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Lifting Seminormality

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Suppose R is a local Noetherian ring and y is a regular element contained in the maximal ideal of R. If R satisfies some nice property $(\star)$ then R/yR frequently does not satisfy $(\star)$, although there are exceptions—for example, when $(\star)$ is the Cohen–Macaulay property. On the other hand, many theorems state that $(\star)$ can be lifted from R/yR to R. If R/yR is an integral domain, respectfully reduced, then so is R. If $(\star)$ is regularity, the result is trivial. If $(\star)$ is normality, the result is well known and easy to prove; we will include a proof here simply to illustrate the relative levels of difficulty of this and our main result. However, when David Jaffe asked what happened when $(\star)$ was seminormality, a quick answer was not forthcoming. The purpose of this article is to show that seminormality can be lifted.

We should remark that the requirement for R to be a local Noetherian ring is important for this result and virtually all results of this type. There are non-Noetherian rings with a single maximal principal ideal yR and all kinds of pathological behavior, and the fact that R/yR is a field yields little. Likewise, if R has more than one maximal ideal, then passing to R/yR can "improve" R by removing maximal ideals P from the prime spectrum when R_P fails to satisfy $(\star)$.

Throughout this article, all rings are commutative with unity. Local rings are always Noetherian. The total quotient ring of R will be denoted by $Q(R)$, and the integral closure of R in $Q(R)$ will be denoted by R'. We will primarily be concerned with Noetherian rings, but excellence is not assumed and so R' need not be Noetherian. We begin with a quick proof of the well-known result that normality lifts. Here we consider only the domain case, but allowing R/yR to be reduced merely makes the proof slightly longer; the ideas in the proof remain the same. The same is true of the proof of our main theorem: restricting to the domain case does not make the problem any easier.

Theorem. *If R is a local integral domain, yR is a prime ideal in R, and R/yR is normal, then R is normal.*

Proof. We will show R to be normal by showing that it satisfies the Serre conditions (R1) and (S2). Suppose P is a height-1 prime ideal of R. If $P = yR$, then P principal implies R_P regular. If $P \neq yR$, then there exists a height-2 prime ideal Q of R that contains P and yR. Since R/yR satisfies (R1), it follows that

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$(Q/yR)(R/yR)_{Q/yR}$ is principally generated and so QR_Q requires only two generators. Thus R_Q is regular and so is its localization R_P.

Next suppose P is a prime ideal of R of height > 1. If $y \in P$, the facts that y is regular and $P \notin \mathrm{Ass}(yR)$ imply depth $P \geq 2$. If $y \notin P$, there exists a $Q \in \mathrm{Spec}(R)$ that is minimal over $P + yR$. Since $\mathrm{ht}(Q/yR) \geq 2$ and R/yR satisfies (S2), we have $\mathrm{depth}(Q/yR) \geq 2$. Thus depth $Q \geq 3$. Since $\mathrm{ht}(Q/P) = 1$, it follows that depth $Q \leq$ depth $P + 1$ and so depth $P \geq 2$ as desired. □

Next we review the notion of seminormality. In [T], Traverso defined a ring R to be seminormal if $R = \{x \in R' \mid \bar{x} \in R_P + J(R'_P) \text{ for each } P \in \mathrm{Spec}(R)\}$, where $J(R'_P)$ is the Jacobson radical of $(R')_{R-P}$. The major results in this area were developed for rings with finite integral closure by Schanuel (see [Ba]), Traverso, and Hamann [Ha]. The restrictive hypothesis was removed in [GH], [BC], and [S].

THEOREM. *The following statements are equivalent for a reduced Noetherian ring R.*

(1) Pic $R \cong$ Pic $R[X]$ *for an indeterminate X.*
(2) Pic $R \cong$ Pic $R[X]$ *for a family of indeterminates X.*
(3) *If $x \in Q(R)$ and $x^2, x^3 \in R$, then $x \in R$.*
(4) *R is seminormal.*

Conditions (3) and (4) both imply that R is reduced and so are fully equivalent in the Noetherian case. If R is not reduced, then conditions (1) and (2) will hold precisely when R_{red} is seminormal. As it happens, in the non-Noetherian case, the equivalence of (1) and (4) fails for reduced rings with infinitely many minimal prime ideals [GH; S]. Swan [S] addressed this problem by offering a new definition of seminormality that is always equivalent to (1) and (2) for reduced rings. This new definition was a modification of (3), not of Traverso's original definition. (Swan deleted the hypothesis $x \in Q(R)$ from (3) and then rephrased it so it would make sense.) In this article, we will use condition (3) rather than Traverso's original definition of seminormality.

