \rfloor E_i$ is full. (As usual, for any $\rho \in \mathbb{R}$, $\lfloor \rho \rfloor$ denotes the greatest integer $\le \rho$.)

Conjecture 1.3. *If $D = \sum_{i=1}^r n_i E_i$ is a full divisor, then*

$$\mathrm{H}^i_E(X, \mathcal{O}_X(D)) = 0 \quad \textit{for all } i \neq d.$$

This holds, obviously, when $i \le 0$ or $i > d$.

We assume henceforth that f is a composition

$$X = X_n \to X_{n-1} \to \cdots \to X_0 = \mathrm{Spec}(\alpha), \tag{1.3.1}$$

where each $X_{i+1} \to X_i$ $(i < n)$ is the blowup of a regular closed subscheme of X_i.

Example 1.3.2. For f as in (1.3.1), the conjecture holds when $D = 0$, in which case it is usually referred to as (an instance of) Grauert–Riemenschneider vanishing.

Indeed, for this to hold, [L4, p. 153, Lemma 4.2] shows it is enough that the natural derived-category map $\tau : \alpha \to \mathbf{R}\Gamma(X, \mathcal{O}_X)$ be an isomorphism; and a straightforward induction, using the natural isomorphism $\mathbf{R}\Gamma(X, \mathcal{O}_X) \cong \mathbf{R}\Gamma(Z, \mathbf{R}h_*\mathcal{O}_X)$ associated to a suitable factorization of f as $X \xrightarrow{h} Z \xrightarrow{g} Y$, reduces proving that τ is an isomorphism to the case of a single blowup, where it follows from [GD1, (2.1.14) and (4.2.1)] (since the fibers of τ are single points or projective spaces) or from [L4, Thms. 4.1 and 5] (since regular local rings are pseudo-rational [LT, Sec. 4]).

Set $U := \mathrm{Spec}(\alpha) - \{\mathfrak{m}_\alpha\}$ and $V := f^{-1}U$. From Example 1.3.2 one gets a natural isomorphism $\mathcal{O}_U \xrightarrow{\sim} \mathbf{R}f_*\mathcal{O}_V$, whence $\mathrm{H}^i(V, \mathcal{O}_V) \cong \mathrm{H}^i(U, \mathcal{O}_U)$ for all i. But $\mathrm{H}^0(U, \mathcal{O}_U) \cong \alpha$, and for $0 < i < d-1$ we have $\mathrm{H}^i(U, \mathcal{O}_U) \cong \mathrm{H}^{i+1}_{\mathfrak{m}_\alpha}(\alpha) = 0$. Hence, for $D := \sum_{i=1}^r n_i E_i$ with $n_i \ge 0$ (so that $\mathcal{O}_X(D)|_V = \mathcal{O}_V$ and $H^0(\mathcal{O}_X(D)) = \alpha$), the natural exact sequences

$$\begin{aligned}\mathrm{H}^{i-1}(X, \mathcal{O}_X(D)) \longrightarrow \mathrm{H}^{i-1}(V, \mathcal{O}_V) \longrightarrow \mathrm{H}^i_E(X, \mathcal{O}_X(D))& \\ \xrightarrow{\psi^i} \mathrm{H}^i(X, \mathcal{O}_X(D)) \to \mathrm{H}^i(V, \mathcal{O}_V)&\end{aligned}$$

show that ψ^i is an isomorphism for $0 < i < d-1$ and that ψ^{d-1} is injective.

Furthermore, if $\mathfrak{m}_\alpha\mathcal{O}_X$ is invertible and if we take the harmless liberty of identifying the closed fiber E with the corresponding divisor, so that $\mathfrak{m}_\alpha\mathcal{O}_X = \mathcal{O}_X(-E)$, then by applying $\varinjlim_n$ to the exact row of the natural diagram

$$\begin{array}{ccccc}\mathrm{Ext}^{d-1}(\mathcal{O}_{nE}, \mathcal{O}_X(D)) & \longrightarrow & \mathrm{Ext}^{d-1}(\mathcal{O}_X, \mathcal{O}_X(D)) & \longrightarrow & \mathrm{Ext}^{d-1}(\mathcal{O}_X(-nE), \mathcal{O}_X(D)) \\ & & \simeq\downarrow & & \downarrow\simeq \\ & & \mathrm{H}^{d-1}(X, \mathcal{O}_X(D)) & & \mathrm{H}^{d-1}(X, \mathcal{O}_X(D+nE))\end{array}$$

we deduce a natural exact sequence

$$0 \to \mathrm{H}^{d-1}_E(X, \mathcal{O}_X(D)) \xrightarrow{\psi} \mathrm{H}^{d-1}(X, \mathcal{O}_X(D)) \to \varinjlim_n \mathrm{H}^{d-1}(X, \mathcal{O}_X(D+nE)),$$

where, one verifies, ψ is the injective map ψ^{d-1} described previously.

Thus for f as in (1.3.1) such that, further, $\mathfrak{m}_\alpha \mathcal{O}_X = \mathcal{O}_X(-E)$ is invertible, Conjecture 1.3 may be written as follows.

CONJECTURE 1.4. *If $D = \sum_{i=1}^{r} n_i E_i$ is a full divisor then*

$$\mathrm{H}^i(X, \mathcal{O}_X(D)) = 0 \quad \textit{for } 0 < i < d-1.$$

For all $n > 0$, the natural map is an injection

$$\mathrm{H}^{d-1}(X, \mathcal{O}_X(D)) \hookrightarrow \mathrm{H}^{d-1}(X, \mathcal{O}_X(D + nE)).$$

2. A Special Case

We prove Conjectures 1.3 and 1.4 in a special case.

