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The Core of Ideals in Arbitrary Characteristic

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Dedicated to Mel Hochster on the occasion of his sixty-fifth birthday

1. Introduction

In this paper we provide explicit formulas for the core of an ideal. Recall that for an ideal I in a Noetherian ring R, the *core* of I, $\operatorname{core}(I)$, is the intersection of all reductions of I. For a subideal $J \subset I$ we say that J is a *reduction* of I, or that I is *integral* over J, if $I^{r+1} = JI^r$ for some $r \geq 0$; the smallest such r is called the *reduction number* of I with respect to J and is denoted by $r_J(I)$. If $(R, \mathfrak{m})$ is local with infinite residue field k then every ideal has a *minimal reduction,* which is a reduction minimal with respect to inclusion. Minimal reductions of a given ideal I are far from unique, but they all share the same minimal number of generators, called the *analytic spread* of I and written $\ell(I)$. Minimal reductions arise from Noether normalizations of the *special fiber ring* $\mathcal{F}(I) = \operatorname{gr}_I(R) \otimes k$ of I, and therefore $\ell(I) = \dim \mathcal{F}(I)$. From this one readily sees that $\operatorname{ht} I \leq \ell(I) \leq \dim R$; these inequalities are equalities for any $\mathfrak{m}$-primary ideal, and if the first inequality is an equality then I is called *equimultiple*. Obviously, the core can be obtained as an intersection of minimal reductions of a given ideal.

Through the study of the core one hopes to better understand properties shared by all reductions. The notion was introduced by Rees and Sally for the purpose of generalizing the Briançon–Skoda Theorem [17]. As an a priori infinite intersection of reductions, the core is difficult to compute, and there have been considerable efforts to find explicit formulas; see [3; 4; 9; 10; 11; 12; 15]. We quote the following result from [15].

Theorem 1.1. *Let R be a local Gorenstein ring with infinite residue field k, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. Assume that I satisfies G_ℓ, that* $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ *for* $1 \leq j \leq \ell - g$, *and that either* $\operatorname{char} k = 0$ *or* $\operatorname{char} k > r - \ell + g$. *Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

The property G_ℓ in Theorem 1.1 is a rather weak requirement on the local number of generators of I: it means that the minimal number of generators $\mu(I_\mathfrak{p})$ is at most

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dim $R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ containing I with dim $R_\mathfrak{p} \leq \ell - 1$. Both hypotheses, the G_ℓ condition and the depth assumption on the powers, are automatically satisfied if I is equimultiple. They also hold for one-dimensional generic complete intersection ideals or, more generally, for Cohen–Macaulay generic complete intersections with $\ell = g+1$. In the presence of the G_ℓ property, the depth inequalities for the powers hold if I is perfect with $g = 2$, if I is perfect Gorenstein with $g = 3$ or, more generally, if I is in the linkage class of a complete intersection [8, 1.11].

Theorem 1.1 is not true in general without the assumption on the characteristic, as was shown in [15, 4.9]. Hence in this paper we study the case of arbitrary characteristic. Explicit formulas for the core that are valid in any characteristic and for any reduction number are known for equimultiple ideals of height 1 [15, 3.4(a)] and for powers of the homogeneous maximal ideal of standard graded reduced Cohen–Macaulay rings over an infinite perfect field [12, 4.1]. In this paper we clarify the latter result and generalize it to ideals generated by forms of the same degree that are not necessarily zero-dimensional or even equimultiple.

THEOREM 1.2. *Let k be an infinite field, R' a positively graded geometrically reduced Cohen–Macaulay k-algebra, and R the localization of R' at the homogeneous maximal ideal. Let I be an R-ideal generated by forms in R' of the same degree with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. If $\ell > g$, further assume that R' is Gorenstein, I satisfies G_ℓ, and* depth $R/I^j \geq \dim R/I - j + 1$ *for* $1 \leq j \leq \ell - g$. *Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

Recall that the k-algebra R' is said to be *geometrically reduced* if, after tensoring with the algebraic closure $\bar{k}$ of k, the ring $R' \otimes_k \bar{k}$ is reduced.

Theorem 1.2 is a special case of a considerably more general result in which the assumption on the grading is replaced by the condition that the residue field is perfect and the special fiber ring $\mathcal{F}(I)$ is reduced or, still more generally, has embedding dimension ≤ 1 locally at every minimal prime of maximal dimension (Theorem 3.3). We identify further instances where the assumption on the special fiber ring is satisfied. More generally than in Theorem 1.2, it suffices to require that $I = (K, f)$, where K is generated by forms of the same degree and either f is integral over K (Theorem 4.1) or else $\ell(K) \leq \ell(I)$ and $\mathcal{F}(K)$ satisfies Serre's condition R_1 (Theorem 4.3). We give a series of examples showing that our hypotheses are sharp: Theorem 1.2 fails to hold without the assumption of geometric reducedness, even when R' is a domain and $I = \mathfrak{m}$ (Example 5.1); this also shows that an assumption needs to be added in [12, 4.1]. Likewise, in Theorem 3.3 it does not suffice to suppose that the generic embedding dimension of $\mathcal{F}(I)$ be at most 2 (Example 5.2), and in Theorems 4.1 and 4.3 we must require f to be integral over K or $\mathcal{F}(K)$ to satisfy R_1 (Example 5.3).

Our approach, which differs from that in [12], can be outlined as follows. Write $\ell = \ell(I)$, let $f_1, \dots, f_{\ell+1}$ be general elements in I, set $J = (f_1, \dots, f_\ell)$, and let $\bar{}$ denote reduction modulo the "geometric residual intersection" $(f_1, \ldots, f_{\ell-1}) : I$. Because $\bar{I}$ is an equimultiple ideal of height 1, we can apply the formula of [15, 3.4(a)]; this formula states that, regardless of characteristic,

$$\operatorname{core}(\bar{I}) = \bar{J}^{n+1} : \sum_{y \in \bar{I}} (\bar{J}, y)^n \quad \text{for } n \gg 0.$$

The problem is that the formula does not "lift" from $\bar{I}$ to I. On the other hand, according to one of our main technical results, the equality $\operatorname{core}(\bar{I}) = \bar{J}^{n+1} : \bar{I}^n$ does lift (Lemma 3.2; cf. also [15, 4.2]). Thus the task becomes to show that

$$\sum_{y \in \bar{I}} (\bar{J}, y)^n = \bar{I}^n \quad \text{for } n \gg 0.$$

This follows from a general "decomposition formula" for powers that may be interesting in its own right. In fact, we prove that if R is a Noetherian local ring with infinite perfect residue field and if $\mathcal{F}(I)$ has embedding dimension ≤ 1 locally at every minimal prime of maximal dimension, then

$$I^n = (f_1, \ldots, f_{\ell-1}) I^{n-1} + (f_\ell, f_{\ell+1})^n \quad \text{for } n \gg 0$$

(special case of Theorem 2.7).

