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Sally Modules of Rank One

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1. Introduction

Let A be a Cohen–Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. We assume the residue class field $k = A/\mathfrak{m}$ of A is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and choose a minimal reduction $Q = (a_1, a_2, \dots, a_d)$ of I. Let

$$R = \mathrm{R}(I) := A[It] \quad \text{and} \quad T = \mathrm{R}(Q) := A[Qt] \subseteq A[t],$$

respectively, denote the Rees algebras of I and Q, where t stands for an indeterminate over A. We put

$$R' = \mathrm{R}'(I) := A[It, t^{-1}], \qquad T' = \mathrm{R}'(Q) := A[Qt, t^{-1}],$$

and

$$G = \mathrm{G}(I) := R'/t^{-1}R' \cong \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

Let $B = T/\mathfrak{m}T$, which is the polynomial ring with d indeterminates over the field k. Following Vasconcelos [13], we then define

$$\mathrm{S}_Q(I) = IR/IT$$

and call it the *Sally module* of I with respect to Q. We observe that the Sally module $S = \mathrm{S}_Q(I)$ is a finitely generated graded T-module, since R is a module-finite extension of the graded ring T.

Let $\ell_A(\cdot)$ stand for the length and consider the Hilbert function

$$H_I(n) = \ell_A(A/I^{n+1})$$

$(n \geq 0)$ of I. Then we have the integers $\{\mathrm{e}_i = \mathrm{e}_i(I)\}_{0 \leq i \leq d}$ such that the equality

$$H_I(n) = \mathrm{e}_0\binom{n+d}{d} - \mathrm{e}_1\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_d$$

holds for all $n \gg 0$.

The Sally module S was introduced by Vasconcelos [13], where he gave an elegant review (in terms of his Sally module) of Sally's works [10; 11; 12] about the structure of $\mathfrak{m}$-primary ideals I with interaction to the structure of G and Hilbert coefficients e_i. Sally first investigated those ideals I satisfying the equality $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ and gave several important results, among which one

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can find the following characterization of ideals I with $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ and $\mathrm{e}_2 \neq 0$, where $B(-1)$ stands for the graded B-module whose grading is given by $[B(-1)]_n = B_{n-1}$ for all $n \in \mathbb{Z}$. (The reader may also wish to consult [2] and [14] for further ingenious use of Sally modules.)

Theorem 1.1 [12; 13]. *The following three conditions are equivalent.*

(1) $S \cong B(-1)$ *as graded T-modules.*
(2) $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ *and if* $d \geq 2$ *then* $\mathrm{e}_2 \neq 0$.
(3) $I^3 = QI^2$ *and* $\ell_A(I^2/QI) = 1$.

When this is the case, the following assertions hold true:

(i) $\mathrm{e}_2 = 1$ *if* $d \geq 2$;
(ii) $\mathrm{e}_i = 0$ *for all* $3 \leq i \leq d$ *if* $d \geq 3$;
(iii) $\operatorname{depth} G \geq d - 1$.

This research is a continuation of [12; 13] and aims at similar understanding of the structure of Sally modules of ideals I that satisfy the equality $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ but $\mathrm{e}_2 = 0$. When $\mathfrak{m}S = (0)$, we denote by $\mu_B(S)$ the number of elements in a minimal homogeneous system of generators of the graded B-module S. Let

$$\tilde{I} = \bigcup_{n \geq 1} [I^{n+1} : I^n] = \bigcup_{n \geq 1} [I^{n+1} : (a_1^n, a_2^n, \dots, a_d^n)]$$

denote the Ratliff–Rush closure of I (cf. [8]), which is the largest $\mathfrak{m}$-primary ideal of A such that $I \subseteq \tilde{I}$ and

$$\mathrm{e}_i(\tilde{I}) = \mathrm{e}_i(I) \quad \text{for all } \ 0 \leq i \leq d.$$

With this notation, the main result of this paper is stated as follows.

Theorem 1.2. *Suppose $d \geq 2$. Then the following four conditions are equivalent.*

(1) $\mathfrak{m}S = (0)$, $\operatorname{rank}_B S = 1$, *and* $\mu_B(S) = 2$.
(2) *There exists an exact sequence*

$$0 \to B(-2) \to B(-1) \oplus B(-1) \to S \to 0$$

of graded T-modules.

(3) $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$, $\mathrm{e}_2 = 0$, *and* $\operatorname{depth} G \geq d - 2$.
(4) $I^3 = QI^2$, $\ell_A(I^2/QI) = 2$, $\mathfrak{m}I^2 \subseteq QI$, *and* $\ell_A(I^3/Q^2I) < 2d$.

When $d = 2$, one can add the following condition:

(5) $\ell_A(\tilde{I}/I) = 1$ *and* $\tilde{I}^2 = Q\tilde{I}$.

When any of conditions (1), (2), (3), *or* (4) *is satisfied, the following assertions hold true*:

(i) $\operatorname{depth} G = d - 2$;
(ii) $\mathrm{e}_3 = -1$ *if* $d \geq 3$;
(iii) $\mathrm{e}_i = 0$ *for all* $4 \leq i \leq d$ *if* $d \geq 4$;
(iv) $\ell_A(I^3/Q^2I) = 2d - 1$.

Moreover, when $d = 2$ and condition (5) *is satisfied, the graded rings G, R, and R' are all Buchsbaum rings with the same Buchsbaum invariants*

$$\mathbb{I}(G) = \mathbb{I}(R) = \mathbb{I}(R') = 2.$$

Combined with Theorem 1.1, this theorem gives, for $d = 2$, a complete structure theorem of Sally modules of those ideals I with $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ (cf. Theorem 3.1). We could similarly describe the structure of Sally modules in higher-dimensional cases also provided one could show that $I^3 = QI^2$ if $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$, which we surmise holds true although we could not prove the implication.

Let us now briefly explain how this paper is organized. We shall prove Theorem 1.2 in Section 3. The key for our proof of Theorem 1.2 is Theorem 2.4, whose applications we will closely discuss in Section 2. Section 2 also includes some auxiliary facts on Sally modules. If $\mathrm{e}_1 = 2$ but $I^2 \neq QI$, the ideal I naturally satisfies the equality $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$. In Section 4 we shall explore those ideals I with $\mathrm{e}_1 = 2$ but $I^2 \neq QI$ in connection with the Buchsbaum property of the graded rings R, G, and R' associated to I. We shall explore in Section 5 one example in order to illustrate our theorems.

In what follows, unless otherwise specified, let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring with $d = \dim A > 0$. We assume that the field $A/\mathfrak{m}$ is infinite. Let I be an $\mathfrak{m}$-primary ideal in A and let S be the Sally module of I with respect to a minimal reduction $Q = (a_1, a_2, \ldots, a_d)$ of I. We put $R = A[It]$, $T = A[Qt]$, $R' = A[It, t^{-1}]$, $T' = A[Qt, t^{-1}]$, and $G = R'/t^{-1}R'$. Let $M = \mathfrak{m}T + T_+$ be the unique graded maximal ideal in T. We denote by $\mathrm{H}^i_M(\cdot)$ $(i \in \mathbb{Z})$ the ith local cohomology functor of T with respect to M. Let L be a graded T-module. For each $n \in \mathbb{Z}$, let $[\mathrm{H}^i_M(L)]_n$ stand for the homogeneous component of $\mathrm{H}^i_M(L)$ with degree n. We denote by $L(\alpha)$, for each $\alpha \in \mathbb{Z}$, the graded T-module whose grading is given by $[L(\alpha)]_n = L_{\alpha+n}$ for all $n \in \mathbb{Z}$.

2. Preliminaries

The purpose of this section is to summarize some auxiliary results on Sally modules that we will use throughout this paper. Some of the results are known, but we include brief proofs for the sake of completeness.

Lemma 2.1. *The following assertions hold true.*

(1) $\mathfrak{m}^{\ell} S = (0)$ *for integers* $\ell \gg 0$.

(2) *The homogeneous components* $\{S_n\}_{n \in \mathbb{Z}}$ *of the graded T-module S are given by*

$$S_n \cong \begin{cases} (0) & \text{if } n \leq 0, \\ I^{n+1}/IQ^n & \text{if } n \geq 1. \end{cases}$$

(3) $S = (0)$ *if and only if* $I^2 = QI$.

(4) *Suppose that* $S \neq (0)$ *and put* $V = S/MS$. *Let* V_n $(n \in \mathbb{Z})$ *denote the homogeneous component of the finite-dimensional graded* (T/M)*-space* V

with degree n *and put* $\Lambda = \{n \in \mathbb{Z} \mid V_n \neq (0)\}$. *Let* $q = \max \Lambda$. *Then* $\Lambda = \{1, 2, \dots, q\}$ *and* $\mathrm{r}_Q(I) = q + 1$, *where* $\mathrm{r}_Q(I)$ *stands for the reduction number of* I *with respect to* Q.

(5) $S = TS_1$ *if and only if* $I^3 = QI^2$.

Proof. Let $u = t^{-1}$ and note that $S = IR/IT \cong IR'/IT'$ as graded T-modules. We then have $u^\ell \cdot (IR'/IT') = (0)$ for some $\ell \gg 0$, because the graded T'-module IR'/IT' is finitely generated and $[IR'/IT']_n = (0)$ for all $n \leq 0$. Hence $\mathfrak{m}^\ell \cdot S = (0)$ for $\ell \gg 0$, because $Q^\ell = (Qt^\ell)u^\ell \subseteq u^\ell T' \cap A$ and $\mathfrak{m} = \sqrt{Q}$. This proves assertion (1).

Because $[IR]_n = (I^{n+1})t^n$ and $[IT]_n = (IQ^n)t^n$ for all $n \geq 0$, assertion (2) follows from the definition of the Sally module $S = IR/IT$. Assertion (3) readily follows from assertion (2).

To show assertion (4), we observe that $V_1 \cong S_1/\mathfrak{m}S_1 \neq (0)$ since $S = \sum_{n \geq 1} S_n$ and $S_1 \cong I^2/QI \neq (0)$. Hence $1 \in \Lambda$. Let $i \in \Lambda$ and put $\alpha_i = \dim_k V_i$, where $k = T/M$. We choose elements $\{\xi_{i,j}\}_{1 \leq j \leq \alpha_i}$ of S_i so that the images of $\{\xi_{i,j}\}_{1 \leq j \leq \alpha_i}$ in V form a k-basis of V_i. Hence, thanks to graded Nakayama's lemma, we have

$$S = \sum_{i \in \Lambda} \left(\sum_{j=1}^{\alpha_i} T\xi_{i,j} \right).$$

Let $\xi_{i,j}$ be the image of $x_{i,j}t^i$ in S with $x_{i,j} \in I^{i+1}$.

