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Intersection Multiplicities, the Canonical Element Conjecture, and the Syzygy Problem

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In honor of Melvin Hochster on the occasion of his 65th birthday

Introduction

In this paper we shall concentrate on the canonical element conjecture due to M. Hochster as well as several of its ramifications. In [17] Hochster introduced a number of equivalent forms of this conjecture and proved it in the equicharacteristic case. One of the earliest forms, the direct summand conjecture, was proved by Hochster [15] a decade earlier under the same hypothesis (see also [16]). In 1981, Evans and Griffith [10] gave an affirmative answer to the syzygy problem for equicharacteristic local rings. In the course of their proof, they implicitly established a result for finite free complexes [10] that Hochster explicitly isolated in his article [17]. He referred to the new result as the "improved new intersection theorem" (henceforth INIT) because it is clear that INIT implies the new intersection theorem [17]. Of course, INIT remains a conjecture in the case of mixed characteristic. In the same article [17] Hochster pointed out that INIT was a consequence of the canonical element conjecture, and later the first author [3] showed that the two conjectures are equivalent. Over the years, several special cases of the canonical element conjecture have been proved and new equivalent forms have been introduced (see [2; 3; 5; 6; 7; 8; 14]). The four main equivalent versions of this conjecture, that is, the direct summand conjecture, the monomial conjecture, the canonical element conjecture, and the improved new intersection conjecture along with the statement of the syzygy problem are given at the end of this section.

In Section 1 our main focus will be on a formulation of the monomial conjecture in terms of comparing the lengths "Tor_0" and "Tor_1" of a pair of finitely generated modules over a regular local ring. Given a local ring A and a pair of finitely generated modules M and N such that $\ell(M \otimes_A N) < \infty$ (where ℓ denotes length), we raise the following question:

$$\text{Is } \ell(M \otimes_A N) > \ell(\mathrm{Tor}_1^A(M, N))? \qquad \text{(Q)}$$

It is clear that (Q) has obvious negative answers—for example, when $M = N = K$, the residue field of A, or when $M = K$ and $N = m$, the maximal ideal of A. To get to the heart of the matter for A, a regular local ring, we first observe that (Q) boils down to the following question:

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$$\text{Is } \chi^A(M,N) > \chi_2^A(M,N)?$$

See Section 1 for definitions of χ and χ_i. In Proposition 1.1(ii) we argue that, if A is an equicharacteristic or unramified regular local ring, then (Q) has a negative answer when $\dim M + \dim N < \dim A$ (i.e., when vanishing holds) and $\operatorname{depth} M + \operatorname{depth} N < \dim A - 1$. If $\dim M + \dim N = \dim A$, then Proposition 1.1(iii) gives a positive answer when M or N is a perfect module and the other module satisfies $\dim - \operatorname{depth} = 1$. Even when $N = A/I$ and I is generated by an A-sequence (the best scenario for both χ and χ_i), we are unable to get a definite answer for (Q) when $\dim A/I + \dim M = \dim A$. The best we can prove in this situation is the following.

THEOREM 1.4. *Let (A, m) be a local ring and let I be an ideal of A generated by an A-sequence. Let M be a finitely generated A-module such that $\ell(M/IM) < \infty$, and let* $\dim A/I + \dim M = \dim A$. *Then there exists an integer $t > 0$ and a minimal set of generators $x_1, \dots, x_n$ of I such that* (Q) *has a positive answer for the pair $(A/I_{n,s}, M)$, where $I_{n,s} = (x_1, \dots, x_{n-1}, x_n^s)$ for $s \geq t$.*

Our proof of this theorem requires (a) an extension of the definition and properties of superficial elements (introduced by Samuel [25]) from local rings to finitely generated modules over local rings and (b) several properties of Hilbert–Samuel multiplicity (see Lemma 1.2 and Proposition 1.3).

Our next theorem connects a positive answer for a special case of (Q) with an affirmative answer for the monomial conjecture (MC).

THEOREM 1.5. *The monomial conjecture is valid for all local rings if and only if, for every unramified and equicharacteristic regular local ring A,* (Q) *has a positive answer for the pair $(A/I, A/J)$, where every pair of ideals I, J of A is such that* (i) *I is a complete intersection ideal,* (ii) *J is an almost complete intersection ideal,* (iii) $\operatorname{ht} I + \operatorname{ht} J = \dim A$, *and* (iv) *$I + J$ is m-primary.*

Theorem 1.5 demonstrates in a direct way why a definitive answer to (Q), even in the best (but nonobvious) case, is difficult to come by.

As a follow-up we prove in Theorem 1.6 that, in order to prove the monomial conjecture, it is enough to have a positive answer for (Q) when $I + J$ is not a complete intersection ideal in A.

In our final result of this section we describe the cases for which we can assert a positive answer for (Q).

THEOREM 1.7. *Let A, I, J satisfy conditions* (i)–(iv) *as in Theorem 1.5. Then* (Q) *has a positive answer for the pair $(A/I, A/J)$ in the following cases*:

(a) $\operatorname{Tor}_1^A(A/I, A/J)$ *is cyclic*;
(b) $\operatorname{Tor}_1^A(A/I, A/J)$ *is decomposable*;
(c) $\operatorname{Tor}_1^A(A/I, A/J)^\vee$ *is not cyclic, where $(\cdot)^\vee$ denotes the Matlis dual*; *and*
(d) *the mixed characteristic p represents a nonzero divisor on A/J, in particular if J is a prime ideal.*

Note that by Theorem 1.4 there exists an integer $t > 0$ such that (Q) has a positive answer for the pair $(A/I_{n,s}, A/J)$ for $s \geq t$.

The proof of Theorem 1.5 includes the reduction of MC on any local ring to MC on almost complete intersections. As a corollary to Theorem 1.7 we have the following.

COROLLARY. *Let C be an almost complete intersection local ring and let $x_1, \ldots, x_n$ be a system of parameters for C. Then $x_1, \ldots, x_n$ satisfies MC in the following cases*:

(i) *$H_1(\underline{x}; C)$ is cyclic*;
(ii) *$H_1(\underline{x}; C)$ is decomposable*;
(iii) *$H_1(\underline{x}; C)^\vee$ is not cyclic*; *and*
(iv) *p is not a nonzero divisor on C, in particular if C is an almost complete intersection local domain.*

These results grew out of the first author's work in [7]. The question (Q), Theorem 1.4, and parts (b) and (c) of Theorem 1.7 are generalizations and reformulations of results in [7] in the broader spectrum of regular local rings. This generality is proposed with the hope that the greater latitude of regular local rings together with the techniques for studying intersection multiplicities might shed more light on understanding this group of conjectures. The first author has been trying to prove Theorem 1.7(a) for a long time—that is, the validity of the monomial and canonical element conjectures when $H_1(\underline{x}; C)$, where $C = A/J$, (the first Koszul homology for a system of parameters $x_1, \ldots, x_n$ of C) is cyclic. Note that Theorem 1.7(b) states that the conjectures are valid when $H_1(\underline{x}; C)$ is decomposable. Finally, techniques of proving various aspects of the canonical element conjecture (see Theorem 1.6) and a theorem of Kunz [20] came to the rescue. This result makes the first author hopeful about prospects for the approach outlined in this section.

Section 2 is devoted to a re-examination of Koh's proof [19] of his significant result on validity of the direct summand conjecture for degree-p extensions. Namely: If R is a regular local ring of mixed characteristic p such that R/pR is again regular, then a ring extension of the form $R \hookrightarrow R[u^{1/p}]'$ must be R-split (here $[\cdot]'$ denotes "integral closure"). To support the relevance of his result, Koh recalls from Hochster's article [15] that the direct summand conjecture reduces to the case of finite ring extensions $R \hookrightarrow B \hookrightarrow B[u^{1/p}]'$, where B is an intermediate normal domain. Thus Koh's main result addresses the special case $R = B$. Koh's argument requires a good deal of computation and somewhat tedious linear algebra. With the aid of 20/20 hindsight we propose here to give a more conceptual proof that relies on Galois theory and basic concepts of eigenvalues from linear algebra. After noting a simple criteria for a finite extension of normal domains $R \hookrightarrow A$ to be R-split (Proposition 2.1), we proceed to construct, within the context of Koh's hypothesis, a canonical free R-subalgebra S of A such that $pA \subseteq S$ (Theorem 2.2 and Theorem 2.3). The existence of such an S allows us to conclude $p\,\mathrm{Ext}^1_R(A/R, R) = 0$. Since the short exact sequence $e\colon 0 \to R \to A \to A/R \to 0$ necessarily splits modulo p (the class $[e]$ represents an element of $p\,\mathrm{Ext}^1_R(A/R, R)$), it follows that $R \hookrightarrow A$ is R-split.