MAIN THEOREM. *If (R, M) is a local ring, y is a regular element in M, and R/yR is seminormal, then R is seminormal.*

The theorem will be proved by contradiction starting with a sequence of lemmas. Throughout, R will be a local ring with maximal ideal M. Elements denoted by Greek letters will always be elements of R. We will assume R/yR is seminormal and R is not seminormal, the incompatible assumptions that will lead to our contradiction. The seminormal ring R/yR is of course reduced and so yR is a radical ideal. Since R is not seminormal, there must exist an element $x \in R' - R$ such that $x^2, x^3 \in R$.

LEMMA 1. *We may assume that $(R :_R x) = Q$ is a prime ideal of R and that M is minimal over $Q + yR$.*

Proof. Let Q be any prime divisor of $(R :_R x)$. Then $Q = (R :_R rx)$ for some regular $r \in R$ and, since $(rx)^2, (rx)^3 \in R$, we may harmlessly replace x by rx.

Furthermore, if P is any minimal prime divisor of $Q + yR$, then $x \notin R_P$ and R_P/yR_P is seminormal. Hence we may replace R by R_P. □

LEMMA 2. *We may assume R is complete in the yR-adic topology.*

Proof. Let S be the yR-adic completion of R. Trivially, S/yS is isomorphic to R/yR and so is seminormal. This isomorphism also tells us that S is local. Certainly $x^2, x^3 \in S$, and x cannot be in S because it is not even an element of the full completion; hence S is not seminormal. Finally, if Q_1 is a prime ideal of S such that $Q_1 \cap R = Q$, then Q_1 is a minimal prime divisor of $(S :_S x)$ and so $Q_1 = (S :_S sx)$ for some regular $s \in S$. Thus we may harmlessly replace R by S, x by sx, and Q by Q_1, so the lemma holds. □

Let $B = \{t \in R' \mid Qt \subset R\}$. Because B is integral over R, any element of QB has a power contained in Q. Since Q is a prime ideal in R, this gives $QB = Q$ and so B is a subring of R'. Since B is isomorphic to qB for any regular $q \in Q$, it follows that B is a finite R-module. (If Q did not contain a regular element, x would not be an element of $Q(R)$.)

LEMMA 3. *Let K be the quotient ring of R/yR. Then we have a commutative diagram of ring homomorphisms with injective rows*:

$$\begin{array}{ccccc} R/yR & \longrightarrow & B/yB & \longrightarrow & K \\ \uparrow & & \uparrow & & \\ R & \longrightarrow & B. & & \end{array}$$

Proof. The injection $R \subset B$ induces the commutative square on the left. Since $Q = (R :_R B)$ contains a regular element and is not a minimal prime divisor of the radical ideal yR, there exists an element $c \in Q$ that is regular on both R and R/yR. Thus we have $R \subset B \subset B[c^{-1}] = R[c^{-1}] \to (R/yR)[\bar{c}^{-1}] \subseteq K$ and hence a map $\theta\colon B \to K$. Moreover, under this map $yB \to yR[c^{-1}] \to 0$ and so θ factors through B/yB, yielding the entire diagram.

It remains only to see that the upper maps are injective. Since the composition is injective, the left map certainly is and so $yB \cap R = yR$. If $\bar{b} \in \operatorname{Ker}(B/yB \to K)$ then so is $\bar{c}\bar{b}$. However, this gives $cb \in \operatorname{Ker}(R \to K) = yR \subset yB$. Hence the right map is injective if (and only if) c is regular on B/yB.

Suppose we have $b \in B$ with $cb \in yB$. Since $cb \in R$ and $yB \cap R = yR$, we actually have $cb \in yR$. Thus $(cb)^n \in y^nR$ for every positive integer n. But $(cb)^n = c^{n-1}(cb^n)$, $cb^n \in R$, and c is regular on R/yR. Hence $cb^n \in y^nR$ for every n, which implies $b/y \in R'$. Finally, $Qb \subset R$ and $cQb \subset yR$ yield $Qb \subset yR$. So $Q(b/y) \subset R$ and $b \in yB$, demonstrating the desired regularity of c on B/yB. □

LEMMA 4. *$MB = M + yB$. If $b \in B$ with $b^m, b^{m+1} \in R$ for some positive integer m, then $b \in R + yB$.*

Proof. We prove the second statement first. Let $\bar{b}$ denote the image of b in B/yB. By Lemma 3, we may regard $\bar{b}$ as an element of K. We have $\bar{b}^m, \bar{b}^{m+1} \in R/yR$ and so, by seminormality, $\bar{b} \in R/yR$. Thus $b \in R + yB$.