THEOREM 2.1. *With α as before, suppose the map $f: X \to \operatorname{Spec}(\alpha)$ factors as*

$$X = X_r \to X_{r-1} \to \cdots \to X_0 = \operatorname{Spec}(\alpha) \quad (r > 0),$$

where for $0 \le i < r$ the map $X_{i+1} \to X_i$ is the blowup of a closed point of X_i. Then Conjecture 1.4—and thus Conjecture 1.3—holds true.

Proof. We proceed by induction on r. We often write $\mathrm{H}^i(–)$ for $\mathrm{H}^i(X, –)$.

Suppose $r = 1$, so that (with our preceding notation) $E = E_1$ and $D = n_1 E$, $n_1 \ge 0$. For any $q \ge 0$ there is a standard exact sequence with $\mathcal{O}_E(mE) := \mathcal{O}_E \otimes \mathcal{O}_X(mE)$:

$$0 \to \mathcal{O}_X(qE) \to \mathcal{O}_X((q+1)E) \to \mathcal{O}_E((q+1)E) \to 0.$$

Here $E \cong \mathbb{P}^{d-1}$, the $(d-1)$-dimensional projective space over the field $\alpha/\mathfrak{m}_\alpha$, and $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-1)$; hence $\mathrm{H}^i(\mathcal{O}_E((q+1)E)) = 0$ for $i < d-1$. Thus, for $0 < i < d-1$, there are natural isomorphisms

$$\mathrm{H}^i(\mathcal{O}_X(qE)) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{O}_X((q+1)E)),$$

and since (by Example 1.3.2) $\mathrm{H}^i(\mathcal{O}_X) = 0$ it follows that $\mathrm{H}^i(\mathcal{O}_X(n_1 E)) = 0$.

Moreover, for every q the natural map

$$\mathrm{H}^{d-1}(\mathcal{O}_X(qE)) \to \mathrm{H}^{d-1}(\mathcal{O}_X((q+1)E)) = \mathrm{H}^{d-1}(\mathcal{O}_X(qE + E))$$

is injective, whence so is $\mathrm{H}^{d-1}(\mathcal{O}_X(n_1 E)) \to \mathrm{H}^{d-1}(\mathcal{O}_X(n_1 E + nE))$.

Next, when $r > 1$, let $g: Y \to \operatorname{Spec}(\alpha)$ be the composition of $r-1$ closed-point blowups and let $h: X \to Y$ be the blowup of a closed point $y \in Y$. Make the indexing such that E_1 is the closed fiber of h. With v_i as in Section 1 and $2 \le i \le r$, let E'_i be the center of v_i on Y. Arrange further that $E'_2, \dots, E'_s$ are all of the E'_i that pass through y. Fullness of $D = \sum_{i=1}^{r} n_i E_i$ entails $n_1 \ge n_2 + \cdots + n_s$.

Let $D' := n_2 E'_2 + \cdots + n_r E'_r$ and let $h^{-1}D'$ be the divisor

$$h^{-1}D' := (n_2 + \cdots + n_s)E_1 + n_2 E_2 + \cdots + n_r E_r,$$

so that $\mathcal{O}_X(h^{-1}D') = h^*\mathcal{O}_Y(D')$. Fullness of D' follows from that of D because, for $i > 1$, β_i is not proximate to β_1. By induction, then, Conjecture 1.4 holds for D', and it follows that Conjecture 1.4 holds also for $h^{-1}D'$. Indeed, since $\mathbf{R}h_*\mathcal{O}_X = \mathcal{O}_Y$ (cf. Example 1.3.2), the standard projection isomorphism gives

$$\begin{aligned}\mathbf{R}\Gamma(X, \mathcal{O}_X(h^{-1}D')) &= \mathbf{R}\Gamma(Y, \mathbf{R}h_*(\mathcal{O}_X \otimes h^*\mathcal{O}_Y(D')))\\ &\cong \mathbf{R}\Gamma(Y, \mathbf{R}h_*(\mathcal{O}_X) \otimes \mathcal{O}_Y(D')) = \mathbf{R}\Gamma(Y, \mathcal{O}_Y(D')),\end{aligned}$$

and similarly for the full divisor $D' + nE'$, where E' is the full divisor such that $\mathcal{O}_Y(-E') = \mathfrak{m}_\alpha\mathcal{O}_Y$ (see Example 1.2.2(a)), so $h^{-1}(D' + nE') = h^{-1}(D') + nE$. As a result,

$$\mathrm{H}^i(X, \mathcal{O}_X(h^{-1}D')) \cong \mathrm{H}^i(Y, \mathcal{O}_Y(D')) = 0 \quad (0 < i < d-1),$$

and the natural map

$$\mathrm{H}^{d-1}(X, \mathcal{O}_X(h^{-1}D')) \to \mathrm{H}^{d-1}(X, \mathcal{O}_X(h^{-1}D' + nE))$$

is isomorphic to the natural injection

$$\mathrm{H}^{d-1}(Y, \mathcal{O}_Y(D')) \hookrightarrow \mathrm{H}^{d-1}(Y, \mathcal{O}_Y(D' + nE')).$$

It will therefore be enough to show the following lemma.

Lemma 2.2. *If Conjecture 1.4 holds for a divisor* $D_\nu := \nu E_1 + n_2E_2 + \cdots + n_rE_r$, *where* $\nu \geq n_2 + \cdots + n_s$, *then it holds also for* $D_{\nu+1}$.