In Section 3 we return to the syzygy theorem of Evans–Griffith [10] and review its proof in light of the recent proof by Heitmann [14] of the direct summand conjecture in dimension 3. Although Heitmann's result is for dimension 3, its impact by way of INIT allows us to argue that a nonfree $(n-2)$th syzygy of finite projective dimension over a local ring A of dimension n must have rank $\geq n-2$ in any characteristic. Recall that the syzygy theorem of Evans and Griffith is valid in the equicharacteristic case. The results described so far lead to a proof of the syzygy conjecture for rings up to dimension ≤ 5 in mixed characteristic (Corollary 3.6). A graded version of the syzygy conjecture was shown in [13] to have an affirmative answer for mixed characteristic.

This is not a joint work in the usual sense of the term. The first author's work is described in Section 1 and the second author's work is described in Sections 2 and 3. When the authors found out that they were each writing a paper on the same group of equivalent conjectures from different perspectives for the same volume honoring Melvin Hochster, they decided to submit their work as a single paper.

We start out by stating the syzygy conjecture and four equivalent versions of the canonical element conjecture that will be used in the main body of this paper. Throughout, by a local ring we mean a Noetherian local ring. The subscript (resp. superscript) A will be omitted from the notation of Ext (resp. Tor) when there is no danger of ambiguity. For a module M over a ring A, $\dim M$ will denote its Krull dimension; for a local ring A, E will denote the injective hull of the residue field.

A. Syzygy Conjecture. Let A be a local ring, and let M be a finitely generated nonfree kth syzygy over A with finite projective dimension. Then $\mathrm{rank}_A M \geq k$.

B. Direct Summand Conjecture (DSC). Let R be a regular local ring, and let $i\colon R \hookrightarrow A$ be a module-finite extension of R. Then i splits as an R-module map.

C. Monomial Conjecture (MC). Let A be a local ring of dimension n, and let $x_1, \dots, x_n$ be a system of parameters of A. Then, for every integer $t > 0$,

$$(x_1 \dots x_n)^t \notin (x_1^{t+1}, \dots, x_n^{t+1}).$$

D. Canonical Element Conjecture (CEC). Let A be a local ring of dimension n with maximal ideal m and residue field K. Let S_i denote the ith syzygy of K in a minimal resolution of K over A, and let $\theta_n\colon \mathrm{Ext}^n_A(K, S_n) \to H^n_m(S_n)$ denote the direct limit map. Then θ_n (class of the identity map on S_n) $\neq 0$.

E. Improved New Intersection Conjecture (INIC). Let A be as before. Let $F_\bullet$ be a complex of finitely generated free A-modules,

$$F_\bullet\colon 0 \to F_s \to F_{s-1} \to \cdots \to F_1 \to F_0 \to 0,$$

such that $\ell(H_i(F_\bullet)) < \infty$ for $i > 0$ and $H_0(F_\bullet)$ has a minimal generator annihilated by a power of the maximal ideal m. Then $\dim A \leq s$.

Section 1

Let $(R, m, K = R/m)$ be a regular local ring, and let M and N be two finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$. Following Serre [26], we define $\chi^R(M, N) = \sum_{j \geq 0} (-1)^j \ell(\mathrm{Tor}_j^R(M, N))$ and, for $i > 0$, $\chi_i^R(M, N) = \sum_{j \geq 0} (-1)^j \ell(\mathrm{Tor}_{i+j}^R(M, N))$. We drop R from the notation when there is no danger of ambiguity. Note that these definitions make sense over any local ring A provided at least one of the modules has finite projective dimension. In [15] Hochster proved that the direct summand conjecture is valid provided we assume that the regular local ring is unramified.

Now we prove our first proposition. This will involve several results from Serre [26] and Lichtenbaum [21] on intersection multiplicities.

PROPOSITION 1.1. *Let R be an equicharacteristic or unramified regular local ring and let M, N be two finitely generated R-modules such that $\ell(M \otimes_R N) < \infty$.*

(i) *If* $\dim M + \dim N < \dim R$ *and* $\operatorname{depth} M + \operatorname{depth} N = \dim R - 1$, *then* $\ell(M \otimes_R N) = \ell(\mathrm{Tor}_1^R(M, N))$.

(ii) *If* $\dim M + \dim N < \dim R$ *and* $\operatorname{depth} M + \operatorname{depth} N < \dim R - 1$, *then* (Q) *has a negative answer.*

(iii) *If* $\dim M + \dim N = \dim R$ *and* $\operatorname{depth} M + \operatorname{depth} N = \dim R - 1$, *then* (Q) *has a positive answer.*

Proof. By our definitions we have

$$\ell(M \otimes_R N) - \ell(\mathrm{Tor}_1^R(M, N)) = \chi(M, N) - \chi_2(M, N).$$

Recall now the following results on equicharacteristic/unramified regular local rings when $\ell(M \otimes_R N) < \infty$:

(a) $\dim M + \dim N \leq \dim R$ [26, Chap. V, Thm. 3] (true for any regular local ring);

(b) $\chi(M, N) \geq 0$, where equality holds if and only if $\dim M + \dim N < \dim R$ [26, Chap. V, Thm. 1, Lemma];

(c) $\chi_i(M, N) \geq 0$, where equality holds if and only if $\mathrm{Tor}_j(M, N) = 0$ for $j \geq i$ [21; 26]; and

(d) if $i = \dim R - \operatorname{depth} M - \operatorname{depth} N$, then $\mathrm{Tor}_j(M, N) = 0$ for $j > i$ [26, Chap. V, Thm. 4] (true in more generality).

The proof of the proposition is a direct consequence of (a)–(d), as follows.

(i) If $\dim M + \dim N < \dim R$ and $\operatorname{depth} M + \operatorname{depth} N = \dim R - 1$ then, by (b) and (d), $0 = \chi(M, N) = \ell(M \otimes_R N) - \ell(\mathrm{Tor}_1(M, N))$.

(ii) If $\dim M + \dim N < \dim R$ and $\operatorname{depth} M + \operatorname{depth} N < \dim R - 1$ then, by (b) and (c) and (d), $0 = \chi(M, N) = \ell(M \otimes N) - \ell(\mathrm{Tor}_1(M, N)) + \chi_2(M, N)$ and $\chi_2(M, N) > 0$.

(iii) If $\dim M + \dim N = \dim R$ and $\operatorname{depth} M + \operatorname{depth} N = \dim R - 1$ then, by (b) and (d), $0 < \chi(M, N) = \ell(M \otimes N) - \ell(\mathrm{Tor}_1(M, N))$. □

Remarks. 1. The answer to (Q) is clear when $\dim M + \dim N < \dim R$—that is, when vanishing holds. For the positivity case (i.e., when $\dim M + \dim N = \dim R$) it will be highly significant to understand the situation when $\dim R - \operatorname{depth} M - \operatorname{depth} N = 2$. In this case, by Proposition 1.1, $\chi(M,N) = \ell(M \otimes N) - \ell(\operatorname{Tor}_1(M,N)) + \ell(\operatorname{Tor}_2(M,N))$. Recall the observation in [9] that the general question on positivity can be reduced to this case.

2. Proposition 1.1(iii) is valid over any Cohen–Macaulay local ring for a pair of modules of which one is perfect and the other has (dim − depth) = 1.

3. Serre observed in [26] that in the equicharacteristic case, for the positivity part, $\chi(M,N) \geq e_m(M)e_m(N)$, where $e_m(T)$ denotes the Hilbert–Samuel multiplicity of a finitely generated R-module T. This brings up the following difficult question: Is $\chi_2(M,N) < e_m(M)e_m(N)$? We do not know the answer even in the best possible nonobvious situation.