2. A Decomposition Formula for Powers

In this section we show our decomposition formula for powers of ideals. The proof is based on Theorem 2.3, a generalization of the primitive element theorem. We begin by reviewing two lemmas.

LEMMA 2.1. *Let k be an infinite field, $A = k[X_1, \ldots, X_n]$ a polynomial ring with quotient field K, and B an A-algebra essentially of finite type. Then*

$$\dim B \otimes_A A/(\{X_i - \lambda_i\}) \leq \dim B \otimes_A K$$

for $(\lambda_1, \ldots, \lambda_n) \in k^n$ general.

Proof. By the generic flatness lemma, there exists an element $0 \neq f \in A$ such that $A_f \to B_f$ is flat and hence satisfies going down [14, 24.1]. For every $(\lambda_1, \ldots, \lambda_n) \in k^n \setminus V(f)$,

$$\dim B \otimes_A A/(\{X_i - \lambda_i\}) \leq \dim B \otimes_A K$$

by [14, 15.3]. $\square$

LEMMA 2.2. *Let k be an infinite field, C a finitely generated k-algebra, and $I = (f_1, \ldots, f_n)$ a C-ideal. Let $\mathfrak{a}$ be a C-ideal generated by t general k-linear combinations of $f_1, \ldots, f_n$. Then*

$$\dim C/(\mathfrak{a} : I^\infty) \leq \dim C - t.$$

In particular, $\dim C/\mathfrak{a} \leq \max\{\dim C - t, \dim C/I\}$.

Proof. Let X_{ij} be variables over k, where $1 \leq i \leq t$ and $1 \leq j \leq n$, set $R = C[\{X_{ij}\}]$, and write $\mathfrak{A}$ for the R-ideal generated by the t generic linear combinations $\sum_{j=1}^{n} X_{ij} f_j$, where $1 \leq i \leq t$. We first show that $\mathfrak{A} : I^{\infty}$ has height $\geq t$ in R or, equivalently, that $IR \subset \sqrt{\mathfrak{A}}$ locally in codimension $\leq t-1$. So, let Q be a prime ideal of R that has height $\leq t-1$ and does not contain I. Replacing C by $C_{Q \cap C}$, we may assume that C is local and $I = C$; after applying a C-automorphism of R we are in the situation where $f_1, \dots, f_n = 1, 0, \dots, 0$. But then $\mathfrak{A} = (X_{11}, \dots, X_{t1})$, which cannot be contained in Q because Q has height $\leq t-1$.

Next, consider the map $A = k[\{X_{ij}\}] \to B = R/\mathfrak{A} : I^{\infty}$. Write K for the quotient field of A, and put $S = R \otimes_A K = C \otimes_k K$. Notice that $\dim S = \dim C \otimes_k K = \dim C$, because C is a finitely generated k-algebra, and that $\operatorname{ht}(\mathfrak{A} : I^{\infty})S \geq \operatorname{ht}(\mathfrak{A} : I^{\infty}) \geq t$. Therefore,

$$\dim B \otimes_A K = \dim S/(\mathfrak{A} : I^{\infty})S \leq \dim S - \operatorname{ht}(\mathfrak{A} : I^{\infty})S \leq \dim C - t.$$

Finally, for a point $(\lambda_{ij}) \in k^{tn}$, let $\mathfrak{a}$ denote the C-ideal generated by the t elements $\sum_{j=1}^{n} \lambda_{ij} f_j$. Observe that $B \otimes_A A/(\{X_{ij} - \lambda_{ij}\})$ maps onto $C/(\mathfrak{a} : I^{\infty})$. Hence Lemma 2.1 shows that if (λ_{ij}) is general then $\dim C/(\mathfrak{a} : I^{\infty}) \leq \dim B \otimes_A K \leq \dim C - t$. □

THEOREM 2.3. *Let k be an infinite perfect field, $B = k[y_1, \dots, y_n]$ a finitely generated k-algebra of dimension d, and s a positive integer. Let A be a k-subalgebra generated by $d + s$ general k-linear combinations of $y_1, \dots, y_n$. Then B is a finite A-module, and* $\dim_A B/A < d$ *if and only if B has embedding dimension $\leq s$ locally at every minimal prime of dimension d.*

Proof. Clearly B is a finite A-module by Lemma 2.2.

First assume that $\dim_A B/A < d$. Let $\mathfrak{q} \in \operatorname{Spec}(B)$ with $\dim B/\mathfrak{q} = d$, and let $\mathfrak{p} = \mathfrak{q} \cap A$. Notice that $\dim A/\mathfrak{p} = d$. Since $\dim_A B/A < d$, it follows that $A_{\mathfrak{p}} = B_{\mathfrak{p}} = B_{\mathfrak{q}}$. Write A as an epimorphic image of the polynomial ring $k[X_1, \dots, X_{d+s}]$, and let $\mathfrak{P}$ be the preimage of $\mathfrak{p}$ in $k[X_1, \dots, X_{d+s}]$. Then

$$\begin{aligned} \dim k[X_1, \dots, X_{d+s}]_{\mathfrak{P}} &= d + s - \dim k[X_1, \dots, X_{d+s}]/\mathfrak{P} \\ &= d + s - \dim A/\mathfrak{p} = s. \end{aligned}$$

Hence $B_{\mathfrak{q}} = A_{\mathfrak{p}}$ has embedding dimension $\leq s$.

We now assume that B has embedding dimension $\leq s$ locally at every minimal prime of dimension d. Let $x_1, \dots, x_{d+s}$ be general k-linear combinations of $y_1, \dots, y_n$, and consider the exact sequence

$$0 \longrightarrow \mathbb{D} \longrightarrow C = B \otimes_k B \xrightarrow{\text{mult}} B \longrightarrow 0.$$

Notice that $\Omega_k(B) = \mathbb{D}/\mathbb{D}^2$ is the module of differentials of B over k. The C-ideal $\mathbb{D}$ is generated by $c_i = y_i \otimes 1 - 1 \otimes y_i$ for $1 \leq i \leq n$. Thus, setting $a_i = x_i \otimes 1 - 1 \otimes x_i$, we have that $a_1, \dots, a_{d+s}$ are general k-linear combinations of the generators $c_1, \dots, c_n$ of $\mathbb{D}$. Write $\mathfrak{a}$ for the C-ideal generated by $a_1, \dots, a_{d+s}$. According to Lemma 2.2,

$$\dim C/(\mathfrak{a} : \mathbb{D}^{\infty}) \le \dim C - d - s \le d - 1.$$

Hence, for every $Q \in \operatorname{Spec}(C)$ with $\dim C/Q = d$, $Q \in V(\mathfrak{a})$ if and only if $Q \in V(\mathbb{D})$. Observe that there are only finitely many such primes because they are all minimal over $\mathbb{D}$.

