Michigan Math. J. 57 (2008)

F-Thresholds, Tight Closure, Integral Closure, and Multiplicity Bounds

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Dedicated to Professor Mel Hochster
on the occasion of his sixty-fifth birthday

Introduction

Let R be a Noetherian ring of positive characteristic p. For every ideal $\mathfrak{a}$ in R, and for every ideal J whose radical contains $\mathfrak{a}$, one can define asymptotic invariants that measure the containment of the powers of $\mathfrak{a}$ in the Frobenius powers of J. These invariants were introduced in the case of a regular local F-finite ring in [MTW], where it was shown that they coincide with the jumping exponents for the generalized test ideals of Hara and Yoshida [HaY]. In this paper we work in a general setting and show that the F-thresholds still capture interesting and subtle information. In particular, we relate them to tight closure and integral closure and also to multiplicities.

Given $\mathfrak{a}$ and J as just described, we define for every positive integer e

$$\nu_{\mathfrak{a}}^{J}(p^e) := \max\{r \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\},$$

where $J^{[q]}$ is the ideal generated by the p^e-powers of the elements of J. We put

$$\mathrm{c}_{+}^{J}(\mathfrak{a}) := \limsup_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e} \quad \text{and} \quad \mathrm{c}_{-}^{J}(\mathfrak{a}) := \liminf_{e\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(p^e)}{p^e},$$

and if these two limits coincide then we denote their common value by $\mathrm{c}^{J}(\mathfrak{a})$ and call it the *F-threshold of* $\mathfrak{a}$ *with respect to* J.

Our first application of this notion is to the description of the tight closure and of the integral closure of parameter ideals. Suppose that $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring of positive characteristic and that J is an ideal in R generated by a full system of parameters. We show that, under mild conditions, for every ideal $I \supseteq J$ we have $I \subseteq J^*$ if and only if $\mathrm{c}_{+}^{I}(J) = d$ (and in this case $\mathrm{c}_{-}^{I}(J) = d$, too). We similarly show that, under suitable mild hypotheses, if $I \supseteq J$, then

Received August 20, 2007. Revision received November 26, 2007.
The first author was partially supported by NSF grant DMS-0244405. The second author was partially supported by NSF grant DMS-0500127 and by a Packard Fellowship. The third and fourth authors were partially supported by Grant-in-Aid for Scientific Research 17740021 and 17540043, respectively. The third author was also partially supported by Special Coordination Funds for Promoting Science and Technology from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

$I \subseteq \bar{J}$ if and only if $c^J_+(I) = d$. For the precise statements, see Corollary 3.2 and Theorem 3.3.

As we have mentioned, for R regular and F-finite it was shown in [MTW] that the F-thresholds of an ideal $\mathfrak{a}$ coincide with the jumping exponents for the generalized test ideals of [HaY]. In order to recover such a result in a more general setting, we develop a notion of F-threshold for the ideal $\mathfrak{a}$ corresponding to a submodule N of a module M and such that $\mathfrak{a}^n N = 0$ for some n. We then show that, under suitable hypotheses on a local ring R, one can again recover the jumping exponents for the generalized test ideals of an ideal $\mathfrak{a}$ in R from the F-thresholds of $\mathfrak{a}$ with respect to pairs (E, N), where N is a submodule of the injective hull E of the residue field (see Corollary 4.4).

We study the connection between F-thresholds and multiplicity and then formulate the following conjecture: If $(R, \mathfrak{m})$ is a d-dimensional Noetherian local ring of characteristic $p > 0$ and if $\mathfrak{a}$ and J are $\mathfrak{m}$-primary ideals in R, with J generated by a system of parameters, then

$$e(\mathfrak{a}) \geq \frac{d^d}{c^J_-(\mathfrak{a})^d} e(J).$$

The case $J = \mathfrak{m}$ (when R is in fact regular) was proved in [TW]. We mention that, in this case, $c^{\mathfrak{m}}(\mathfrak{a})$ is related via reduction modulo p to a fundamental invariant in birational geometry, the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ (see [TW] for the precise relation between these two invariants). The corresponding inequality between the multiplicity and the log canonical threshold of $\mathfrak{a}$ was proved in [dFEM2] and plays a key role in proving that, for small values of n, no smooth hypersurface of degree n in $\mathbb{P}^n$ is rational (see [Co; dFEM1]).

We prove our conjecture when both $\mathfrak{a}$ and J are generated by homogeneous systems of parameters in a graded Cohen–Macaulay k-algebra (cf. Corollary 5.9). Moreover, we prove it also when R is regular and $J = (x_1^{a_1}, \ldots, x_n^{a_n})$ for a regular system of parameters $x_1, \ldots, x_n$. The proof of this latter case follows the ideas in [TW] and [dFEM2], reducing to the case of a monomial ideal $\mathfrak{a}$ and then using the explicit interpretation of the invariants involved in terms of the Newton polyhedron of $\mathfrak{a}$.

On the other hand, the proof of the homogeneous case is based on new ideas that we expect to be useful also in attacking the general case of the conjecture. In fact, we prove the following stronger statement: Suppose that $\mathfrak{a}$ and J are ideals generated by homogeneous systems of parameters in a d-dimensional graded Cohen–Macaulay k-algebra, where k is a field of *arbitrary* characteristic; if $\mathfrak{a}^N \subseteq J$ for some N, then

$$e(\mathfrak{a}) \geq \left(\frac{d}{d+N-1}\right)^d e(J).$$

The paper is structured as follows. In Section 1, we recall some basic notions of tight closure theory and review the definition of generalized test ideals from [HaY]. In Section 2 we introduce the F-thresholds and discuss some basic properties. Section 3 is devoted to the connections with tight closure and integral closure.

We introduce the F-thresholds with respect to pairs of modules in Section 4 and relate them to the jumping exponents for the generalized test ideals. In Section 5, we discuss inequalities involving F-thresholds and multiplicities; in particular, we state our conjecture and prove the aforementioned special cases.

1. Preliminaries

In this section we review some definitions and notation that will be used throughout the paper. All rings are Noetherian commutative rings with unity. For a ring R, we denote by R° the set of elements of R that are not contained in any minimal prime ideal. Elements $x_1, \ldots, x_r$ in R are called *parameters* if they generate an ideal of height r. The integral closure of an ideal $\mathfrak{a}$ is denoted by $\bar{\mathfrak{a}}$. The order of a nonzero element f in a Noetherian local ring $(R, \mathfrak{m})$ is the largest r such that $f \in \mathfrak{m}^r$. For a real number u, we denote by $\lfloor u \rfloor$ the largest integer $\leq u$ and by $\lceil u \rceil$ the smallest integer $\geq u$.

Let R be a ring of characteristic $p > 0$, and let $F\colon R \to R$ denote the Frobenius map that sends $x \in R$ to $x^p \in R$. The ring R viewed as an R-module via the e-times iterated Frobenius map $F^e\colon R \to R$ is denoted by ${}^eR$. We say that R is *F-finite* if ${}^1R$ is a finitely generated R-module. We also say that R is *F-pure* if the Frobenius map is pure; that is, $F_M = 1_M \otimes F\colon M = M \otimes_R R \to M \otimes_R {}^1R$ is injective for any R-module M. For every ideal I in R and for every $q = p^e$, we denote by $I^{[q]}$ the ideal generated by the qth powers of all elements of I.

If M is an R-module then we put $\mathbb{F}^e(M) := {}^eR \otimes_R M$. Hence, in $\mathbb{F}^e(M)$ we have $u \otimes (ay) = ua^{p^e} \otimes y$ for every $a \in R$. Note that the e-times iterated Frobenius map $F_M^e\colon M \to \mathbb{F}^e(M)$ is an R-linear map. The image of $z \in M$ via this map is denoted by $z^q := F_M^e(z)$. If N is a submodule of M, then we denote by $N_M^{[q]}$ (or simply by $N^{[q]}$) the image of the canonical map $\mathbb{F}^e(N) \to \mathbb{F}^e(M)$ (note that, if $N = I$ is a submodule of $M = R$, then this is consistent with our previous notation for $I^{[q]}$).

First, we recall the definitions of classical tight closure and related notions. Our references for classical tight closure theory and for F-rational rings are [HHu] and[FW], respectively; see also the book [HuH].

Definition 1.1. Let I be an ideal in a ring R of characteristic $p > 0$.

(i) The *Frobenius closure* I^F of I is defined as the ideal of R consisting of all elements $x \in R$ such that $x^q \in I^{[q]}$ for some $q = p^e$. If R is F-pure, then $J = J^F$ for all ideals $J \subseteq R$. The *tight closure* I^* of I is defined to be the ideal of R consisting of all elements $x \in R$ for which there exists a $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all large $q = p^e$.

(ii) We say that $c \in R^\circ$ is a *test element* if, for all ideals $J \subseteq R$ and all $x \in J^*$, we have $cx^q \in I^{[q]}$ for all $q = p^e \geq 1$. Every excellent and reduced ring R has a test element.

(iii) If $N \subseteq M$ are R-modules, then the *tight closure* N_M^* of N in M is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists a $c \in R^\circ$ such that $cz^q \in N_M^{[q]}$ for all large $q = p^e$. The *test ideal*

$\tau(R)$ of R is defined to be $\tau(R) = \bigcap_M \operatorname{Ann}_R(0_M^*)$, where M runs over all finitely generated R-modules. If $M = R/I$, then $\operatorname{Ann}_R(0_M^*) = (I : I^*)$. In other words, $\tau(R)J^* \subseteq J$ for all ideals $J \subseteq R$. We say that R is *F-regular* if $\tau(R_P) = R_P$ for all prime ideals P of R.

(iv) R is called *F-rational* if $J^* = J$ for every ideal $J \subseteq R$ generated by parameters. If R is an excellent equidimensional local ring, then R is F-rational if and only if $I = I^*$ for *some* ideal I generated by a full system of parameters for R.

We now recall the definition of $\mathfrak{a}^t$-tight closure and of the generalized test ideal $\tau(\mathfrak{a}^t)$. The reader is referred to [HaY] for details.

DEFINITION 1.2. Let $\mathfrak{a}$ be a fixed ideal in a reduced ring R of characteristic $p > 0$ such that $\mathfrak{a} \cap R^\circ \neq \emptyset$, and let I be an arbitrary ideal in R.