Recall that both Serre's work on intersection multiplicities [26] and Lichtenbaum's work on the χ_i problem [21] depended heavily on the following observation: Let (A,m,K) be a local ring, M a finitely generated A-module, and I an ideal of A generated by an A-sequence such that $\ell(M/IM) < \infty$; then

- $\chi(A/I, M) \geq 0$, where equality holds if and only if $\dim M < \operatorname{ht} I$, and
- $\chi_i(A/I, M) \geq 0$, where equality holds if and only if $\operatorname{Tor}_j(A/I, M) = 0$ for $j \geq i$.

However—even when A is regular, I is as described previously, and $\dim M + \dim A/I = \dim A$—we do not know, in general, the answer to (Q) for the pair $(A/I, M)$. The best result we can prove in this situation is stated in Theorem 1.4. For our proof of this theorem first we define superficial elements for a finitely generated module over a local ring A by extending the original definition of Samuel [25] for local rings.

Definition. Let (A,m,K) be a local ring, M a finitely generated A-module, and I an ideal of A. An element $x \in I^s$ is called a *superficial element* of order s for M in I if there exists an integer a such that $(I^nM : x) \cap I^aM = I^{n-s}M$ whenever $n \geq s + a$.

The existence and properties of superficial elements are described in the following lemma.

Lemma 1.2. *With A, I, M as just defined, the following statements hold.*

(i) *There exists a superficial element $x \in I$ for M of some order s.*
(ii) *If* $\operatorname{depth}_I M > 0$, *then x can be chosen to be a nonzero divisor.*
(iii) *If $K = A/m$ is infinite, then there exist superficial elements for M of any given order. Henceforth we assume that K is infinite.*
(iv) *Assume that $\ell(M/IM) < \infty$. If x is a superficial element of order s, then $e(I; M/xM) = se(I; M)$. Here $e(I; M)$ denotes the Hilbert–Samuel multiplicity of M with respect to the ideal I.*

(v) *If I is minimally generated by a system of parameters of M, then one can choose $x_1, \ldots, x_n$ in I such that $I = (x_1, \ldots, x_n)$, the $x_1, \ldots, x_{n-1}$ are superficial elements of M, and*

$$e(I; M) = e((x_2, \ldots, x_n); M/x_1M) = \cdots = e(x_n; M/(x_1, \ldots, x_{n-1})M).$$

Proof. The proof of these facts is essentially as outlined by Samuel [25] for local rings. We leave this as an exercise for the reader.

PROPOSITION 1.3. *Let (A, m, K) be a local ring and M a finitely generated A-module, and let $x_1, \ldots, x_n$ be an A-sequence that is a system of parameters for M. Assume*

$$\begin{aligned} e((x_1, \ldots, x_n); M) &= e((x_2, \ldots, x_n); M/x_1M) \\ &\vdots \\ &= e(x_n; M/(x_1, \ldots, x_{n-1})M). \end{aligned}$$

Then $\ell(H_i(x_1, \ldots, x_{n-1}; M)) < \infty$ for $i > 0$.

Proof. We write $\underline{x}$ for the ideal $(x_1, \ldots, x_n)$ and $\underline{x}_{n-1}$ for the ideal $(x_1, \ldots, x_{n-1})$. Let $P_1, \ldots, P_r$ denote the minimal primes in $\mathrm{Ass}_A(M/\underline{x}_{n-1}M)$. Then

$$e(x_n; M/\underline{x}_{n-1}M) = \sum_{i=1}^{r} \ell((M/x_{n-1}M)_{P_i}) e(x_n; A/P_i). \tag{1}$$

On the other hand, by the associativity formula for Hilbert–Samuel multiplicity we have

$$e(\underline{x}; M) = \sum_{i=1}^{r} e(x_n; A/P_i) e(\underline{x}_{n-1}; M_{P_i}) \tag{2}$$

(see [22] and [25] for (1) and (2), respectively).

Recall that

$$e(\underline{x}_{n-1}; M_{P_i}) = \sum_{j=0}^{n-1} (-1)^j \ell(H_j(\underline{x}_{n-1}; M_{P_i}) \tag{3}$$

(see [25]). By the result on χ_i mentioned earlier [21] we know that $\chi_1^{(i)} = \sum_{j=0}^{n-2} (-1)^j \ell(H_{j+1}(\underline{x}_{n-1}, M_{P_i})) \geq 0$, where equality holds if and only if $H_j(\underline{x}_{n-1}; M_{P_i}) = 0$ for $j \geq 1$. Given our assumptions, subtracting (2) from (1) now yields

$$0 = \sum_{i=1}^{r} e(x_n; A/P_i) \chi_1^{(i)}.$$

Since $e(x_n; A/P_i) > 0$, we must have $\chi_1^{(i)} = 0$ for $i = 1, 2, \ldots, r$. This implies, by the result stated at the beginning of Lemma 1.2, that $H_j(x_{n-1}; M_{P_i}) = 0$ for every $j \geq 1$ and $i = 1, \ldots, r$. Hence $\ell(H_j(x_{n-1}, M)) < \infty$ for every $j \geq 1$. $\square$

THEOREM 1.4. *Let (A, m, K) be a local ring, and let M be a finitely generated A-module. Let I be an ideal of A generated by an A-sequence of length n such* *that* $\ell(M/IM) < \infty$ *and* $\dim A/I + \dim M = \dim A$. *Then there exists an integer* $t > 0$ *and a minimal set of generators* $x_1, \dots, x_n$ *of* I *such that* (Q) *has a positive answer for pairs* $(A/I_{n,s}, M)$, *where* $I_{n,s} = (x_1, \dots, x_{n-1}, x_n^s)$ *for* $s \geq t$.

Proof. Let $I = (y_1, \dots, y_n)$, where the $y_1, \dots, y_n$ form an A-sequence. By assumption, $y_1, \dots, y_n$ form a system of parameters for M. By Lemma 1.2(v) we can construct $x_1, \dots, x_{n-1} \in I$ such that $I = (x_1, \dots, x_{n-1}, y_n)$ and

$$e(I, M) = e((x_2, \dots, x_{n-1}, y_n); M/x_1M) = \cdots = e(y_n; M/(x_1, \dots, x_{n-1})M).$$

Write $\underline{x} = (x_1, \dots, x_{n-1})$, and observe that $\operatorname{Tor}_i^A(A/I, M) = H_i(\underline{x}, y_n; M)$ for $i \geq 0$. We have the short exact sequence

$$0 \to \frac{H_1(\underline{x}; M)}{y_n^t H_1(\underline{x}; M)} \to H_1(\underline{x}, y_n^t; M) \to (0 : y_n^t)_{M/\underline{x}M} \to 0.$$

By Proposition 1.3, $\ell(H_j(\underline{x}; M)) < \infty$ for $j > 0$. Then, for $t \gg 0$, it follows that $y_n^t H_1(\underline{x}; M) = 0$ and $\ell((0 : y_n^t)_{M/\underline{x}M})$ is constant. However, $\ell(M/(\underline{x}, y_n^t)M)$ is a strictly increasing function of t. Hence $\ell(M/(\underline{x}, y_n^t)M) > \ell(H_1(\underline{x}, y_n^t; M)) = \ell(\operatorname{Tor}_1^A(M, A/(\underline{x}, y_n^t)))$ for $t \gg 0$, and the proof is complete. □

Our next theorem demonstrates why a definite answer to (Q), even when one of the modules is a complete intersection, is so difficult to comprehend. This reveals the relation between MC and a special case of a question on intersection multiplicities in terms of χ and χ_2.

THEOREM 1.5. *The monomial conjecture is valid for all local rings* A *if and only if, for every unramified and equicharacteristic regular local ring* R,

$$\ell(R/(I + J)) > \ell(\operatorname{Tor}_1^R(R/I, R/J))$$

for every pair of ideals I, J *of* R *such that* (i) I *is a complete intersection ideal,* (ii) J *is an almost complete intersection ideal* (*i.e., minimally generated by* $(\operatorname{ht} J + 1)$ *elements*), (iii) $\operatorname{ht} I + \operatorname{ht} J = \dim R$, *and* (iv) $I + J$ *is primary to the maximal ideal of* R.

Proof. The proof will be completed in a sequence of three steps. Note that for the monomial conjecture (or for any other equivalent form) we can always assume A is a complete local normal domain. Hence we write $A = R/\tilde{P}$, where R is an unramified or equicharacteristic complete regular local ring. Let $S = R/\underline{\xi}$, where $\underline{\xi}$ is the ideal generated by a maximal R-sequence $\xi_1, \dots, \xi_r$ contained in $\tilde{P}$. Then $A = S/P$, where $P = \tilde{P}/\underline{\xi}$. Write $\Omega = \operatorname{Hom}_S(A, S)$, the canonical module for A; here Ω is an ideal of S. Let E denote the injective hull of the residue field of S.