Let Q be one of these primes and write $\mathfrak{q} = Q/\mathbb{D}$. Now $\dim B/\mathfrak{q} = d$ and hence $\operatorname{edim} B_{\mathfrak{q}} \le s$. Consider the exact sequence

$$\mathfrak{q}B_{\mathfrak{q}}/\mathfrak{q}^2 B_{\mathfrak{q}} \longrightarrow \Omega_k(B_{\mathfrak{q}}) \otimes_{B_{\mathfrak{q}}} k(\mathfrak{q}) \longrightarrow \Omega_k(k(\mathfrak{q})) \longrightarrow 0.$$

In this sequence we have $\mu(\mathfrak{q}B_{\mathfrak{q}}/\mathfrak{q}^2 B_{\mathfrak{q}}) \le s$, and $\mu(\Omega_k(k(\mathfrak{q}))) = \operatorname{trdeg}_k k(\mathfrak{q}) = \dim B/\mathfrak{q} = d$ because k is perfect. As a result, $\mu(\Omega_k(B_{\mathfrak{q}}) \otimes_{B_{\mathfrak{q}}} k(\mathfrak{q})) \le d + s$. Notice that $\Omega_k(B_{\mathfrak{q}}) = \mathbb{D}_Q/\mathbb{D}_Q^2$ and hence $\mu(\mathbb{D}_Q) \le d + s$ by Nakayama's lemma. Therefore, $\mathbb{D}_Q = \mathfrak{a}_Q$ by the general choice of $a_1, \dots, a_{d+s}$. In summary, we obtain $\mathbb{D}_Q = \mathfrak{a}_Q$ for every $Q \in V(\mathfrak{a})$ with $\dim C/Q = d$.

Write $A = k[x_1, \dots, x_{d+s}]$ and consider the exact sequence

$$0 \longrightarrow \mathbb{D}' = \mathbb{D}/\mathfrak{a} \longrightarrow C' = C/\mathfrak{a} = B \otimes_A B \longrightarrow B \longrightarrow 0.$$

From our previous discussion it follows that $\mathbb{D}'_Q = 0$ for every $Q \in \operatorname{Spec}(C')$ with $\dim C'/Q = d$. The homomorphism $A \to C' = B \otimes_A B$ makes C' a finite A-module. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\dim A/\mathfrak{p} = d$. Let Q be any prime of C' lying over $\mathfrak{p}$. Since $\dim C'/Q = \dim A/\mathfrak{p} = d$, we obtain $\mathbb{D}'_Q = 0$. Because this holds for any such Q, we have $\mathbb{D}'_{\mathfrak{p}} = 0$. Thus $B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} \cong B_{\mathfrak{p}}$. Computing numbers of generators as $A_{\mathfrak{p}}$-modules, we conclude that $B_{\mathfrak{p}} = A_{\mathfrak{p}}$ and hence $(B/A)_{\mathfrak{p}} = 0$. $\square$

Theorem 2.3 enables us to prove various versions of our decomposition formula, as follows.

LEMMA 2.4. *Let k be a field and B a standard graded k-algebra of dimension d with homogeneous maximal ideal $\mathfrak{m}$. Let A be a k-subalgebra generated by $d + s$ linear forms $x_1, \dots, x_{d+s}$ in B. Then $\mathfrak{m}^n = (x_1, \dots, x_{d-1})\mathfrak{m}^{n-1} + (x_d, \dots, x_{d+s})^n$ for $n \gg 0$ if and only if B/A is a finite module over $k[x_1, \dots, x_{d-1}]$.*

Proof. Write $C = B/A$. Mapping variables $X_i \mapsto x_i$, we obtain homogeneous homomorphisms

$$k[X_d, \dots, X_{d+s}] \twoheadrightarrow A/(x_1, \dots, x_{d-1})A \to B/(x_1, \dots, x_{d-1})B.$$

Their composition is surjective in large degrees if and only if $C/(x_1, \dots, x_{d-1})C$ is a finite-dimensional k-vector space, which by the graded Nakayama lemma means that C is a finite module over $k[x_1, \dots, x_{d-1}]$. $\square$

PROPOSITION 2.5. *Let k be an infinite perfect field, B a standard graded k-algebra of dimension d with homogeneous maximal ideal $\mathfrak{m}$, and s a positive integer. Let A be a k-subalgebra generated by $d + s$ general linear forms $x_1, \dots, x_{d+s}$ in B. Then $\mathfrak{m}^n = (x_1, \dots, x_{d-1})\mathfrak{m}^{n-1} + (x_d, \dots, x_{d+s})^n$ for $n \gg 0$ if and only if B has embedding dimension $\le s$ locally at every minimal prime of dimension d.*

Proof. The assertion is an immediate consequence of Lemma 2.4 and Theorem 2.3. □

Corollary 2.6. *Let R be a Noetherian local ring and I an R-ideal of analytic spread ℓ. Let $f_1, \dots, f_{\ell+s}$ be elements in I, let $\mathfrak{a} = (f_1, \dots, f_{\ell-1})$ and $K = (f_1, \dots, f_{\ell+s})$, and consider the natural map of special fiber rings $\varphi\colon \mathcal{F}(K) \to \mathcal{F}(I)$. Then $I^n = (f_1, \dots, f_{\ell-1})I^{n-1} + (f_\ell, \dots, f_{\ell+s})^n$ for $n \gg 0$ if and only if* $\operatorname{coker}(\varphi)$ *is a finite $\mathcal{F}(\mathfrak{a})$-module.*

Proof. Apply Lemma 2.4 with $B = \mathcal{F}(I)$ and $A = \varphi(\mathcal{F}(K))$; then use Nakayama's lemma. □

We are now ready to prove the main result of this section. Let I be an ideal in a Noetherian local ring R with infinite residue field k. Elements $f_1, \dots, f_t$ in I are said to be *general* if the image of the tuple $(f_1, \dots, f_t)$ is a general point of the affine space $(I \otimes k)^t$. Recall that $t \geq \ell(I)$ general elements in I generate a reduction and hence give $I^n = (f_1, \dots, f_t)I^{n-1}$ for $n \gg 0$. The next result provides, under suitable assumptions, a different type of decomposition formula for the powers of I.

Theorem 2.7. *Let R be a Noetherian local ring with infinite perfect residue field, I an R-ideal of analytic spread ℓ, and s a positive integer. Let $f_1, \dots, f_{\ell+s}$ be general elements in I, let $\mathfrak{a} = (f_1, \dots, f_{\ell-1})$ and $K = (f_1, \dots, f_{\ell+s})$, and consider the natural map of special fiber rings $\varphi\colon \mathcal{F}(K) \to \mathcal{F}(I)$. Then the following statements are equivalent*:

(i) $I^n = (f_1, \dots, f_{\ell-1})I^{n-1} + (f_\ell, \dots, f_{\ell+s})^n$ *for* $n \gg 0$;
(ii) $\operatorname{coker}(\varphi)$ *is a finite $\mathcal{F}(\mathfrak{a})$-module*;
(iii) *$\mathcal{F}(I)$ has embedding dimension $\leq s$ locally at every minimal prime of dimension ℓ.*

Proof. Apply Corollary 2.6 and Proposition 2.5. □

3. The Main Theorem

In this section we prove our main theorem about the core in arbitrary characteristic. The proof uses reduction to the case of equimultiple height 1 ideals, which we treat by means of the results in the previous section. The reduction step requires the following two technical lemmas.