(i) Let $N \subseteq M$ be R-modules. Given a rational number $t \geq 0$, the *$\mathfrak{a}^t$-tight closure* $N_M^{*\mathfrak{a}^t}$ of N in M is defined to be the submodule of M consisting of all elements $z \in M$ for which there exists a $c \in R^\circ$ such that $cz^q\mathfrak{a}^{\lceil tq \rceil} \subseteq N_M^{[q]}$ for all large $q = p^e$.

(ii) The *generalized test ideal* $\tau(\mathfrak{a}^t)$ is defined to be $\tau(\mathfrak{a}^t) = \bigcap_M \operatorname{Ann}_R(0_M^{*\mathfrak{a}^t})$, where M runs through all finitely generated R-modules. If $\mathfrak{a} = R$, then the generalized test ideal $\tau(\mathfrak{a}^t)$ is nothing but the test ideal $\tau(R)$.

(iii) Assume that R is an F-regular ring and that J is an ideal containing $\mathfrak{a}$ in its radical. The *F-jumping exponent* of $\mathfrak{a}$ with respect to J is defined by

$$\xi^J(\mathfrak{a}) = \sup\{c \in \mathbb{R}_{\geq 0} \mid \tau(\mathfrak{a}^c) \not\subseteq J\}.$$

If $(R, \mathfrak{m})$ is local, then we call the smallest F-jumping exponent $\xi^{\mathfrak{m}}(\mathfrak{a})$ the *F-pure threshold* of $\mathfrak{a}$ and denote it by $\operatorname{fpt}(\mathfrak{a})$.

In characteristic 0, one defines multiplier ideals and their jumping exponents using resolution of singularities (see [La, Chap. 9]). It is known that, for a given ideal in characteristic 0 and for a given t, the reduction modulo $p \gg 0$ of the multiplier ideal $\mathcal{J}(\mathfrak{a}^t)$ coincides with the generalized test ideal $\tau(\mathfrak{a}_p^t)$ of the reduction $\mathfrak{a}_p$ of $\mathfrak{a}$. Therefore, the F-jumping exponent $\xi^J(\mathfrak{a})$ is a characteristic p analogue of jumping exponent of multiplier ideals. See [BMS2; HaMo; HaY; MTW; TW] for further discussions.

2. Basic Properties of F-Thresholds

The F-thresholds are invariants of singularities of a given ideal $\mathfrak{a}$ in positive characteristic that are obtained by comparing the powers of $\mathfrak{a}$ with the Frobenius powers of other ideals. They were introduced and studied in [MTW] for the case of an ambient regular ring. In this section, we recall the definition of F-thresholds and study their basic properties when the ring is not necessarily regular.

Let R be a Noetherian ring of dimension d and of characteristic $p > 0$. Let $\mathfrak{a}$ be a fixed proper ideal of R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$. To each ideal J of R such that $\mathfrak{a} \subseteq \sqrt{J}$, we associate an F-threshold as follows. For every $q = p^e$, let

$$\nu_{\mathfrak{a}}^{J}(q) := \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq J^{[q]}\}.$$

Since $\mathfrak{a} \subseteq \sqrt{J}$, this is a nonnegative integer (if $\mathfrak{a} \subseteq J^{[q]}$, then we put $\nu_{\mathfrak{a}}^{J}(q) = 0$). We put

$$\mathrm{c}_{+}^{J}(\mathfrak{a}) = \limsup_{q\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(q)}{q}, \qquad \mathrm{c}_{-}^{J}(\mathfrak{a}) = \liminf_{q\to\infty} \frac{\nu_{\mathfrak{a}}^{J}(q)}{q}.$$

When $\mathrm{c}_{+}^{J}(\mathfrak{a}) = \mathrm{c}_{-}^{J}(\mathfrak{a})$, we call this limit the *F-threshold* of the pair $(R, \mathfrak{a})$ (or simply of $\mathfrak{a}$) with respect to J, and we denote it by $\mathrm{c}^{J}(\mathfrak{a})$.

REMARKS 2.1. (a) (cf. [MTW, Rem. 1.2]) One has

$$0 \le \mathrm{c}_{-}^{J}(\mathfrak{a}) \le \mathrm{c}_{+}^{J}(\mathfrak{a}) < \infty.$$

In fact, if $\mathfrak{a}$ is generated by l elements and if $\mathfrak{a}^N \subseteq J$, then

$$\mathfrak{a}^{N(l(p^e-1)+1)} \subseteq (\mathfrak{a}^{[p^e]})^N = (\mathfrak{a}^N)^{[p^e]} \subseteq J^{[p^e]}.$$

Therefore, $\nu_{\mathfrak{a}}^{J}(p^e) \le N(l(p^e-1)+1)-1$. Dividing by p^e and taking the limit gives $\mathrm{c}_{+}^{J}(\mathfrak{a}) \le Nl$.

(b) Question 1.4 in [MTW] asked whether the F-threshold $\mathrm{c}^{J}(\mathfrak{a})$ is a rational number (when it exists). A positive answer was given in [BMS1; BMS2] for a regular F-finite ring that is essentially of finite type over a field as well as for every regular F-finite ring when the ideal $\mathfrak{a}$ is principal. For a proof in the case of a principal ideal in a complete regular ring (that is not necessarily F-finite), see [KLZ]. However, this question remains open in general.

Recall that a ring extension $R \hookrightarrow S$ is *cyclic pure* if, for every ideal I in R, we have $IS \cap R = I$.

PROPOSITION 2.2 (cf. [MTW, Prop. 1.7]). *Let $\mathfrak{a}$ and J be ideals as described previously.*

(i) *If $I \supseteq J$ then $\mathrm{c}_{\pm}^{I}(\mathfrak{a}) \le \mathrm{c}_{\pm}^{J}(\mathfrak{a})$.*

(ii) *If $\mathfrak{b} \subseteq \mathfrak{a}$ then $\mathrm{c}_{\pm}^{J}(\mathfrak{b}) \le \mathrm{c}_{\pm}^{J}(\mathfrak{a})$; moreover, if $\mathfrak{a} \subseteq \bar{\mathfrak{b}}$ then $\mathrm{c}_{\pm}^{J}(\mathfrak{b}) = \mathrm{c}_{\pm}^{J}(\mathfrak{a})$.*

(iii) $\mathrm{c}_{\pm}^{J}(\mathfrak{a}^r) = \frac{1}{r}\,\mathrm{c}_{\pm}^{J}(\mathfrak{a})$ *for every integer $r \ge 1$.*

(iv) $\mathrm{c}_{\pm}^{J^{[q]}}(\mathfrak{a}) = q\,\mathrm{c}_{\pm}^{J}(\mathfrak{a})$ *for every $q = p^e$.*

(v) *If $R \hookrightarrow S$ is a cyclic pure extension, then*

$$\mathrm{c}_{\pm}^{J}(\mathfrak{a}) = \mathrm{c}_{\pm}^{JS}(\mathfrak{a}S).$$

(vi) *Let $R \hookrightarrow S$ be an integral extension. If the conductor ideal $\mathfrak{c}(S/R) := \mathrm{Ann}_R(S/R)$ contains the ideal $\mathfrak{a}$ in its radical, then*

$$\mathrm{c}_{\pm}^{J}(\mathfrak{a}) = \mathrm{c}_{\pm}^{JS}(\mathfrak{a}S).$$

(vii) $\mathrm{c}_{+}^{J}(\mathfrak{a}) \le c$ *(resp. $\mathrm{c}_{-}^{J}(\mathfrak{a}) \ge c$) if and only if, for every power q_0 of p, we have $\mathfrak{a}^{\lceil cq\rceil + q/q_0} \subseteq J^{[q]}$ (resp. $\mathfrak{a}^{\lceil cq\rceil - q/q_0} \not\subseteq J^{[q]}$) for all $q = p^e \gg q_0$.*

Proof. For (i)–(iv), see [MTW] (the proofs therein do not use the fact that R is regular). If $R \hookrightarrow S$ is cyclic pure, then $\nu_{\mathfrak{a}S}^{JS}(q) = \nu_{\mathfrak{a}}^{J}(q)$ for every q, and this yields (v).

For (vi), we fix a positive integer m such that $\mathfrak{a}^m \subseteq \mathfrak{c}(S/R)$. By the definition of the conductor ideal $\mathfrak{c}(S/R)$, if $(\mathfrak{a}S)^n \subseteq (JS)^{[q]}$ for some $n \in \mathbb{N}$ and some $q = p^e$ then $\mathfrak{a}^{m+n} \subseteq J^{[q]}$. This implies that

$$\nu^{JS}_{\mathfrak{a}S}(q) \leq \nu^J_{\mathfrak{a}}(q) \leq \nu^{JS}_{\mathfrak{a}S}(q) + m,$$

and these inequalities imply (vi).

In order to prove (vii), suppose first that $\mathrm{c}^J_+(\mathfrak{a}) \leq c$. It follows from the definition of $\mathrm{c}^J_+(\mathfrak{a})$ that, for every power q_0 of p, we can find a q_1 such that $\nu^J_{\mathfrak{a}}(q)/q < c + 1/q_0$ for all $q = p^e \geq q_1$. Thus, $\nu^J_{\mathfrak{a}}(q) < \lceil cq \rceil + q/q_0$; that is,

$$\mathfrak{a}^{\lceil cq \rceil + q/q_0} \subseteq J^{[q]} \tag{1}$$

for all $q = p^e \geq q_1$. Conversely, suppose that (1) holds for every $q \geq q_1$. This implies that $\nu^J_{\mathfrak{a}}(q) \leq \lceil cq \rceil + q/q_0 - 1$. Dividing by q and taking the limit gives $\mathrm{c}^J_+(\mathfrak{a}) \leq c + 1/q_0$. If this holds for every q_0, we conclude that $\mathrm{c}^J_+(\mathfrak{a}) \leq c$. The assertion regarding $\mathrm{c}^J_-(\mathfrak{a})$ follows from a similar argument. □

We now give a variant of the definition of F-threshold. If $\mathfrak{a}$ and J are ideals in R such that $\mathfrak{a} \cap R^\circ \neq \emptyset$ and $\mathfrak{a} \subseteq \sqrt{J}$, then we put

$$\tilde{\nu}^J_{\mathfrak{a}}(q) := \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq (J^{[q]})^F\}.$$

It follows from the definition of Frobenius closure that, if $u \notin (J^{[q]})^F$, then $u^p \notin (J^{[pq]})^F$. This means that

$$\frac{\tilde{\nu}^J_{\mathfrak{a}}(pq)}{pq} \geq \frac{\tilde{\nu}^J_{\mathfrak{a}}(q)}{q}$$

for all $q = p^e$. Therefore,

$$\lim_{q \to \infty} \frac{\tilde{\nu}^J_{\mathfrak{a}}(q)}{q} = \sup_{q = p^e} \frac{\tilde{\nu}^J_{\mathfrak{a}}(q)}{q}.$$

We denote this limit by $\tilde{\mathrm{c}}^J(\mathfrak{a})$. Observe that $\tilde{\mathrm{c}}^J(\mathfrak{a}) \leq \mathrm{c}^J_-(\mathfrak{a})$.