Let $n \geq 1$ be an integer and assume that $n \notin \Lambda$. Choose $x \in I^{n+1}$ and let ξ be the image of xt^n in S. We write

$$\xi = \sum_{i \in \Lambda, i < n} \left(\sum_{j=1}^{\alpha_i} \varphi_{i,j} \xi_{i,j} \right)$$

with $\varphi_{i,j} \in T_{n-i}$. Then, letting $\varphi_{i,j} = b_{i,j}t^{n-i}$ with $b_{i,j} \in Q^{n-i}$, we obtain

$$x \equiv \sum_{i \in \Lambda, i < n} \left(\sum_{j=1}^{\alpha_i} b_{i,j} x_{i,j} \right) \bmod Q^n I,$$

whence $x \in QI^n$ because $\sum_{j=1}^{\alpha_i} b_{i,j}x_{i,j} \in Q^{n-i}I^{i+1} \subseteq QI^n$ for all $i \in \Lambda$ such that $i < n$. Thus $I^{n+1} = QI^n$. Suppose now $n \leq q$. Then $I^{q+1} = QI^q$, whence $S_q \subseteq T_+ S$ and so $V_q = (0)$, which is impossible. Therefore, $\Lambda = \{1, 2, \dots, q\}$. If we choose $n = q + 1$ then the preceding observation shows that $I^{q+2} = QI^{q+1}$, whence $\mathrm{r}_Q(I) \leq q + 1$. If $r = \mathrm{r}_Q(I) < q + 1$ then $I^{q+1} = QI^q$ and so $S_q \subseteq T_+ S$, which is absurd. Thus $\mathrm{r}_Q(I) = q + 1$. This proves assertion (4). Assertion (5) is now clear. □

PROPOSITION 2.2. *Let* $\mathfrak{p} = \mathfrak{m}T$. *Then the following assertions hold true.*

(1) $\mathrm{Ass}_T S \subseteq \{\mathfrak{p}\}$; *hence* $\dim_T S = d$ *if* $S \neq (0)$.

(2) $\ell_A(A/I^{n+1}) = \mathrm{e}_0\binom{n+d}{d} - (\mathrm{e}_0 - \ell_A(A/I)) \cdot \binom{n+d-1}{d-1} - \ell_A(S_n)$ *for all* $n \geq 0$.

(3) $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$; *hence* $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ *if and only if* $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B S = 1$.

(4) *Suppose that* $S \neq (0)$ *and let* $s = \operatorname{depth}_T S$; *then* $\operatorname{depth} G = s - 1$ *if* $s < d$, *and* S *is a Cohen–Macaulay* T*-module if and only if* $\operatorname{depth} G \geq d - 1$.

Proof. (1) Let $P \in \operatorname{Ass}_T S$. Then $\mathfrak{p} = \mathfrak{m}T \subseteq P$, since $\mathfrak{m}^\ell S = 0$ for some $\ell \gg 0$ by Lemma 2.1(1). Since $\operatorname{ht}_T \mathfrak{p} = 1$, it is enough to show that $\operatorname{ht}_T P \leq 1$. Consider the exact sequence

$$0 \to IT_P \to IR_P \to S_P \to 0$$

of T_P-modules. We recall that IT is a Cohen–Macaulay T-module with $\dim_T IT = d + 1$ because

$$T/IT = (A/I) \otimes_{A/Q} (T/QT)$$

is the polynomial ring with d indeterminates over A/I and T is a Cohen–Macaulay ring with $\dim T = d+1$. Notice now that $a_1 \in P$ is a nonzero divisor on IR, whence $\operatorname{depth}_{T_P} IR_P > 0$. Thanks to the depth lemma, it follows from the previous exact sequence that $\dim_{T_P} IT_P = 1$, since $\operatorname{depth}_{T_P} IR_P > 0$ and $\operatorname{depth}_{T_P} S_P = 0$. Hence $\dim T_P = 1$, because IT is a Cohen–Macaulay T-module with $(0) :_T IT = (0)$. Thus $P = \mathfrak{p}$ and so we have $\operatorname{Ass}_T S = \{\mathfrak{p}\}$ as claimed.

(2) Let $n \geq 0$ be an integer. Then, by the exact sequence

$$0 \to S_n \to A/Q^nI \to A/I^{n+1} \to 0$$

of A-modules (Lemma 2.1(2)), we have

$$\ell_A(A/I^{n+1}) = \ell_A(A/Q^nI) - \ell_A(S_n),$$

and by the exact sequence

$$0 \to Q^n/Q^nI \to A/Q^nI \to A/Q^n \to 0$$

we have

$$\begin{aligned} \ell_A(A/Q^nI) &= \ell_A(A/Q^n) + \ell_A(Q^n/Q^nI) \\ &= \ell_A(A/Q) \cdot \binom{n+d-1}{d} + \ell_A(Q^n/Q^nI) \\ &= \mathrm{e}_0\binom{n+d-1}{d} + \ell_A(Q^n/Q^nI) \\ &= \mathrm{e}_0\binom{n+d}{d} - \mathrm{e}_0\binom{n+d-1}{d-1} + \ell_A(Q^n/Q^nI) \end{aligned}$$

because $\mathrm{e}_0 = \ell_A(A/Q)$ (recall that $Q = (a_1, a_2, \dots, a_d)$ is a minimal reduction of I). By virtue of the isomorphisms

$$Q^n/Q^nI \cong (A/I) \otimes_A (Q^n/Q^{n+1}) \cong (A/I) \otimes_A [(A/Q)^{\binom{n+d-1}{d-1}}] \cong (A/I)^{\binom{n+d-1}{d-1}},$$

we also have the equality

$$\ell_A(Q^n/Q^nI) = \ell_A(A/I) \cdot \binom{n+d-1}{d-1}.$$

Thus,

$$\begin{aligned}
&\ell_A(A/I^{n+1})\\
&\quad= \ell_A(A/Q^nI) - \ell_A(S_n)\\
&\quad= \left[\mathrm{e}_0\binom{n+d}{d} - \mathrm{e}_0\binom{n+d-1}{d-1} + \ell_A(Q^n/Q^nI)\right] - \ell_A(S_n)\\
&\quad= \left[\mathrm{e}_0\binom{n+d}{d} - \mathrm{e}_0\binom{n+d-1}{d-1} + \ell_A(A/I)\cdot\binom{n+d-1}{d-1}\right] - \ell_A(S_n)\\
&\quad= \mathrm{e}_0\binom{n+d}{d} - (\mathrm{e}_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n)
\end{aligned}$$

for all $n \geq 0$.

(3) If $S = (0)$, then $\mathrm{e}_1 = e_0 - \ell_A(A/I)$ by assertion (2). We may thus assume that $S \neq (0)$. Take a filtration

$$S = L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_q = (0)$$

of the graded T-module S such that each L_i is a graded T-submodule of S and

$$L_i/L_{i+1} \cong (T/P_i)(-\alpha_i)$$

with some integer α_i for all $0 \leq i < q$, where P_i is a graded prime ideal of T. Then, because $\mathrm{Ass}_T\, S = \mathrm{Min}_T\, S = \{\mathfrak{p}\}$, we see that $\mathfrak{p} \subseteq P_i$ for all $0 \leq i < q$. Furthermore,

$$\ell_{T_\mathfrak{p}}(S_\mathfrak{p}) = \#\{i \mid 0 \leq i < q,\ \mathfrak{p} = P_i\},$$

since

$$\ell_{T_\mathfrak{p}}(S_\mathfrak{p}) = \sum_{i=0}^{q-1} \ell_{T_\mathfrak{p}}((L_i/L_{i+1})_\mathfrak{p}) = \sum_{i=0}^{q-1} \ell_{T_\mathfrak{p}}(T_\mathfrak{p}/P_iT_\mathfrak{p})$$

and

$$T_\mathfrak{p}/P_iT_\mathfrak{p} = \begin{cases} B_\mathfrak{p} & \text{if } \mathfrak{p} = P_i,\\ (0) & \text{if } \mathfrak{p} \subsetneq P_i. \end{cases}$$

On the other hand,

$$\ell_A(S_n) = \sum_{i=0}^{q-1} \ell_A([L_i/L_{i+1}]_n) = \sum_{i=0}^{q-1} \ell_A([(T/P_i)(-\alpha_i)]_n)$$

for all $n \in \mathbb{Z}$. When $\mathfrak{p} = P_i$ we have

$$\begin{aligned}
\ell_A([(T/P_i)(-\alpha_i)]_n) = \ell_A(B_{n-\alpha_i}) &= \binom{n-\alpha_i+d-1}{d-1}\\
&= \binom{n+d-1}{d-1} - \alpha_i\binom{n+d-2}{d-2} + \text{(lower terms)},
\end{aligned}$$

and when $\mathfrak{p} \subsetneq P_i$ we have $\dim T/P_i < d$, so the degree of the Hilbert polynomial of T/P_i is less than $d-1$. Consequently, the normalized coefficient in degree $d-1$ of the Hilbert polynomial of the graded T-module S is exactly equal to $\ell_{T_\mathfrak{p}}(S_\mathfrak{p})$; then, by assertion (2), we get the equality $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + \ell_{T_\mathfrak{p}}(S_\mathfrak{p})$.

To show the second part of assertion (3), we recall that $\mathrm{Ass}_T\, S = \{\mathfrak{p}\}$. If $\ell_{T_\mathfrak{p}}(S_\mathfrak{p}) = 1$ then $\mathfrak{p}S_\mathfrak{p} = (0)$, so that $\mathfrak{p}S = (0)$; hence $\mathfrak{m}S = (0)$ and $\mathrm{rank}_B\, S = \ell_{T_\mathfrak{p}}(S_\mathfrak{p}) = 1$. The reverse implication is clear.

(4) Recall that $s \le d = \dim_T S$. Because IT is a Cohen–Macaulay T-module with $\dim_T IT = d + 1$, it follows from the exact sequence

$$0 \to IT \to IR \to S \to 0 \tag{a}$$

that $\mathrm{depth}_T\, IR \ge d$ if $s = d$ and $\mathrm{depth}_T\, IR = s$ if $s < d$ (by the depth lemma). We put $L = R_+$ and note that $IR \cong L(1)$ as graded R-modules. Therefore, since A is a Cohen–Macaulay ring with $\dim A = d$, from the exact sequence

$$0 \to L \to R \to A \to 0 \tag{b}$$

it follows that $\mathrm{depth}\, R \ge d$ if $s = d$ and $\mathrm{depth}\, R = s$ if $s < d$. Hence, the exact sequence

$$0 \to IR \to R \to G \to 0 \tag{c}$$

implies that $\mathrm{depth}\, G \ge d - 1$ if $s = d$. If $s < d$, then $\mathrm{depth}\, R = s$ and so, by [4, Thm. 2.1], we obtain $\mathrm{depth}\, G = s - 1$.

Suppose that $\mathrm{depth}\, G \ge d - 1$. Then $\mathrm{depth}\, R \ge d$ by [4, Thm. 2.1]; whence, by the exact sequence (b) we have $\mathrm{depth}_T\, L \ge d$ and so $\mathrm{depth}_T\, S \ge d$ by the exact sequence (a). Therefore, S is a Cohen–Macaulay T-module. □

Combining Lemma 2.1(3) and Proposition 2.2 yields the following result of Northcott and Huneke.