Proof. Denote the residue field of y by $\kappa(y)$, so that $E_1 \cong \mathbb{P}^{d-1}_{\kappa(y)}$. For any $n \geq 0$, there is the usual exact sequence

$$0 \to \mathcal{O}_X(D_\nu + nE) \to \mathcal{O}_X(D_{\nu+1} + nE) \to \mathcal{O}_{E_1} \otimes \mathcal{O}_X(D_{\nu+1} + nE) \to 0.$$

Moreover, with $N := n_2 + \cdots + n_s - \nu - 1$,

$$\mathcal{O}_{E_1} \otimes \mathcal{O}_X(D_{\nu+1} + nE) \cong \mathcal{O}_{E_1}(N).$$

To see this, note (with $\lambda\colon E_1 \hookrightarrow X$ the inclusion) that $\lambda^*\mathcal{O}_X(E_1) \cong \mathcal{O}_{E_1}(-1)$; that, since E_j' is a regular subscheme of Y passing through y, if $2 \leq j \leq s$ then

$$\lambda^*\mathcal{O}_X(E_j) \cong \lambda^*(h^*\mathcal{O}_Y(E_j') \otimes \mathcal{O}_X(-E_1)) \cong \lambda^*(\mathcal{O}_X(-E_1)) \cong \mathcal{O}_{E_1}(1);$$

that if $j > s$ then $\lambda^*\mathcal{O}_X(E_j) = \mathcal{O}_{E_1}$; and that, since $\mathcal{O}_Y(E')|_U \cong \mathcal{O}_U$ for some open U containing y (with E' as before), $\lambda^*\mathcal{O}_X(E) \cong \lambda^*h^*\mathcal{O}_Y(E') \cong \mathcal{O}_{E_1}$.

Since $N < 0$, it follows in case $n = 0$ that

$$\mathrm{H}^i(\mathcal{O}_X(D_\nu)) \cong \mathrm{H}^i(\mathcal{O}_X(D_{\nu+1})) \quad (0 \leq i < d-1),$$

so that $\mathrm{H}^i(\mathcal{O}_X(D_\nu)) = 0$ implies $\mathrm{H}^i(\mathcal{O}_X(D_{\nu+1})) = 0$. Furthermore, for any $n \geq 0$ there is a natural injection $\mathrm{H}^{d-1}(\mathcal{O}_X(D_\nu + nE)) \hookrightarrow \mathrm{H}^{d-1}(\mathcal{O}_X(D_{\nu+1} + nE))$. We then have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{H}^{d-1}(\mathcal{O}_X(D_\nu)) & \longrightarrow & \mathrm{H}^{d-1}(\mathcal{O}_X(D_{\nu+1})) & \longrightarrow & \mathrm{H}^{d-1}(\mathcal{O}_{E_1}(N)) \\
 & \downarrow{\scriptstyle \psi_\nu} & & \downarrow{\scriptstyle \psi_{\nu+1}} & & \| \\
0 \longrightarrow & \mathrm{H}^{d-1}(\mathcal{O}_X(D_\nu + nE)) & \longrightarrow & \mathrm{H}^{d-1}(\mathcal{O}_X(D_{\nu+1} + nE)) & \longrightarrow & \mathrm{H}^{d-1}(\mathcal{O}_{E_1}(N)).
\end{array}$$

Hence, if ψ_ν is injective then so is $\psi_{\nu+1}$. This completes the proof of Lemma 2.2 and thus of Theorem 2.1. □

REMARK 2.3. With f as in Theorem 2.1 and with E_i, β_i as before, set

$$E_i^* := \sum_{\{j \mid \beta_j \supset \beta_i\}} \operatorname{ord}_{\beta_j}(\mathfrak{m}_{\beta_i}) E_j.$$

So if $\mathbf{p}$ is the $r \times r$ proximity matrix with $p_{ii} = 1$ and with $p_{ji} = -1$ if $\beta_j \prec \beta_i$ and $p_{ji} = 0$ otherwise, then by [L3, p. 301, (4.6)] (whose proof is valid in any dimension) we have

$$(E_1^*, \dots, E_r^*)^{\mathrm{t}} = \mathbf{p}^{-1}(E_1, \dots, E_r)^{\mathrm{t}}, \tag{2.3.1}$$

where "t" means "transpose". Then, for any $n_1, \dots, n_r \in \mathbb{Z}$, premultiplying both sides of (2.3.1) by $(n_1, \dots, n_r)\mathbf{p}$ yields

$$\sum_{i=1}^{r} \Bigl(n_i - \sum_{\{j \mid \beta_j \prec \beta_i\}} n_j \Bigr) E_i^* = \sum_{i=1}^{r} n_i E_i.$$

Hence, the monoid of full divisors is freely generated by $E_1^*, \dots, E_r^*$.

For example, the relative canonical divisor $K_f := (d-1)(E_1^* + \cdots + E_r^*)$ is full. Note that $\mathcal{J}_f := \mathcal{O}_X(-K_f)$ is the relative Jacobian ideal of f by [LS, pp. 201–202] and that $\omega_f := \mathcal{J}_f^{-1} = \mathcal{O}_X(K_f)$ is a canonical dualizing sheaf for f by [LS, p. 206, (2.3)]. (In fact, since f is a local complete intersection map, $\omega_f \cong f^!\mathcal{O}_{\operatorname{Spec}(\alpha)}$.)