The F-threshold $\mathrm{c}^J(\mathfrak{a})$ exists in many cases.

LEMMA 2.3. *Let $\mathfrak{a}$ and J be as before.*

(i) *If $J^{[q]} = (J^{[q]})^F$ for all large $q = p^e$, then the F-threshold $\mathrm{c}^J(\mathfrak{a})$ exists; that is, $\mathrm{c}^J_+(\mathfrak{a}) = \mathrm{c}^J_-(\mathfrak{a})$. In particular, if R is F-pure, then $\mathrm{c}^J(\mathfrak{a})$ exists.*

(ii) *If the test ideal $\tau(R)$ contains $\mathfrak{a}$ in its radical, then the F-threshold $\mathrm{c}^J(\mathfrak{a})$ exists and $\mathrm{c}^J(\mathfrak{a}) = \mathrm{c}^{J^*}(\mathfrak{a})$.*

(iii) *If $\mathfrak{a}$ is principal, then $\mathrm{c}^J(\mathfrak{a})$ exists.*

Proof. Part (i) follows from the previous discussion, since in that case we have $\tilde{\nu}^J_{\mathfrak{a}}(q) = \nu^J_{\mathfrak{a}}(q)$ for all $q \gg 0$.

In order to prove (ii), we take an integer $m \geq 1$ such that $\mathfrak{a}^m \subseteq \tau(R)$. Then, by the definition of $\tau(R)$, one has $\mathfrak{a}^{2m}((J^*)^{[q]})^F \subseteq \mathfrak{a}^m (J^*)^{[q]} \subseteq J^{[q]}$ for all $q = p^e$. This means that

$$\tilde{\nu}^{J^*}_{\mathfrak{a}}(q) \leq \nu^{J^*}_{\mathfrak{a}}(q) \leq \nu^J_{\mathfrak{a}}(q) \leq \tilde{\nu}^{J^*}_{\mathfrak{a}}(q) + 2m.$$

Because $\tilde{c}^{J^*}(\mathfrak{a})$ always exists, $c^J(\mathfrak{a})$ and $c^{J^*}(\mathfrak{a})$ also exist and these three limits are all equal.

For part (iii), note that if $\mathfrak{a}$ is principal and $\mathfrak{a}^r \subseteq J^{[q]}$, then $a^{pr} \subseteq J^{[pq]}$. Hence, we have

$$\frac{\nu_{\mathfrak{a}}^J(pq)+1}{pq} \leq \frac{\nu_{\mathfrak{a}}^J(q)+1}{q}$$

for every $q = p^e$. This implies that

$$\lim_{q\to\infty} \frac{\nu_{\mathfrak{a}}^J(q)}{q} = \lim_{q\to\infty} \frac{\nu_{\mathfrak{a}}^J(q)+1}{q} = \inf_{q=p^e} \frac{\nu_{\mathfrak{a}}^J(q)}{q}. \qquad \square$$

As shown in [MTW, Prop. 2.7], the F-threshold $c^J(\mathfrak{a})$ coincides with the F-jumping exponent $\xi^J(\mathfrak{a})$ when the ring is F-finite and regular. The statement in [MTW] requires the ring to be local; however, the proof easily generalizes to the nonlocal case (see [BMS1]). More precisely, we have the following statement.

PROPOSITION 2.4. *Let R be an F-finite regular ring of characteristic $p > 0$. If $\mathfrak{a}$ is a nonzero ideal contained in the radical of J, then $\tau(\mathfrak{a}^{c^J(\mathfrak{a})}) \subseteq J$. Going the other way, if $\alpha \in \mathbb{R}_+$, then $\mathfrak{a}$ is contained in the radical of $\tau(\mathfrak{a}^\alpha)$ and $c^{\tau(\mathfrak{a}^\alpha)}(\mathfrak{a}) \leq \alpha$. In particular, the F-threshold $c^J(\mathfrak{a})$ coincides with the F-jumping exponent $\xi^J(\mathfrak{a})$.*

REMARK 2.5. The F-threshold $c^J(\mathfrak{a})$ sometimes coincides with the F-jumping exponent $\xi^J(\mathfrak{a})$ even when R is singular. For example, let $R = k[[X,Y,Z,W]]/(XY - ZW)$, and let $\mathfrak{m}$ be the maximal ideal of R. Then the F-threshold $c^{\mathfrak{m}}(\mathfrak{m})$ of $\mathfrak{m}$ with respect to $\mathfrak{m}$ and the F-pure threshold (i.e., the smallest F-jumping exponent) $\mathrm{fpt}(\mathfrak{m})$ of $\mathfrak{m}$ are both equal to 2.

However, $c^J(\mathfrak{a})$ does not agree with $\xi^J(\mathfrak{a})$ in general. For example, let $R = k[[X,Y,Z]]/(XY - Z^2)$ be a rational double point of type A_1 over a field k of characteristic $p > 2$, and let $\mathfrak{m}$ be the maximal ideal of R. Then $\mathrm{fpt}(\mathfrak{m}) = 1$ (see [TW, Ex. 2.5]) whereas $c^{\mathfrak{m}}(\mathfrak{m}) = 3/2$.

REMARK 2.6. Suppose that $\mathfrak{m}$ is a maximal ideal in any Noetherian ring R and that J is an $\mathfrak{m}$-primary ideal. For every $q = p^e$ we have $J^{[q]}R_{\mathfrak{m}} \cap R = J^{[q]}$, so for every ideal $\mathfrak{a} \subseteq \mathfrak{m}$ we have $\nu_{\mathfrak{a}}^J(q) = \nu_{\mathfrak{a}R_{\mathfrak{m}}}^{JR_{\mathfrak{m}}}(q)$. In particular, $c_{\pm}^J(\mathfrak{a}) = c_{\pm}^{JR_{\mathfrak{m}}}(\mathfrak{a}R_{\mathfrak{m}})$.

EXAMPLE 2.7. (i) Let R be a Noetherian local ring of characteristic $p > 0$, and let $J = (x_1, \ldots, x_d)$, where the sequence $x_1, \ldots, x_d$ forms a full system of parameters in R. It follows from the monomial conjecture (which is a theorem in this setting; see [H, Prop. 3]) that $(x_1 \cdots x_d)^{q-1} \notin J^{[q]}$ for every q. Hence $\nu_J^J(q) \geq d(q-1)$ for every q and therefore $c_-^J(J) \geq d$. On the other hand, $c_+^J(J) \leq d$ by Remark 2.1(a), and we conclude that $c^J(J) = d$.

(ii) Let $R = k[x_1, \ldots, x_d]$ be a d-dimensional polynomial ring over a field k of characteristic $p > 0$, and let $\mathfrak{a}, J \subseteq R$ be zero-dimensional ideals generated by monomials. In order to compute $c^J(\mathfrak{a})$, we may assume that k is perfect; hence we may use Proposition 2.4.

Let $P(\mathfrak{a}) \subseteq \mathbb{R}^d_{\geq 0}$ denote the Newton polyhedron of $\mathfrak{a}$. That is, $P(\mathfrak{a})$ is the convex hull of those $u = (u_1, \ldots, u_d) \in \mathbb{N}^d$ such that $x^u = x_1^{u_1} \cdots x_d^{u_d} \in \mathfrak{a}$. It follows from [HaY, Thm. 6.10] that

$$\tau(\mathfrak{a}^c) = (x^u \mid u + e \in \mathrm{Int}(c \cdot P_{\mathfrak{a}})),$$

where $e = (1, 1, \ldots, 1)$. We deduce that, if $\lambda(u)$ is defined by the condition $u + e \in \partial(\lambda(u) \cdot P(\mathfrak{a}))$, then

$$\mathrm{c}^J(\mathfrak{a}) = \max\{\lambda(u) \mid u \in \mathbb{N}^d,\ x^u \notin J\}$$

(note that, since J is zero-dimensional, this maximum is over a finite set). In particular, we see that if $J = (x_1^{a_1}, \ldots, x_d^{a_d})$ then $\mathrm{c}^J(\mathfrak{a})$ is characterized by $a = (a_1, \ldots, a_d) \in \partial(\mathrm{c}^J(\mathfrak{a}) \cdot P(\mathfrak{a}))$.

(iii) Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring of characteristic $p > 0$, and let $J \subset R$ be an $\mathfrak{m}$-primary ideal. We claim that

$$\mathrm{c}^J(\mathfrak{m}) = \max\{r \in \mathbb{Z}_{\geq 0} \mid \mathfrak{m}^r \not\subseteq J\} + d. \tag{2}$$

In particular, $\mathrm{c}^J(\mathfrak{m})$ is an integer $\geq d$.

Indeed, if $u \notin J$ then $(J : u) \subseteq \mathfrak{m}$; hence $J^{[q]} : u^q = (J : u)^{[q]} \subseteq \mathfrak{m}^{[q]}$ and therefore $u^q \mathfrak{m}^{d(q-1)} \not\subseteq J^{[q]}$. If $u \in \mathfrak{m}^r$, then $\nu_{\mathfrak{m}}^J(q) \geq rq + d(q-1)$. Dividing by q and passing to the limit yields $\mathrm{c}^J(\mathfrak{m}) \geq r + d$, so we have "$\geq$" in (2). For the reverse inequality, note that if $\mathfrak{m}^{r+1} \subseteq J$ then

$$\mathfrak{m}^{(r+d)q} \subseteq (\mathfrak{m}^{r+1})^{[q]} \subseteq J^{[q]}$$

for every $q = p^e$. Hence $\nu_{\mathfrak{m}}^J(q) \leq (r + d)q - 1$ for all q, and we obtain $\mathrm{c}^J(\mathfrak{m}) \leq r + d$.