Corollary 2.3 [5; 7]. *We have* $\mathrm{e}_1 \ge \mathrm{e}_0 - \ell_A(A/I)$. *The equality* $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I)$ *holds true if and only if* $I^2 = QI$. *When this is the case,* $\mathrm{e}_i = 0$ *for all* $2 \le i \le d$, *provided* $d \ge 2$.

The following result is the heart of our paper.

Theorem 2.4. *The following conditions are equivalent.*

(1) $\mathfrak{m}S = (0)$ *and* $\mathrm{rank}_B\, S = 1$.
(2) *Either* $S \cong B(-1)$ *as graded T-modules, or* $S \cong \mathfrak{a}$ *as graded T-modules for some graded ideal* $\mathfrak{a}$ $(\ne B)$ *of* B *with* $\mathrm{ht}_B\, \mathfrak{a} \ge 2$.

Proof. We have only to show (1) ⇒ (2). Because $S_1 \ne (0)$ and $S = \sum_{n\ge1} S_n$ by Lemma 2.1, we have $S \cong B(-1)$ as graded B-modules once S is B-free.

Suppose that S is not B-free. The B-module S is torsion-free, since $\mathrm{Ass}_T\, S = \{\mathfrak{m}T\}$ by Proposition 2.2(1). Therefore, since $\mathrm{rank}_B\, S = 1$, it follows that $d \ge 2$ and $S \cong \mathfrak{a}(m)$ as graded B-modules for some integer m and some graded ideal $\mathfrak{a}$ $(\ne B)$ in B, so that we obtain the exact sequence

$$0 \to S(-m) \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules. We may assume that $\mathrm{ht}_B\, \mathfrak{a} \ge 2$, since $B = k[X_1, X_2, \dots, X_d]$ is the polynomial ring over the field $k = A/\mathfrak{m}$. We then have $m \ge 0$, because $\mathfrak{a}_{m+1} = [\mathfrak{a}(m)]_1 \cong S_1 \ne (0)$ and $\mathfrak{a}_0 = (0)$. We want to show $m = 0$.

Because $\dim B/\mathfrak{a} \leq d-2$, the Hilbert polynomial of $B/\mathfrak{a}$ has degree at most $d-3$. Hence

$$\begin{aligned}\ell_A(S_n) &= \ell_A(B_{m+n}) - \ell_A([B/\mathfrak{a}]_{m+n})\\ &= \binom{m+n+d-1}{d-1} - \ell_A([B/\mathfrak{a}]_{m+n})\\ &= \binom{n+d-1}{d-1} + m\binom{n+d-2}{d-2} + \text{(lower terms)}\end{aligned}$$

for $n \gg 0$. Consequently,

$$\begin{aligned}\ell_A(A/I^{n+1}) &= \mathrm{e}_0\binom{n+d}{d} - (\mathrm{e}_0 - \ell_A(A/I))\cdot\binom{n+d-1}{d-1} - \ell_A(S_n)\\ &= \mathrm{e}_0\binom{n+d}{d} - (\mathrm{e}_0 - \ell_A(A/I)+1)\cdot\binom{n+d-1}{d-1}\\ &\quad - m\binom{n+d-2}{d-2} + \text{(lower terms)}\end{aligned}$$

by Proposition 2.2(2), so $\mathrm{e}_2 = -m$. Thus $m = 0$, because $\mathrm{e}_2 \geq 0$ by Narita's theorem [6]. □

We note some consequences of Theorem 2.4.

COROLLARY 2.5. *Suppose* $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ *and* $I^3 = QI^2$. *Let* $c = \ell_A(I^2/QI)$. *Then the following assertions hold true.*

(1) $0 < c \leq d$ *and* $\mu_B(S) = c$.
(2) $\operatorname{depth} G \geq d-c$ *and* $\operatorname{depth}_B S = d-c+1$.
(3) $\operatorname{depth} G = d-c$ *for* $c \geq 2$.
(4) *If* $c < d$, *then* $\ell_A(A/I^{n+1}) = \mathrm{e}_0\binom{n+d}{d} - \mathrm{e}_1\binom{n+d-1}{d-1} + \binom{n+d-c-1}{d-c-1}$ *for all* $n \geq 0$ *and*

$$\mathrm{e}_i = \begin{cases} 0 & \text{if } i \neq c+1,\\ (-1)^{c+1} & \text{if } i = c+1\end{cases}$$

for $2 \leq i \leq d$.
(5) *If* $c = d$, *then* $\ell_A(A/I^{n+1}) = \mathrm{e}_0\binom{n+d}{d} - \mathrm{e}_1\binom{n+d-1}{d-1}$ *for all* $n \geq 1$. *Also,* $\mathrm{e}_i = 0$ *for* $2 \leq i \leq d$.

Proof. We have $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 1$ by Proposition 2.2(3), while $S = TS_1$ since $I^3 = QI^2$ (cf. Lemma 2.1(5)). Thus by Theorem 2.4 we have $S \cong \mathfrak{a}$ as graded B-modules, where $\mathfrak{a} = (X_1, X_2, \dots, X_c)$ is an ideal in B generated by linear forms $\{X_i\}_{1\leq i\leq c}$. Hence $0 < c \leq d$, $\mu_B(S) = c$, and $\operatorname{depth}_B S = d-c+1$, so assertions (1), (2), and (3) follow (cf. Proposition 2.2(4)). Considering the exact sequence

$$0 \to S \to B \to B/\mathfrak{a} \to 0$$

of graded B-modules, we have

$$\begin{aligned}\ell_A(S_n) &= \ell_A(B_n) - \ell_A([B/\mathfrak{a}]_n)\\ &= \binom{n+d-1}{d-1} - \binom{n+d-c-1}{d-c-1}\end{aligned}$$

for all $n \geq 0$ (resp. $n \geq 1$) if $c < d$ (resp. $c = d$). Thus assertions (4) and (5) follow (cf. Proposition 2.2(2)). □

Let $\tilde{I} = \bigcup_{n \geq 1}[I^{n+1} : I^n]$ be the Ratliff–Rush closure of I [8], which is the largest $\mathfrak{m}$-primary ideal in A such that $I \subseteq \tilde{I}$ and $\mathrm{e}_i(\tilde{I}) = \mathrm{e}_i$ for all $0 \leq i \leq d$.

Corollary 2.6. *Suppose that $d \geq 2$. Then the following three conditions are equivalent.*

(1) *$S \cong B_+$ as graded T-modules.*
(2) $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$, $I^3 = QI^2$, *and* $\mathrm{e}_i = 0$ *for all* $2 \leq i \leq d$.
(3) $I^3 = QI^2$, $\ell_A(\tilde{I}/I) = 1$, *and* $\tilde{I}^2 = Q\tilde{I}$.

When these conditions hold, $\operatorname{depth} G = 0$.

Proof. Let $c = \ell_A(I^2/QI)$.

(1) ⇒ (2) and the last assertion: These follow from Corollary 2.5. Notice that $c = \ell_A(S_1) = d$ and $I^3 = QI^2$ because $S \cong B_+$.

(2) ⇒ (1): We have $c = d$ by Corollary 2.5(4) and (5) because $\mathrm{e}_i = 0$ for all $2 \leq i \leq d$, so $S \cong B_+$ (see the proof of Corollary 2.5).

(2) ⇒ (3): We have $\operatorname{depth} G = 0$ by Corollary 2.5(3), since $c = d$. Now we apply local cohomology functors $\mathrm{H}^i_M(\cdot)$ of T with respect to the graded maximal ideal $M = \mathfrak{m}T + T_+$ to the exact sequences

$$0 \to IR \to R \to G \to 0 \quad \text{and} \quad 0 \to IT \to IR \to S \to 0$$

of graded T-modules and so derive the monomorphism

$$\mathrm{H}^0_M(G) \hookrightarrow \mathrm{H}^1_M(IR)$$

and the isomorphisms

$$\mathrm{H}^1_M(IR) \cong \mathrm{H}^1_M(S) \cong B/B_+$$

of graded T-modules (recall that $S \cong B_+$ and IT is a Cohen–Macaulay T-module with $\dim_T IT = d + 1$). Consequently, because $\mathrm{H}^0_M(G) \neq (0)$ and $\ell_A(B/B_+) = 1$, we have

$$\mathrm{H}^0_M(G) \cong \mathrm{H}^1_M(IR) \cong \mathrm{H}^1_M(S) \cong B/B_+,$$

whence $\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_0 \neq (0)$. Then $\ell_A(\tilde{I}/I) = 1$ because $[\mathrm{H}^0_M(G)]_0 \cong \tilde{I}/I$. Therefore, it follows from the equality $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ that

$$\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}),$$

since $\mathrm{e}_i(\tilde{I}) = \mathrm{e}_i$ for $i = 0, 1$ and $\ell_A(A/I) = \ell_A(A/\tilde{I}) + 1$. Hence $\tilde{I}^2 = Q\tilde{I}$ by Corollary 2.3.

(3) ⇒ (2): We have $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ and $\mathrm{e}_i = 0$ for all $2 \leq i \leq d$, since $\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) = \mathrm{e}_0 - \ell_A(A/I) + 1$ and $\mathrm{e}_i(\tilde{I}) = 0$ for all $2 \leq i \leq d$ (cf. Corollary 2.3). □

We include a proof of Theorem 1.1 in this context in order to show how our arguments work.

Proof of Theorem 1.1. (1) $\Rightarrow$ (3): See parts (2) and (5) of Lemma 2.1.

(3) $\Rightarrow$ (1): By Lemma 2.1(5) we have $S = TS_1$, whence $\mathfrak{m}S = (0)$ because $S_1 \cong I^2/QI$ and $\ell_A(I^2/QI) = 1$. We thus have an epimorphism $B(-1) \to S \to 0$, which must be an isomorphism because $\dim_T S = d$.

(1) $\Rightarrow$ (2) and the last assertions: We have $I^3 = QI^2$ since $S = TS_1$, so the assertions now follow from Corollary 2.5 (note that $c = 1$).

(2) $\Rightarrow$ (1): We have $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 1$ by Proposition 2.2(3), and the B-module S is torsion-free by Proposition 2.2(1). Hence S is B-free if $d = 1$ and so $S \cong B(-1)$ as graded T-modules (note that $S_1 \neq (0)$).

Assume that $d = 2$. Then we have the exact sequence

$$0 \to B(-1) \to S \to C \to 0 \tag{d}$$

of graded B-modules with $\dim_B C \leq 1$. Hence $\ell_A(S_n) = \ell_A(B_{n-1}) + \ell_A(C_n) = \binom{n}{1} + \ell_A(C_n)$ for all $n \geq 1$ and so, by Proposition 2.2(2),

$$\ell_A(A/I^{n+1}) = \mathrm{e}_0\binom{n+2}{2} - (\mathrm{e}_0 - \ell_A(A/I) + 1)\binom{n+1}{1} + (1 - \ell_A(C_n)).$$

Consequently, $\mathrm{e}_2 = 1 - \ell_A(C_n) > 0$ by Narita's theorem [6] and so $\ell_A(C_n) = 0$ for all $n \geq 1$. Thus $\ell_A(C) \leq 1$, so that $C = (0)$ by the exact sequence (d).