Step 1. We first sketch a short proof of the following theorem due to Strooker and Stückrad on a characterization of MC (the first author independently proved a similar characterization for DSC [7]).

THEOREM [27]. *With notation as before,* A *satisfies MC if and only if, for every system of parameters* $x_1, \dots, x_n$ *of* S, *we have* $\Omega \not\subset (x_1, \dots, x_n)$.

Proof. Let $y_1, \dots, y_n$ be a system of parameters of A. We can lift it to a system of parameters $x_1, \dots, x_n$ for S such that $\mathrm{im}(x_i) = y_i, 1 \le i \le n$. Conversely, any system of parameters for S is a system of parameters for A. Write $\underline{x} = (x_1, \dots, x_n)$ and $\underline{y} = (y_1, \dots, y_n)$. The monomial conjecture for A is equivalent to the assertion that, for every system of parameters $y_1, \dots, y_n$ of A, the direct limit map $\alpha\colon A/\underline{y} \to H^n_m(A)$ is nonnull [15]. Because S is a complete intersection, the direct limit map $\beta\colon S/\underline{x} \to H^n_{m_S}(S)$ is nonnull [2]. We write $T^\vee = \mathrm{Hom}_S(T, E)$ for any S-module T and have the commutative diagram

$$\begin{tikzcd} S/\underline{x} \arrow[r, "\beta"] \arrow[d, "\eta"] \arrow[dr] & H^n_{m_s}(S) \arrow[d, "\gamma"] \\ A/\underline{y} \arrow[r, "\alpha"] & H^n_m(A), \end{tikzcd}$$

where η denotes the natural surjection and $\gamma = H^n_{m_s}(\eta)$. This implies that

$$\begin{aligned} \alpha \text{ is nonnull } &\iff \alpha \circ \eta = \gamma \circ \beta \text{ is nonnull} \\ &\iff H^n_m(A)^\vee \to (S/\underline{x})^\vee \text{ is nonnull} \\ &\iff \mathrm{Im}(\Omega \to S/\underline{x}) \text{ is nonnull} \\ &\iff \Omega \not\subset \underline{x} \end{aligned}$$

(recall that, by local duality, $H^n_m(A)^\vee = \Omega$). □

Step 2. In this step we reduce MC for all local rings to MC for local almost complete intersections. We prove that

> *The monomial conjecture is valid for all local rings if and only if it holds for all local almost complete intersections.*

Proof. Suppose MC holds for all local almost complete intersections. Let A be a complete local domain. Then we have $A = R/\tilde{P}$, where R is a complete regular local ring. Since $R_{\tilde{P}}$ is a regular local ring, one can choose a maximal R-sequence $\xi_1, \dots, \xi_r$ in $\tilde{P}$ such that $\tilde{P}R_{\tilde{P}} = (\xi_1, \dots, \xi_r)R_{\tilde{P}}$. Write $S = R/\underline{\xi}$, where $\underline{\xi} = (\xi_1, \dots, \xi_r)$ and $P = \tilde{P}/\underline{\xi}$. Then S is a complete intersection, $A = S/P$, $\dim S = \dim A$, and $PS_P = 0$. Let $\Omega = \mathrm{Hom}_S(A, S)$, the canonical module of S. Consider the primary decomposition in S: $0 = P \cap q_2 \cap \cdots \cap q_h$, where q_i is P_i-primary and $\mathrm{ht}\, P_i = \mathrm{ht}\, P = 0$ for $2 \le i \le h$. It can be checked easily that $\Omega = q_2 \cap \cdots \cap q_h$. Choose $\lambda \in P - \bigcup_{i \ge 2} P_i$. Then $\Omega = \mathrm{Hom}(S/\lambda S, S)$, and $S/\lambda S$ is an almost complete intersection. Since $S/\lambda S$ satisfies MC by assumption, it follows from Step 1 that Ω is not contained in the ideal generated by any system of parameters in S. Hence, again by Step 1, A satisfies MC. □

Step 3. Now we prove the assertion of Theorem 1.5. First assume that every pair (I, J) in the theorem satisfies the length inequality:

$$\ell(R/(I + J)) > \ell(\mathrm{Tor}_1^R(R/I, R/J)).$$

By Step 2, we can assume that A is an almost complete intersection ring of the form $S/\lambda S$, where S is a complete intersection and $\dim S = \dim A$. Write $\Omega = \mathrm{Hom}(S/\lambda S, S)$, the canonical module for A. Consider the short exact sequence

$$0 \longrightarrow S/\Omega \xrightarrow{f} S \longrightarrow S/\lambda S \longrightarrow 0, \tag{4}$$

where $f(\bar{1}) = \lambda$. Let $x_1'', \dots, x_n''$ be a system of parameters for A. We can lift $x_1'', \dots, x_n''$ to $x_1', \dots, x_n'$ in S in such a way that $\{x_1', \dots, x_n'\}$ form a system of parameters in S. Write $\underline{x}'' = (x_1'', \dots, x_n'')$ and $\underline{x}' = (x_1', \dots, x_n')$. Tensoring (4) with $S/\underline{x}'$ yields the exact sequence

$$0 \longrightarrow \mathrm{Tor}_1^S(S/\underline{x}', S/\lambda S) \longrightarrow S/(\Omega + \underline{x}') \xrightarrow{\bar{f}} S/\underline{x}' \longrightarrow S/(\underline{x}' + \lambda S) \longrightarrow 0, \tag{5}$$

where $\bar{f}$ is induced by f and $\mathrm{Tor}_1^S(S/\underline{x}', S/\lambda S) = H_1(\underline{x}'; S/\lambda S) = H_1(\underline{x}'; A)$. Then

$$\Omega \not\subset \underline{x}' \iff \ell(S/(\underline{x}' + \lambda S)) > \ell(\mathrm{Tor}_1(S/\underline{x}', S/\lambda S)). \tag{6}$$

As in Step 2, let R be a complete regular local ring mapping onto S. Now lift $x_1', \dots, x_n'$ to an R-sequence $x_1, \dots, x_n$ in R. Write $I = \underline{x}$ and $J = (\underline{\xi}, \lambda)$. Then (6) translates to $\ell(R/(I + J)) > \ell(\mathrm{Tor}_1(R/I, R/J))$, as required in our statement.

For the converse part of the theorem, write $I = (x_1, \dots, x_n)$ and $J = (y_1, \dots, y_r, \lambda)$, where $x_1, \dots, x_n$ and $y_1, \dots, y_r$ form R-sequences such that $n + r = \dim R$. Let $S = R/(y_1, \dots, y_r)$ and $A = S/\lambda S$, and let $x_i' = \mathrm{im}(x_i)$ in S for $1 \le i \le n$. Write $\underline{x}' = (x_1', \dots, x_n')$ and $\underline{x} = (x_1, \dots, x_n)$. Since $I + J$ is primary to the maximal ideal and since both $R/\underline{x}$ and S are complete intersections, it follows from [26] that $\mathrm{Tor}_i^R(R/\underline{x}, S) = 0$ for $i > 0$. This implies that $\mathrm{Tor}_i^R(R/I, R/J) = \mathrm{Tor}_i^S(S/\underline{x}', A)$ for $i \ge 0$. Let $\Omega = \mathrm{Hom}_S(A, S)$, the canonical module for A. Now (4)–(6) and the subsequent arguments complete the proof. □

As a corollary we derive the following.

Corollary 1. *With notation as in the theorem,*

$$\ell(R/(I + J)) \ge \ell(\mathrm{Tor}_1^R(R/I, R/J)).$$

The proof follows from the exact sequence (5).

Corollary 2. *Let A be a complete local domain, and let $x_1, \dots, x_n$ be a system of parameters for A. Then there exist $y_1, \dots, y_{n-1} \in (x_1, \dots, x_n)$ such that $(y_1, \dots, y_{n-1}, x_n) = (x_1, \dots, x_{n-1}, x_n)$ and $(y_1, \dots, y_{n-1}, x_n^t)$ satisfies MC for $t \gg 0$.*

The proof follows from Theorem 1.4 and the equivalence (6).