3. Connections with Tight Closure and Integral Closure

THEOREM 3.1. *Let $(R, \mathfrak{m})$ be an excellent analytically irreducible Noetherian local domain of positive characteristic p. Set $d = \dim(R)$, let $J = (x_1, \ldots, x_d)$ be an ideal generated by a full system of parameters in R, and let $I \supseteq J$ be another ideal. Then I is not contained in the tight closure J^* of J if and only if there exists a $q_0 = p^{e_0}$ such that $x^{q_0 - 1} \in I^{[q_0]}$, where $x = x_1 x_2 \cdots x_d$.*

Proof. After passing to completion, we may assume that R is a complete local domain. Suppose first that $x^{q_0-1} \in I^{[q_0]}$ and, by way of contradiction, suppose also that $I \subseteq J^*$. Let $c \in R^\circ$ be a test element. Then, for all $q = p^e$, one has $cx^{q(q_0-1)} \in cI^{[qq_0]} \subset J^{[qq_0]}$ and so $c \in J^{[qq_0]} : x^{q(q_0-1)} \subseteq (J^{[q]})^*$ by colon-capturing [HHu, Thm. 7.15a]. Therefore, c^2 lies in $\bigcap_{q=p^e} J^{[q]} = (0)$—a contradiction.

Conversely, suppose that $I \not\subseteq J^*$, and choose an element $f \in I \setminus J^*$. We choose a coefficient field k and let $B = k[[x_1, \ldots, x_d, f]]$ be the complete subring of R generated by $x_1, \ldots, x_d, f$. Note that B is a hypersurface singularity and hence is Gorenstein. Furthermore, by persistence of tight closure [HHu, Lemma 4.11a], $f \notin ((x_1, \ldots, x_d)B)^*$. If we prove that there exists a $q_0 = p^{e_0}$ such that $x^{q_0-1} \in ((x_1, \ldots, x_d, f)B)^{[q_0]}$, then clearly x^{q_0-1} is also in $I^{[q_0]}$. Hence we can reduce

to the case in which R is Gorenstein. Since $I \not\subseteq J^*$, it follows from a result of Aberbach [A] that $J^{[q]} : I^{[q]} \subseteq \mathfrak{m}^{n(q)}$, where $n(q)$ is a positive integer with $\lim_{q\to\infty} n(q) = \infty$. In particular, we can find $q_0 = p^{e_0}$ such that $J^{[q_0]} : I^{[q_0]} \subseteq J$. As a result, $x^{q_0-1} \in J^{[q_0]} : J \subseteq J^{[q_0]} : (J^{[q_0]} : I^{[q_0]}) = I^{[q_0]}$, where the last equality follows from the fact that R is Gorenstein. □

COROLLARY 3.2. *Let $(R, \mathfrak{m})$ be a d-dimensional excellent analytically irreducible Noetherian local domain of characteristic $p > 0$, and let $J = (x_1, \ldots, x_d)$ be an ideal generated by a full system of parameters in R. Given an ideal $I \supseteq J$, we have $I \subseteq J^*$ if and only if $\mathrm{c}^I_+(J) = d$ (and in this case $\mathrm{c}^I(J)$ exists). In particular, R is F-rational if and only if $\mathrm{c}^I_+(J) < d$ for every ideal $I \supsetneq J$.*

Proof. Note first that, by Remark 2.1(a), for every $I \supseteq J$ we have $\mathrm{c}^J_+(I) \leq d$. Suppose now that $I \subseteq J^*$. It follows from Theorem 3.1 that $J^{d(q-1)} \not\subseteq I^{[q]}$ for every $q = p^e$. This gives $\nu^I_J(q) \geq d(q-1)$ for all q, and therefore $\mathrm{c}^I_-(J) \geq d$. We conclude that, in this case, $\mathrm{c}^I_+(J) = \mathrm{c}^I_-(J) = d$.

Conversely, suppose that $I \not\subseteq J^*$. By Theorem 3.1, we can find a $q_0 = p^{e_0}$ such that

$$\mathfrak{b} := (x_1^{q_0}, \ldots, x_d^{q_0}, (x_1 \cdots x_d)^{q_0-1}) \subseteq I^{[q_0]}.$$

If $(x_1, \ldots, x_d)^r \not\subseteq \mathfrak{b}^{[q]}$, then

$$r \leq (qq_0 - 1)(d-1) + q(q_0 - 1) - 1 = qq_0 d - q - d.$$

Hence $\nu^{\mathfrak{b}}_J(q) \leq qq_0 d - q - d$ for every q, which implies $\mathrm{c}^{\mathfrak{b}}(J) \leq q_0 d - 1$. Because q_0 is a fixed power of p, we deduce that

$$\mathrm{c}^I_+(J) = \frac{1}{q_0}\mathrm{c}^{I^{[q_0]}}_+(J) \leq \frac{1}{q_0}\mathrm{c}^{\mathfrak{b}}(J) \leq d - \frac{1}{q_0} < d. \qquad \square$$

THEOREM 3.3. *Let $(R, \mathfrak{m})$ be a d-dimensional formally equidimensional Noetherian local ring of characteristic $p > 0$. If I and J are ideals in R, with J generated by a full system of parameters, then the following statements hold.*

(i) *$\mathrm{c}^J_+(I) \leq d$ if and only if $I \subseteq \bar{J}$.*

(ii) *If, in addition, $J \subseteq I$, then $I \subseteq \bar{J}$ if and only if $\mathrm{c}^J_+(I) = d$. Moreover, if these equivalent conditions hold then $\mathrm{c}^J(I) = d$.*

Proof. Observe that, by Example 2.7(i), if $J \subseteq I$ then $\mathrm{c}^J_-(I) \geq \mathrm{c}^J_-(J) = \mathrm{c}^J(J) = d$. Hence both assertions in Theorem 3.3(ii) follow from the assertion in (i).

One implication in (i) is easy: if $I \subseteq \bar{J}$ then, by Proposition 2.2(ii), $\mathrm{c}^J_+(I) \leq \mathrm{c}^J_+(\bar{J}) = \mathrm{c}^J(J) = d$. Conversely, suppose that $\mathrm{c}^J_+(I) \leq d$. In order to show that $I \subseteq \bar{J}$, we may assume that R is complete and reduced. Indeed, first note that the inverse image of $J\hat{R}_{\mathrm{red}}$ in R is contained in $\bar{J}$; hence it is enough to show that $I\hat{R}_{\mathrm{red}} \subseteq \overline{J\hat{R}_{\mathrm{red}}}$. Since $J\hat{R}_{\mathrm{red}}$ is again generated by a full system of parameters and since we (trivially) have

$$\mathrm{c}^{J\hat{R}_{\mathrm{red}}}(I\hat{R}_{\mathrm{red}}) \leq \mathrm{c}^J(I) \leq d,$$

we may replace R by $\hat{R}_{\mathrm{red}}$.

Since R is complete and reduced, we can find a test element c for R. By Proposition 2.2(vii), the assumption $\mathrm{c}^J_+(I) \leq d$ implies that, for all $q_0 = p^{e_0}$ and for all large $q = p^e$,

$$I^{q(d+(1/q_0))} \subseteq J^{[q]}.$$

Hence $I^q I^{q(d-1+(1/q_0))} \subseteq J^{[q]}$ and thus

$$I^q \subseteq J^{[q]} : I^{q(d-1+(1/q_0))} \subseteq (J^{q-d+1-(q/q_0)})^*,$$

where the last containment follows from the colon-capturing property of tight closure [HHu, Thm. 7.15a]. By the Artin–Rees lemma we then obtain $cI^q \subseteq cR \cap J^{q-d+1-(q/q_0)} \subseteq cJ^{q-d+1-(q/q_0)-l}$ for some fixed integer l that is independent of q. Since c is a nonzero divisor in R, it follows that

$$I^q \subseteq J^{q-d+1-(q/q_0)-l}. \tag{3}$$

If ν is a discrete valuation with center in $\mathfrak{m}$, then we may apply ν to (3) and so deduce $q\nu(I) \geq (q-d+1-q/q_0-l)\nu(J)$. Dividing by q and letting q go to infinity yields $\nu(I) \geq (1-1/q_0)\nu(J)$. We now let q_0 go to infinity to obtain $\nu(I) \geq \nu(J)$. Since this holds for every ν, we have $I \subseteq \bar{J}$. □

EXAMPLE 3.4. Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$ with $\dim(R) = d$, and let J be an ideal of R generated by a full system of parameters. We define a to be the maximal integer n such that $\mathfrak{m}^n \not\subseteq J$. Then $\mathfrak{m}^s \subseteq \bar{J}$ if and only if $s \geq \frac{a}{d} + 1$ because $\mathrm{c}^J(\mathfrak{m}^s) = \frac{a+d}{s}$ by Example 2.7(iii) and Proposition 2.2(iii).

QUESTIONS 3.5. Does this statement hold in a more general setting? Can we replace "regular" by "Cohen–Macaulay"?

4. F-Thresholds of Modules

In this section we give a generalization of the notion of F-thresholds in which we replace the auxiliary ideal in the definition by a submodule of a given module. We have seen in Proposition 2.4 that, in a regular F-finite ring, the F-thresholds of an ideal $\mathfrak{a}$ coincide with the F-jumping exponents of $\mathfrak{a}$. This might fail in nonregular rings; in fact, it is often the case that $\mathrm{fpt}(\mathfrak{a}) < \mathrm{c}^J(\mathfrak{a})$ for every ideal J. However, as Corollary 4.4 shows, we can remedy this situation if we consider the following more general notion of F-thresholds.

Suppose now that $\mathfrak{a}$ is a fixed ideal in a Noetherian ring R of characteristic $p > 0$. Let M be an R-module, and let $N \subseteq M$ be a submodule such that $\mathfrak{a}^n N = 0$ for some $n > 0$.

- For $q = p^e$, let $\nu^N_{M,\mathfrak{a}}(q) = \max\{r \in \mathbb{N} \mid \mathfrak{a}^r N^{[q]}_M \neq 0\}$ (we put $\nu^N_{M,\mathfrak{a}}(q) = 0$ if $\mathfrak{a} N^{[q]}_M = 0$).
- We define

$$\mathrm{c}^N_{M,+}(\mathfrak{a}) = \limsup_{q\to\infty} \frac{\nu^N_{M,\mathfrak{a}}(q)}{q} \quad \text{and} \quad \mathrm{c}^N_{M,-}(\mathfrak{a}) = \liminf_{q\to\infty} \frac{\nu^N_{M,\mathfrak{a}}(q)}{q}.$$

When $\mathrm{c}^N_{M,+}(\mathfrak{a}) = \mathrm{c}^N_{M,-}(\mathfrak{a})$, we call this limit the *F-threshold* of $\mathfrak{a}$ with respect to (N, M) and denote it by $\mathrm{c}^N_M(\mathfrak{a})$.