For the first statement, it is clear that $M + yB \subseteq MB$. To prove the reverse inequality, we first note that $M^k \subseteq Q + yR$ for some positive integer k. Then, for any $b \in MB$, we have $b^k, b^{k+1} \in (Q + yR)B \subset R + yB$. By the second statement, $b \in R + yB$. Because b and y are both in the Jacobson radical of B, necessarily $b \in M + yB$ as desired. □

REMARK. In this argument, proving $b \in M + yB$ required only that b be in the radical of MB. Thus $MB = M + yB$ is in fact the Jacobson radical of B and B/MB is a direct sum of fields.

LEMMA 5. *Suppose $u, s \in B$ and $\delta \in R$ are such that $xu = \delta + y^e s$ and $xs \in R$. Then $s \in MB$.*

Proof. Multiply the given equation by x. Since $x^2 \in Q$ and $xs \in R$, we obtain $\delta x \in R$ and so $\delta \in Q$. Then, because $y^e s = xu - \delta$, for any $k > 1$ we have $(y^e s)^k \in Q$. Now $yB \cap R = yR$ gives $s^k \in R$ and so $s^k \in Q$. Thus s is in the Jacobson radical of B and, by the previous remark, $s \in MB$. □

We now prove the theorem.

Proof of Main Theorem. Let $B_i = R + xMB + y^i B$. Clearly we have a descending chain of R-modules $B \supseteq B_1 \supseteq B_2 \supseteq \cdots \supseteq R + xMB$. Since $B/(R + xMB)$ is a finite R-module and $y \in M$, we have $\bigcap B_i = R + xMB$ by the Krull intersection theorem. Let $U_i = \{t \in B \mid xt \in B_i\}$ and $U = \bigcap U_i = \{t \in B \mid xt \in R + xMB\}$. Again we have a descending chain $B \supseteq U_1 \supseteq U_2 \supseteq \cdots \supseteq U$. Moreover, for any $t \in B$, $x^2 t^2, x^3 t^3 \in R$ because $x^2, x^3 \in Q$. Thus $xt \in R + yB$ by Lemma 4 and so $U_1 = B$. Also, because $MB \subseteq U$, it follows that B/U is Artinian and $U = U_m$ for some m.

Next consider the map $B \to B/MB$ and let $\bar{U}$, $\bar{U}_i$ denote the images under this map. Then we have an ascending chain $\bar{U} = \bar{U}_m \subseteq \bar{U}_{m-1} \subseteq \cdots \subseteq \bar{U}_1 = \bar{B}$. Now we arbitrarily choose a basis for the R/M vector space $\bar{B}$ that contains a basis for $\bar{U}_i$ for each i. We lift this basis to a generating set for B in the following manner. Let $\bar{b}$ be an element of the basis and let b' be a particular lifting of $\bar{b}$ to B. If $b' \in U$ (independent of lifting, since $MB \subseteq U$), we have $xb' \in R + xMB$. Since adding an element of MB to a lifting gives another lifting, we may lift $\bar{b}$ to an element b so that $xb \in R$. If $b' \notin U$, let j be the largest integer such that $b' \in U_j$ (again independent of lifting). Here we have $xb' \in R + xMB + y^j B$ and, as before, we may choose our lifting b so that $xb \in R + y^j B$. We enumerate the elements in our generating set $u_1, u_2, \ldots, u_n$ so that, if $\dim \bar{U}_j = n - k_j > 0$, then $\bar{u}_{k_j+1}, \ldots, \bar{u}_n$ is a basis for $\bar{U}_j$. In particular, if $\dim \bar{U} = n - k \geq 0$ then $\bar{u}_{k+1}, \ldots, \bar{u}_n$ is a basis for $\bar{U}$. For each $i \leq k$ we have a generator u_i with $u_i \in U_{e_i} - U_{e_i+1}$. We may write $xu_i = \alpha_i + y^{e_i} s_i$. Since $u_i \notin U_{e_i+1}$, it follows that $s_i \notin R + yB$ and so $s_i \notin MB$ by Lemma 4. By this process we construct a sequence of elements $s_1, \ldots, s_k$. Each s_i is unique only up to an element of R. We claim that $\bar{1}, \bar{s}_1, \ldots, \bar{s}_k$ is a linearly independent set. If not, choose j minimal so that $\bar{1}, \bar{s}_1, \ldots, \bar{s}_j$ is linearly dependent. Then we have elements $\rho_i \in R$ such that $s_j - \sum_{i<j} \rho_i s_i \in R + MB$. Next let $f_i = e_j - e_i \geq 0$ for $i \leq j$ and set $u = u_j - \sum_{i<j} y^{f_i} \rho_i u_i$. Then

$$xu = \left(\alpha_j - \sum_{i<j} y^{f_i}\rho_i\alpha_i\right) + y^{e_j}\left(s_j - \sum_{i<j} \rho_i s_i\right) \in R + y^{e_j}(R + MB)$$
$$= R + y^{e_j}(R + yB) = R + y^{e_j+1}B.$$