As mentioned in the Introduction, for the past several years we have been trying to prove that MC/CEC holds for a system of parameters $x_1, \dots, x_n$ of local ring A if $H_1(\underline{x}; A)$ is cyclic. We could prove the conjecture if $H_1(\underline{x}; A)$ is decomposable, but a proof for the first important case of indecomposability—that is, the cyclic case—always eluded us. Finally, we are now able to prove this case when A is an almost complete intersection ring. The following theorem plays a crucial role in the proof of this result. Our proof of the theorem will involve several results from

[4] and [6], where the first author studied various aspects of the canonical element conjecture.

THEOREM 1.6. *The monomial conjecture holds for all local rings if and only if, for every regular local ring R and every pair of ideals I, J of R,*

$$\ell(R/(I+J)) > \ell(\mathrm{Tor}_1^R(R/I,\ R/J)),$$

where (i) *I is a complete intersection,* (ii) *J is an almost complete intersection,* (iii) $\mathrm{ht}\, I + \mathrm{ht}\, J = \dim R$, (iv) *$I+J$ is m-primary, and* (v) *$I+J$ is not a complete intersection ideal in R.*

Proof. We will prove this theorem in three steps as follows.

Step 1. Let (A, m, K) be a local ring of dimension n, and let S_i denote the ith syzygy of K in a minimal resolution of K over A. Let $\theta_i : \mathrm{Ext}^i(K, S_i) \to H_m^i(S_i)$ denote the direct limit map, and let $\eta_i = \theta_i$ (the image of the identity map on S_i). Now CEC demands that $\eta_n \neq 0$ [17].

Consider a minimal resolution $F_\bullet = \{A^{t_i}, d_i\}_{i \geq 0}$ of K and break it up into short exact sequences:

$$\begin{aligned} &0 \to S_n \to A^{t_{n-1}} \to S_{n-1} \to 0,\\ &0 \to S_{n-1} \to A^{t_{n-2}} \to S_{n-2} \to 0, \ldots,\\ &0 \to S_1 \to A \to K \to 0. \end{aligned} \tag{7}$$

These sequences give rise to the following commutative diagram:

$$\begin{tikzcd} K \arrow[r, "\delta_0"] \arrow[d, equal, "\theta_0=\mathrm{Id}"] & \mathrm{Ext}_A^1(K, S_1) \arrow[r, "\delta_1"] \arrow[d, "\theta_1"] & \mathrm{Ext}_A^2(K, S_2) \arrow[r, "\delta_2"] \arrow[d, "\theta_2"] & \cdots \arrow[r] & \mathrm{Ext}_A^{n-1}(K, S_{n-1}) \arrow[r, "\delta_{n-1}"] \arrow[d, "\theta_{n-1}"] & \mathrm{Ext}_A^n(k, S_n) \arrow[d, "\theta_n"] \\ H_m^0(K) \arrow[r, "\tilde{\delta}_0"] & H_m^1(S_1) \arrow[r, "\tilde{\delta}_1"] & H_m^2(S_2) \arrow[r, "\tilde{\delta}_2"] & \cdots \arrow[r] & H_m^{n-1}(S_{n-1}) \arrow[r, "\tilde{\delta}_{n-1}"] & H_m^n(S_n), \end{tikzcd} \tag{8}$$

where all the horizontal maps are connecting homomorphisms obtained from the preceding short exact sequences. It follows that $\tilde{\delta}_i(\eta_i) = \eta_{i+1}$ for $0 \leq i < n$, so η_n is nothing but the image of $1 \in K$ at the upper left-hand corner. We have the following theorem.

THEOREM [4]. *With notation as before, $\eta_i \neq 0$ for $0 \leq i \leq n-1$.*

Since the techniques involved in the proof of this theorem are completely different from those in this paper, we refer the reader to [4] for a proof.

COROLLARY. *Let A be a local ring with notation as before, and suppose that $H_m^{n-1}(A) = 0$. Then CEC holds for A.*

Proof. Since $H_m^{n-1}(A) = 0$, from the first short exact sequence in (7) it follows that there exists a short exact sequence $0 \to H_m^{n-1}(S_{n-1}) \to H_m^n(S_n)$. Now, by the preceding theorem, the proof is complete. □

Recall that, for the validity of CEC et cetera, we can assume without loss of generality that the given local ring A is a complete local normal domain [17].

Step 2. Let A be a complete local normal domain. Then, as described previously, we can write $A = S/P$ for S a complete intersection such that $\dim S = \dim A$. Write $\Omega = \mathrm{Hom}_S(A, S)$, the canonical module for A. Then S/Ω satisfies CEC. This was proved in [6] using dualizing complexes.

Since A is normal domain, $\mathrm{Hom}_S(\Omega, S) = \mathrm{Hom}_S(\Omega, \Omega) = \mathrm{Hom}_A(\Omega, \Omega) \simeq A$. We can construct S in such a way that $\mathrm{Hom}_S(S/\Omega, S) = P$ (cf. Step 2 in the proof of Theorem 1.5). Consider the short exact sequence

$$0 \to \Omega \to S \to S/\Omega \to 0.$$

Applying $\mathrm{Hom}_S(\cdot, S)$ to this sequence yields the following short exact sequence:

$$0 \to P \hookrightarrow S \to A \to \mathrm{Ext}^1_S(S/\Omega, S) \to 0.$$

Since $A = S/P$, this implies that $\mathrm{Ext}^1_S(S/\Omega, S) = 0$ and therefore, by local duality, $H^{n-1}_m(S/\Omega) = 0$. Hence we are done by the corollary in Step 1.

Step 3. In this step we show that, given a system of parameters $x_1, \dots, x_n$ of S, we can choose $\lambda \in P$ in such a way that $\lambda \notin (x_1, \dots, x_n)$ (cf. Step 2 in the proof of Theorem 1.5).

As before, we can choose S in such a way that $PS_P = 0$. Since S/Ω satisfies CEC, by Step 1 of Theorem 1.5 we conclude that $P = \mathrm{Hom}_S(S/\Omega, S)$, the canonical module for S/Ω, is not contained in the ideal generated by any system of parameters of S. Thus, given $x_1, \dots, x_n$, a system of parameters of S, we can choose $\lambda \in P - \left[(x_1, \dots, x_n) \cup \left(\bigcup P_i, i \geq 2\right)\right]$, where the P_i are as in Step 2 of Theorem 1.5.

Now, by Step 3 of Theorem 1.5, the proof of Theorem 1.6 is complete. □

Now we are ready to prove our final theorem of this section. With notation as in Theorem 1.5, we can assume by Theorem 1.6 that $I + J$ is not a complete intersection ideal. This assumption will be used only for the proof of part (a) of our theorem; for other parts, no such assumption is necessary.

Theorem 1.7. *Let R, I, J satisfy conditions* (i)–(iv) *of Theorem 1.5. Then we have $\ell(R/(I + J)) > \ell(\mathrm{Tor}_1(R/I, R/J))$ in the following cases*:

(a) $\mathrm{Tor}_1(R/I, R/J)$ *is cyclic*;
(b) $\mathrm{Tor}_1(R/I, R/J)$ *is decomposable*;
(c) $[\mathrm{Tor}_1(R/I, R/J)]^\vee$ *is not cyclic, where* $[\cdot]^\vee = \mathrm{Hom}([\cdot], E)$; *and*
(d) *the mixed characteristic p is not a zero divisor on R/J, in particular if J is a prime ideal.*

Proof. Let $I = (x_1, \dots, x_n)$, $J = (y_1, \dots, y_r, \lambda)$, $\underline{y} = (y_1, \dots, y_r)$, $S = R/\underline{y}$, and $A = R/J$. Then $\Omega = \mathrm{Hom}_S(S/\lambda S, S) = \mathrm{Hom}_S(A, S)$. Write $\xi_i = \mathrm{im}(x_i)$ in S;

then $\xi_1, \ldots, \xi_n$ is a system of parameters for the complete intersection S. Write $\underline{\xi} = (\xi_1, \ldots, \xi_n)$.