REMARK 4.1. If J is an ideal of R with $\mathfrak{a} \subseteq \sqrt{J}$, then clearly $\nu^{A/J}_{\mathfrak{a},A/J}(q) = \nu^J_{\mathfrak{a}}(q)$ and so $\mathrm{c}^{A/J}_{A/J,\pm}(\mathfrak{a}) = \mathrm{c}^J_{\pm}(\mathfrak{a})$. Thus the notion of F-threshold with respect to modules extends our previous definition of F-thresholds with respect to ideals.

LEMMA 4.2. *Let R, $\mathfrak{a}$, M, and N be as defined previously.*

(i) *If $\mathfrak{b} \subseteq \mathfrak{a}$ is an ideal, then $\mathrm{c}^N_{M,\pm}(\mathfrak{b}) \leq \mathrm{c}^N_{M,\pm}(\mathfrak{a})$.*
(ii) *If $N' \subseteq N$, then $\mathrm{c}^{N'}_{M,\pm}(\mathfrak{a}) \leq \mathrm{c}^N_{M,\pm}(\mathfrak{a})$.*
(iii) *If $\phi: M \to M'$ is a homomorphism of R-modules and if $N' = \phi(N)$, then $\mathrm{c}^{N'}_{M',\pm}(\mathfrak{a}) \leq \mathrm{c}^N_{M,\pm}(\mathfrak{a})$. If R is regular and ϕ is injective, then $\mathrm{c}^{N'}_{M',\pm}(\mathfrak{a}) = \mathrm{c}^N_{M,\pm}(\mathfrak{a})$.*
(iv) *If R is F-pure, then $\nu^N_{M,\mathfrak{a}}(q)/q \leq \nu^N_{M,\mathfrak{a}}(qq')/qq'$ for every q, q'. Hence, in this case the limit $\mathrm{c}^N_M(\mathfrak{a})$ exists and is equal to $\sup_q(\nu^N_{M,\mathfrak{a}}(q)/q)$.*

Proof. The assertions in (i) and (ii) follow by definition. For (iii), note that ϕ induces a surjection $N^{[q]} \to N'^{[q]}$, which gives the first statement. Moreover, if R is regular and ϕ is injective, then the flatness of the Frobenius morphism implies $N^{[q]} \simeq N'^{[q]}$, and we have equality.

Suppose now that R is F-pure; hence $M \otimes_R {}^eR$ is a submodule of $M \otimes_R {}^{ee'}R$. If $q = p^e$ and $q' = p^{e'}$ and if $\mathfrak{a}^r N^{[q]} \neq 0$, then $\mathfrak{a}^{q'r} N^{[qq']} \supseteq (\mathfrak{a}^r)^{[q']} N^{[qq']} \neq 0$. Therefore, $\nu^N_{M,\mathfrak{a}}(qq') \geq q' \cdot \nu^N_{M,\mathfrak{a}}(q)$. □

Our next proposition gives an analogue of Proposition 2.4 for the nonregular case.

PROPOSITION 4.3. *Let $\mathfrak{a}$ be a proper nonzero ideal in a local normal $\mathbb{Q}$-Gorenstein ring $(R, \mathfrak{m})$. Suppose that R is F-finite and F-pure and that the test ideal $\tau(R)$ is $\mathfrak{m}$-primary. We denote by E the injective hull of $R/\mathfrak{m}$.*

(i) *If N is a submodule of E such that $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}_R(N)}$ and if $\alpha = \mathrm{c}^N_E(\mathfrak{a})$, then $N \subseteq (0)^{*\mathfrak{a}^\alpha}_E$.*
(ii) *If α is a nonnegative real number and if $N = (0)^{*\mathfrak{a}^\alpha}_E$, then $\mathrm{c}^N_E(\mathfrak{a}) \leq \alpha$.*
(iii) *There is an order-reversing bijection between the F-thresholds of $\mathfrak{a}$ with respect to the submodules of E and the ideals of the form $\tau(\mathfrak{a}^\alpha)$.*

Proof. For (i), observe that since R is F-pure, we have $\nu^N_E(q) \leq \alpha q$ for every $q = p^e$. This implies

$$\mathfrak{a}^{\lceil \alpha q \rceil + 1} N^{[q]}_E = 0;$$

hence, for every nonzero $d \in \mathfrak{a}$, we have $d\mathfrak{a}^{\lceil \alpha q \rceil} N^{[q]}_E = 0$ for all q. By definition, $N \subseteq (0)^{*\mathfrak{a}^\alpha}_E$.

Suppose now that $\alpha \geq 0$ and that $N = (0)^{*\mathfrak{a}^\alpha}_E$. By hypothesis, we can find an m such that $\mathfrak{a}^m \subseteq \tau(R)$. It follows from [HaT, Cor. 2.4] that every element in $\tau(R)$ is an $\mathfrak{a}^\alpha$-test element. Therefore, $\mathfrak{a}^{m+\lceil \alpha q \rceil} N^{[q]}_E = 0$ and so $\nu^N_{E,\mathfrak{a}}(q) < m + \alpha q$ for all $q \gg 0$. Dividing by q and then taking the limit as q goes to infinity yields $\mathrm{c}^N_E(\mathfrak{a}) \leq \alpha$.

We assume that R is F-finite, normal, and $\mathbb{Q}$-Gorenstein; hence for every nonnegative t we have $\tau(\mathfrak{a}^t) = \operatorname{Ann}_R(0^{*\mathfrak{a}^t}_E)$. Note also that, by [HaT, Prop. 3.2], taking the generalized test ideal commutes with completion. This shows that the set of ideals of the form $\tau(\mathfrak{a}^\alpha)$ is in bijection with the set of submodules of E of the form $(0)^{*\mathfrak{a}^\alpha}_E$. Hence in order to prove (iii) it is enough to show that the map

$$\{(0)^{*\mathfrak{a}^\alpha}_E \mid \alpha \geq 0\} \to \left\{\mathrm{c}^N_E(\mathfrak{a}) \mid N \subseteq E,\ \mathfrak{a} \subseteq \sqrt{\operatorname{Ann}_R(N)}\right\}$$

taking N to $\mathrm{c}^N_E(\mathfrak{a})$ is bijective, with the inverse map taking α to $(0)^{*\mathfrak{a}^\alpha}_E$.

Suppose first that $N = (0)^{*\mathfrak{a}^\alpha}_E$ and let $\beta = \mathrm{c}^N_E(\mathfrak{a})$. It follows from (ii) that $\beta \leq \alpha$, hence $(0)^{*\mathfrak{a}^\beta}_E \subseteq N$. On the other hand, (i) gives $N \subseteq (0)^{*\mathfrak{a}^\beta}_E$ and so we have equality.

Let us now start with $\alpha = \mathrm{c}^N_E(\mathfrak{a})$ and let $N' = (0)^{*\mathfrak{a}^\alpha}_E$. We deduce from (i) that $N \subseteq N'$, so $\mathrm{c}^{N'}_E(\mathfrak{a}) \geq \alpha$. Since (ii) implies $\mathrm{c}^{N'}_E(\mathfrak{a}) \leq \alpha$ it follows that $\alpha = \mathrm{c}^{N'}_E(\mathfrak{a})$, which completes the proof of (iii). □

COROLLARY 4.4. *Let $\mathfrak{a}$ be a proper nonzero ideal in a local normal $\mathbb{Q}$-Gorenstein ring $(R, \mathfrak{m})$. If R is F-finite and F-regular then, for every ideal J in R,*

$$\xi^J(\mathfrak{a}) = \mathrm{c}^N_E(\mathfrak{a}),$$

where E is the injective hull of $R/\mathfrak{m}$ and $N = \operatorname{Ann}_E(J)$. In particular, the F-pure threshold $\operatorname{fpt}(\mathfrak{a})$ *is equal to* $\mathrm{c}^Z_E(\mathfrak{a})$, *where $Z = (0 :_E \mathfrak{m})$ is the socle of E.*

Proof. Let $\beta := \mathrm{c}^N_E(\mathfrak{a})$. Given $\alpha \geq 0$, Matlis duality implies that $\tau(\mathfrak{a}^\alpha) \subseteq J$ if and only if $N \subseteq (0)^{*\mathfrak{a}^\alpha}_E$. If this holds, then part (ii) of the proposition gives

$$\alpha \geq \mathrm{c}_E^{(0)^{*\mathfrak{a}^\alpha}_E}(\mathfrak{a}) \geq \mathrm{c}^N_E(\mathfrak{a}) = \beta.$$

Conversely, if $\alpha \geq \beta$ then

$$(0)^{*\mathfrak{a}^\alpha}_E \supseteq (0)^{*\mathfrak{a}^\beta}_E \supseteq N$$

by part (i) of the proposition. This shows that $\mathrm{c}^N_E(\mathfrak{a}) = \xi^J(\mathfrak{a})$, and the last assertion in the corollary follows by taking $J = \mathfrak{m}$. □

REMARK 4.5. Let $\mathfrak{a}$ be an ideal in the local ring $(R, \mathfrak{m})$. We have seen that $\mathrm{c}^I(\mathfrak{a}) \geq \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$ for every proper ideal I. Note also that applying Proposition 4.2(iii) to the embedding $R/m \simeq Z \hookrightarrow E = E_R(R/\mathfrak{m})$ yields $\mathrm{c}^{\mathfrak{m}}(\mathfrak{a}) = \mathrm{c}^{R/\mathfrak{m}}_{R/\mathfrak{m}}(\mathfrak{a}) \geq \mathrm{c}^Z_E(\mathfrak{a}) = \operatorname{fpt}(\mathfrak{a})$. Thus we always have $\operatorname{fpt}(\mathfrak{a}) \leq \mathrm{c}^I(\mathfrak{a})$, and equality is possible only if $\operatorname{fpt}(\mathfrak{a}) = \mathrm{c}^{\mathfrak{m}}(\mathfrak{a})$. Although this equality holds in some nonregular examples (see Remark 2.5), this seems to happen rather rarely.

5. Connections between F-Thresholds and Multiplicity

For an $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in a regular local ring $(R, \mathfrak{m})$ that is essentially of finite type over a field of characteristic 0, an inequality involving the log canonical threshold $\operatorname{lct}(\mathfrak{a})$ and the multiplicity $e(\mathfrak{a})$ was proved in [dFEM2]. Later, a

characteristic p analogue of this result—replacing the log canonical threshold $\mathrm{lct}(\mathfrak{a})$ by the F-pure threshold $\mathrm{fpt}(\mathfrak{a})$—was proved in [TW]. We propose to generalize this inequality via the following conjecture.