COROLLARY 2.4. *Under the hypotheses of Theorem 2.1, the following statements hold for any full divisor D on X.*

(i) $\mathrm{H}^i(X, \mathcal{O}_X(K_f - D)) = 0$ *for all* $i \neq 0$.
(ii) $\mathrm{H}^i_E(X, \mathcal{O}_X(K_f - D)) = 0$ *for all* $i \neq 1, d$.
(iii) $\mathrm{H}^1_E(X, \mathcal{O}_X(K_f - D)) \cong \alpha / \mathrm{H}^0(X, \mathcal{O}_X(K_f - D))$.
(iv) $\mathrm{H}^d_E(X, \mathcal{O}_X(K_f - D))$ *is an injective hull of* $\alpha/\mathfrak{m}_\alpha$.

Proof. For any invertible $\mathcal{O}_X$-module L, the α-module $\mathrm{H}^i(X, L \otimes \omega_f)$ is Matlis-dual to $\mathrm{H}^{d-i}_E(X, L^{-1})$ [L1, p. 188, Thm.]; so (i) and (ii) follow from Theorem 2.1 by duality (via Conjectures 1.3 and 1.4, respectively). Similarly, (iv) is dual to the obvious statement that $\mathrm{H}^0(X, \mathcal{O}_X(D)) = \alpha$. Assertion (iii) results from the natural exact sequence

$$\begin{aligned} 0 = \mathrm{H}^0_E(X, \mathcal{O}_X(K_f - D)) \to \mathrm{H}^0(X, \mathcal{O}_X(K_f - D)) &\to \alpha = \mathrm{H}^0(V, \mathcal{O}_V) \\ &\to \mathrm{H}^1_E(X, \mathcal{O}_X(K_f - D)) \\ &\to \mathrm{H}^1(X, \mathcal{O}_X(K_f - D)) \overset{\text{(i)}}{=} 0. \end{aligned}$$

□

3. Finitely Supported Ideals

Recall that an α-ideal I is *finitely supported* if there is a map $f\colon X \to \operatorname{Spec}(\alpha)$ that factors as in Theorem 2.1 and such that the $\mathcal{O}_X$-module $I\mathcal{O}_X$ is invertible. In this situation $I\mathcal{O}_X = \mathcal{O}_X(-D)$, where (as in Example 1.2.2(a)) D is a full divisor. (For more on finitely supported ideals, see [CGL; C; Ga; To].

Also, $\mathrm{H}^0(X, I\mathcal{O}_X) = \bar{I}$, the integral closure of I. With $\omega_f = \mathcal{O}_X(K_f)$ as in Remark 2.3, $\mathrm{H}^0(X, \mathcal{O}_X(K_f - D)) = \mathrm{H}^0(X, I\omega_f)$ is the *adjoint ideal* $\tilde{I}$; see [L5, p. 742, (1.3.1)].

The vanishing conjecture and various consequences hold for finitely supported ideals (but see Remark 4.1).

Corollary 3.1. *If $f : X \to \mathrm{Spec}(\alpha)$ is as in Theorem 2.1 and if I is an α-ideal such that $\mathcal{I} := I\mathcal{O}_X$ is invertible, then the following statements hold.*

(i) $\mathrm{H}^i_E(X, \mathcal{I}^{-1}) = \mathrm{H}^i_E(X, \mathcal{I}^{-1}\omega_f) = 0$ *for all* $i \neq d$.
(i′) $\mathrm{H}^i_E(X, \mathcal{I}) = \mathrm{H}^i_E(X, \mathcal{I}\omega_f) = 0$ *for all* $i \neq 1, d$.
(ii) $\mathrm{H}^i(X, \mathcal{I}\omega_f) = \mathrm{H}^i(X, \mathcal{I}) = 0$ *for all* $i \neq 0$.
(ii′) $\mathrm{H}^i(X, \mathcal{I}^{-1}\omega_f) = \mathrm{H}^i(X, \mathcal{I}^{-1}) = 0$ *for all* $i \neq d-1, 0$.
(iii) $\mathrm{H}^{d-1}(X, \mathcal{I}^{-1}\omega_f)$ *is Matlis-dual to* $\mathrm{H}^1_E(X, \mathcal{I}) \cong \alpha/\bar{I}$.
(iv) $\mathrm{H}^{d-1}(X, \mathcal{I}^{-1})$ *is Matlis-dual to* $\mathrm{H}^1_E(X, \mathcal{I}\omega_f) \cong \alpha/\tilde{I}$.
(v) $\mathrm{H}^0(X, \mathcal{I}^{-1}\omega_f) = \mathrm{H}^0(X, \mathcal{I}^{-1}) \cong \alpha$.
(vi) $\mathrm{H}^d_E(X, \mathcal{I}) = \mathrm{H}^d_E(X, \mathcal{I}\omega_f)$ *is an injective hull of* $\alpha/\mathfrak{m}_\alpha$.