Lemma 3.1. *Let R be a Noetherian local ring with infinite residue field k, I an R-ideal, and J a reduction of I. Let x be a general element in J, write x^* for the image of x in $[\mathcal{F}(I)]_1$, and let $\bar{\ }$ denote images in $\bar{R} = R/(x)$.*

(a) *The kernel of the natural map $\mathcal{F}(I)/x^*\mathcal{F}(I) \to \mathcal{F}(\bar{I})$ is a finite-dimensional k-vector space.*

(b) *Let $\mathfrak{a} \subset K$ be R-ideals with $x \in \mathfrak{a}$ and $K \subset I$. Consider the natural map of special fiber rings $\varphi\colon \mathcal{F}(K) \to \mathcal{F}(I)$, and write $\bar{\varphi}$ for the induced map from $\mathcal{F}(\bar{K})$ to $\mathcal{F}(\bar{I})$. Then* $\operatorname{coker}(\bar{\varphi})$ *is a finite $\mathcal{F}(\bar{\mathfrak{a}})$-module if and only if* $\operatorname{coker}(\varphi)$ *is a finite $\mathcal{F}(\mathfrak{a})$-module.*

Proof. To prove part (a) let $\mathcal{G}(I)$ and $\mathcal{G}(\bar{I})$ denote the associated graded ring of I and $\bar{I}$, respectively. Consider the exact sequence

$$0 \longrightarrow C \longrightarrow \mathcal{G}(I)/(x + I^2)\mathcal{G}(I) \longrightarrow \mathcal{G}(\bar{I}) \longrightarrow 0.$$

Since x is general in J and since J is a reduction of I, it follows that x is a superficial element of I. Thus C vanishes in large degrees. Tensoring the preceding sequence with the residue field k, we deduce that

$$C \otimes_R k \longrightarrow \mathcal{F}(I)/x^*\mathcal{F}(I) \longrightarrow \mathcal{F}(\bar{I}) \longrightarrow 0$$

is exact and that $C \otimes_R k$ is a finite-dimensional k-vector space.

To prove part (b), observe that part (a) and the snake lemma show that the kernel of the natural map

$$\operatorname{coker}(\varphi)/(x^* \operatorname{coker}(\varphi)) \twoheadrightarrow \operatorname{coker}(\bar{\varphi})$$

is a finite-dimensional k-vector space as well. Hence $\operatorname{coker}(\bar{\varphi})$ is finitely generated as a $\mathcal{F}(\bar{\mathfrak{a}})$-module if and only if $\operatorname{coker}(\varphi)/(x^* \operatorname{coker}(\varphi))$ is finitely generated as a $\mathcal{F}(\mathfrak{a})$-module. By the graded Nakayama lemma, the latter condition means that $\operatorname{coker}(\varphi)$ is a finite $\mathcal{F}(\mathfrak{a})$-module. □

The following two results use, in an essential way, the theory of residual intersections. Let R be a local Cohen–Macaulay ring, I an R-ideal, and s an integer. Recall that $\mathfrak{a} : I$ is a *geometric s-residual intersection* of I if $\mathfrak{a}$ is an s-generated R-ideal properly contained in I and if $\operatorname{ht} \mathfrak{a} : I \geq s$ and $\operatorname{ht}(I, \mathfrak{a} : I) \geq s + 1$. The ideal I has the *Artin–Nagata property* AN_s^- if $R/\mathfrak{a} : I$ is Cohen–Macaulay for every geometric i-residual intersection $\mathfrak{a} : I$ and every $i \leq s$.

Lemma 3.2. *Let R be a local Cohen–Macaulay ring with infinite residue field, and assume that R has a canonical module. Let I be an R-ideal with analytic spread $\ell > 0$, and suppose that I satisfies G_ℓ and* $\mathrm{AN}_{\ell-1}^-$. *Let J be a minimal reduction of I and let K be an R-ideal with $J \subset K \subset I$. Consider the natural map of special fiber rings $\varphi\colon \mathcal{F}(K) \to \mathcal{F}(I)$. Assume that* $\operatorname{coker}(\varphi)$ *has dimension $\leq \ell - 1$ as a module over $\mathcal{F}(J)$. Write $\mathcal{A} = \mathcal{A}(J)$ for the set consisting of all ideals $\mathfrak{a}$ such that $\mathfrak{a} : J$ is a geometric $(\ell - 1)$-residual intersection of J, $\mu(J/\mathfrak{a}) = 1$, and* $\operatorname{coker}(\varphi)$ *is a finite $\mathcal{F}(\mathfrak{a})$-module. For t a positive integer, let H be an R-ideal with* $\operatorname{ht}(J, J^t : H) \geq \ell$. *Then*

$$H \cap \bigcap_{\mathfrak{a} \in \mathcal{A}} (J^t, \mathfrak{a}) \subset J^t.$$

Proof. We prove the lemma by induction on ℓ. First let $\ell = 1$. Since I satisfies G_1, so does J and hence $0 : J$ is a geometric 0-residual intersection of J. Therefore, $\mathcal{A} = \{0\}$ and the assertion is clear. We may thus assume that $\ell \geq 2$. Let $b \in H$

and suppose that $b \notin J^t$. We shall prove the existence of an ideal $\mathfrak{a} \in \mathcal{A}$ with $b \notin (J^t, \mathfrak{a})$. Since $(J^t, \mathfrak{a}) \subset J$, we may assume that $b \in J$.

We first reduce to the case where I has positive height. Let $\bar{\ }$ denote images in $\bar{R} = R/0 : I$. Notice that $0 : I$ is a geometric 0-residual intersection of I since I satisfies G_1. Therefore, $\bar{R}$ is Cohen–Macaulay by the AN_0^- condition, and $\operatorname{ht} \bar{I} > 0$. Furthermore, $I \cap (0 : I) = 0$ by [18, 1.7(c)]. Hence the canonical epimorphism $R \twoheadrightarrow \bar{R}$ induces isomorphisms $I^m \simeq \bar{I}^m$ and $J^m \simeq \bar{J}^m$ for every $m \geq 1$. Therefore $\bar{b} \notin \bar{J}^t$. Moreover, $[\mathcal{G}(I)]_m \simeq [\mathcal{G}(\bar{I})]_m$ for $m \geq 1$ and $\mathcal{F}(I) \simeq \mathcal{F}(\bar{I})$. Hence $\ell(\bar{I}) = \ell(I)$ and $\bar{J}$ is a minimal reduction of $\bar{I}$. As $\operatorname{ht} 0 : I = 0$, it follows that $\bar{I}$ satisfies G_ℓ; since $I \cap (0 : I) = 0$, the ideal $\bar{I}$ satisfies $\mathrm{AN}_{\ell-1}^-$ according to [13, 2.4(b)]. Obviously, the cokernel of the induced map $\bar{\varphi}\colon \mathcal{F}(\bar{K}) \to \mathcal{F}(\bar{I})$ has dimension $\leq \ell - 1$ as an $\mathcal{F}(\bar{J})$-module.