Now let $d \geq 3$ and assume that our assertion holds for $d - 1$. Choose the element $a_1 \in Q$ so that a_1 is a superficial element of I (this choice is possible because the field $A/\mathfrak{m}$ is infinite). Let $\bar{A} = A/(a_1)$, $\bar{I} = I/(a_1)$, and $\bar{Q} = Q/(a_1)$. Then all the assumptions of condition (2) are safely fulfilled for the ideal $\bar{I}$ in $\bar{A}$, since $\mathrm{e}_i(\bar{A}) = \mathrm{e}_i$ for all $0 \leq i \leq d - 1$. As a result, the hypothesis of induction yields that $\operatorname{depth} \mathrm{G}(\bar{I}) \geq (d-1) - 1 = d - 2 > 0$ and so, thanks to Sally's technique [12], we see that a_1t is a nonzero divisor for G; from this it follows that $I^3 = QI^2$ because $\bar{I}^3 = \bar{Q}\bar{I}^2$. Thus $S \cong B(-1)$ as graded B-modules by Corollary 2.5 (note that $c = 1$). $\square$

3. Proof of Theorem 1.2

We begin with the following statement.

THEOREM 3.1. *Suppose that $d = 2$. Then the following three conditions are equivalent.*

(1) $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$.

(2) *Either $S \cong B(-1)$ as graded T-modules or $S \cong B_+$ as graded T-modules.*

(3) *Either* (a) $I^3 = QI^2$ *and* $\ell_A(I^2/QI) = 1$ *or* (b) $\ell_A(\tilde{I}/I) = 1$ *and* $\tilde{I}^2 = Q\tilde{I}$.

We obtain $\mathrm{e}_2 = 1$ (*resp.* $\mathrm{e}_2 = 0$) *if condition* (3)(a) (*resp. condition* (3)(b)) *is satisfied and also have the following results.*

e_2	$\mathrm{r}_Q(I)$	$\operatorname{depth}_B S$	$\operatorname{depth} G$	
1	2	2	2	*if* $Q \not\supseteq I^2$
1	2	2	1	*if* $Q \supseteq I^2$
0	2	1	0	*G is a Buchsbaum ring with* $\mathbb{I}(G) = 2$

Proof. (1) $\Rightarrow$ (2): In view of Corollary 2.5 and its proof, we need only show that $I^3 = QI^2$. This equality follows directly from a result of Rossi [9, Cor. 1.5]. We present a proof in our context for the sake of completeness.

We have $\mathfrak{m}S = (0)$ and $\operatorname{rank}_B S = 1$. Assume that $S \not\cong B(-1)$ as graded B-modules. Then, by Theorem 2.4, we have $S \cong \mathfrak{a}$ as graded B-modules for some graded ideal $\mathfrak{a} \neq B$ with $\operatorname{ht}_B \mathfrak{a} = 2$. We will show that $\mathfrak{a} = B_+$. Because $\mathfrak{a}_1 \cong S_1 \neq (0)$, the ideal $\mathfrak{a}$ contains a linear form $f \neq 0$ of B and so the ideal $\mathfrak{a}/(f)$ of $B/(f)$ is principal, since $B/(f)$ is the polynomial ring with one indeterminate over the field $k = A/\mathfrak{m}$. We write $\mathfrak{a} = (f, g)$ with a form $g \in B$. Then f, g is a regular sequence in B, since $\operatorname{ht}_B \mathfrak{a} = 2$. Let $\alpha = \deg g$; then $\alpha \leq 2$ by Lemma 2.1(4). We will show that $\alpha = 1$.

Assume that $\alpha = 2$. Then, since $S \cong \mathfrak{a} = (f, g)$, the graded B-module S has a resolution of the form

$$0 \to B(-3) \xrightarrow{\binom{g}{f}} B(-1) \oplus B(-2) \xrightarrow{\varphi = (\xi\ \eta)} S \to 0,$$

in which the homomorphism φ is defined by $\varphi(\mathbf{e}_1) = \xi \in S_1$ and $\varphi(\mathbf{e}_2) = \eta \in S_2$ (here $\{\mathbf{e}_1, \mathbf{e}_2\}$ denotes the standard basis of $B(-1) \oplus B(-2)$). Let $a \in Q$, $c \in Q^2$, $x \in I^2$, and $y \in I^3$ be such that f and g are (respectively) the images of at and ct^2 in B and ξ and η are (respectively) the images of xt and yt^2 in S. We observe that $a \notin \mathfrak{m}Q$, so $Q = (a, b)$ for some $b \in Q$. Hence $c = a^2 z_1 + abz_2 + b^2 z_3$ for some $z_1, z_2, z_3 \in A$.

We now consider the relation $g\xi + f\eta = 0$ in S_3; that is, $cx + ay \in Q^3 I$. Write $cx + ay = (a^2 z_1 + abz_2 + b^2 z_3)x + ay = a^2 i + b^2 j$ with $i, j \in QI$ (recall that $Q^3 = (a^2, b^2)Q$). We then have $ay' = b^2 x'$, where $y' = y + az_1 x + bz_2 x - ai$ and $x' = j - z_3 x$. Hence $x' = ah$ and $y' = b^2 h$ for some $h \in A$, because the sequence a, b^2 is A-regular. Therefore, $h \in I^3 : (a^2, b^2) \subseteq \tilde{I}$ because $a^2 h = ax' \in I^3$ and $b^2 h = y' \in I^3$. Now take note that $S = B\xi + B\eta$. We thus have $S_1 = B_0 \xi$ and $S_2 = B_1 S_1 + B_0 \eta$, so $\ell_A(I^2/QI) = 1$ and $I^3 = QI^2 + (y)$.

We shall need the following.

Claim 1. $h \notin I$ *and* $x' = ah \notin QI$.

Proof of Claim 1. Assume that $h \in I$. Then $y' = b^2 h \in Q^2 I$ and so $y = y' - az_1 x - bz_2 x + ai \in QI^2$, whence $I^3 = QI^2 + (y) = QI^2$. This forces $S = BS_1$, which is impossible because $\alpha = 2$. Thus $h \notin I$. Suppose $ah \in QI$ and let $ah = ai_1 + bi_2$ with $i_1, i_2 \in I$. Then $a(h - i_1) = bi_2$ and so $h - i_1 \in (b)$. Hence $h \in I$, which is impossible. □

Because $\ell_A(\tilde{I}/I) \geq 1$ by this claim, we obtain

$$\begin{aligned}
\mathrm{e}_1 &= \mathrm{e}_0 - \ell_A(A/I) + 1 \\
&= \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) - (\ell_A(\tilde{I}/I) - 1) \\
&\leq \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \\
&\leq \mathrm{e}_1(\tilde{I}) \\
&= \mathrm{e}_1,
\end{aligned}$$

where $\mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \le \mathrm{e}_1(\tilde{I})$ is the inequality of Northcott for the ideal $\tilde{I}$ (cf. Corollary 2.3). Then we have $\ell_A(\tilde{I}/I) = 1$ and $\mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I})$, so that $\tilde{I} = I + (h)$ and $\tilde{I}^2 = Q\tilde{I}$ by Corollary 2.3, since Q is also a reduction of $\tilde{I}$. Thus the associated graded ring of $\tilde{I}$ is a Cohen–Macaulay ring and so $(a) \cap \tilde{I}^n = a\tilde{I}^{n-1}$ for all $n \in \mathbb{Z}$, because at is $\mathrm{G}(\tilde{I})$-regular.

Now recall that $x' = ah \notin QI$; then $I^2 = QI + (ah)$ because $\ell_A(I^2/QI) = 1$. Let $\bar{A} = A/(a)$, $\bar{I} = I/(a)$, and $\bar{Q} = Q/(a)$. Then $\bar{I}^2 = \bar{Q}\bar{I}$ and so $\bar{I}^3 = \bar{Q}\bar{I}^2$, whence $I^3 \subseteq QI^2 + (a)$. Thus $I^3 = QI^2 + [(a) \cap I^3]$. On the other hand,

$$(a) \cap I^3 \subseteq (a) \cap \tilde{I}^3 = a\tilde{I}^2 = aQ\tilde{I} = (aQ)(I + (h)) = (aQ)I + x'Q \subseteq QI^2;$$

then $I^3 = QI^2$ and so $\alpha = 1$, which is the required contradiction. Thus $S = BS_1$ and $S \cong B_+$.

(2) ⇒ (3): See Theorem 1.1 and Corollary 2.6.

(3) ⇒ (1): If condition (3)(a) is satisfied, then (1) follows from Theorem 1.1. Suppose condition (3)(b) is satisfied. Then $\mathrm{e}_1 = \mathrm{e}_1(\tilde{I}) = \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) = \mathrm{e}_0 - \ell_A(A/I) + 1$ (cf. Corollary 2.3).

We now consider the theorem's last assertions. Suppose condition (3)(a) is satisfied. Then $\mathrm{e}_2 = 1$ by Theorem 1.1. If $Q \supseteq I^2$ then $I^2 = Q \cap I^2 \ne QI$, so that G is not a Cohen–Macaulay ring. If $Q \not\supseteq I^2$ then $Q \cap I^2 = QI$, because $\ell_A(I^2/QI) = 1$ and $I^2 \supsetneq Q \cap I^2 \supseteq QI$. Because $I^3 = QI^2$, this yields that G is a Cohen–Macaulay ring.

Suppose condition (3)(b) is satisfied. Then, since $\tilde{I}^2 = Q\tilde{I}$, we have $\mathrm{e}_2 = 0$ (by Corollary 2.3; recall that $\mathrm{e}_2(\tilde{I}) = \mathrm{e}_2$) and $\mathrm{R}'(\tilde{I})$ is a Cohen–Macaulay ring. We also have the following.

Claim 2. $\tilde{I}^n = I^n$ *for all* $n \ge 2$.