CONJECTURE 5.1. *Let $(R, \mathfrak{m})$ be a d-dimensional Noetherian local ring of characteristic $p > 0$. If $J \subseteq \mathfrak{m}$ is an ideal generated by a full system of parameters and if $\mathfrak{a} \subseteq \mathfrak{m}$ is an $\mathfrak{m}$-primary ideal, then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{\mathrm{c}^J_-(\mathfrak{a})}\right)^d e(J).$$

REMARKS 5.2. (a) When R is regular and $J = \mathfrak{m}$, Conjecture 5.1 is precisely the inequality in [TW, Prop. 4.5].

(b) When R is a d-dimensional regular local ring that is essentially of finite type over a field of characteristic 0, we can consider an analogous conjecture: *Let $\mathfrak{a}, J$ be $\mathfrak{m}$-primary ideals in R such that J is generated by a full system of parameters*; *then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{\lambda^J(\mathfrak{a})}\right)^d e(J),$$

where $\lambda^J(\mathfrak{a}) := \max\{c > 0 \mid \mathcal{J}(\mathfrak{a}^c) \not\subseteq J\}$. This would generalize the inequality in [dFEM2], which is the special case $J = \mathfrak{m}$. However, this version is also open in general.

(c) The condition in Conjecture 5.1 that J be generated by a system of parameters is crucial, since otherwise there are plenty of counterexamples. Suppose, for example, that $(R, \mathfrak{m})$ is a regular local ring of dimension $d \geq 2$ and of characteristic $p > 0$. Let $\mathfrak{a} = \mathfrak{m}^k$ and $J = \mathfrak{m}^\ell$ for integers $k \geq 1$ and $\ell \geq 2$. It follows from Example 2.7(iii) that $\mathrm{c}^J(\mathfrak{a}) = (d + \ell - 1)/k$. Moreover, we have $e(\mathfrak{a}) = k^d$ and $e(J) = \ell^d$; thus

$$e(\mathfrak{a}) = k^d < (dk\ell/(d + \ell - 1))^d = \left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d e(J).$$

EXAMPLE 5.3. Let $R = k[[X, Y, Z]]/(X^2 + Y^3 + Z^5)$ be a rational double point of type E_8, with k a field of characteristic $p > 0$. Let $\mathfrak{a} = (x, z)$ and $J = (y, z)$. Then $e(\mathfrak{a}) = 3$ and $e(J) = 2$. It is easy to check that $\mathrm{c}^J(\mathfrak{a}) = 5/3$ and $\mathrm{c}^{\mathfrak{a}}(J) = 5/2$. Thus:

$$e(\mathfrak{a}) = 3 > \frac{72}{25} = \left(\frac{2}{\mathrm{c}^J(\mathfrak{a})}\right)^2 e(J);$$

$$e(J) = 2 > \frac{48}{25} = \left(\frac{2}{\mathrm{c}^{\mathfrak{a}}(J)}\right)^2 e(\mathfrak{a}).$$

See Corollary 5.9 for a general statement in the homogeneous case.

We now show that Conjecture 5.1 implies an effective estimate of the multiplicity of complete intersection F-rational rings.

PROPOSITION 5.4. *Let $(R, \mathfrak{m})$ be a d-dimensional F-rational local ring of characteristic $p > 0$ with infinite residue field (resp., a rational singularity over a field of characteristic 0) that is a complete intersection. If Conjecture 5.1 (resp., Remark 5.2*(a)*) holds true for the regular case, then $e(R) \leq 2^{d-1}$.*

Proof. Let $J \subseteq \mathfrak{m}$ be a minimal reduction of $\mathfrak{m}$. Observe that J is generated by a full system of parameters for R. The Briançon–Skoda theorem for F-rational rings (or for rational singularities; see [AHu] or [HyV], respectively) gives $\mathfrak{m}^d \subseteq J$. Taking the quotient of R by J, we reduce the assertion in the proposition to the following claim.

Claim. Let $(A, \mathfrak{m})$ be a complete intersection Artinian local ring of characteristic $p > 0$ (resp., essentially of finite type over a field of characteristic 0). If s is the largest integer s such that $\mathfrak{m}^s \neq 0$, then $e(A) \leq 2^s$.

We now show that the regular case of Conjecture 5.1 implies the claim in positive characteristic (the argument in characteristic 0 is entirely analogous). Write $A = S/I$, where $(S, \mathfrak{n})$ is an n-dimensional regular local ring and $I \subseteq S$ is an ideal generated by a full system of parameters $f_1, \ldots, f_n$ for S. For every i, we denote by α_i the order of f_i. We may assume that $\alpha_i \geq 2$ for all i.

Let $\mathfrak{n} = (y_1, \ldots, y_n)$, and let us write $f_i = \sum_j a_{ij} y_j$. A standard argument relating the Koszul complexes on the f_i and (respectively) the y_i shows that $\det(a_{ij})$ generates the socle of A. In particular, if

$$s := \max\{r \in \mathbb{N} \mid \mathfrak{n}^r \not\subseteq I\},$$

then $s \geq \sum_{i=1}^n (a_i - 1) \geq n$. On the other hand, it follows from Example 2.7(iii) that $\mathrm{c}^I(\mathfrak{m}) = s + n$ (the corresponding formula in characteristic 0 is an immediate consequence of the description of the multiplier ideals of the ideal of a point). Applying Conjecture 5.1 to S, we obtain

$$1 = e(\mathfrak{n}) \geq \left(\frac{n}{\mathrm{c}^I(\mathfrak{m})}\right)^n e(I) = \left(\frac{n}{s+n}\right)^n e(I).$$

Note that $(n/(s+n))^n \geq (s/(s+s))^s = (1/2)^s$ because $s \geq n$. Thus, we have $e(A) = e(I) \leq 2^s$. □

PROPOSITION 5.5. *If $(R, \mathfrak{m})$ is a one-dimensional analytically irreducible local domain of characteristic $p > 0$ and if $\mathfrak{a}, J$ are $\mathfrak{m}$-primary ideals in R, then*

$$\mathrm{c}^J(\mathfrak{a}) = \frac{e(J)}{e(\mathfrak{a})}.$$

In particular, Conjecture 5.1 holds in R.

Proof. By Proposition 2.2(v), we may assume that R is a complete local domain. Since R is one-dimensional, the integral closure $\bar{R}$ is a discrete valuation ring. We thus have

$$\mathrm{c}^{J\bar{R}}(\mathfrak{a}\bar{R}) = \mathrm{ord}_{\bar{R}}(J\bar{R})/\mathrm{ord}_{\bar{R}}(\mathfrak{a}\bar{R}).$$

On the other hand, $e(J\bar{R}) = \operatorname{ord}_{\bar{R}}(J\bar{R})$ and $e(\mathfrak{a}\bar{R}) = \operatorname{ord}_{\bar{R}}(\mathfrak{a}\bar{R})$. Hence, by Proposition 2.2(vi),

$$\mathrm{c}^J(\mathfrak{a}) = \mathrm{c}^{J\bar{R}}(\mathfrak{a}\bar{R}) = \frac{e(J\bar{R})}{e(\mathfrak{a}\bar{R})} = \frac{e(J)}{e(\mathfrak{a})}. \qquad \square$$

THEOREM 5.6. *If $(R, \mathfrak{m})$ is a regular local ring of characteristic $p > 0$, and if $J = (x_1^{a_1}, \ldots, x_d^{a_d})$ with $x_1, \ldots, x_d$ a full regular system of parameters for R and with $a_1, \ldots, a_d$ positive integers, then the inequality given by Conjecture 5.1 holds.*

Proof. The proof follows the idea in [dFEM2] and [TW], reducing the assertion to the case when $\mathfrak{a}$ is a monomial ideal and then using the explicit description of the invariants involved. By definition we have $e(\mathfrak{a}) = \lim_{n\to\infty}(d! \cdot \ell_R(R/\mathfrak{a}^n)/n^d)$; hence it is enough to show that, for every $\mathfrak{m}$-primary ideal $\mathfrak{a}$ of R,

$$\ell_R(R/\mathfrak{a}) \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d e(J). \tag{4}$$

After passing to completion and then using Proposition 2.2(v) and Remark 2.6, we see that it is enough to prove the inequality (4) in the case when $R = k[x_1, \ldots, x_d]$, $\mathfrak{m} = (x_1, \ldots, x_d)$, $\mathfrak{a}$ is $\mathfrak{m}$-primary, and $J = (x_1^{a_1}, \ldots, x_d^{a_d})$.

Observe that $e(J) = a_1 \cdots a_d$. We fix a monomial order λ on the monomials in the polynomial ring and then use λ to take a Gröbner deformation of $\mathfrak{a}$ (see [Ei, Chap. 15]). This is a flat family $\{\mathfrak{a}_s\}_{s\in k}$ such that $R/\mathfrak{a}_s \cong R/\mathfrak{a}$ for all $s \neq 0$ and such that $\mathfrak{a}_0 = \mathrm{in}_\lambda(\mathfrak{a})$, the initial ideal of $\mathfrak{a}$.

For I an ideal generated by monomials, we denote by $P(I)$ the Newton polyhedron of I (see Example 2.7(ii) for definition). We also use $\mathrm{Vol}(P)$ to denote the volume of a region P in $\mathbb{R}^d$ with the Euclidean metric. Since the deformation we consider is flat, it follows that $\mathrm{in}_\lambda(\mathfrak{a})$ is also $\mathfrak{m}$-primary and that

$$\ell_R(R/\mathfrak{a}) = \ell_R(R/\mathrm{in}_\lambda(\mathfrak{a})) \geq \mathrm{Vol}\big(\mathbb{R}^d_{\geq 0} \setminus P(\mathrm{in}_\lambda(\mathfrak{a}))\big),$$

where the inequality follows from [dFEM2, Lemma 1.3].