Every ideal in $\mathcal{A}(\bar{J})$ is of the form $\bar{\mathfrak{a}}$ for some $\mathfrak{a} \in \mathcal{A}(J)$. Indeed, if $\bar{\mathfrak{a}} \in \mathcal{A}(\bar{J})$ then there exists an $(\ell - 1)$-generated ideal $\mathfrak{a} \subset J$ whose image in $\bar{R}$ is $\bar{\mathfrak{a}}$. Since $J \cap (0 : I) = 0$ we have $\mathfrak{a} : J = (0 : I, \mathfrak{a}) : J$, and it follows that $\mathfrak{a} : J$ is a geometric $(\ell - 1)$-residual intersection of J. Notice that a minimal generating set of $\mathfrak{a}$ forms part of a minimal generating set of J, so $\mu(J/\mathfrak{a}) = 1$. Furthermore, $\operatorname{coker}(\varphi)$ is a finite $\mathcal{F}(\mathfrak{a})$-module because $\mathcal{F}(I) \simeq \mathcal{F}(\bar{I})$. Finally, $\operatorname{ht}(\bar{J}, \bar{J}^t : \bar{H}) \geq \operatorname{ht}(J, J^t : H) \geq \ell$ because $\operatorname{ht} 0 : I = 0$. We may thus replace R by $\bar{R}$ and assume that $\operatorname{ht} I > 0$. With this additional assumption we now prove that $b \notin (J^t, \mathfrak{a})$ for some $\mathfrak{a} \in \mathcal{A}$.

Notice that $\operatorname{ht} J : I \geq \ell$ according to [13, 2.7]. Since I satisfies G_ℓ, it follows that J satisfies G_∞. Again, since $\operatorname{ht} J : I \geq \ell$, the property $\mathrm{AN}_{\ell-1}^-$ passes from I to J by [18, 1.12]. Now J satisfies the sliding depth condition according to [18, 1.8(c)]. In particular, $\operatorname{Sym}(J/J^2) \simeq \mathcal{G}(J)$ via the natural map and these algebras are Cohen–Macaulay by [6, 6.1].

The proof of [15, 4.2] shows that $\bar{b} \notin \bar{J}^t$, where now $\bar{\ }$ denotes images in $\bar{R} = R/(x)$ for a general element x in J. By the general choice of x in J and since J is a minimal reduction of I, we have $\ell(\bar{I}) \leq \mu(\bar{J}) = \ell(I) - 1$; then Lemma 3.1(a) shows that $\ell(\bar{I}) = \ell(I) - 1$, in particular $\bar{J}$ is a minimal reduction of $\bar{I}$. Again because x is a general element and $\operatorname{ht} J > 0$, it follows that x is R-regular. For the same reasons and because $\mathcal{G}(J)$ is Cohen–Macaulay, the leading form x^* of x in $\mathcal{G}(J)$ is regular on $\mathcal{G}(J)$, which gives $\mathcal{G}(J)/x^*\mathcal{G}(J) \simeq \mathcal{G}(\bar{J})$. Therefore, $\operatorname{Sym}(\bar{J}/\bar{J}^2) \simeq \mathcal{G}(\bar{J})$, and this forces $\bar{J}$ to satisfy G_∞. Hence $\bar{I}$ satisfies $G_{\ell-1}$, because $\operatorname{ht} \bar{J} : \bar{I} \geq \ell - 1$. Since x is an R-regular element, it is easy to see that $\bar{I}$ is $\mathrm{AN}_{\ell-2}^-$.

Again by the general choice of x, the cokernel of the natural map from $\mathcal{F}(\bar{K})$ to $\mathcal{F}(\bar{I})$ has dimension $\leq \ell - 2$ as an $\mathcal{F}(\bar{J})$-module. Finally, $\operatorname{ht}(\bar{J}, \bar{J}^t : \bar{H}) \geq \ell - 1$ and, by Lemma 3.1(b), every ideal of $\mathcal{A}(\bar{J})$ is of the form $\bar{\mathfrak{a}}$ for some $\mathfrak{a} \in \mathcal{A}$. Thus, by the induction hypothesis, $\bar{b} \notin (\bar{J}^t, \bar{\mathfrak{a}})$ for some $\mathfrak{a} \in \mathcal{A}$. Hence $b \notin (J^t, \mathfrak{a})$. □

We are now ready to prove our main result.

THEOREM 3.3. *Let R be a local Cohen–Macaulay ring with infinite perfect residue field, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a* *minimal reduction of I with $r = r_J(I)$. Suppose that the special fiber ring $\mathcal{F}(I)$ of I has embedding dimension ≤ 1 locally at every minimal prime of dimension ℓ. If $\ell > g$, further assume that R is Gorenstein, that I satisfies G_ℓ, and that* depth $R/I^j \geq \dim R/I - j + 1$ *for* $1 \leq j \leq \ell - g$. *Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

Proof. According to [18, 2.9(a)], the ideals I and $I\hat{R}$ satisfy $\mathrm{AN}_{\ell-1}^-$ and are universally weakly $\ell - 1$ residually S_2 in the sense of [2, p. 203]. Therefore, [3, 4.8] shows that $\operatorname{core}(I)\hat{R} = \operatorname{core}(I\hat{R})$. Thus we may pass to the completion of R and assume that R has a canonical module. Let $f_1, \ldots, f_{\ell+1}$ be general elements in I. The ideal $J^{n+1} : I^n$ for $n \geq \max\{r - \ell + g, 0\}$ is independent of the minimal reduction J and of n, as can be seen from [11, 5.1.6] if $\ell = g$ and from [15, 2.3] if $\ell > g$. Hence we may assume that $J = (f_1, \ldots, f_\ell)$ and $n \gg 0$. We use the notation of Lemma 3.2 with $K = (J, f_{\ell+1})$, $t = n + 1$, and H the intersection of all primary components of J^{n+1} of height $< \ell$. Notice that $\operatorname{coker}(\varphi)$ has dimension $\leq \ell - 1$ as a module over $\mathcal{F}(J)$ according to Theorem 2.7. Hence the assumptions of Lemma 3.2 are satisfied.

Let $\mathfrak{a} \in \mathcal{A}$ be as in Lemma 3.2. Write $\bar{}$ for images in $\bar{R} = R/\mathfrak{a} : I$. Notice that $\bar{R}$ is Cohen–Macaulay and that, by [18, 1.7(a)], $\operatorname{ht} \bar{I} > 0$; consequently, $\operatorname{ht} \bar{I} = \ell(\bar{I}) = 1$. Now [15, 3.4] shows that $\operatorname{core}(\bar{I}) = \bar{J}^{n+1} : \sum_{y \in \bar{I}} (\bar{J}, y)^n$. Notice that $\bar{K}^n \subset \sum_{y \in \bar{I}} (\bar{J}, y)^n \subset \bar{I}^n$ and that $\bar{K}^n = \bar{I}^n$ according to Corollary 2.6. Hence $\operatorname{core}(\bar{I}) = \bar{J}^{n+1} : \bar{I}^n$.