Proof of Claim 2. We have $S \cong B_+$ as graded T-modules, because $\mathrm{e}_2 = 0$. Hence $\mathrm{H}^0_M(G) = [\mathrm{H}^0_M(G)]_0$ by the implication (2) ⇒ (3) in the proof of Corollary 2.6. Let $n \ge 2$ be an integer. We then have

$$[\widetilde{I^n} \cap I^{n-1}]/I^n \cong [\mathrm{H}^0_M(G)]_{n-1} = (0).$$

Consequently, $\tilde{I}^n = I^n$ because $\tilde{I}^n \subseteq \widetilde{I^n} \cap I^{n-1}$ (recall that $\tilde{I}^n = Q^{n-1}\tilde{I}$, since $\tilde{I}^2 = Q\tilde{I}$). Thus $\tilde{I}^n = I^n$ for all $n \ge 2$. □

We put $W = \mathrm{R}'(\tilde{I})/R'$ and look at the exact sequence

$$0 \to R' \to \mathrm{R}'(\tilde{I}) \to \mathrm{R}'(\tilde{I})/R' \to 0 \qquad (\dagger)$$

of graded R'-modules. Observe that $W = W_1 \cong \tilde{I}/I$ by Claim 2, whence $\ell_A(W) = 1$. Let $N = (\mathfrak{m}, R_+, t^{-1})R'$ be the unique graded maximal ideal in R'. Then, because $\mathrm{R}'(\tilde{I})$ is a Cohen–Macaulay ring, applying functors $\mathrm{H}^i_N(\cdot)$ to the exact sequence (†) yields $\mathrm{H}^i_N(R') = (0)$ for all $i \ne 1, 3$ and $\mathrm{H}^1_N(R') = W$. Thus R' is a Buchsbaum ring with the Buchsbaum invariant

$$\mathbb{I}(R') = \sum_{i=0}^{2} \binom{2}{i} \ell_A(\mathrm{H}^i_N(R')) = 2,$$

whence so is the graded ring $G = R'/t^{-1}R'$. We similarly have that R is a Buchsbaum ring with $\mathbb{I}(R) = 2$, because $\mathrm{R}(\tilde{I})$ is a Cohen–Macaulay ring and $\mathrm{R}(\tilde{I})/R = [\mathrm{R}(\tilde{I})/R]_0 \cong \tilde{I}/I$. This completes the proof of Theorem 3.1. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. (1) ⇒ (3): We have $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$ by Proposition 2.2(3) and so $\mathrm{e}_2 = 0$ by Theorem 1.1. Because $S \not\cong B(-1)$, by Theorem 2.4 we get $S \cong \mathfrak{a}$ as graded B-modules for some graded ideal $\mathfrak{a}$ ($\neq B$) in B with $\mathrm{ht}_B\, \mathfrak{a} \geq 2$. Since $\mu_B(\mathfrak{a}) = \mu_B(S) = 2$, the ideal $\mathfrak{a}$ is a complete intersection with $\mathrm{ht}_B\, \mathfrak{a} = 2$ and so $\mathrm{depth}_B\, B/\mathfrak{a} = d - 2$, whence $\mathrm{depth}_B\, S = d - 1$. Therefore, $\mathrm{depth}\, G = d - 2$ by Proposition 2.2(4).

(3) ⇒ (2): We first show that $I^3 = QI^2$. Thanks to Theorem 3.1, we may assume that $d \geq 3$ and our assertion holds true for $d - 1$. Since $\mathrm{depth}\, G \geq d - 2 > 0$, we may choose $a_1 \in Q$ so that $a_1 t$ is a nonzero divisor in G. Let $\bar{A} = A/(a_1)$, $\bar{I} = I/(a_1)$, and $\bar{Q} = Q/(a_1)$. Then, because $\mathrm{G}(\bar{I}) \cong G/a_1 t \cdot G$ and $\mathrm{e}_i(\bar{I}) = \mathrm{e}_i$ for all $0 \leq i \leq d - 1$, condition (3) is satisfied for the ideal $\bar{I}$ and so $\bar{I}^3 = \bar{Q}\bar{I}^2$, whence $I^3 = QI^2$. Therefore, since $\mathrm{e}_2 = 0$, we see by Corollary 2.5 that $c = \mu_B(S) = 2$ and so assertion (2) follows.

(2) ⇒ (4): We have $\mathfrak{m}S = (0)$, $S = TS_1$, and $S_1 \cong B_0^2$. Hence $\mathfrak{m}I^2 \subseteq QI$, $I^3 = QI^2$, and $\ell_A(I^2/QI) = \ell_A(S_1) = 2$. We similarly have

$$\ell_A(I^3/Q^2I) = \ell_A(S_2) = 2\ell_A(B_1) - \ell_A(B_0) = 2d - 1 < 2d.$$

(4) ⇒ (1): We have $S = TS_1$ and so $\mathfrak{m}S = (0)$, since $\mathfrak{m}S_1 = (0)$. Given that $\ell_A(S_1) = 2$, we have an epimorphism $B(-1)^2 \to S \to 0$ of graded B-modules, which cannot be an isomorphism because $\ell_A(S_2) = \ell_A(I^3/IQ^2) < 2d$. Thus $\mathrm{rank}_B\, S = 1$, from which $\mu_B(S) = 2$ follows by Corollary 2.5.

See Theorem 3.1 for the equivalence between condition (5) and the others. See Corollary 2.5 and the proof of Theorem 3.1 for the last assertions. □

We note the following.

EXAMPLE 3.2. Let $A = k[[X, Y, Z_1, Z_2, \ldots, Z_m]]$ ($m \geq 0$) be the formal power series ring over a field k. Hence $\dim A = m + 2$. We put

$$Q = (X^4, Y^4, Z_1, Z_2, \ldots, Z_m) \quad \text{and} \quad I = Q + (X^3Y, XY^3).$$

Then

$$\mathfrak{m}I^2 \subseteq QI, \quad \ell_A(I^2/QI) = 2, \quad \ell_A(I^3/Q^2I) < 2d, \quad I^3 = QI^2,$$

where $d = m + 2$. Hence condition (4) in Theorem 1.2 is satisfied, so that $\mathfrak{m}S = (0)$, $\mathrm{rank}_B\, S = 1$, and $\mu_B(S) = 2$. We have $\ell_A(A/Q) = 16$, $\ell_A(A/I) = 11$, and

$$\ell_A(A/I^{n+1}) = 16\binom{n+2}{2} - 6\binom{n+1}{1}$$

for all $n \geq 1$ if $m = 0$. If $m \geq 1$, then

$$\ell_A(A/I^{n+1}) = 16\binom{n+d}{d} - 6\binom{n+d-1}{d-1} + \binom{n+d-3}{d-3}$$

for all $n \geq 0$. As a result, $\mathrm{e}_3 = -1$ and $\mathrm{e}_i = 0$ ($2 \leq i \leq d$, $i \neq 3$).

Proof. Because $G = \mathrm{G}((X^4, X^3Y, XY^3, Y^4))[Z_1, Z_2, \dots, Z_m]$ (the polynomial ring), the case $m > 0$ follows easily from the case $m = 0$ (see Theorem 1.2(3)). Let $m = 0$; then $I^2 = QI + (X^6Y^2, X^2Y^6)$. It is routine to show that $\mathfrak{m}I^2 \subseteq QI$, $\ell_A(I^2/QI) = 2$, and $I^3 = QI^2$. We have $QI^2 = Q^2I + (X^{10}Y^2, X^6Y^6, X^2Y^{10})$, so $\ell_A(I^3/Q^2I) = 3$. □

Before closing this section, we briefly study ideals with $\mathrm{e}_1 = 2$.

THEOREM 3.3. *Suppose that* $\mathrm{e}_1 = 2$ *and* $I^2 \neq QI$. *Then the following assertions hold.*

- (i) $\ell_A(I/Q) = \ell_A(I^2/QI) = 1$.
- (ii) $I^3 = QI^2$.
- (iii) $S \cong B(-1)$ *as graded T-modules.*
- (iv) $\operatorname{depth} G = d - 1$.
- (v) $\mathrm{e}_2 = 1$ *if* $d \geq 2$ *and* $\mathrm{e}_i = 0$ *for* $3 \leq i \leq d$ *if* $d \geq 3$.

Proof. Since $I^2 \neq QI$, it follows from Corollary 2.3 that

$$0 < \ell_A(I/Q) = \mathrm{e}_0 - \ell_A(A/I) < \mathrm{e}_1 = 2.$$

Therefore, $\ell_A(I/Q) = 1$ and $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$. Let $I = Q + (x)$ with $x \in A$. Then $I^2 = QI + (x^2)$, so that $\ell_A(I^2/QI) = 1$ because $I^2 \neq QI$ and $\mathfrak{m}I \subseteq Q$. We will show by induction on d that $I^3 = QI^2$ and $\operatorname{depth} G \geq d - 1$. Since $\ell_A(S_1) = \ell_A(I^2/QI) = 1$, by Theorems 1.1 and 3.1 we may assume that $d \geq 3$ and then our assertion holds true for $d - 1$. Choose $a_1 \in Q$ so that a_1 is a superficial element of I. Then, passing to the ideals $\bar{I} = I/(a_1)$ and $\bar{Q} = Q/(a_1)$ in the ring $\bar{A} = A/(a_1)$, we obtain $\mathrm{e}_1(\bar{I}) = \mathrm{e}_1 = 2$. We claim that $\bar{I}^2 \neq \bar{Q}\bar{I}$. In fact, if $\bar{I}^2 = \bar{Q}\bar{I}$ then the ring $\mathrm{G}(\bar{I})$ is Cohen–Macaulay. We can thus use Sally's technique [12] to find that a_1t is regular on G; hence $I^2 = QI$, which is impossible. Consequently, the hypothesis of induction shows $\bar{I}^3 = \bar{Q}\bar{I}^2$ and $\operatorname{depth} \mathrm{G}(\bar{I}) \geq (d-1) - 1 = d - 2 > 0$. Thus, again using Sally's technique, we find that a_1t is regular on G and so $I^3 = QI^2$ and $\operatorname{depth} G \geq d - 1$. Since $\mathfrak{m}I \subseteq Q$, it follows that $I^2 \subseteq Q$; hence G is not a Cohen–Macaulay ring, for otherwise $I^2 = Q \cap I^2 = QI$. Therefore, $\operatorname{depth} G = d - 1$. See Theorem 1.1 for assertions (iii) and (v). □

COROLLARY 3.4. *Suppose that* $\mathrm{e}_1 = 2$. *Then* $\operatorname{depth} G \geq d - 1$, *and the ring* G *is Cohen–Macaulay if and only if* $I^2 = QI$.

4. Buchsbaumness in the Graded Rings G Associated to Ideals with $\mathrm{e}_1 = 2$

The purpose of this section is to study the problem of when the associated graded rings G are Buchsbaum for the ideals I with $\mathrm{e}_1 = 2$.

We assume that $\mathrm{e}_1 = 2$ but $I^2 \neq QI$. We have $\operatorname{depth} R = d$ [4, Thm. 2.1] because $\operatorname{depth} G = d - 1$ by Theorem 3.3. Let $N = \mathfrak{m}R + R_+$ and let

$$\mathrm{a}_i(G) = \sup\{n \in \mathbb{Z} \mid [\mathrm{H}^i_N(G)]_n \neq (0)\}$$

for $0 \leq i \leq d$.

Lemma 4.1. *The following assertions hold true.*

(1) $\mathrm{a}_d(G) = 2 - d$ *and* $\ell_A([\mathrm{H}_N^d(G)]_{2-d}) = 1$.

(2) $\mathrm{a}_{d-1}(G) = 1 - d$ *and* $\ell_A([\mathrm{H}_N^{d-1}(G)]_{1-d}) = 1$.

In particular, $\mathrm{H}_N^0(G) = [\mathrm{H}_N^0(G)]_0$ *and* G *is a Buchsbaum ring if* $d = 1$.