Proof. Since the divisors D and $K_f + D$ are both full, (i) and (ii′) follow from Theorem 2.1 via Conjectures 1.3 and 1.4, respectively. Given the duality mentioned in the proof of Corollary 2.4, (iii) and (iv) both result from Corollary 2.4(iii). Statement (v) is obviously true. Finally, (ii), (i′), and (vi) result from their respective dual versions (i), (ii′), and (v). □

Remark 3.2. For the vanishing of $\mathrm{H}^1_E(X, \mathcal{I}^{-1})$ and hence of its dual $\mathrm{H}^{d-1}(X, \mathcal{I}\omega_f)$, it suffices that f factor as in (1.3.1). Indeed, there exists an exact and locally split sequence

$$0 \to C \to \mathcal{O}_X^n \to \mathcal{I} \to 0.$$

As a result, with $(-)^* := \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$, we have the exact sequence

$$0 \to \mathcal{I}^{-1} = \mathcal{I}^* \to \mathcal{O}_X^n \to C^* \to 0; \tag{3.2.1}$$

this gives another exact sequence,

$$0 = \mathrm{H}^0_E(X, C^*) \to \mathrm{H}^1_E(X, \mathcal{I})^{-1} \to \mathrm{H}^1_E(X, \mathcal{O}_X^n) = 0,$$

where the first term vanishes because C^* is locally free and the third vanishes by Example 1.3.2.

Tensoring (3.2.1) with ω_f (a dualizing sheaf, inverse to the relative Jacobian ideal) and noting that $\mathrm{H}^{d-1}(X, \mathcal{O}_X)$ and hence its dual $\mathrm{H}^1_E(X, \omega_f)$ vanish (Example 1.3.2), one shows similarly that $\mathrm{H}^1_E(\mathcal{I}^{-1}\omega_f)$ and its dual $\mathrm{H}^{d-1}(X, \mathcal{I})$ both vanish.

The *point basis* $\mathbf{B}(I)$ of a nonzero α-ideal I is the family of nonnegative integers $(\mathrm{ord}_\beta(I^\beta))$ indexed by the set of all points β infinitely near to α, with I^β the transform of I in β (i.e., $I^\beta := t^{-1}I\beta$, where t is the gcd of the elements in $I\beta$).

Two nonzero α-ideals have the same point basis iff (if and only if) their integral closures are the same; see [L2, p. 209, Prop. (1.10)]. The proof in [L2] shows, moreover, that if I and J are α-ideals such that $\mathrm{ord}_\beta(I^\beta) \le \mathrm{ord}_\beta(J^\beta)$ for all β, then $\bar{I} \supset \bar{J}$ (here the overbar denotes integral closure).

The ideal I is finitely supported iff I has finitely many *base points*—that is, β such that $\mathrm{ord}_\beta(I^\beta) \neq 0$ (see [L2, p. 213, (1.20), p. 215, Remark]). Thus the product of two finitely supported ideals is still finitely supported. (It can be shown, for any α-ideal I and any d-dimensional infinitely–near β, that if I^β is $\mathfrak{m}_\beta$-primary then β is dominated by a Rees valuation of I. Hence I is finitely supported iff every base point of I is d-dimensional. More constructively, I is finitely supported iff $I\mathcal{O}_X$ is invertible, where $X \to \mathrm{Spec}(\alpha)$ is obtained by successively blowing up all the finitely many d-dimensional infinitely–near β such that β is dominated by a Rees valuation of I—in which case, with the notation of Remark 2.3, $I\mathcal{O}_X = \mathcal{O}_X(-\sum_i \mathrm{ord}_{\beta_i}(I^{\beta_i})E_i^*)$.)

Here is the main result in this section (proved for $d = 2$ in [L5, p. 749, (3.1.2)]).

THEOREM 3.3. *Let α be a d-dimensional regular local ring ($d \ge 2$) and I a finitely supported α-ideal with point basis $\mathbf{B}(I) = (r_\beta)$. Then*:

(1) $\mathrm{ord}_\alpha(\tilde{I}) = \max(\mathrm{ord}_\alpha(I) + 1 - d, 0)$; *and*
(2) *for any β infinitely near to α*, $\widetilde{I^\beta} = (\tilde{I})^\beta$.

Hence the adjoint ideal $\tilde{I}$ is the unique integrally closed ideal with point basis $(\max(r_\beta + 1 - d, 0))$. *In particular, $\tilde{I}$ is finitely supported.*

REMARK 3.3.1. Apropos of (2), $\overline{I^\beta} = \overline{\bar{I}^\beta} \supset \bar{I}^\beta$ (see [L2, p. 207, Prop. (1.5)(vi)]); but equality doesn't always hold (see [To, Ex. 1.2]).

COROLLARY 3.3.2. *For any finitely supported α-ideal I, $\tilde{I} = \alpha$ if and only if* $\mathrm{ord}_\alpha(I) < d$.

We also have the following weak subadditivity consequence.

COROLLARY 3.3.3. *For finitely supported α-ideals I and J,*

$$\overline{\tilde{I}\tilde{J}} \supset \widetilde{IJ}.$$

Proof. One may check that, for any nonnegative integers r and s,

$$\max(r + 1 - d, 0) + \max(s + 1 - d, 0) \le \max(r + s + 1 - d, 0).$$

Because "transform" respects products, $\mathrm{ord}_\beta((\tilde{I}\tilde{J})^\beta) \le \mathrm{ord}_\beta((\widetilde{IJ})^\beta)$ for all β. Hence the conclusion follows. □

Our next corollary (and also Proposition 3.4) lies in the opposite direction.

COROLLARY 3.3.4. *For finitely supported α-ideals I and J,*

$$\widetilde{IJ} \supset \overline{I\tilde{J}},$$

where equality holds if and only if $\mathrm{ord}_\beta(J^\beta) \ge d - 1$ *at every base point β of I.*

Proof. The inclusion is a consequence of $\widetilde{IJ} : I = \tilde{J}$ [L5, p. 741, (b) and (d)].