On the other hand, by [3, 4.5] we have $\operatorname{core}(\bar{I}) = (\overline{\alpha_1}) \cap \cdots \cap (\overline{\alpha_\gamma})$ for some integer γ and for γ general principal ideals $(\overline{\alpha_1}), \ldots, (\overline{\alpha_\gamma})$ in $\bar{I}$. Notice that $(\mathfrak{a}, \alpha_i)$ are reductions of I, hence $\operatorname{core}(I) \subset \bigcap_{i=1}^{\gamma} (\mathfrak{a}, \alpha_i)$. Therefore,

$$\overline{\operatorname{core}(I)} \subset \overline{\bigcap_{i=1}^{\gamma} (\mathfrak{a}, \alpha_i)} \subset \bigcap_{i=1}^{\gamma} (\overline{\alpha_i}) = \operatorname{core}(\bar{I}).$$

Because $\operatorname{core}(\bar{I}) = \bar{J}^{n+1} : \bar{I}^n$, we obtain

$$\begin{aligned} \operatorname{core}(I) &\subset (J^{n+1}, \mathfrak{a} : I) : I^n \\ &= (J^{n+1}, (\mathfrak{a} : I) \cap I) : I^n \\ &= (J^{n+1}, \mathfrak{a}) : I^n. \end{aligned}$$

The last equality holds because $(\mathfrak{a} : I) \cap I = \mathfrak{a}$ by [18, 1.7(c)]. It follows that

$$\operatorname{core}(I) \subset \bigcap_{\mathfrak{a} \in \mathcal{A}} (J^{n+1}, \mathfrak{a}) : I^n. \tag{3.1}$$

Next we show that

$$\operatorname{core}(I) \subset H : I^n \tag{3.2}$$

or, equivalently, $(\operatorname{core}(I))_\mathfrak{p} \subset (H : I^n)_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ with $\dim R_\mathfrak{p} < \ell$. Indeed, by [13, 2.7] we have $J_\mathfrak{p} = I_\mathfrak{p}$, and so $J_\mathfrak{p}^n = I_\mathfrak{p}^n$. Thus $(\operatorname{core}(I))_\mathfrak{p} \subset J_\mathfrak{p} \subset J_\mathfrak{p}^{n+1} : J_\mathfrak{p}^n = H_\mathfrak{p} : I_\mathfrak{p}^n$.

Finally, $J^{n+1} : I^n \subset \operatorname{core}(I)$, as can be seen from the proof of [15, 4.5] via [11, 5.1.6] if $\ell = g$ and from [15, 4.8] otherwise. Hence (3.1), (3.2), and Lemma 3.2 together imply that

$$\begin{aligned} J^{n+1} : I^n \subset \operatorname{core}(I) &\subset \Big(H \cap \bigcap_{\mathfrak{a} \in \mathcal{A}} (J^{n+1}, \mathfrak{a})\Big) : I^n \\ &\subset J^{n+1} : I^n. \end{aligned}$$

As a result, $\operatorname{core}(I) = J^{n+1} : I^n$. □

4. Applications

In this section, we collect several instances where the assumption on the generic embedding dimension of the special fiber ring required in Theorem 3.3 holds automatically.

THEOREM 4.1. *Let k be an infinite field, R' a positively graded geometrically reduced Cohen–Macaulay k-algebra, and R the localization of R' at the homogeneous maximal ideal. Let K be an R-ideal generated by forms in R' of the same degree, let f be an element of R integral over K, and write $I = (K, f)$. Set $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. If $\ell > g$, suppose that R' is Gorenstein, that I satisfies G_ℓ, and that* $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ *for* $1 \leq j \leq \ell - g$. *Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

Proof. Observe that

$$R = R'_{R'_+} \hookrightarrow S = (R' \otimes_k \bar{k})_{(R' \otimes_k \bar{k})_+}$$

is a flat local extension. Furthermore, according to [18, 2.9(a)], the ideals I and IS are universally weakly $\ell - 1$ residually S_2. Therefore, [3, 4.8] shows that $\operatorname{core}(I)S = \operatorname{core}(IS)$. Thus, replacing k by $\bar{k}$, we may suppose that k is perfect and that R' is reduced.

Write $K = (f_1, \dots, f_m)$, where $f_1, \dots, f_m$ are forms of the same degree. Now $\mathcal{F}(K) \simeq k[f_1, \dots, f_m]$ is a subalgebra of R' and thus is reduced. Let $\mathfrak{p}$ be a minimal prime of $\mathcal{F}(I)$ of dimension ℓ and write $\mathfrak{q}$ for its contraction to $\mathcal{F}(K)$. Since K is a reduction of I, we have $\ell(K) = \ell(I)$ and so $\dim \mathcal{F}(K) = \ell = \dim \mathcal{F}(I)$. Furthermore, $\mathcal{F}(I)$ is finitely generated as a module over $\mathcal{F}(K)$. It follows that $\mathfrak{q}$ is a minimal prime of $\mathcal{F}(K)$. Because $\mathcal{F}(K)$ is reduced, the localization $\mathcal{F}(K)_q$ is a field, say L. Now $\mathcal{F}(I)_\mathfrak{p}$ is a localization of an L-algebra generated by a single element—namely, the image of f. Hence $\mathcal{F}(I)_\mathfrak{p}$ has embedding dimension $\leq$ 1. Since this holds for every minimal prime $\mathfrak{p}$ of dimension ℓ, the result follows from Theorem 3.3. □

REMARK 4.2. Taking $f = 0$ in Theorem 4.1 yields Theorem 1.2 of the Introduction. There is a graded and a global version of the latter theorem if the ideal I is zero-dimensional. Thus, let I' be a homogeneous R'-ideal with $I'R = I$, and let J' be an R'-ideal generated by $\dim R'$ general k-linear combinations of homogeneous minimal generators of I'. Then

$$\operatorname{gradedcore}(I') = \operatorname{core}(I') = J'^{n+1} : I'^{n} \quad \text{for every } n \geq r,$$

where $\operatorname{gradedcore}(I')$ stands for the intersection of all homogeneous reductions of I'.

In fact, since I' is zero-dimensional and generated by forms of the same degree, the first equality obtains by [3, 4.5] and [16, 2.1]; the second equality follows from Theorem 1.2 and [16, 2.1].