Proof. Suppose $d = 1$. Let $a = a_1$ and $f = at$. Then $I^3 = aI^2$ by Theorem 3.3. Let $n \geq 1$ be an integer and let $x \in I^n$. Then, since $I^{n+2} = aI^{n+1}$, we obtain $x \in I^{n+1}$ if $ax \in I^{n+2}$. Thus $(0) :_G f = [(0) :_G f]_0$. Hence $(0) :_G f^n = (0) :_G f$ for all $n \geq 1$, so

$$\mathrm{H}_N^0(G) = (0) :_G f = [(0) :_G f]_0 \cong \tilde{I}/I.$$

In particular, $\ell_A(\tilde{I}/I) > 0$. Because

$$\begin{aligned} \mathrm{e}_1 &= \mathrm{e}_0 - \ell_A(A/I) + 1 \\ &= \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) - (\ell_A(\tilde{I}/I) - 1) \\ &\leq \mathrm{e}_0(\tilde{I}) - \ell_A(A/\tilde{I}) \\ &\leq \mathrm{e}_1(\tilde{I}) \\ &= \mathrm{e}_1, \end{aligned}$$

it follows that $\ell_A(\tilde{I}/I) = 1$, which proves assertion (2). In particular, $\mathrm{H}_N^0(G) = [\mathrm{H}_N^0(G)]_0$ and G is a Buchsbaum ring. Because $(0) :_G f = \mathrm{H}_N^0(G)$, we have the exact sequence

$$0 \to \mathrm{H}_N^0(G) \to G/fG \to \mathrm{H}_N^1(G)(-1) \xrightarrow{f} \mathrm{H}_N^1(G) \to 0$$

of local cohomology modules. Hence $\mathrm{a}_1(G) = 1$, because $\mathrm{H}_N^0(G) = [\mathrm{H}_N^0(G)]_0$ and $G/fG = A/I \oplus I/Q \oplus I^2/QI$ with $I^2/QI \neq (0)$. We have $[G/fG]_2 \cong [\mathrm{H}_N^1(G)]_1$, whence $\ell_A([\mathrm{H}_N^1(G)]_1) = \ell_A(I^2/QI) = 1$ by Theorem 3.3.

Now we consider the case where $d \geq 2$. Because $\operatorname{depth} G = d - 1 > 0$ by Theorem 3.3, we may assume that $f = a_1 t$ is regular on G. We put $\bar{A} = A/(a_1)$, $\bar{I} = I/(a_1)$, and $\bar{Q} = Q/(a_1)$. Then $\mathrm{e}_1(\bar{I}) = 2$ and $\bar{I}^2 \neq \bar{Q}\bar{I}$ (cf. the proof of Theorem 3.3). The induction hypothesis now yields assertions (1) and (2) for the ideal $\bar{I}$.

We next look at the exact sequence

$$\begin{aligned} 0 \to \mathrm{H}_N^{d-2}(\mathrm{G}(\bar{I})) \to \mathrm{H}_N^{d-1}(G)(-1) \xrightarrow{f} \mathrm{H}_N^{d-1}(G) \to \mathrm{H}_N^{d-1}(\mathrm{G}(\bar{I})) \\ \to \mathrm{H}_N^d(G)(-1) \xrightarrow{f} \mathrm{H}_N^d(G) \to 0 \qquad (*) \end{aligned}$$

of local cohomology modules, which is induced from the canonical exact sequence

$$0 \to G(-1) \xrightarrow{f} G \to \mathrm{G}(\bar{I}) \to 0$$

of graded G-modules. Because $\mathrm{a}_{d-2}(\mathrm{G}(\bar{I})) = 2 - d$, we get a monomorphism $[\mathrm{H}_N^{d-1}(G)]_n \hookrightarrow [\mathrm{H}_N^{d-1}(G)]_{n+1}$ for all $n \geq 2 - d$, whence $[\mathrm{H}_N^{d-1}(G)]_n = (0)$ for all $n \geq 2 - d$. Thus $\mathrm{a}_{d-1}(G) \leq 1 - d$ and

$$[\mathrm{H}_N^{d-2}(\mathrm{G}(\bar{I}))]_{2-d} \cong [\mathrm{H}_N^{d-1}(G)]_{1-d}.$$

Therefore, $\mathrm{a}_{d-1}(G) = 1 - d$ and $\ell_A([\mathrm{H}_N^{d-1}(G)]_{1-d}) = \ell_A([\mathrm{H}_N^{d-2}(\mathrm{G}(\bar{I}))]_{2-d}) = 1$. On the other hand, letting $\mathrm{a} = \mathrm{a}_d(G)$ in the exact sequence $(*)$ shows that

$$[\mathrm{H}_N^{d}(G)(-1)]_{\mathrm{a}+1} = [\mathrm{H}_N^{d}(G)]_{\mathrm{a}} \ \ (\neq (0))$$

is a homomorphic image of $[\mathrm{H}_N^{d-1}(\mathrm{G}(\bar{I}))]_{\mathrm{a}+1}$. Hence $\mathrm{a}+1 \leq \mathrm{a}_{d-1}(\mathrm{G}(\bar{I})) = 3-d$, whence $\mathrm{a} \leq 2-d$. Because $[\mathrm{H}_N^{d-1}(G)]_{3-d} = (0)$ and $[\mathrm{H}_N^{d}(G)]_{3-d} = (0)$, it follows from $(*)$ that $[\mathrm{H}_N^{d-1}(\mathrm{G}(\bar{I}))]_{3-d} \cong [\mathrm{H}_N^{d}(G)]_{2-d}$. Consequently, $\mathrm{a}_d(G) = 2-d$ and $\ell_A([\mathrm{H}_N^{d}(G)]_{2-d}) = 1$, as claimed. □

We can now state the main result of this section. See Theorem 5.1 for an example whose associated graded ring G is a Buchsbaum ring.

THEOREM 4.2. *The following two conditions are equivalent.*

(1) *G is a Buchsbaum ring.*
(2) $\mathrm{H}_N^{d-1}(G) = [\mathrm{H}_N^{d-1}(G)]_{1-d}$.

When $d \geq 2$, one can add the following:

(3) *R is a Buchsbaum ring.*

Proof. (2) ⇒ (1): By Lemma 4.1 we have $N \cdot \mathrm{H}_N^{d-1}(G) = 0$, since

$$\mathfrak{m} \cdot [\mathrm{H}_N^{d-1}(G)]_{1-d} = (0).$$

Hence G is a Buchsbaum ring, because $\operatorname{depth} G = d-1$ by Theorem 3.3.

(1) ⇒ (2): By Lemma 4.1 we may assume that $d \geq 2$ and that our assertion holds true for $d-1$. Because $\operatorname{depth} G = d-1 > 0$, we may assume that $f = a_1 t$ is regular on G. Similarly as before, let $\bar{A} = A/(a_1)$, $\bar{I} = I/(a_1)$, and $\bar{Q} = Q/(a_1)$. Then $\mathrm{G}(\bar{I}) = G/fG$ is a Buchsbaum ring with $\operatorname{depth} \mathrm{G}(\bar{I}) = d-2$. Hence, by induction we derive $\mathrm{H}_N^{d-2}(\mathrm{G}(\bar{I})) = [\mathrm{H}_N^{d-2}(\mathrm{G}(\bar{I}))]_{2-d}$. Thus $\mathrm{H}_N^{d-1}(G) = [\mathrm{H}_N^{d-1}(G)]_{1-d}$, because $\mathrm{H}_N^{d-2}(\mathrm{G}(\bar{I})) \cong \mathrm{H}_N^{d-1}(G)(-1)$ (see the exact sequence $(*)$ in the proof of Lemma 4.1).

(3) ⇒ (1): We continue to suppose that $d \geq 2$. Apply functors $\mathrm{H}_N^i(\cdot)$ to the exact sequences

$$0 \to R_+ \to R \to A \to 0 \quad \text{and} \quad 0 \to R_+(1) \to R \to G \to 0.$$

Then, since $\operatorname{depth} R = d$ (cf. [4, Thm. 2.1]), we have the exact sequences

$$\begin{aligned} &0 \to \mathrm{H}_N^{d}(R_+) \to \mathrm{H}_N^{d}(R) \to \mathrm{H}_{\mathfrak{m}}^{d}(A) \quad \text{and} \\ &0 \to \mathrm{H}_N^{d-1}(G) \to \mathrm{H}_N^{d}(R_+)(1) \to \mathrm{H}_N^{d}(R) \to \mathrm{H}_N^{d}(G). \end{aligned} \tag{$**$}$$

Because R is a Buchsbaum ring, $N \cdot \mathrm{H}_N^{d}(R) = (0)$ and so $N \cdot \mathrm{H}_N^{d}(R_+) = (0)$. Thus $N \cdot \mathrm{H}_N^{d-1}(G) = (0)$, whence G is a Buchsbaum ring.

(2) ⇒ (3): Consider the exact sequences $(**)$. Then

$$[\mathrm{H}_N^{d}(R_+)]_{n+1} \twoheadrightarrow [\mathrm{H}_N^{d}(R)]_n$$

for all $n > \mathrm{a}_d(G) = 2-d$. Hence

$$[\mathrm{H}_N^d(R)]_n \cong [\mathrm{H}_N^d(R_+)]_n = (0)$$

for all $n > 2 - d$. We have

$$[\mathrm{H}_N^d(R_+)]_n \cong [\mathrm{H}_N^d(R)]_n$$

for all $n < 0$ and

$$[\mathrm{H}_N^d(R_+)]_n = [\mathrm{H}_N^d(R_+)(1)]_{n-1} \hookrightarrow [\mathrm{H}_N^d(R)]_{n-1}$$

for all $n < 2 - d$, since $\mathrm{H}_N^{d-1}(G) = [\mathrm{H}_N^{d-1}(G)]_{1-d}$. Therefore, since $d \geq 2$, it follows that $[\mathrm{H}_N^d(R)]_n$ is embedded into $[\mathrm{H}_N^d(R)]_{n-1}$ for all $n < 2 - d$. Hence $[\mathrm{H}_N^d(R)]_n = (0)$ for all $n < 2 - d$, because $\mathrm{H}_N^d(R)$ is a finitely graded R-module (cf. [1]; recall that G is a Buchsbaum ring). As a result,

$$\mathrm{H}_N^d(R) = [\mathrm{H}_N^d(R)]_{2-d}.$$

Because $[\mathrm{H}_N^d(R_+)]_{3-d} = (0)$, by the exact sequence $(**)$ we have

$$[\mathrm{H}_N^d(R)]_{2-d} \hookrightarrow [\mathrm{H}_N^d(G)]_{2-d}$$

and so $\ell_A(\mathrm{H}_N^d(R)) = 1$, since $\ell_A([\mathrm{H}_N^d(G)]_{2-d}) = 1$ by Lemma 4.1 and depth $R = d$ by [4, Thm. 2.1]. Thus $N \cdot \mathrm{H}_N^d(R) = (0)$, whence R is a Buchsbaum ring. $\square$

5. An Example

In this section we explore the following example, which satisfies the conditions in Theorem 1.1(1) and Theorem 4.2(1). The example is a generalization of an example given by the first author [3], where the case $\Lambda = \emptyset$ is explored.