(a) By Theorem 1.6 we can assume that $I + J$ is not a complete intersection ideal in R. Consider the exact sequence

$$0 \longrightarrow S/\Omega \xrightarrow{f} S \longrightarrow S/\lambda S \longrightarrow 0.$$

Tensoring with R/I, we obtain the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^R(R/I, R/J) \longrightarrow S/(\Omega + \underline{\xi}) \xrightarrow{\bar{f}} S/\underline{\xi} \longrightarrow S/(\underline{\xi} + \lambda S) \longrightarrow 0.$$

If MC does not hold on $S/\lambda S$ with respect to the system of parameters $\xi_1, \ldots, \xi_n$, then $\Omega \subset (\xi_1, \ldots, \xi_n)$ and $\bar{f}$ boils down to multiplication by λ on $S/\underline{\xi}$. Hence

$$H_1(\underline{\xi}; S/\lambda S) = \operatorname{Tor}_1^R(R/I, R/J) = (0 : \lambda)_{S/\underline{\xi}} = E_{S/(\underline{\xi}+\lambda S)}(K),$$

where K is the residue field of S. Now consider the 0-dimensional complete intersection ring $B = R/(I + \underline{y}) = S/\underline{\xi}$. Write $\mu = \operatorname{Im}(\lambda)$ in B. Then $\Omega_{B/\mu B} = (0 : \mu)_B = (0 : \lambda)_{S/\underline{\xi}}$, where $B/\mu B = S/(\underline{\xi} + \lambda S)$. By assumption, $\operatorname{Tor}_1^R(R/I, R/J)$ is cyclic; hence $(0 : \lambda)_{S/\underline{\xi}}$ is cyclic. Since $\ell((0 : \lambda)_{S/\underline{\xi}}) = \ell(S/(\underline{\xi} + \lambda S))$, it follows that $(0 : \lambda)_{S/\underline{\xi}} \simeq S/(\underline{\xi} + \lambda s)$. Thus $\Omega_{B/\mu B} \simeq B/\mu B$ (i.e., $B/\mu B$ is Gorenstein). By a theorem of Kunz [21], this implies that $B/\mu B = S/(\underline{\xi} + \lambda S)$ is a complete intersection, which contradicts the fact that $I + J$ is not a complete intersection.

(b) As pointed out in part (a), if MC fails then $\operatorname{Tor}_1(R/I, R/J) = H_1(\underline{\xi}; A) = (0 : \lambda)_{S/\underline{\xi}}$, the injective hull of K over the local ring $S/(\underline{\xi} + \lambda S)$. Hence it follows that $\operatorname{Tor}_1(R/I, R/J)$ is indecomposable.

(c) If MC fails then $[\operatorname{Tor}_1^R(R/I, R/J)]^\vee = H_1(\underline{\xi}; A)^\vee = [(0 : \lambda)_{S/\underline{\xi}}]^\vee$ is a cyclic module, because $(0 : \lambda)_{S/\underline{\xi}}$ is the injective hull of K over the 0-dimensional local ring $S/(\underline{\xi} + \lambda S)$.

(d) We need the following lemma and the subsequent theorem on CEC in order to prove our assertion.

LEMMA. *Let A be an almost complete intersection ring—that is, let $A = S/\lambda S$ as before with S a complete intersection and* $\dim S = \dim A$. *Let x be a nonzero divisor on S and A. Then x is a nonzero divisor on* $\operatorname{Ext}_S^1(A, S)$.

Proof. Consider the short exact sequence

$$\theta \longrightarrow S \xrightarrow{x} S \xrightarrow{\eta} S/xS \longrightarrow 0.$$

Applying $\operatorname{Hom}_S(A, \cdot)$ we obtain the exact sequence

$$0 \longrightarrow \frac{\Omega}{x\Omega} \xrightarrow{\bar{\eta}} \Omega_{A/xA} \longrightarrow \operatorname{Ext}_S^1(A, S) \xrightarrow{x} \operatorname{Ext}_S^1(A, S) \longrightarrow \cdots,$$

where $\Omega = \operatorname{Hom}_S(A, S)$, $\Omega_{A/xA} = \operatorname{Hom}_S(A, S/xS) = \operatorname{Hom}_S(A/xA, S/xS)$.

Claim. The map $\bar{\eta}$ is onto.

Proof of Claim. Let $y \in S$ be such that $\lambda y \in xS$, and write $\lambda y = x\mu$. Since x is a nonzero divisor on $A = S/\lambda S$, we have $\mu = \lambda b$ for $b \in S$. Hence $\lambda y = x\mu = x\lambda b$; that is, $\lambda(y - xb) = 0$. Thus $y - xb \in \Omega$, and the claim follows. □

The proof of the lemma now follows from the preceding exact sequence. □

THEOREM. *Let A be a local ring of the form S/I, where S is a complete intersection, such that* $\dim S = \dim A$. *Let x be a nonzero divisor on A and on* $\operatorname{Ext}^1_S(A, S)$. *Then A satisfies CEC if and only if A/xA does so.*

The "if" part is due to Hochster [15], and the "only if" part is due to the first author [5]. Because the complete proof uses the dualizing complex, we refrain from giving it here and instead refer the reader to [5].

Since A/pA satisfies CEC [15], part (d) now follows from the lemma and theorem mentioned above. Hence our proof of Theorem 1.7 is complete.

COROLLARY. *Let A be an almost complete intersection ring, and let $x_1, \dots, x_n$ be a system of parameters for A. Then $x_1, \dots, x_n$ satisfies MC in the following cases*:

(i) $H_1(\underline{x}; A)$ *is cyclic*;
(ii) $H_1(\underline{x}; A)$ *is decomposable*;
(iii) $H_1(\underline{x}; A)^\vee$ *is not cyclic*; *and*
(iv) *p is not a zero divisor on A—in particular, A is an almost complete intersection domain.*

The proof is immediate from Theorem 1.7. Recall that we reduced the validity of MC over all local rings to its validity on almost complete intersection rings in the proof of Theorem 1.5 (cf. [7, Prop. 1.2]).

Section 2

As noted in the Introduction, Koh's result [19] provides an affirmative answer to the direct summand conjecture for the case $R \hookrightarrow A$ when A represents the integral closure of a pth-root extension of R. We begin with a general observation that allows one to conclude that a finite ring extension $A \hookrightarrow B$ of normal domains is A-split; in other words, the short exact sequence $e\colon 0 \to A \to B \to C \to 0$ is A-split exact where $C = B/A$.

PROPOSITION 2.1. *Suppose that $A \hookrightarrow B$ is a finite extension of integral domains for which A is local and integrally closed. Let $x \in m_A - \{0\}$, and consider the short exact sequence $e\colon 0 \to A \to B \to C \to 0$ and its class $[e] \in \operatorname{Ext}^1_A(C, A)$.*

(i) *If $A/xA \to B/xB$ is A-split* (*or, equivalently, A/xA-split*), *then* $[e] \in x\operatorname{Ext}^1_A(C, A)$.
(ii) *If B contains an A-free submodule F such that $A \subseteq F \subseteq B$ and $xB \subseteq F$, then* $x\operatorname{Ext}^1_A(C, \cdot) \equiv 0$.
(iii) *If the hypotheses of both* (i) *and* (ii) *hold simultaneously, then $A \hookrightarrow B$ is A-split.*

Proof. (i) Since A is integrally closed and since B is an integral domain, it follows that $C = B/A$ is necessarily a torsion-free A-module (i.e., a relation $xb = a$ means that b is in the fraction field of A and hence $b \in A$). So the element $x \in m_A - \{0\}$ is necessarily regular on C, whence the induced map $A/xA \to B/xB$ is an injective ring homomorphism. In addition, the short exact sequence $0 \to A \xrightarrow{x} A \to \bar{A} \to 0$ induces the "change of rings" long exact sequence (see the discussion in [12, p. 5]) on cohomology:

$$\begin{aligned} 0 &\to \operatorname{Hom}_A(C, A) \to \operatorname{Hom}_A(C, A) \to \operatorname{Hom}_{\bar{A}}(\bar{C}, \bar{A}) \\ &\xrightarrow{\delta} \operatorname{Ext}^1_A(C, A) \xrightarrow{x} \operatorname{Ext}^1_A(C, A) \to \operatorname{Ext}^1_{\bar{A}}(\bar{C}, \bar{A}) \to \cdots \end{aligned}$$

(here the overbar indicates "modulo x"). Under the assumption of part (i), we have that the class $[e]$ is sent to zero in $\operatorname{Ext}^1_{\bar{A}}(\bar{C}, \bar{A})$. It readily follows that $[e] \in x \operatorname{Ext}^1_A(C, A)$.