THEOREM 4.3. *Let R be a local Cohen–Macaulay ring with infinite perfect residue field, let I be an R-ideal with $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. Suppose that $I = (K, f)$, where $\ell(I) \geq \ell(K)$ and the special fiber ring $\mathcal{F}(K)$ satisfies Serre's condition R_1. If $\ell > g$, further assume that R is Gorenstein, that I satisfies G_ℓ, and that* $\operatorname{depth} R/I^j \geq \dim R/I - j + 1$ *for $1 \leq j \leq \ell - g$. Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \geq \max\{r - \ell + g, 0\}$.

Proof. According to Theorem 3.3 it suffices to prove that $\mathcal{F}(I)$ has embedding dimension ≤ 1 locally at every minimal prime of dimension $\ell = \dim \mathcal{F}(I)$.

Let $A = \mathcal{F}(K)$ and $B = \mathcal{F}(I)$. Let $\mathfrak{q}$ be a prime ideal of B of dimension ℓ and write $\mathfrak{p}$ for the preimage of $\mathfrak{q}$ in A. We claim that $\dim A_\mathfrak{p} \leq 1$. The affine domain $B/\mathfrak{q}$ is generated by one element as an algebra over $A/\mathfrak{p}$. Therefore,

$$\dim A/\mathfrak{p} \geq \dim B/\mathfrak{q} - 1 = \dim B - 1.$$

Hence

$$\dim A_\mathfrak{p} \leq \dim A - \dim A/\mathfrak{p} \leq \dim A - \dim B + 1 = \ell(K) - \ell + 1 \leq 1.$$

Since $\dim A_\mathfrak{p} \leq 1$, our assumption gives that $A_\mathfrak{p}$ is regular. Now we consider the following exact sequence of modules of differentials:

$$B_\mathfrak{q} \otimes_{A_\mathfrak{p}} \Omega_k(A_\mathfrak{p}) \longrightarrow \Omega_k(B_\mathfrak{q}) \longrightarrow \Omega_{A_\mathfrak{p}}(B_\mathfrak{q}) \longrightarrow 0.$$

Since $A_\mathfrak{p}$ is regular and k is perfect, it follows that $\mu_{A_\mathfrak{p}}(\Omega_k(A_\mathfrak{p})) \leq \dim A_\mathfrak{p} + \operatorname{trdeg}_k A/\mathfrak{p}$ and so $\mu_{B_\mathfrak{q}}(B_\mathfrak{q} \otimes_{A_\mathfrak{p}} \Omega_k(A_\mathfrak{p})) \leq \dim A_\mathfrak{p} + \operatorname{trdeg}_k A/\mathfrak{p}$. Because B is generated by one element as an A-algebra, the $B_\mathfrak{q}$-module $\Omega_{A_\mathfrak{p}}(B_\mathfrak{q})$ is cyclic. By computing numbers of generators along the preceding exact sequence, we obtain

$$\mu_{B_\mathfrak{q}}(\Omega_k(B_\mathfrak{q})) \leq \dim A_\mathfrak{p} + \operatorname{trdeg}_k A/\mathfrak{p} + 1.$$

On the other hand, by [1, Satz 1(a)],

$$\mu_{B_\mathfrak{q}}(\Omega_k(B_\mathfrak{q})) = \operatorname{edim} B_\mathfrak{q} + \operatorname{trdeg}_k B/\mathfrak{q}.$$

We conclude that

$$\begin{aligned} \operatorname{edim} B_{\mathfrak{q}} &\le \dim A_{\mathfrak{p}} + \operatorname{trdeg}_k A/\mathfrak{p} - \operatorname{trdeg}_k B/\mathfrak{q} + 1 \\ &= \dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} - \dim B + 1 \\ &\le \dim A - \dim B + 1 \\ &\le 1. \end{aligned}$$ □

Theorem 4.3 can be considered as a generalization of the case of second analytic deviation 1 treated in [15, 4.8]. In this case, the minimal number of generators of I exceeds ℓ by at most 1 and so we can choose K to be J in Theorem 4.3. But then $\ell(K) = \ell$ and $\mathcal{F}(K)$ satisfies R_1, being a polynomial ring over k.

Also observe that the condition $\ell \ge \ell(K)$ in Theorem 4.3 is always satisfied if I is primary to the maximal ideal. We now describe another situation where this inequality holds automatically.

THEOREM 4.4. *Let k be an infinite perfect field, R' a positively graded Cohen–Macaulay k-algebra, and R the localization of R' at the homogeneous maximal ideal. Let K be an R-ideal generated by forms in R' of the same degree e, let f be a form in R' of degree at least e, write $I = (K, f)$, and assume that the subalgebra $k[K_e]$ of R' satisfies Serre's condition R_1. Set $g = \operatorname{ht} I > 0$ and $\ell = \ell(I)$, and let J be a minimal reduction of I with $r = r_J(I)$. If $\ell > g$, further suppose that R' is Gorenstein, that I satisfies G_ℓ, and that* $\operatorname{depth} R/I^j \ge \dim R/I - j + 1$ *for* $1 \le j \le \ell - g$. *Then*

$$\operatorname{core}(I) = J^{n+1} : I^n$$

for every $n \ge \max\{r - \ell + g, 0\}$.

Proof. After rescaling the grading, we can identify the subalgebra $k[K_e]$ of R' with $\mathcal{F}(K)$, which shows that the latter ring satisfies R_1. Thus, to apply Theorem 4.3, it suffices to verify that $\ell(I) \ge \ell(K)$. Comparing Hilbert functions shows that $\ell(I) = \dim \mathcal{F}(I) \ge \dim \mathcal{F}(K) = \ell(K)$, once we have proved the injectivity of the natural map $\varphi\colon \mathcal{F}(K) \to \mathcal{F}(I)$. To show the latter, write $\mathfrak{m}$ for the maximal ideal of R. Let F be a form of degree s in $\mathcal{F}(K)$ such that $\varphi(F) = 0$. Then $F \in \mathfrak{m} I^s$ as an element of $R' \subset R$. In R', the form F has degree se whereas the nonzero homogeneous elements of $\mathfrak{m} I^s$ have degrees at least $se + 1$. Therefore, $F = 0$. □

5. Examples

In this section we present several examples showing that the various assumptions in our theorems are, in fact, necessary. We will always use zero-dimensional ideals in local Gorenstein rings, so that the property G_ℓ as well as the depth conditions for the powers of the ideal hold automatically.

The first example illustrates that Theorem 1.2 is no longer true if the ring R' fails to be geometrically reduced, even if it is a domain and all the other assumptions of the theorem are satisfied.

EXAMPLE 5.1. Let k_0 be a field of characteristic $p > 0$, and let $k = k_0(s, t)$ be the rational function field in two variables. Consider the ring

$$R' = k[x, y, z]/(x^p - sz^p, y^p - tz^p),$$

which is a one-dimensional standard graded Gorenstein domain. Indeed, the elements x^p, y^p, z^p generate an ideal of grade 3 in $k_0[x, y, z]$, and $x^p - sz^p$ and $y^p - tz^p$ are obtained from generic linear combinations of these elements by localization, change of variables, and descent. Thus [7, Thm. (b)] shows that $x^p - sz^p$ and $y^p - tz^p$ generate a prime ideal in the ring $k[x, y, z]$.