Let $m \geq d > 0$ be integers. Let Λ be a subset of $\{1, 2, \ldots, m\}$ such that $\Lambda \cap \{1, 2, \ldots, d\} = \emptyset$. Let

$$U = k[[X_1, X_2, \ldots, X_m, V, Y_1, Y_2, \ldots, Y_d]]$$

be the formal power series ring over a field k, and let

$$\mathfrak{a} = (X_1, X_2, \ldots, X_m) \cdot (X_1, X_2, \ldots, X_m, V) + \left(V^2 - \sum_{i=1}^{d} X_i Y_i\right).$$

We put $A = U/\mathfrak{a}$ and denote the images of X_i, V, and Y_j in A by x_i, v, and a_j, respectively. Then $\dim A = d$, since $\sqrt{\mathfrak{a}} = (X_1, X_2, \ldots, X_m, V)$. Let $\mathfrak{m} = (x_j \mid 1 \leq j \leq m) + (v) + (a_i \mid 1 \leq i \leq d)$ be the maximal ideal in A. We put

$$I = (a_1, a_2, \ldots, a_d) + (x_\alpha \mid \alpha \in \Lambda) + (v) \quad \text{and} \quad Q = (a_1, a_2, \ldots, a_d).$$

Then $\mathfrak{m}^2 = Q\mathfrak{m}$, $I^2 = QI + (v^2) \neq QI$, and $I^3 = QI^2$ (cf. Lemma 5.3), whence Q is a minimal reduction of both $\mathfrak{m}$ and I and the series $a_1, a_2, \ldots, a_d$ is a system of parameters for A.

We are now interested in the Hilbert coefficients e_i' of the ideal I as well as the structure of the associated graded ring and the Sally module of I. We maintain the same notation as in the previous sections. Our first result is as follows.

THEOREM 5.1. *The following assertions hold true.*

1. *A is a Cohen–Macaulay local ring with* $\dim A = d$.
2. *$S \cong B(-1)$ as graded T-modules.*
3. $\mathrm{e}_0 = m+2$ *and* $\mathrm{e}_1 = \#\Lambda + 2$. *Hence,* $\mathrm{e}_1 = 2$ *but* $I^2 \neq QI$ *if* $\Lambda = \emptyset$.
4. $\mathrm{e}_2 = 1$ *if* $d \geq 2$, *and* $\mathrm{e}_i = 0$ *for all* $3 \leq i \leq d$ *if* $d \geq 3$.
5. *G is a Buchsbaum ring with* $\operatorname{depth} G = d-1$, *and* $\ell_A(\mathrm{H}_N^{d-1}(G)) = 1$.

We divide the proof of Theorem 5.1 into several steps as follows.

PROPOSITION 5.2. *Let* $\mathfrak{p} = \sqrt{(X_1, X_2, \dots, X_m, V)}$ *in U. Then* $\ell_{U_\mathfrak{p}}(A_\mathfrak{p}) = m+2$.

Proof. Let $\tilde{k} = k[Y_1, 1/Y_1]$ and $\tilde{U} = U[1/Y_1]$. We put $Z_i = X_i/Y_1$ for $1 \leq i \leq m$, $T_j = Y_j/Y_1$ for $2 \leq j \leq d$, and $W = V/Y_1$. Then $\tilde{U} = \tilde{k}[Z_1, Z_2, \dots, Z_m, V, T_2, T_3, \dots, T_d]$ and

$$\mathfrak{a}\tilde{U} = (Z_1, Z_2, \dots, Z_m) \cdot (Z_1, Z_2, \dots, Z_m, W) + \left(W^2 - \sum_{j=2}^{d} T_j Z_j - Z_1\right).$$

Since the elements $\{Z_i\}_{1 \leq i \leq m}$, W, and $\{T_j\}_{2 \leq j \leq d}$ are algebraically independent over $\tilde{k}$, it follows that

$$\tilde{U}/\mathfrak{a}\tilde{U} \cong \bar{U} = \frac{\tilde{k}[Z_2, Z_3, \dots, Z_m, W, T_2, T_3, \dots, T_d]}{(W^2, Z_2, Z_3, \dots, Z_m) \cdot (Z_2, Z_3, \dots, Z_m, W)},$$

where we have replaced Z_1 with $W^2 - \sum_{j=2}^{d} T_j Z_j$. Then the ideal $\mathfrak{p}\tilde{U}/K\tilde{U}$ corresponds to the prime ideal $P = (Z_2, Z_3, \dots, Z_m, W)$. Thus $\ell_{U_\mathfrak{p}}(A_\mathfrak{p}) = \ell_{\bar{U}_P}(\bar{U}_P) = m+2$. $\square$

Now we have $\mathrm{e}_0(Q) = \ell_{U_\mathfrak{p}}(A_\mathfrak{p}) \cdot \mathrm{e}_0^{A/\mathfrak{p}A}((Q + \mathfrak{p}A)/\mathfrak{p}A) = m+2$ by the associative formula of multiplicity, because $\mathfrak{p} = \sqrt{\mathfrak{a}}$ and $U/\mathfrak{p} \cong k[Y_1, Y_2, \dots, Y_d]$. On the other hand, $\ell_A(A/Q) = m+2$ because

$$A/Q \cong \frac{k[[X_1, X_2, \dots, X_m, V]]}{(X_1, X_2, \dots, X_m) \cdot (X_1, X_2, \dots, X_m, V) + (V^2)}.$$

Hence $\mathrm{e}_0(Q) = \ell_A(A/Q)$, so A is a Cohen–Macaulay ring and $\mathrm{e}_0(Q) = m+2$.

LEMMA 5.3. *The following assertions hold true.*

1. $\mathfrak{m}^2 = Q\mathfrak{m}$, $I^2 = QI + (v^2) \neq QI$, *and* $I^3 = QI^2$.
2. $(a_1, a_2, \dots, \check{a}_i, \dots, a_d) \cap I^2 = (a_1, a_2, \dots, \check{a}_i, \dots, a_d)I$ *for all* $1 \leq i \leq d$.
3. $(a_\alpha \mid \alpha \in \Gamma) \cap I^n = (a_\alpha \mid \alpha \in \Gamma)I^{n-1}$ *for all subsets* $\Gamma \subsetneq \{1, 2, \dots, d\}$ *and for all integers* $n \in \mathbb{Z}$.
4. $(a_1^2, a_2^2, \dots, a_d^2) \cap I^n = (a_1^2, a_2^2, \dots, a_d^2)I^{n-2}$ *for all* $3 \leq n \leq d+1$.

Proof. (1) It is routine to check that $\mathfrak{m}^2 = Q\mathfrak{m}$ and $I^2 = QI + (v^2)$. We have $I^3 = QI^2$, since $v^3 = 0$. Let us check that $v^2 \notin QI$. Suppose $v^2 \in QI$ and write

$$v^2 = \sum_{i=1}^{d} a_i x_i = \sum_{i=1}^{d} a_i \xi_i$$

with $\xi_i \in I$. Then $a_d(x_d - \xi_d) \in (a_1, a_2, \ldots, a_{d-1})$ and so $x_d - \xi_d \in (a_1, a_2, \ldots, a_{d-1})$, because $a_1, a_2, \ldots, a_d$ is a regular sequence. Hence $x_d \in I$ so that $X_d \in \mathfrak{a} + (Y_1, Y_2, \ldots, Y_d)U + (X_\alpha \mid \alpha \in \Lambda)U + VU$, which is impossible because $\Lambda \cap \{1, 2, \ldots, d\} = \emptyset$.

(2) Let $1 \le i \le d$ be an integer and put $Q_i = (a_1, a_2, \ldots, \check{a}_i, \ldots, a_d)$. Then

$$\begin{aligned} Q_i \cap I^2 &= Q_i \cap (QI + (v^2)) \\ &= Q_i \cap (Q_i I + a_i I + (v^2)) \\ &= Q_i I + Q_i \cap [a_i I + (v^2)]. \end{aligned}$$

Let $\varphi \in Q_i \cap (a_i I + v^2 A)$ and write $\varphi = a_i \rho + v^2 \xi$ with $\rho \in I$ and $\xi \in A$. Then $\varphi = a_i \rho + \sum_{j=1}^{d} a_j x_j \xi = a_i(\rho + x_i \xi) + \sum_{j \ne i} a_j x_j \xi$. Hence $a_i(\rho + x_i \xi) \in Q_i$ and so $\rho + x_i \xi \in Q_i$; thus $x_i \xi \in I$. Therefore, $\xi \in \mathfrak{m} = I + (x_\alpha \mid \alpha \notin \Lambda)$. Let $\xi = \xi' + \xi''$ with $\xi' \in I$ and $\xi'' \in (x_\alpha \mid \alpha \notin \Lambda)$. Notice that $x_j \xi = x_j(\xi' + \xi'') = x_j \xi' + x_j \xi'' = x_j \xi'$ for all $1 \le j \le d$, since $x_j \xi'' \in (x_1, x_2, \ldots, x_m)^2 = (0)$. Consequently, $\varphi = a_i(\rho + x_i \xi') + \sum_{j \ne i} a_j x_j \xi' \in Q_i I$, since $\xi' \in I$ and $\rho + x_i \xi' = \rho + x_i \xi \in Q_i$. Thus $Q_i \cap I^2 \subseteq Q_i I$, so we have $Q_i \cap I^2 = Q_i I$.

(3) Let $\tau = \#\Gamma$; we will prove assertion (3) by descending induction on τ. Suppose that $\tau = d - 1$ and let $\Gamma = \{1, 2, \ldots, \check{i}, \ldots, d\}$ with $1 \le i \le d$. If $n \le 2$, assertion (3) is obvious and follows from assertion (2). So assume that $n \ge 3$ and that our assertion holds true for $n - 1$. Then, since $I^3 = QI^2$, we have

$$\begin{aligned} Q_i \cap I^n &= Q_i \cap QI^{n-1} \\ &= Q_i \cap (Q_i I^{n-1} + a_i I^{n-1}) \\ &= Q_i I^{n-1} + [Q_i \cap a_i I^{n-1}] \\ &= Q_i I^{n-1} + a_i [Q_i \cap I^{n-1}]. \end{aligned}$$

Since $Q_i \cap I^{n-1} = Q_i I^{n-2}$, it follows by induction on n that

$$a_i [Q_i \cap I^{n-1}] = a_i [Q_i I^{n-2}] \subseteq Q_i I^{n-1}.$$

Thus $Q_i \cap I^n \subseteq Q_i I^{n-1}$, whence $Q_i \cap I^n = Q_i I^{n-1}$.