(ii) First we observe that $A \subseteq F$ is necessarily A-split because "1" cannot lie in $m_A F$; hence $G = F/A$ is an A-free submodule of C. Moreover, our hypothesis in (ii) guarantees $xC \subseteq G \subseteq C$. A standard argument in elementary homological algebra now shows that $x \operatorname{Ext}^1_A(C, \cdot) \equiv 0$.

(iii) The claim is a trivial consequence of (i) and (ii), since $[e] \in x \operatorname{Ext}^1_A(C, A)$ and $x \operatorname{Ext}^1_A(C, A) = 0$. □

In order to apply the preceding criteria in our proof of Koh's theorem, we must first set up some notation and a construction. Since Koh reduces his argument quickly to the case where R is a complete local ring, we assume that V is a complete DVR (discrete valuation ring) of mixed characteristic p in which p generates the maximal ideal in V. We set $R = V[[X_1, \ldots, X_n]]$ and consider a finite extension $R \hookrightarrow A$, where $A = R[u^{1/p}]'$. We intend to construct a free R-algebra S in A such that $pA \subseteq S$. Toward this end, let ζ be a primitive pth root of unity in a field extension of the fraction field of V, and let V' represent the integral closure of $V[\zeta]$. So V' is a complete DVR. Now ζ is a root of the polynomial $f(X) = 1 + X + X^2 + \cdots + X^{p-1} = (X^p - 1)/(X - 1)$. Because p is a prime element in V one obtains $f(X)$ is irreducible in $V[X]$ by noticing that $f(X+1) = [(X+1)^p - 1]/X$ is irreducible from Eisenstein's criteria. Therefore, $[V' : V] = p - 1$. We set $R' = V'[[X_1, \ldots, X_n]]$ and observe that R' is a complete regular local ring such that R' is R-free of rank $p - 1$. From the associated diagram

$$\begin{array}{ccc} L & \xrightarrow{p-1} & L' = K[u^{1/p}, \zeta] \\ {\scriptstyle p}\uparrow & & {\scriptstyle p}\uparrow \\ K & \xrightarrow[p-1]{} & K' = K[\zeta] \end{array}$$

of fraction fields, where K and L are the fraction fields of R and A, respectively, one sees that L'/K and L'/K' are Galois extensions. In particular, L'/K' is a Kummer extension. From the commutative diagram of fraction fields we get a corresponding diagram

$$\begin{tikzcd} A \arrow[r, "p-1"] & A' \\ R \arrow[u, "p"] \arrow[r, "p-1"'] & R' \arrow[u, "p"'] \end{tikzcd}$$

of finite ring extensions, where R and R' are regular local rings and A' is the integral closure of R in L'. We denote the Galois group of L'/K by G and the corresponding (necessarily) normal subgroup $H = \mathrm{Gal}(L'/K')$. Of course $H = \langle\sigma\rangle$ is a cyclic group of order p. As a K'-endomorphism of the K'-vector space, L' has an eigenspace decomposition $L' = L'_0 \oplus \cdots \oplus L'_{p-1}$ with respect to σ, where $L'_i = \{\ell' \in L' \mid \sigma(\ell') = \zeta^i \ell'\}$. Moreover, the contractions $S'_i = A' \cap L'_i$ are rank-1 R'-modules in A' such that A'/S'_i is R'-torsion free; thus the S'_i are isomorphic to R' for each i, since the S'_i must satisfy the Serre (S_2) condition and since R' is a UFD (unique factorization domain). We observe that $S'_0 = R'$ and that there is a natural ring structure on $S' = S'_0 \oplus \cdots \oplus S'_{p-1}$ when $x \in S'_i$ and $y \in S'_j$ have the property that $xy \in S'_k$ with $k = (i+j) \bmod p$.

In the next theorem we summarize properties and draw additional conclusions about the foregoing construction.

THEOREM 2.2. *The notation F, R', A, A', and S' represents the setup as described previously.*

(i) *The R'-subalgebra S' of A' is R'-free.*

(ii) *If $a' \in A'$ and $\omega = \zeta^i$ for $0 \le i < p$, then the Lagrange resolvant*

$$\Delta(a', \omega) = a' + \omega^{-1}\sigma(a') + \cdots + \omega^{-(p-1)}\sigma^{p-1}(a')$$

is an element of S'_i.

(iii) $pa' = \sum_{i=0}^{p-1} \Delta(a', \zeta^i) \in S'$.

(iv) *If $\tau \in G = \mathrm{Gal}(L'/K)$ then $\tau(S') \subseteq S'$.*

Proof. Part (i) was established in the discussion preceding the statement of Theorem 2.2. Part (ii) is a standard calculation of $\sigma(\Delta(a', \omega))$; observe that $\sigma(\omega x) = \omega\sigma(x)$ for $x \in A'$. To see part (iii) we note that $\Delta(a', \zeta^i) = \mathrm{tr}'(a')$ when $i = p$, where $\mathrm{tr}'\colon A' \to R'$ is the standard trace map. From the array of calculations

$$\begin{aligned}
\Delta(a', 1) &= a' + \sigma(a') + \sigma^2(a') + \cdots + \sigma^{p-1}(a'),\\
\Delta(a', \zeta) &= a' + \zeta^{-1}\sigma(a') + \zeta^{-2}\sigma^2(a') + \cdots + \zeta^{-(p-1)}\sigma^{p-1}(a'),\\
\Delta(a', \zeta^2) &= a' + \zeta^{-2}\sigma(a') + \zeta^{-4}\sigma^2(a') + \cdots,\\
&\ \ \vdots\\
\Delta(a', \zeta^{p-1}) &= a' + \zeta^{-(p-1)}\sigma(a') + \zeta^{-2(p-1)}\sigma^2(a') + \cdots,
\end{aligned}$$

one can see that the right-hand side sums to pa' because

$$1 + \omega + \omega^2 + \cdots + \omega^{p-1} = 0 \quad \text{for } \omega = \zeta^i,$$

where $0 \le i \le p-1$.

Finally, to justify part (iv) it suffices to argue that $\tau(s_i') \in S_j'$ for some j, where $s_i' \in S_i'$. We mention that the initial eigenspace decomposition for σ is similarly an eigenspace decomposition for each σ^k. The equations $\tau\sigma\tau^{-1}(\tau s_i') = \sigma^j(\tau s_i')$ and $\tau\sigma\tau^{-1}(\tau s_i') = \tau\sigma(s_i') = \tau(\zeta^i s_i') = \omega\tau(s_i')$, where $\omega = \tau(\zeta^i) = \zeta^m$ for some m, show that $\tau(s_i') \in S_k'$ for some k, since we have argued $\tau(s_i')$ is an eigenvector for σ^i. The observation here results from the facts that $\langle\sigma\rangle$ is normal in G and the $\{\zeta^i\}_{i=0}^{p-1}$ are all conjugate under the action of G.

THEOREM 2.3. *With notation as before, define S in A by $S = A \cap S'$. Then S is a free R-algebra in A such that $R \subseteq S$ and $pA \subseteq S$.*

Proof. Because S' is invariant as a set under the action of the entire Galois group G, we see that $S' \cap A = S$ is invariant under the subgroup that corresponds to A. Therefore, $\mathrm{tr}(S') = S$, where $\mathrm{tr}\colon A' \to A$ is the trace map. Since $[A' : A] = p-1$ represents a unit in R, we actually get that S is an S-direct summand of S'; thus S is a free R-module. Finally, we observe that $pA \subseteq S$ since $pA' \subseteq S'$. □

THEOREM 2.4 (Koh's theorem [22]). *Let notation be as before. Then $R \hookrightarrow A$ is necessarily R-split.*

Proof. Koh's result now follows from Proposition 2.1, where the R-free module F is taken to be the R-subalgebra S described in Theorem 2.3 and where the element x is taken to be $x = p$. We note that $R/pR \to A/pA$ is injective because R is integrally closed (see the proof of Theorem 2.1(i)) and that $R/pR \to A/pA$ is R/pR-split because the equicharacteristic case of the direct summand conjecture is known to be true (see Hochster's article [15]). □

REMARK. When $u^{1/p}$ is replaced by u^{1/p^n} one constructs, in the foregoing spirit, a Kummer extension L'/K'; in this case, ζ represents a primitive p^nth root of unity. Thus one obtains a free R'-subalgebra S' in the same way. However, technical problems arise when one contracts S' to A, since $[A' : A] = [R' : R] = \varphi(p^n)$ is divisible by p for $n > 1$. Moreover, one merely obtains that $p^nA' \subseteq S'$ and likewise $p^nA \subseteq S$ (one does not know whether $R/p^nR \to A/p^nA$ is split).