On the other hand, R' is not geometrically reduced. After tensoring with the algebraic closure $\bar{k}$ of k, we obtain

$$R' \otimes_k \bar{k} \simeq \bar{k}[x, y, z]/((x - zs^{1/p})^p, (y - zt^{1/p})^p),$$

which is not a reduced ring.

Let $(R, \mathfrak{m})$ denote the localization of R' at the homogenous maximal ideal. We claim that

$$\operatorname{core}(\mathfrak{m}) \neq J^{n+1} : \mathfrak{m}^n \quad \text{for every } n \gg 0 \text{ and every minimal reduction } J \text{ of } \mathfrak{m}.$$

Indeed, the pth power of any general linear form in R generates the ideal Rz^p. Since the core of $\mathfrak{m}$ is a finite intersection of principal ideals generated by general linear forms [3, 4.5], it follows that $Rz^p \subset \operatorname{core}(\mathfrak{m})$. On the other hand, by [15, 3.2(a)] we have $J^{n+1} : \mathfrak{m}^n = Rz^{n+1} : \mathfrak{m}^n$ because Rz is a minimal reduction of $\mathfrak{m}$. Since $R'/R'z^{n+1}$ is a standard graded Artinian Gorenstein ring with a-invariant $n + 2p - 2$, it follows that $Rz^{n+1} : \mathfrak{m}^n = \mathfrak{m}^{2p-1}$, which does not contain Rz^p.

The next example shows that the assumption in Theorem 3.3 on the local embedding dimension of the special fiber ring is sharp: If we allow the local embedding dimension to be 2, then the statement of the theorem is no longer true even in the presence of the other conditions.

EXAMPLE 5.2. Let k be an infinite perfect field of characteristic 2, let $R = k[x, y]_{(x,y)}$ be a localized polynomial ring, and let $I = (x^6, x^5y^3, x^4y^4, x^2y^8, y^9)$. Using Macaulay 2 [5], the special fiber ring of I may be computed as

$$\mathcal{F}(I) \simeq k[a, b, c, d, e]/(b^2, bd, cd, d^2, c^2 - ad),$$

where a, b, c, d, e are variables over k. This ring has a unique minimal prime ideal $\mathfrak{p}$, which is generated by the images of b, c, d, and one easily sees that $[\mathcal{F}(I)]_{\mathfrak{p}}$ has embedding dimension 2.

We claim that

$$\operatorname{core}(I) \neq J^{n+1} : I^n \quad \text{for every } n \gg 0 \text{ and every minimal reduction } J \text{ of } I.$$

Indeed, $H = (x^6, y^9)$ is a minimal reduction of I with $r_H(I) = 2$. Hence, by [15, 2.3], $J^{n+1} : I^n = H^3 : I^2$. On the other hand, the algorithm of [16, 3.6] has been used to show that $\operatorname{core}(I) \neq H^3 : I^2$ [16, 3.9].

The next example shows that in Theorem 4.1 and Theorem 4.3 it is essential to assume that either f is integral over K or else the special fiber ring $\mathcal{F}(K)$ satisfies Serre's condition R_1.

EXAMPLE 5.3. Let k be an infinite perfect field of characteristic 2, and let $R = k[x, y]_{(x,y)}$ be a localized polynomial ring. Let $K = (x^9, x^5y^4, x^3y^6, x^2y^7)$, which is an ideal generated by monomials of the same degree, and let $f = y^8$ and $I = (K, f)$. Again we claim that

$$\operatorname{core}(I) \neq J^{n+1} : I^n \quad \text{for every } n \gg 0 \text{ and every minimal reduction } J \text{ of } I.$$

The ideal $H = (x^9, y^8)$ is a minimal reduction of I with $r_H(I) = 2$, so $J^{n+1} : I^n = H^3 : I^2$ by [15, 2.3]. On the other hand, using the algorithm of [16, 3.6] and Macaulay 2 [5], we can compute

$$\operatorname{core}(I) = H^3 : I^2 + (xy^{12}, y^{13}) \supsetneq H^3 : I^2.$$

References

[1] R. Berger and E. Kunz, *Über die Struktur der Differentialmoduln von diskreten Bewertungsringen,* Math. Z. 77 (1961), 314–338.

[2] M. Chardin, D. Eisenbud, and B. Ulrich, *Hilbert functions, residual intersections, and residually S_2 ideals,* Compositio Math. 125 (2001), 193–219.

[3] A. Corso, C. Polini, and B. Ulrich, *The structure of the core of ideals,* Math. Ann. 321 (2001), 89–105.

[4] ———, *Core and residual intersections of ideals,* Trans. Amer. Math. Soc. 354 (2002), 2579–2594.

[5] D. Grayson and M. Stillman, *Macaulay 2: A computer algebra system for computing in algebraic geometry and commutative algebra,* ⟨http://www.math.uiuc.edu/Macaulay2⟩.

[6] J. Herzog, A. Simis, and W. Vasconcelos, *Approximation complexes of blowing-up rings, II,* J. Algebra 82 (1983), 53–83.

[7] M. Hochster, *Properties of Noetherian rings stable under general grade reduction,* Arch. Math. (Basel) 24 (1973), 393–396.

[8] C. Huneke, *Linkage and Koszul homology of ideals,* Amer. J. Math. 104 (1982), 1043–1062.

[9] C. Huneke and I. Swanson, *Cores of ideals in 2-dimensional regular local rings,* Michigan Math. J. 42 (1995), 193–208.

[10] C. Huneke and N. V. Trung, *On the core of ideals,* Compositio Math. 141 (2005), 1–18.

[11] E. Hyry and K. Smith, *On a non-vanishing conjecture of Kawamata and the core of an ideal,* Amer. J. Math. 125 (2003), 1349–1410.

[12] ———, *Core versus graded core, and global sections of line bundles,* Trans. Amer. Math. Soc. 356 (2004), 3143–3166.

[13] M. Johnson and B. Ulrich, *Artin–Nagata properties and Cohen–Macaulay associated graded rings,* Compositio Math. 103 (1996), 7–29.

[14] H. Matsumura, *Commutative ring theory,* Cambridge Stud. Adv. Math., 8, Cambridge Univ. Press, Cambridge, 1989.

[15] C. Polini and B. Ulrich, *A formula for the core of an ideal,* Math. Ann. 331 (2005), 487–503.
[16] C. Polini, B. Ulrich, and M. Vitulli, *The core of zero-dimensional monomial ideals,* Adv. Math. 211 (2007), 72–93.
[17] D. Rees and J. Sally, *General elements and joint reductions,* Michigan Math. J. 35 (1988), 241–254.
[18] B. Ulrich, *Artin–Nagata properties and reductions of ideals,* Commutative algebra: Syzygies, multiplicities, and birational algebra (South Hadley, 1992), Contemp. Math., 159, pp. 373–400, Amer. Math. Soc., Providence, RI, 1994.

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