We now consider the case where $\tau < d - 1$. Assume that $n \ge 2$ and that our assertion holds true for $n - 1$. Let $\varphi \in (a_\alpha \mid \alpha \in \Gamma) \cap I^n$ and let $\beta \in \{1, 2, \ldots, d\} \setminus \Gamma$. Then

$$(a_\alpha \mid \alpha \in \Gamma) \cap I^n \subseteq [(a_\alpha \mid \alpha \in \Gamma) + (a_\beta)] \cap I^n = [(a_\alpha \mid \alpha \in \Gamma) + (a_\beta)] I^{n-1}$$

by the hypothesis on τ. We write $\varphi = \varphi' + a_\beta \rho$ with $\varphi' \in (a_\alpha \mid \alpha \in \Gamma) I^{n-1}$ and $\rho \in I^{n-1}$. Then $a_\beta \rho \in (a_\alpha \mid \alpha \in \Gamma)$ and so $\rho \in (a_\alpha \mid \alpha \in \Gamma) \cap I^{n-1}$, while $(a_\alpha \mid \alpha \in \Gamma) \cap I^{n-1} = (a_\alpha \mid \alpha \in \Gamma) I^{n-2}$ by the hypothesis on n. Hence $\rho \in (a_\alpha \mid \alpha \in \Gamma) I^{n-2}$ and so $\varphi \in (a_\alpha \mid \alpha \in \Gamma) I^{n-1}$. Thus $(a_\alpha \mid \alpha \in \Gamma) \cap I^n \subseteq (a_\alpha \mid \alpha \in \Gamma) I^{n-1}$ as claimed.

(4) We put $J = (a_1^2, a_2^2, \ldots, a_d^2)$. Assume that $J \cap I^n \ne JI^{n-2}$ for some $3 \le n \le d + 1$ and choose d as small as possible among such counterexamples. Hence $d \ge 2$. Let $\varphi \in J \cap I^n$ such that $\varphi \notin JI^{n-2}$.

We begin with the following.

Claim 3.

$$I^{d+1} = JI^{d-1} + a_1a_2\cdots a_dI + \sum_{i=1}^{d} a_1a_2\cdots \check{a}_i\cdots a_dv^2A.$$

Proof of Claim 3. Since $I^2 = QI + (v^2)$ and $I^3 = QI^2$, we have

$$I^{d+1} = Q^{d-1}I^2 = Q^{d-1}(QI + (v^2)) = Q^dI + v^2Q^{d-1}.$$

On the other hand, since

$$Q^d = JQ^{d-2} + (a_1a_2\cdots a_d) \quad \text{and}$$

$$Q^{d-1} = JQ^{d-3} + \sum_{i=1}^{d} a_1a_2\cdots \check{a}_i\cdots a_dA,$$

it follows that

$$Q^dI = JQ^{d-2}I + a_1a_2\cdots a_dI \subseteq JI^{d-1} + a_1a_2\cdots a_dI$$

and

$$\begin{aligned} v^2Q^{d-1} &= v^2JQ^{d-3} + v^2\Bigl(\sum_{i=1}^{d} a_1a_2\cdots \check{a}_i\cdots a_dA\Bigr) \\ &\subseteq JI^{d-1} + \sum_{i=1}^{d} a_1a_2\cdots \check{a}_i\cdots a_dv^2A \end{aligned}$$

(notice that $v \in I$). Therefore,

$$I^{d+1} \subseteq JI^{d-1} + a_1a_2\cdots a_dI + \sum_{i=1}^{d} a_1a_2\cdots \check{a}_i\cdots a_dv^2A. \qquad \square$$

Suppose that $n = d + 1$. Then by Claim 3 we may write

$$\varphi = \varphi' + a_1a_2\cdots a_d\eta + \sum_{i=1}^{d} c_ia_1a_2\cdots \check{a}_i\cdots a_dv^2$$

with $\varphi' \in JI^{d-1}$, $\eta \in I$, and $c_i \in A$. Because $v^2 = \sum_{i=1}^{d} a_ix_i$, we see that

$$\sum_{i=1}^{d} c_ia_1a_2\cdots \check{a}_i\cdots a_dv^2 \equiv a_1a_2\cdots a_d\Bigl(\sum_{i=1}^{d} c_ix_i\Bigr) \bmod J,$$

whence

$$a_1a_2\cdots a_d\Bigl(\eta + \sum_{i=1}^{d} c_ix_i\Bigr) \equiv a_1a_2\cdots a_d\eta + \sum_{i=1}^{d} c_ia_1a_2\cdots \check{a}_i\cdots a_dv^2 \equiv 0 \bmod J$$

because

$$\varphi = \varphi' + a_1a_2\cdots a_d\eta + \sum_{i=1}^{d} c_ia_1a_2\cdots \check{a}_i\cdots a_dv^2 \in J.$$

As a result, $\eta + \sum_{i=1}^{d} c_i x_i \in Q$ because $a_1, a_2, \ldots, a_d$ is a regular sequence in A; thus we have

$$\sum_{i=1}^{d} c_i x_i \in I = (a_i \mid 1 \le i \le d) + (x_\alpha \mid \alpha \in \Lambda) + (v).$$

Since $\{x_i\}_{1 \le i \le m}$, v, and $\{a_i\}_{1 \le i \le d}$ constitute a minimal basis of the maximal ideal $\mathfrak{m}$ of A and since $\Lambda \cap \{1, 2, \ldots, d\} = \emptyset$, this forces $c_i \in \mathfrak{m}$ for all $1 \le i \le d$. We write $c_i = c_i' + c_i''$ with $c_i' \in Q$ and $c_i'' \in (x_1, x_2, \ldots, x_m, v)$. Then, since $(x_1, x_2, \ldots, x_m, v) \cdot (x_1, x_2, \ldots, x_m) = (0)$, it follows that $c_i'' x_i = 0$ and so

$$c_i x_i = c_i' x_i + c_i'' x_i = c_i' x_i \in Q$$

because $c_i' \in Q$. Consequently, since $\eta + \sum_{i=1}^{d} c_i x_i \in Q$, we have

$$\eta \equiv \eta + \sum_{i=1}^{d} c_i' x_i = \eta + \sum_{i=1}^{d} c_i x_i \equiv 0 \bmod Q.$$

Hence $\eta \in Q$ and so

$$a_1 a_2 \cdots a_d \eta \in Q^{d+1} = (a_1^2, a_2^2, \ldots, a_d^2) Q^{d-1} \subseteq J I^{d-1}.$$

On the other hand, we have $c_i'' v^2 = 0$ since $c_i'' \in (x_1, x_2, \ldots, x_m, v)$, so that $c_i v^2 = c_i' v^2 + c_i'' v^2 = c_i' v^2 \in Q^2$ because $c_i', v^2 \in Q$. Thus

$$c_i a_1 a_2 \cdots \check{a}_i \cdots a_d v^2 = a_1 a_2 \cdots \check{a}_i \cdots a_d \cdot c_i' v^2 \in Q^{d+1} \subseteq J I^{d-1}$$

for all $1 \le i \le d$, so that

$$\varphi = \varphi' + a_1 a_2 \cdots a_d \eta + \sum_{i=1}^{d} c_i a_1 a_2 \cdots \check{a}_i \cdots a_d v^2 \in J I^{d-1},$$

which is a contradiction. Therefore, $3 \le n \le d$.

We put $\bar{A} = A/(a_d)$ and $\bar{I} = I/(a_d)$. For each $x \in A$, let $\bar{x}$ denote the image of x in $\bar{A}$. We then have, by the minimality of d, that

$$(\overline{a_1^2}, \overline{a_2^2}, \ldots, \overline{a_{d-1}^2}) \cap \bar{I}^n = (\overline{a_1^2}, \overline{a_2^2}, \ldots, \overline{a_{d-1}^2}) \bar{I}^{n-2}$$

for all $3 \le n \le d$. Hence $\bar{\varphi} \in (\overline{a_1^2}, \overline{a_2^2}, \ldots, \overline{a_{d-1}^2}) \bar{I}^{n-2}$, so that

$$\varphi \in (a_1^2, a_2^2, \ldots, a_{d-1}^2) I^{n-2} + [(a_d) \cap I^n].$$

Since $(a_d) \cap I^n = a_d I^{n-1}$ by assertion (3), we have $\varphi = \varphi' + a_d \xi$ for some $\varphi' \in (a_1^2, a_2^2, \ldots, a_{d-1}^2) I^{n-2}$ and $\xi \in I^{n-1}$; hence $a_d \xi \in J$, because $\varphi, \varphi' \in J$. We write $a_d \xi = \sum_{i=1}^{d} a_i^2 \xi_i$ with $\xi_i \in A$. Then $a_d(\xi - a_d \xi_d) \in (a_1^2, a_2^2, \ldots, a_{d-1}^2)$ and so $\xi - a_d \xi_d \in (a_1^2, a_2^2, \ldots, a_{d-1}^2)$. Consequently,

$$\bar{\xi} \in (\overline{a_1^2}, \overline{a_2^2}, \ldots, \overline{a_{d-1}^2}) \cap \bar{I}^{n-1} = (\overline{a_1^2}, \overline{a_2^2}, \ldots, \overline{a_{d-1}^2}) \bar{I}^{n-3}$$

by the minimality of d. Therefore,

$$\xi \in (a_1^2, a_2^2, \ldots, a_{d-1}^2) I^{n-3} + [(a_d) \cap I^{n-1}].$$

However, since $(a_d) \cap I^{n-1} = a_d I^{n-2}$ by assertion (3), we have

$$a_d\xi \in a_d(a_1^2, a_2^2, \ldots, a_{d-1}^2)I^{n-3} + a_d^2 I^{n-2} \subseteq JI^{n-2}.$$

As a result, $\varphi = \varphi' + a_d\xi \in JI^{n-2}$, which is the required contradiction. Thus

$$J \cap I^n = JI^{n-2}$$

for all $3 \leq n \leq d+1$, as we wanted. □

We are now in a position to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. We have $\ell_A(I^2/QI) = 1$, since $\mathfrak{m}v^2 \subseteq QI$ (recall that $I^2 \neq QI$ and $I^2 = QI + (v^2)$ by Lemma 5.3(1)). Because $I^3 = QI^2$, by Theorem 1.1 we have $S \cong B(-1)$ as graded T-modules, so that $\mathrm{e}_1 = \mathrm{e}_0 - \ell_A(A/I) + 1$, $\mathrm{e}_2 = 1$ if $d \geq 2$, and $\mathrm{e}_i = 0$ for all $3 \leq i \leq d$ if $d \geq 3$. Because $\ell_A(A/I) = m - \#\Lambda + 1$ and $\mathrm{e}_0 = m + 2$, we obtain $\mathrm{e}_1 = \#\Lambda + 2$; hence $\mathrm{e}_1 = 2$ if $\Lambda = \emptyset$.

Observe that G is not a Cohen–Macaulay ring. In fact, $Q \cap I^2 \neq QI$ (recall that $I^2 \subseteq Q$ since $\mathfrak{m}^2 = Q\mathfrak{m}$). The ring G is Buchsbaum by parts (1), (2), and (4) of Lemma 5.3 and [3, Prop. 9.1], so $\mathrm{H}_N^{d-1}(G) = [\mathrm{H}_N^{d-1}(G)]_{1-d}$ and $\ell_A([\mathrm{H}_N^{d-1}(G)]_{1-d}) = 1$ both follow by induction on d similarly as in the proofs of Lemma 4.1 and Theorem 4.2. □

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