The point basis $\mathbf{B}(I\tilde{J}) =: (r_\beta)$ satisfies

$$r_\beta = \mathrm{ord}_\beta(I^\beta) + \max(\mathrm{ord}_\beta(J^\beta) - d + 1, 0)$$

[L2, p. 212, (1.15)], from which it follows that

$$\begin{aligned} r_\beta &= \max(\mathrm{ord}_\beta((IJ)^\beta) - d + 1, 0) \\ &\iff \text{ either } \mathrm{ord}_\beta(I^\beta) = 0 \text{ or } [\mathrm{ord}_\beta(I^\beta) > 0 \text{ and } \mathrm{ord}_\beta(J^\beta) \geq d-1]. \quad \square \end{aligned}$$

Proof of Theorem 3.3. We begin by proving Corollary 3.3.2.

Any β with $I^\beta \neq \beta$ is d-dimensional [L2, p. 214, (1.22)] and so, because α is regular, the residue field of β is finite over that of α; see [GD2, (5.6.4)]. Hence, by [Hi, p. 217, Lemma 8],

(∗) if $\mathrm{ord}_\alpha(I) < d$ then $\mathrm{ord}_\beta(I^\beta) < d$ for any infinitely near β.

With this in mind, recall from Remark 2.3 that, with $f: X \to \mathrm{Spec}(\alpha)$ as at the beginning of this section and with $K_f =: \sum_i c_i E_i$, we have

$$\tilde{I} = \mathrm{H}^0(X, I\omega_f) = \mathrm{H}^0\Big(X, \mathcal{O}_X\Big(\sum_i (c_i - \mathrm{ord}_{\beta_i}(I))E_i\Big)\Big)$$

and

$$c_i = \sum_{\beta_j \subset \beta_i} \mathrm{ord}_{\beta_i}(\mathfrak{m}_j^{d-1}) \geq \sum_{\beta_j \subset \beta_i} \mathrm{ord}_{\beta_j}(I^{\beta_j})\,\mathrm{ord}_{\beta_i}(\mathfrak{m}_j) = \mathrm{ord}_{\beta_i}(I),$$

the last equality by [L2, pp. 209–210, Lemma (1.11)]. The “if” part of Corollary 3.3.2 results.

Furthermore, if (say) E_1 corresponds to the valuation ring of ord_α, then

$$\mathrm{ord}_\alpha(\tilde{I}) \geq \mathrm{ord}_\alpha(I) - c_1 = \mathrm{ord}_\alpha(I) - (d-1).$$

In particular, if $\tilde{I} = \alpha$ then $\mathrm{ord}_\alpha(I) \leq d-1$. This gives the “only if” part of Corollary 3.3.2.

Now Corollary 3.3.2 and (∗) show that Theorem 3.3 holds if $\mathrm{ord}_\alpha(I) < d$. For the rest, we need the following key fact.

PROPOSITION 3.4. *Let I and J be finitely supported α-ideals such that, for each β infinitely near to α, $\mathrm{ord}_\beta(J^\beta) \geq (d-1)\,\mathrm{ord}_\beta(I^\beta)$. Then*

$$\widetilde{IJ} = I\tilde{J}.$$

Proof. In the proof of Lemma 1.1 applied to the present situation, $I_i = I^{\beta_i}$; hence the condition “$\mathrm{ord}_\beta(J^\beta) \geq (d-1)\,\mathrm{ord}_\beta(I^\beta)$ for each β” may be written as

$$\Big(v_i(J) - \sum_{\{j \mid \beta_j \prec \beta_i\}} v_j(J)\Big) \geq (d-1)\Big(v_i(I) - \sum_{\{j \mid \beta_j \prec \beta_i\}} v_j(I)\Big) \quad (1 \leq i \leq r). \tag{3.4.1}$$

This implies that if $f: X \to \mathrm{Spec}(\alpha)$ is a composite of closed-point blowups such that $I\mathcal{O}_X$ and $J\mathcal{O}_X$ are both invertible (such an f exists because IJ is finitely supported) then, for $0 \leq k \leq d-1$, we have $I^k(J\mathcal{O}_X)^{-1} = \mathcal{O}_X(D_k)$ with D_k a full divisor on X.

Hence, by Corollary 2.4(i),

$$\mathrm{H}^{d-i}(X,(I\mathcal{O}_X)^{-k}J\omega_f) = \mathrm{H}^{d-i}(X,\mathcal{O}_X(K_f - D_k)) = 0 \quad (0 \le i,k \le d-1).$$

This being so, we see that the case $J = I^n$ $(n \ge d-1)$ is treated in [L5, Sec. 2.3]; the proof for arbitrary J is essentially the same. (Here, a principalization of I is given to begin with, so the fact—of which a special case is used in [L5]—that the $\sim$ operation on α-ideals commutes with smooth base change follows from commutativity with H^0 and with formation of ω.) □

COROLLARY 3.4.2. *If J is a finitely supported α-ideal with* $\mathrm{ord}_\alpha(J) \ge d-1$, *then*

$$\widetilde{\mathfrak{m}_\alpha J} = \mathfrak{m}_\alpha \tilde{J}.$$

Now we can prove Theorem 3.3 by induction on the least number of closed-point blowups needed to principalize I. Set $\mathrm{ord}_\alpha(I) := a$. Since we have already disposed of the case $a < d$, it will clearly be enough to show that:

(1) if $a \ge d-1$ then $\mathrm{ord}_\alpha(\tilde{I}) = a+1-d$; and
(2) if $g\colon X_1 \to \mathrm{Spec}(\alpha)$ is the blowup of $\mathfrak{m} := \mathfrak{m}_\alpha$ and if β is the local ring of a closed point on X_1, then

$$\widetilde{I^\beta} = (\tilde{I})^\beta.$$

Let $h\colon X = X_r \to X_1$ be as in Theorem 2.1. For any $\mathcal{O}_X$-module L, the natural map is an isomorphism $\mathfrak{m}h_*(L) \xrightarrow{\sim} h_*(\mathfrak{m}L)$. (Since the assertion is local on X_1, one can assume that $\mathfrak{m}\mathcal{O}_{X_1} \cong \mathcal{O}_{X_1}\dots$.) Furthermore, from Remark 2.3 one deduces that $\omega_h = \omega_f(\mathfrak{m}\mathcal{O}_X)^{d-1}$. Hence, with $\mathcal{I}_1 := I(\mathfrak{m}\mathcal{O}_{X_1})^{-a}$, it follows that

$$\begin{aligned}(\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1}\tilde{\mathcal{I}}_1 &:= (\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1}h_*(\mathcal{I}_1\omega_h) = (\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1}h_*(\mathcal{I}_1\omega_f(\mathfrak{m}\mathcal{O}_X)^{d-1}) \\ &= (\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1}h_*(I\omega_f(\mathfrak{m}\mathcal{O}_X)^{-a+d-1}) = h_*(I\omega_f).\end{aligned}$$

Using induction on $s > 0$, from Corollary 3.4.2 we deduce that

$$\mathfrak{m}^s\tilde{I} = \widetilde{\mathfrak{m}^s I} := \mathrm{H}^0(X,\mathfrak{m}^s I\omega_f) = \mathrm{H}^0(X_1, h_*(\mathfrak{m}^s I\omega_f)) = \mathrm{H}^0(X_1,\mathfrak{m}^s h_*(I\omega_f)).$$

The invertible $\mathcal{O}_{X_1}$-module $\mathfrak{m}\mathcal{O}_{X_1}$ is g-ample and $h_*(I\omega_f)$ is coherent, so $\mathfrak{m}^s h_*(I\omega_f)$ is generated by its global sections for all $s \gg 0$. In other words, we have shown that

$$\mathfrak{m}^s h_*(I\omega_f) = \mathfrak{m}^s\tilde{I}\mathcal{O}_{X_1}$$

and so

$$(\mathfrak{m}\mathcal{O}_{X_1})^{a-d+1}\tilde{\mathcal{I}}_1 = h_*(I\omega_f) = \tilde{I}\mathcal{O}_{X_1}.$$

Since $\tilde{\mathcal{I}}_1 \not\subset \mathfrak{m}\mathcal{O}_{X_1}$, this implies statement (1). Then—since it is straightforward to check for any $z \in X_1$ that the stalk $(\tilde{\mathcal{I}}_1)_z$ is just $\widetilde{(\mathcal{I}_1)_z}$—localizing at β gives (2). This completes the proof of Theorem 3.3. □

4. Additional Observations

Let $J = (\xi_1,\dots,\xi_d)$ $(d := \dim\alpha)$ be a finitely supported and hence $\mathfrak{m}_\alpha$-primary α-ideal. Proposition 4.2(ii) shows, as mentioned in the Introduction, that $J\widetilde{J^{n-1}} = \widetilde{J^n}$

for all $n \geq d$ but that $J\widetilde{J^{s-1}} \neq \widetilde{J^s}$ for $1 \leq s < d$. Proposition 4.2(i) shows (via Corollary 3.3.2) that, for $s > 0$, $J\overline{J^{s-1}} = \overline{J^s}$ if and only if $s > d(1 - 1/\mathrm{ord}_\alpha(J))$. In particular, $J\overline{J^{d-1}} = \overline{J^d}$.

REMARK 4.1. Bernd Ulrich informed me of an example of Huneke and Huckaba [HH, p. 88] in which α can be taken to be the localization at (x, y, z) of the polynomial ring $\mathbb{C}[x, y, z]$ (so $d = 3$), $J = (x^4, y(y^3 + z^3), z(y^3 + z^3))$, and $J\overline{J^2} \neq \overline{J^3}$. This J cannot, then, be finitely supported. In fact, it has a curve of base points in the blowup of $\mathfrak{m}_\alpha$. Moreover, analysis of the proof of Proposition 4.2(i) shows that, if $f : X \to \mathrm{Spec}(\alpha)$ is a principalization of J by a sequence of smoothly centered blowups, then $\mathrm{H}^1(X, J\mathcal{O}_X) \supset \overline{J^3}/J\overline{J^2} \neq 0$. Thus, for instance, Corollary 3.1(ii) does not hold for principalizations of arbitrary $\mathfrak{m}_\alpha$-primary ideals.