On the other hand, by [dF, Prop. 5.3] we have $\tau(\mathrm{in}_\lambda(\mathfrak{a})^t) \subseteq \mathrm{in}_\lambda(\tau(\mathfrak{a}^t))$ for all $t > 0$. This implies that $\mathrm{c}^J(\mathfrak{a}) \geq \mathrm{c}^{\mathrm{in}_\lambda(J)}(\mathrm{in}_\lambda(\mathfrak{a}))$. Note also that, since J is generated by monomials, $\mathrm{in}_\lambda(J) = J$. We can therefore reduce to the case when $\mathfrak{a}$ is generated by monomials in $x_1, \ldots, x_d$. That is, it is enough to show that, for every $\mathfrak{m}$-primary monomial ideal $\mathfrak{a} \subseteq R$,

$$\mathrm{Vol}(\mathbb{R}^d_{\geq 0} \setminus P(\mathfrak{a})) \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d a_1 \cdots a_d.$$

It follows from the description of $\mathrm{c}^J(\mathfrak{a})$ in Example 2.7(ii) that $(a_1, \ldots, a_d) \in \partial(\mathrm{c}^J(a) \cdot P(\mathfrak{a}))$. We can find a hyperplane $H_q := u_1/b_1 + \cdots + u_d/b_d = 1$ passing through the point $(a_1, \ldots, a_d)$ such that

$$H^+ := \left\{(u_1, \ldots, u_d) \in \mathbb{R}^d_{\geq 0} \;\middle|\; \frac{u_1}{b_1} + \cdots + \frac{u_d}{b_d} \geq 1\right\} \supseteq \mathrm{c}^J(\mathfrak{a}) \cdot P(\mathfrak{a}).$$

We thus have

$$\mathrm{Vol}(\mathbb{R}^d_{\geq 0} \setminus P(\mathfrak{a})) \geq \mathrm{Vol}\left(\mathbb{R}^d_{\geq 0} \setminus \frac{1}{\mathrm{c}^J(\mathfrak{a})} H^+\right) = \frac{b_1 \cdots b_d}{d! \cdot \mathrm{c}^J(\mathfrak{a})^d}.$$

On the other hand, since H passes through $(a_1, \ldots, a_d)$, it follows that $a_1/b_1 + \cdots + a_d/b_d = 1$. Comparing the arithmetic and geometric means of $\{a_i/b_i\}_i$, we see that

$$b_1 \cdots b_d \geq d^d \cdot a_1 \cdots a_d.$$

We can therefore combine these two inequalities to obtain

$$\mathrm{Vol}(\mathbb{R}^d_{\geq 0} \setminus P(\mathfrak{a})) \geq \frac{b_1 \cdots b_d}{d! \cdot \mathrm{c}^J(\mathfrak{a})^d} \geq \frac{1}{d!}\left(\frac{d}{\mathrm{c}^J(\mathfrak{a})}\right)^d a_1 \cdots a_d,$$

as required. □

REMARK 5.7. It might seem that this proves a stronger assertion than the one in Conjecture 5.1, one that involves the length instead of the multiplicity. However, the two assertions are equivalent; this follows from [Mu Cor. 3.8], which states that, for every zero-dimensional ideal $\mathfrak{a}$ in a d-dimensional regular local ring R,

$$\ell_R(R/\mathfrak{a}) \geq \frac{e(\mathfrak{a})}{d!}.$$

We can prove a graded version of Conjecture 5.1. In fact, we prove a more precise statement that is valid independently of the characteristic.

THEOREM 5.8. *Let $R = \bigoplus_{d \geq 0} R_d$ be an n-dimensional graded Cohen–Macaulay ring with R_0 a field of arbitrary characteristic. If $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters for R and if $\mathfrak{a}^N \subseteq J$, then*

$$e(\mathfrak{a}) \geq \left(\frac{n}{n+N-1}\right)^n e(J).$$

COROLLARY 5.9. *Let R be as in the theorem, with $\mathrm{char}(R_0) = p > 0$. If $\mathfrak{a}$ and J are ideals generated by full homogeneous systems of parameters for R, then*

$$e(\mathfrak{a}) \geq \left(\frac{n}{\mathrm{c}^J_-(\mathfrak{a})}\right)^n e(J).$$

Proof. Note that each $J^{[q]}$ is again generated by a full homogeneous system of parameters. It follows from the theorem and from the definition of $\nu^J_{\mathfrak{a}}(q)$ that, for every $q = p^e$,

$$e(\mathfrak{a}) \geq \left(\frac{n}{n + \nu^J_{\mathfrak{a}}(q)}\right)^n e(J^{[q]}) = \left(\frac{qn}{n + \nu^J_{\mathfrak{a}}(q)}\right)^n e(J).$$

On the right-hand side we can take a subsequence converging to $(n/\mathrm{c}^J_-(\mathfrak{a}))^n e(J)$, and this yields the inequality of the corollary. □

Proof of Theorem 5.8. Suppose that $\mathfrak{a}$ is generated by a full homogeneous system of parameters $x_1, \ldots, x_n$ of degrees $a_1 \leq \cdots \leq a_n$ and that J is generated by another homogeneous system of parameters $f_1, \ldots, f_n$ of degrees $d_1 \leq \cdots \leq d_n$.

Define nonnegative integers $t_1, \ldots, t_{n-1}$ inductively as follows: t_1 is the smallest integer t such that $x_1^t \in J$; if $2 \leq i \leq n-1$, then t_i is the smallest integer t such that $x_1^{t_1-1} \cdots x_{i-1}^{t_{i-1}-1} x_i^t \in J$. Note that $N \geq t_1 + \cdots + t_{n-1} - n + 1$ by assumption.

We first show the following inequality for every $i = 1, \ldots, n-1$:

$$t_1 a_1 + \cdots + t_i a_i \geq d_1 + \cdots + d_i. \tag{5}$$

Let I_i be the ideal of R generated by $x_1^{t_1}, x_1^{t_1-1} x_2^{t_2}, \ldots, x_1^{t_1-1} \cdots x_{i-1}^{t_{i-1}-1} x_i^{t_i}$, and note that the definition of the integers t_j implies that $I_i \subseteq J$. The natural surjection of R/I_i onto R/J induces a comparison map between their free resolutions (we resolve R/J by the Koszul complex and R/I_i by a Taylor-type complex). We remark that the ith step in the Taylor complex for the monomials $X_1^{t_1}, X_1^{t_1-1} X_2^{t_2}, \ldots, X_1^{t_1-1} \cdots X_{i-1}^{t_{i-1}-1} X_i^{t_i}$ in a polynomial ring with variables $X_1, \ldots, X_n$, is a free module of rank 1 and with a generator corresponding to the monomial

$$\operatorname{lcm}(X_1^{t_1}, X_1^{t_1-1} X_2^{t_2}, \ldots, X_1^{t_1-1} \cdots X_{i-1}^{t_{i-1}} X_i^{t_i}) = X_1^{t_1} \cdots X_{i-1}^{t_{i-1}} X_i^{t_i}$$

(see [Ei, Exer. 17.11]). It follows that the map between the ith steps in the resolutions of R/I_i and R/J is of the form

$$R(-t_1 a_1 - \cdots - t_i a_i) \to \bigoplus_{1 \leq v_1 < \cdots < v_i \leq n} R(-d_{v_1} - \cdots - d_{v_i}).$$

In particular, unless this map is zero we have

$$t_1 a_1 + \cdots + t_i a_i \geq \min_{1 \leq v_1 < \cdots < v_i \leq n} (d_{v_1} + \cdots + d_{v_i}) = d_1 + \cdots + d_i.$$

We now show that this map cannot be zero. If it is zero, then also the induced map

$$\operatorname{Tor}_i^R(R/I_i, R/\mathfrak{b}_i) \to \operatorname{Tor}_i^R(R/J, R/\mathfrak{b}_i) \tag{6}$$

is zero, where $\mathfrak{b}_i$ is the ideal generated by $x_1, \ldots, x_i$. On the other hand, using the Koszul complex on $x_1, \ldots, x_i$ to compute the preceding Tor modules, we see that the map (6) can be identified with the natural map

$$(I_i : \mathfrak{b}_i)/I_i \to (J : \mathfrak{b}_i)/J.$$

Since $x_1^{t_1-1} \cdots x_i^{t_i-1} \in (I_i : \mathfrak{b}_i)$, it follows that $x_1^{t_1-1} \cdots x_i^{t_i-1}$ lies in J—a contradiction. This proves (5).

We next prove the following inequality:

$$t_1 a_1 + \cdots + t_{n-1} a_{n-1} + (N - t_1 - \cdots - t_{n-1} + n - 1) a_n \geq d_1 + \cdots + d_n. \tag{7}$$

Since $\mathfrak{a}^N \subseteq J$, we have

$$(x_1^N, \ldots, x_n^N) : J \subseteq (x_1^N, \ldots, x_n^N) : \mathfrak{a}^N = (x_1^N, \ldots, x_n^N) + \mathfrak{a}^{(n-1)(N-1)}. \tag{8}$$

On the other hand, the ideal $(x_1^N, \ldots, x_n^N) : J$ can be described as follows. After writing $x_i^N = \sum_{j=1}^n b_{ij} f_j$ and then using the Koszul resolutions of R/J and $R/(x_1^N, \ldots, x_n^N)$, it is clear that multiplication by $D = \det(b_{ij})$ gives an injection $R/J \hookrightarrow R/(x_1^N, \ldots, x_n^N)$ and hence $J = (x_1^N, \ldots, x_n^N) : D$. Moreover, we also have

$$(x_1^N, \dots, x_n^N) : J = (x_1^N, \dots, x_n^N, D)$$

(see e.g. [PS, Prop. 2.6]; note that the statement therein requires R to be regular, but this condition is not used). It follows from the foregoing description that D is homogeneous and that $\deg(D) = N(a_1 + \cdots + a_n) - (d_1 + \cdots + d_n)$.

It follows from (8) that, after possibly adding to D an element in $(x_1^N, \dots, x_n^N)$, we may write

$$D = \sum_{m_1 + \cdots + m_n = (n-1)(N-1)} c_{m_1, \dots, m_n} x_1^{m_1} \cdots x_n^{m_n},$$

where all $c_{m_1, \dots, m_n}$ are homogeneous. Since $x_1^{t_1 - 1} \cdots x_{n-1}^{t_{n-1}-1} \notin J = (x_1^N, \dots, x_n^N) : D$, we see that

$$D \notin (x_1^N, \dots, x_n^N) : x_1^{t_1 - 1} \cdots x_{n-1}^{t_{n-1}-1} = (x_1^{N - t_1 + 1}, \dots, x_{n-1}^{N - t_{n-1} + 1}, x_n^N).$$

Hence there is some $(m_1, \dots, m_n)$, with $\sum_j m_j = (n-1)(N-1)$ and $m_j \leq N - t_j$ for all $j \leq n-1$, such that $c_{m_1, \dots, m_m} \neq 0$. We deduce that the degree of D is at least as large as the smallest degree of such a monomial $x_1^{m_1} \cdots x_n^{m_n}$ and so

$$\begin{aligned} \deg D &= N(a_1 + \cdots + a_n) - (d_1 + \cdots + d_n) \\ &\geq (N - t_1)a_1 + \cdots + (N - t_{n-1})a_{n-1} + (t_1 + \cdots + t_{n-1} - n + 1)a_n, \end{aligned}$$

which implies the inequality (7).