Section 3

As noted in the Introduction, a cornerstone for constructing a proof of the syzygy theorem as given in [10] or [12, pp. 58–59] (see also [2, pp. 370–371] for a more recent treatment) is the improved new intersection theorem (INIT). In fact, there is an important sequence of implications that one can derive from Hochster [18] and Evans–Griffith [12, pp. 56–58]:

theorem on canonical element

$\underset{\text{Hochster}}{\Longrightarrow}$ INIT

$\underset{\text{Evans–Griffith}}{\Longrightarrow}$ order ideal theorem for syzygies of finite projective dimension

$\underset{\text{Evans–Griffith}}{\Longrightarrow}$ syzygy theorem.

The various conjectures and theorems cited in this sequence of implications are described in precise terms in Theorems 3.1 and 3.3. Heitmann [14] established the direct summand theorem for local rings of dimension ≤ 3. At first glance one might guess that a low-dimensional result of this type would have little impact on the syzygy conjecture. But in fact the implication of Heitmann's result with respect to INIT allows us to establish a less obvious case of the syzygy conjecture; see Corollary 3.5.

We remind the reader of a few basic definitions and facts that are taken for the most part from [12]. Let M be a finitely generated module over a local ring (R, m_R). Suppose $\mathrm{pd}_R\, M < \infty$ (i.e., suppose M has finite projective dimension), and let $F_\bullet \to M$ represent a finite free resolution of M. Then $\operatorname{rank} M = \sum_i (-1)^i \operatorname{rank} F_i$. The ith kernel Z_i in $F_\bullet$ is called the ith syzygy module for M; the notation $\operatorname{syz} Z \geq i$ means that Z is at least an ith syzygy for some R-module. For $e \in M$ one defines the order ideal, $O_M(e)$, by

$$O_M(e) = \{f(e) \mid f \in \operatorname{Hom}_R(M, R)\}.$$

Observe that $e \in O_M(e)M$ when M is a free R-module, since $O_M(e)$ is generated by the coordinate projections evaluated at e in this case.

The usual statement of the improved new intersection theorem goes as follows.

Theorem 3.1 (INIT; see [17] or [12, Thm. 1.13]). *Let $F_\bullet$ be a finite free complex over the equicharacteristic local ring (R, m_R) such that:*

(i) length $H_i(F_\bullet) < \infty$ *for* $i > 0$; *and*

(ii) *there exists an* $e \in H_0(F_\bullet) - m_R H_0(F_\bullet)$ *with* $m_R^t e = 0$ *for* $t \gg 0$.

Then $\operatorname{length}(F_\bullet) \geq \dim R$.

Remark 3.2. Our application of INIT requires a slightly more special form. Namely, our complex will have the additional property that $F_\bullet$ is locally trivial on $X_R = \operatorname{Spec} R - m_R$; that is, $H_0(F_\bullet)$ is a locally free module on X_R. That for the validity of INIT it is enough to prove this special form was demonstrated in [3]. In this case, $\operatorname{length}(F_\bullet) \geq \dim R - 1$ even when R is not equicharacteristic (e.g., when R is of mixed characteristic p, where $\dim R/pR < \dim R$). This observation follows because $F_\bullet/pF_\bullet$ satisfies the conditions of Theorem 3.1 over R/pR.

Theorem 3.1 and Remark 3.2 allow us to prove the order ideal theorem for syzygy modules of finite projective dimension as stated next. (For a more general version see [12, Thm. 3.14].)

Theorem 3.3. *Let (R, m_R) be a catenary local ring of dimension $n > 0$ and suppose that INIT holds for all homomorphic images of R having dimension not exceeding $\ell + 1$, where $\ell > 0$. Suppose E is a finitely generated nonfree R-module that is locally free on X_R. If* $\operatorname{pd} E < \infty$ *and if $e \in E - m_R E$, then*

$$\operatorname{codim} O_E(e) \geq n - \ell$$

when $\operatorname{syz} E \geq n - \ell$.

Proof. The argument given here is much like those given in [10]. We suppose that E satisfies the required hypothesis as stated previously. The Auslander–Buchsbaum theorem [1] provides the inequalities

$$\operatorname{pd} E + \operatorname{syz} E \leq \operatorname{pd} E + \operatorname{depth} E \leq \dim R = n.$$

Therefore, $\operatorname{pd} E \leq n - (n - \ell) = \ell$. Let $F_\bullet \to E$ be a minimal R-free resolution of E. Since R is catenary one can show, for $I = O_E(e)$, that $\operatorname{codim} I \geq n - \ell$ by establishing $\ell \geq \dim R/I$. Base-changing to the factor ring $R/I = \bar{R}$ gives a finite free $\bar{R}$-complex $\bar{F}_\bullet = F_\bullet/IF_\bullet$ that satisfies the hypothesis of INIT (Theorem 3.1), since $\operatorname{length}(\operatorname{Tor}_i^R(R/I, E)) < \infty$ for $i > 0$ and since $\operatorname{Supp}(R\bar{e}) = \{m_R\}$, where $\bar{e} = e + IE$. Remark 3.2 applies in this context, so at worst one has $\ell \geq \operatorname{length}(\bar{F}_\bullet) \geq \dim \bar{R} - 1$ or (what is the same) $\ell + 1 \geq \dim R/I$. Thus, by our assumption that INIT holds for homomorphic images of R with dimension $\leq \ell + 1$, we conclude that $\ell \geq \dim R/I$. □

COROLLARY 3.4. *Let (R, m_R) be a local ring of dimension n. Let ℓ be a positive integer such that all homomorphic images of S with dimension $\leq \ell + 1$ have INIT, where S is the completion of any R-algebra essentially of finite type. If E is a nonfree* kth *syzygy of finite projective dimension, where $k \geq n - \ell$, then* $\operatorname{rank} E \geq k$.

Proof. Here our argument is similar to the one given in [11, pp. 7–10]. As before, we may assume that E is locally free on X_R (via localization) and that R is complete. There is no harm in assuming $\operatorname{syz} E = k = n - \ell$. By Theorem 3.3 we know that $\operatorname{codim} O_E(e) \geq n - \ell$. Therefore, if $n - \ell > \operatorname{rank} E$, then we contradict the lemma [11, p. 7] that claims there is a minimal generator e in E (after possibly a finite residue field extension) with $\operatorname{codim} O_E(e) \leq \operatorname{rank} E$.

Although the conditions of Corollary 3.4 appear rather technical, the direct summand result of Heitmann [14] for local rings of dimension 3—together with Hochster's result [17, Sec. 2] that the direct summand conjecture implies INIT—shows that the conditions of Corollary 3.4 are valid for $\ell = 2$ ("every" local ring of dimension $\leq 3 = 2 + 1$ has the property INIT). □

COROLLARY 3.5. *Let (R, m_R) be a local ring of dimension n. If E is a nonfree* kth *syzygy of finite projective dimension such that $k \geq n - 2$, then* $\operatorname{rank} E \geq k$.

Proof. Apply Corollary 2.4 with $\ell = 2$. □

COROLLARY 3.6. *The syzygy theorem holds for all regular local rings of dimension ≤ 5.*

Proof. The first serious case one must confront is that of $\operatorname{syz} E \geq 2$, for which $\operatorname{rank} E = 1$. However, such a module E is isomorphic to a reflexive ideal. Hence $E \cong R$ because R is a UFD. The remaining case of consequence is when $\operatorname{syz} E \geq 3$ and $\operatorname{rank} E = 2$, and this case is covered by Corollary 3.5. Any additional case would have $\operatorname{pd} E \leq 1$, which has been known since the initial statement of the problem (see [11]). □

As long as one restricts to modules of finite projective dimension, Corollary 3.6 holds also when one replaces the regular ring R by any integrally closed local domain (dimension $R \leq 5$).

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