However, this implies that $u \in U_{e_j+1}$ and so $\bar{u}_1, \dots, \bar{u}_j$ are not linearly independent modulo $\bar{U}_{e_j+1}$, contradicting our choice of generating set. Thus the claim holds: $\bar{1}, \bar{s}_1, \dots, \bar{s}_k$ is a linearly independent set. We have shown that if $C = \{1, s_i\}R$ then $\bar{C}$ is a $(k+1)$-dimensional subspace of B/MB. Next we point out how we shall take advantage of the nonuniqueness in the choice of the s_i. Suppose $s \in C$ is a fixed element such that $\bar{s} \notin R/M$, so $s = \gamma + \sum_{i\leq k} \gamma_i s_i$ with some $\gamma_j \notin M$. Then, altering our choice of s_j to $s_j + \gamma/\gamma_j$ yields $s = \sum_{i\leq k} \gamma_i s_i$.

The remainder of the proof is a bit technical, so we give an overview of the idea behind it. If $s_i = u_j$ for some i, j, then the element $s = s_i$ would yield a contradiction to Lemma 5. It is, in fact, possible to create this situation. Define $T = \{t \in B \mid xt \in R\}$. Since $u_{k+1}, \dots, u_n \in T$, we know that $\bar{T} = \bar{U}$ is an $(n-k)$-dimensional vector space. By a dimension argument, $\bar{C}$ must intersect $\bar{T}$ nontrivially. We will find s as a lifting to $C \cap T$ of an element in that nontrivial vector space intersection. We shall also see that $\bar{s} \notin R/M$, allowing us to write s as a linear combination of the s_i. It should be mentioned that we do not lift an arbitrary element of the intersection; we show only that some element *can* be lifted.

Next we will show by contradiction that $\bar{1} \notin \bar{T}$. If $t \in T$, then $xt \in R$ gives $(xt)^2 = x^2t^2 \in QB = Q$ and so $xt \in Q$. Thus $T = (Q :_B x)$, an ideal of B. If $\bar{1} \in \bar{T}$ then the ideal $T + MB$ is all of B and, by Nakayama's lemma, $T = B$, contradicting $x \notin R$; so $\bar{1} \notin \bar{T}$ as desired. Since $\bar{C} + \bar{T} \subseteq \bar{B}$ and $\dim \bar{B} = n$, the dimension of $\bar{C} + \bar{T}$ is $n - d$ for some $d \geq 0$. Now we compute the dimension of $\bar{C} \cap \bar{T}$ from the dimensions of $\bar{T}$, $\bar{C}$, and $\bar{C} + \bar{T}$ to be $(n-k) + (k+1) - (n-d) = d + 1 > 0$. Choose elements $r_j \in C$ for $j = 1, \dots, d+1$ that map to a basis of $\bar{C} \cap \bar{T}$. Let $E = \{r_j\}R$; hence $E \subset C$ and $\bar{E} = \bar{C} \cap \bar{T}$. Next we have a relatively long proof of a critical claim.

CLAIM. *There exists an element $z \in E \cap (T + M + yC)$ that is not in MB.*

Proof. Let $F = \{z_1, \dots, z_g\}$ be a subset of E that satisfies the following properties:

(1) $\bar{z}_1, \dots, \bar{z}_g$ is a linearly independent subset of B/MB;
(2) For each $i = 1, \dots, g$ we have $z_i = b_i + y^{e_i}t_i$, where $b_i \in T + M + yC$, $e_i \in \mathbf{Z}^+$, $t_i \in B$, and $\bar{\bar{t}}_1, \dots, \bar{\bar{t}}_g$ is a linearly independent subset of $(B/MB)/(\bar{C} + \bar{T})$;
(3) $g \geq 0$ is maximal for sets satisfying (1) and (2);
(4) $\sum e_i$ is minimal among sets satisfying (1), (2), and (3);
(5) $e_1 \leq e_2 \leq \cdots \leq e_g$.

It is easy to see that we can choose such an F. Because the empty set satisfies (1) and (2), the collection of sets satisfying these two properties is nonempty. Since g is bounded, we can restrict to the subcollection with maximal g. Next we pick any set in this collection with minimal $\sum e_i$ and reorder the elements if necessary so that $e_1 \leq e_2 \leq \cdots \leq e_g$.