To finish the proof, we will use the following claim.

Claim. Let $\alpha_i, \beta_i, \gamma_i$ be real numbers for $1 \leq i \leq n$. If $1 = \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n$ and if $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i \geq \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for all $i = 1, \dots, n$, then $\alpha_1 + \cdots + \alpha_n \geq \beta_1 + \cdots + \beta_n$.

Proof of Claim. Let $\lambda_i = \alpha_i - \beta_i$ for $1 \leq i \leq n$, so that $\gamma_1\lambda_1 + \cdots + \gamma_i\lambda_i \geq 0$ for all $i = 1, \dots, n$. We prove that $\lambda_1 + \cdots + \lambda_n \geq 0$ by induction on n (the case $n = 1$ is trivial). Suppose that $n > 1$ and that there is an i such that $\lambda_i < 0$ (otherwise, the assertion to prove is clear). We must have $i \geq 2$, and since $\gamma_i \geq \gamma_{i-1}$ it follows that $\gamma_i\lambda_i \leq \gamma_{i-1}\lambda_i$. Let us put $\gamma_j' = \gamma_j$ for $1 \leq j \leq i-1$ and $\gamma_j' = \gamma_{j+1}$ for $i \leq j \leq n-1$. Define also $\lambda_j' = \lambda_j$ for $1 \leq j \leq i-2$ and define $\lambda_{i-1}' = \lambda_{i-1} + \lambda_i$ and $\lambda_j' = \lambda_{j+1}$ for $i \leq j \leq n-1$. It is straightforward to check that $\gamma_1'\lambda_1' + \cdots + \gamma_j'\lambda_j' \geq 0$ for all $j = 1, \dots, n-1$; hence the induction hypothesis implies $\lambda_1 + \cdots + \lambda_n = \lambda_1' + \cdots + \lambda_{n-1}' \geq 0$. □

We now set $\alpha_i = t_i$ for $1 \leq i \leq n-1$ and set $\alpha_n = N - t_1 - \cdots - t_{n-1} + n - 1$. We put $\beta_i = d_i/a_i$ and $\gamma_i = a_i/a_1$ for $1 \leq i \leq n$. Since $a_1 \leq \cdots \leq a_n$, we deduce that $1 = \gamma_1 \leq \cdots \leq \gamma_n$. Moreover, (5) together with (7) yields $\gamma_1\alpha_1 + \cdots + \gamma_i\alpha_i \geq \gamma_1\beta_1 + \cdots + \gamma_i\beta_i$ for $1 \leq i \leq n$. Using the claim just proved, we conclude that

$$N + n - 1 = \alpha_1 + \cdots + \alpha_n \geq \beta_1 + \cdots + \beta_n = \left(\frac{d_1}{a_1} + \cdots + \frac{d_n}{a_n}\right).$$

Comparing the arithmetic and geometric means of $\{d_i/a_i\}_i$, we see that

$$(N + n - 1)^n a_1 \cdots a_n \geq n^n d_1 \cdots d_n.$$

Because $e(\mathfrak{a}) = a_1 \cdots a_n$ and $e(J) = d_1 \cdots d_n$, this concludes the proof. □

When J is not necessarily a parameter ideal, we can prove another inequality involving the F-threshold $c^J(\mathfrak{a})$ that generalizes the results in [dFEM2] and [TW].

PROPOSITION 5.10. *If $(R, \mathfrak{m})$ is a d-dimensional regular local ring of characteristic $p > 0$ and if $\mathfrak{a}, J$ are $\mathfrak{m}$-primary ideals in R, then*

$$e(\mathfrak{a}) \geq \left(\frac{d}{c^J(\mathfrak{a})}\right)^d (c^J(\mathfrak{m}) - d + 1).$$

Proof. As in the proof of Theorem 5.6, we do a reduction to the monomial case. We first see that it is enough to show that—if R is the polynomial ring $k[x_1, \ldots, x_d]$, if $\mathfrak{m} = (x_1, \ldots, x_d)$, and if $\mathfrak{a}$ and J are $\mathfrak{m}$-primary ideals—then

$$\ell(R/\mathfrak{a}) \geq \frac{1}{d!}\left(\frac{d}{c^J(\mathfrak{a})}\right)^d (c^J(\mathfrak{m}) - d + 1). \tag{9}$$

Claim. We can find monomial ideals $\mathfrak{a}_1$ and J_1 such that

$$\ell_R(R/\mathfrak{a}) = \ell_R(R/\mathfrak{a}_1), \quad c^J(\mathfrak{a}) \geq c^{J_1}(\mathfrak{a}_1), \quad c^J(\mathfrak{m}) = c^{J_1}(\mathfrak{m}). \tag{10}$$

This reduces the proof of (9) to the case when both $\mathfrak{a}$ and J are monomial ideals.

Proof of Claim. We do a two-step deformation to monomial ideals. We consider first a flat deformation of $\mathfrak{a}$ and J to $\mathfrak{a}'$ and J' (respectively) where, for an ideal $I \subseteq R$, we denote by I' the ideal defining the respective tangent cone at the origin. We then fix a monomial order λ and consider a Gröbner deformation of $\mathfrak{a}'$ and J' to $\mathfrak{a}_1 := \mathrm{in}_\lambda(\mathfrak{a}')$ and $J_1 := \mathrm{in}_\lambda(J')$, respectively. It follows as in the proof of Theorem 5.6 that the first two conditions in (10) are satisfied. For the third condition, in light of Example 2.7(iii) it is enough to show that

$$\mathfrak{m}^r \subseteq J \iff \mathfrak{m}^r \subseteq J_1.$$

It is clear that if $\mathfrak{m}^r \subseteq J$ then $\mathfrak{m}^r \subseteq J'$ and $\mathfrak{m}^r \subseteq J_1$. For the converse, suppose that $\mathfrak{m}^r \subseteq J_1$. Since J' and J_1 are both homogeneous ideals and since $\dim_k(R/J_1)_r = \dim_k(R/J')_r$ (see [Ei, Chap. 15]), it follows that $\mathfrak{m}^r \subseteq J'$. (Note that if I is a homogeneous ideal in R, then $\mathfrak{m}^r \subseteq I$ if and only if $(R/I)_r = 0$.) We know that $\mathfrak{m}^s \subseteq J$ for some s; hence, in order to prove that $\mathfrak{m}^r \subseteq J$, it is enough to show that if $\mathfrak{m}^t \subseteq J'$ and $\mathfrak{m}^{t+1} \subseteq J$ then $\mathfrak{m}^t \subseteq J$. It is easy to check that $(J \cap \mathfrak{m}^t)' = J' \cap \mathfrak{m}^t$, and since $\mathfrak{m}^{t+1} \subseteq J$ it follows that $J \cap \mathfrak{m}^t$ is homogeneous. Therefore,

$$\mathfrak{m}^t \subseteq J' \cap \mathfrak{m}^t = (J \cap \mathfrak{m}^t)' = J \cap \mathfrak{m}^t. \qquad \square$$

Proof of Proposition 5.10 (*cont.*). From now on we assume that $\mathfrak{a}$ and J are $\mathfrak{m}$-primary monomial ideals. Arguing as in the proof of Theorem 5.6 and using Example 2.7(iii), we see that it is enough to show

$$\mathrm{Vol}(\mathbb{R}^d_{\geq 0} \setminus P(\mathfrak{a})) \geq \frac{1}{d!}\left(\frac{d}{c^J(\mathfrak{a})}\right)^d (r + 1),$$

where $r := \max\{s \in \mathbb{Z}_{\geq 0} \mid \mathfrak{m}^s \not\subseteq J\}$. By definition, we can choose a monomial $x_1^{r_1} \cdots x_d^{r_d}$ of degree r that is not contained in J. Because $\tau(\mathfrak{a}^{c^J(\mathfrak{a})}) \subseteq J$ by

Proposition 2.4, this monomial cannot belong to $\tau(\mathfrak{a}^{c^J(\mathfrak{a})})$. Using the description of generalized test ideals of monomial ideals (see [HaY, Thm. 4.8]), this translates as

$$(r_1+1,\dots,r_d+1) \notin \operatorname{Int}(c^J(\mathfrak{a})\cdot P(\mathfrak{a})).$$

Hence we can find a hyperplane $H : u_1/a_1+\cdots+u_d/a_d = c^J(\mathfrak{a})$ passing through the point $(r_1+1,\dots,r_d+1)$ such that

$$H^+ := \left\{(u_1,\dots,u_d)\in\mathbb{R}^d_{\geq 0} \;\middle|\; \frac{u_1}{a_1}+\cdots+\frac{u_d}{a_d}\geq c^J(\mathfrak{a})\right\} \supseteq c^J(\mathfrak{a})\cdot P(\mathfrak{a}). \tag{11}$$

Note that we have $c^J(\mathfrak{a}) = (1+r_1)/a_1+\cdots+(1+r_d)/a_d$. Comparing the arithmetic and geometric means of $\{(1+r_i)/a_i\}_i$, we see that

$$\left(\frac{c^J(\mathfrak{a})}{d}\right)^d = \left(\frac{1+r_1}{da_1}+\cdots+\frac{1+r_d}{da_d}\right)^d \geq \frac{(1+r_1)\cdots(1+r_d)}{a_1\cdots a_d} \geq \frac{1+r}{a_1\cdots a_d}.$$

On the other hand, (11) implies

$$\begin{aligned}\operatorname{Vol}(\mathbb{R}^d_{\geq 0}\setminus P(\mathfrak{a})) &\geq \operatorname{Vol}(\mathbb{R}^d_{\geq 0}\setminus (1/c^J(\mathfrak{a}))H^+)\\ &= \frac{a_1\cdots a_d}{d!}\\ &\geq \frac{1}{d!}\left(\frac{d}{c^J(\mathfrak{a})}\right)^d (r+1).\end{aligned}$$

□

ACKNOWLEDGMENTS. We thank all the participants in the AIM workshop "Integral Closure, Multiplier Ideals and Cores" for valuable comments. Especially, we are indebted to Mel Hochster for the suggestion to introduce the invariant $\tilde{c}^J(\mathfrak{a})$. We are also grateful to Tommaso de Fernex for suggesting Theorem 5.6.

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