It is well known that, for any ideal $I \supset J$, the dual of α/I (i.e., $\mathrm{Hom}_\alpha(\alpha/I, \mathcal{E})$, where $\mathcal{E}$ is an injective hull of $\alpha/\mathfrak{m}_\alpha$) is (isomorphic to) $(J : I)/J$. Indeed, by local duality the dual of $\alpha/I = \mathrm{H}^0_{\mathfrak{m}_\alpha}(\alpha/I)$ is $\mathrm{Ext}^d(\alpha/I, \alpha)$, and since the sequence $(\xi_1, \dots, \xi_d)$ is regular we have the standard isomorphisms

$$\mathrm{Ext}^d(\alpha/I, \alpha) \cong \mathrm{Hom}_\alpha(\alpha/I, \alpha/J) \cong (J : I)/J.$$

PROPOSITION 4.2. *For the preceding J, set $J^t := \alpha$ for all $t \leq 0$. Then, for all $s \in \mathbb{Z}$,*

(i) *$\overline{J^s}/J\overline{J^{s-1}}$ is dual to $\alpha/(\widetilde{J^{d-s}} + J)$ and*
(ii) *$\widetilde{J^s}/J\widetilde{J^{s-1}}$ is dual to $\alpha/(\overline{J^{d-s}} + J)$.*

Consequently, since a finite-length module and its dual have the same annihilator, we have

(iii) *$J\overline{J^{s-1}} : \overline{J^s} = \widetilde{J^{d-s}} + J$ and*
(iv) *$J\overline{J^{s-1}} : \widetilde{J^s} = \overline{J^{d-s}} + J$.*

A proof is given at the end of the paper.

COROLLARY 4.3. *For any $s \in \mathbb{Z}$, the following conditions are equivalent.*

(i) $J\widetilde{J^{s-1}} = \widetilde{J^s} \cap J$.
(ii) $J\overline{J^{d-s-1}} : \overline{J^{d-s}} = J : \overline{J^{d-s}}$.
(iii) $J\overline{J^{d-s-1}} = \overline{J^{d-s}} \cap J$.
(iv) $J\widetilde{J^{s-1}} : \widetilde{J^s} = J : \widetilde{J^s}$.

Proof. Since, clearly, $J\widetilde{J^{s-1}} \subset \widetilde{J^s} \cap J$, therefore Proposition 4.2(ii) makes condition (i) hold if and only if $\widetilde{J^s}/(\widetilde{J^s} \cap J)$ is dual to $\alpha/(\overline{J^{d-s}} + J)$—that is, isomorphic to $(J : \overline{J^{d-s}})/J$. All these modules have finite length, so the natural isomorphism

$$\widetilde{J^s}/(\widetilde{J^s} \cap J) \cong (\widetilde{J^s} + J)/J \overset{4.2(\mathrm{iii})}{=} (J\overline{J^{d-s-1}} : \overline{J^{d-s}})/J$$

shows that (i) $\Leftrightarrow$ (ii).

The proof of (iii) $\Leftrightarrow$ (iv) is analogous: replace s by $d - s$ and then interchange $\widetilde{\ }$ and $\overline{\ }$. The implications (i) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (ii) are obvious. □

COROLLARY 4.4. *The following statements hold.*

(i) $J\widetilde{J^{d-2}} = \widetilde{J^{d-1}} \cap J$.

(ii) $J\overline{J^{d-2}} = \overline{J^{d-1}} \cap J$.

(iii) $J\widetilde{J^{d-3}} = \widetilde{J^{d-2}} \cap J$.

Proof. For $s \geq d-1$, Corollary 4.3(ii) obviously holds and so Corollary 4.4(i) does also. Similarly, (ii) is a consequence of Corollary 4.3(iv) entailing 4.3(iii) with $s = 1$.

As pointed out by Bernd Ulrich, (iii) results similarly from the fact that Corollary 4.3(iii) holds for $s = d-2$, a special case of the main result in [It]. □

Proof of Proposition 4.2. Let $f : X \to \mathrm{Spec}(\alpha)$ be a composition of closed-point blowups such that $L := J\mathcal{O}_X$ is invertible. Then, for all $s \in \mathbb{Z}$, $\mathrm{H}^0(X, L^s) = \overline{J^s}$ and $\mathrm{H}^0(X, L^s\omega_f) = \widetilde{J^s}$.

Corollary 3.1(ii) and (ii′) give, for all $j \geq 0$,

$$\mathrm{H}^i(X, L^j) = 0 \quad (i \neq 0),$$
$$\mathrm{H}^i(X, L^{-j}) = 0 \quad (0 < i < d-1).$$

Arguing as in [LT, p. 112], one finds then that $\overline{J^s}/J\overline{J^{s-1}}$ is isomorphic to the kernel of the map

$$\mathrm{H}^{d-1}(X, L^{s-d}) \xrightarrow{\xi_1 \oplus \cdots \oplus \xi_d} \underbrace{\mathrm{H}^{d-1}(X, L^{s+1-d}) \oplus \cdots \oplus \mathrm{H}^{d-1}(X, L^{s+1-d})}_{d \text{ times}}.$$

Hence $\overline{J^s}/J\overline{J^{s-1}}$ is dual to the cokernel of the dual map

$$\underbrace{\mathrm{H}^1_E(X, L^{d-s-1}\omega_f) \times \cdots \times \mathrm{H}^1_E(X, L^{d-s-1}\omega_f)}_{d \text{ times}} \xrightarrow{(\xi_1, \ldots, \xi_d)} \mathrm{H}^1_E(X, L^{d-s}\omega_f).$$

Corollary 3.1(i) and (iv) give $\mathrm{H}^1_E(X, L^j\omega_f) \cong \alpha/\widetilde{J^j}$ for all $j \in \mathbb{Z}$. Accordingly, one verifies that $\overline{J^s}/J\overline{J^{s-1}}$ is dual to the cokernel of the map

$$\alpha^d \xrightarrow{(\xi_1, \ldots, \xi_d)} \alpha/\widetilde{J^{d-s}},$$

which proves part (i) of the proposition. The proof of (ii) is analogous, except that one begins by tensoring the complex $K(F, \sigma)$ in [LT, p. 112] with $L^s\omega_f$ instead of with L^s. □

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