The dimension of $(B/MB)/(\bar{C}+\bar{T})$ is d, so $g \le d < d+1 = \dim \bar{E}$. We see that $\bar{z}_1, \dots, \bar{z}_g$ has too few elements to span $\bar{E}$; hence we can find $v_1 \in E$ such that $\bar{z}_1, \dots, \bar{z}_g, \bar{v}_1$ is a linearly independent subset of $\bar{E}$.

Next we shall inductively find a sequence $v_1, v_2, \dots$ of elements in E such that, for all i, $\bar{z}_1, \dots, \bar{z}_g, \bar{v}_i$ is a linearly independent subset of $\bar{E}$ and $v_i \in T + M + yC + y^iB$ while $v_{i+1} - v_i \in y^{i-e_g}B$ for $i > e_g$. (If $g = 0$, set $e_g = 0$.)

We have already chosen v_1 such that $\bar{z}_1, \dots, \bar{z}_g, \bar{v}_1$ is linearly independent. Since $\bar{E} \subseteq \bar{T}$, it follows that $v_1 \in T + MB = T + M + yB$ as desired. Next suppose we have satisfactorily chosen v_j. We can write $v_j = a + y^j t$ with $a \in T + M + yC$ and $t \in B$. By the maximality of g in the choice of F, $\bar{\bar{t}}_1, \dots, \bar{\bar{t}}_g, \bar{\bar{t}}$ is not a linearly independent subset of $(B/MB)/(\bar{C}+\bar{T})$. Moreover, by the minimality of $\sum e_i$, $\bar{\bar{t}}_1, \dots, \bar{\bar{t}}_h, \bar{\bar{t}}$ is linearly dependent if $e_{h+1} > j$. Hence we may write $\bar{\bar{t}} = \sum_{i \le h} \bar{\bar{\rho}}_i \bar{\bar{t}}_i$. This gives $t - \sum_{i \le h} \rho_i t_i \in C + T + MB = C + T + yB$ since $MB = M + yB \subset C + yB$. Now, the equation $v_j = a + y^j t$ does not uniquely determine a and t. We may therefore adjust t by an element of $C + T$ and correspondingly adjust a by an element of $y^j(C+T)$, thereby reducing to the case $t - \sum_{i \le h} \rho_i t_i \in yB$. Now let $f_i = j - e_i$ and set

$$
\begin{aligned}
v_{j+1} &= v_j - \sum_{i \le h} y^{f_i} \rho_i z_i \\
&= \Big(a - \sum_{i \le h} y^{f_i} \rho_i b_i\Big) + y^j \Big(t - \sum_{i \le h} \rho_i t_i\Big) \in T + M + yC + y^j(yB) \\
&= T + M + yC + y^{j+1}B.
\end{aligned}
$$

Since $f_i \ge j - e_g$ for all i, we have $v_{j+1} - v_j \in y^{j-e_g}B$. Finally, since v_{j+1} is simply v_j plus a linear combination of $\{z_1, \dots, z_g\}$, it follows that $\bar{z}_1, \dots, \bar{z}_g, \bar{v}_{j+1}$ is linearly independent.

Next, since R is complete in the yR-adic topology, so is the finite R-module B. Thus the Cauchy sequence $v_1, v_2, \dots$ has a limit in B, a limit we designate as z. Clearly, for all i we have $z \in T + M + yC + y^iB$ and $z \in E + y^iB$; hence, by the Krull intersection theorem, $z \in (T + M + yC) \cap E$. Finally, $\bar{z} = \bar{v}_{e_g+1}$ gives $z \notin MB$ and so the Claim is proved. □

To complete the proof of the Main Theorem, we write $z = s + c$ with $s \in T$ and $c \in M + yC$. Because $z \in E - MB \subseteq C - MB$ and $c \in C \cap MB$, we have $s \in C \cap T - MB$. Since $\bar{1} \notin \bar{T}$, it follows that $\bar{s} \notin R/M$ and we may write $s = \sum_{i \le k} \gamma_i s_i$. Recall that $xu_i = \alpha_i + y^{e_i} s_i$ for each $i \le k$. Let $f_i = e_k - e_i \ge 0$ and set $u = \sum_{i \le k} y^{f_i} \gamma_i u_i$. Then $xu = \sum_{i \le k} y^{f_i} \gamma_i \alpha_i + y^{e_k} s = \delta + y^{e_k} s$ for $\delta = \sum_{i \le k} y^{f_i} \gamma_i \alpha_i \in R$. These elements u, s, δ directly contradict Lemma 5, so the theorem is proved. □

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