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Complete Intersections on General Hypersurfaces

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1. Introduction

Many problems in classical projective geometry ask about the nature of special subvarieties of some given family of varieties. For example: How many isolated singular points can a surface of degree d in $\mathbb{P}^3$ have? When is it true that the members of a certain family of varieties contain a line or contain a linear space of any positive dimension? The reader can easily supply other examples of such questions.

This is the kind of problem we consider in this paper: What types of complete intersection varieties of codimension r in $\mathbb{P}^n$ can one find on the generic hypersurface of degree d?

In case $r = 2$ it was known to Severi [Se] that, for $n \geq 4$, the only complete intersections on a general hypersurface are obtained by intersecting that hypersurface with another.

This observation was extended to $\mathbb{P}^3$ by Noether (and Lefschetz) [Le; GrH] for general hypersurfaces of degree ≥ 4. These ideas were further generalized by Grothendieck [Gro].

Our approach to the question just posed uses a mix of projective geometry and commutative algebra and is much more elementary and accessible than, for example, the approach of Grothendieck. We are able to give a complete answer to the question we raised for complete intersections of codimension r in $\mathbb{P}^n$ that lie on a general hypersurface of degree d whenever $2r \leq n + 2$. In particular, we treat the case of complete intersections of small codimension. The case of complete intersection curves on hypersurfaces (i.e., complete intersection of small dimension) was treated and solved by Szabó in [Sz].

The paper is organized as follows. In Section 2 we lay out the question we want to consider and explain what are the interesting parameters for a response.

In Section 3 we collect some necessary technical information about varieties of reducible forms and their joins. In order to find the dimensions of these joins (using Terracini's lemma), we calculate the tangent space at a point of any variety of reducible forms. We also recall some information about Artinian complete intersection quotients of a polynomial ring.

In Section 4, we use the technical facts collected in Section 3 to reformulate our original question. We illustrate the utility of this reformulation to discuss complete

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intersections of codimension r in $\mathbb{P}^n$ on a general hypersurface when $2r < n+1$. We further use our approach to give a new proof for the existence of a line on the general hypersextic of $\mathbb{P}^5$.

In Section 5 we state and prove our main theorem, which gives a complete description of all complete intersections of codimension r in $\mathbb{P}^n$ that lie on a generic hypersurface when $2r \le n+2$.

2. Question

The objects of study of this paper are complete intersection subschemes of projective space. Recall that $Y \subset \mathbb{P}^n$ is a complete intersection scheme if its ideal is generated by a regular sequence; more precisely, $I(Y) = (F_1, \dots, F_r)$, $F_i \in S = \mathbb{C}[x_0, \dots, x_n]$, and $F_1, \dots, F_r$ form a regular sequence in S. If $\deg F_i = a_i$ for all i, we say that such a Y is a $\mathrm{CI}(a_1, \dots, a_r)$ and will assume that $a_1 \le \dots \le a_r$ (note that Y is unmixed of codimension r in $\mathbb{P}^n$). With this notation, we can rephrase the statement

the degree-d hypersurface X contains a $\mathrm{CI}(a_1, \dots, a_r)$

in terms of ideals as

$I(X) = (F) \subset (F_1, \dots, F_r)$ *for forms F_i forming a regular sequence and such that* $\deg F_i = a_i$ *for all i.*

Clearly, not all choices of the degrees are of interest for us; for example, if $a_i > d$ for all i, then no $\mathrm{CI}(a_1, \dots, a_r)$ can be found on a degree-d hypersurface. On the other hand, any hypersurface of degree d contains a $\mathrm{CI}(a_1, \dots, a_r)$ if $a_i = d$ for some i: simply cut that hypersurface with general hypersurfaces of degrees a_j, $j \ne i$.

So, one need only consider $\mathrm{CI}(a_1, \dots, a_r)$ where none of the $a_i = d$.

LEMMA 2.1. *Let $a_1 \le \dots \le a_i < d < a_{i+1} \le \dots \le a_r$ with $r \le n$. The following are equivalent facts*:

- *there is a* $\mathrm{CI}(a_1, \dots, a_r)$ *on the general hypersurface of degree d in $\mathbb{P}^n$*;
- *there is a* $\mathrm{CI}(a_1, \dots, a_i)$ *on the general hypersurface of degree d in $\mathbb{P}^n$.*

Proof. Let $I(X) = (F)$, where X is a general hypersurface of degree d in $\mathbb{P}^n$. Let $I(Y) = (F_1, \dots, F_r)$ be the ideal of a $\mathrm{CI}(a_1, \dots, a_r)$, with degrees a_i as before. Then $X \supset Y$ if and only if

$$F = \sum_{j=1}^{i} F_j G_j$$

and hence if and only if $X \supset Y'$, where Y' is the complete intersection defined by $F_1, \dots, F_i$. $\square$

From this lemma it is clear that the basic question to be considered is:

For which degrees $a_1, \dots, a_r < d$ does the generic degree-d hypersurface of $\mathbb{P}^n$ contain a $\mathrm{CI}(a_1, \dots, a_r)$? (Q)

If, rather than restricting to the generic case, we asked if *some* hypersurface of degree d contains a $\mathrm{CI}(a_1,\dots,a_r)$, then the answer is trivial. Indeed, the ideal of any $\mathrm{CI}(a_1,\dots,a_r)$ $(a_i < d)$ always contains degree-d elements.

3. Technical Facts

If $\lambda = (\lambda_1,\dots,\lambda_s)$ is a partition of the integer d $\big($i.e., $\sum_{i=1}^{s}\lambda_i = d$ and $\lambda_1 \geq \cdots \geq \lambda_s > 0\big)$, we write $\lambda \vdash d$. For each $\lambda \vdash d$, we define a subvariety $\mathbb{X}_\lambda \subset \mathbb{P}(S_d) \simeq \mathbb{P}^N$ $\big($where $N = \binom{d+n}{n} - 1\big)$ as follows:

$$\mathbb{X}_\lambda := \{[F] \in S_d \mid F = F_1 \cdots F_s,\ \deg F_i = \lambda_i\}.$$

We call $\mathbb{X}_\lambda$ the *variety of reducible forms of type* λ. The dimension of $\mathbb{X}_\lambda$ is easily seen to be $\big[\sum_{i=1}^{s}\binom{\lambda_i+n}{n}\big] - s$. (For other elementary properties of $\mathbb{X}_\lambda$ see [Mam], and for the special case $\lambda_1 = \cdots = \lambda_s = 1$ see [Ca1; Ca2] or [Chi] for the $n = 2$ case.)

If $x_1,\dots,x_r$ are independent points of $\mathbb{P}^N$, then we will call the $\mathbb{P}^{r-1}$ spanned by these points the *join of the points* $x_1,\dots,x_r$ and write

$$J(x_1,\dots,x_r) := \langle x_1,\dots,x_r\rangle.$$

More generally, if $X_1,\dots,X_r$ are varieties in $\mathbb{P}^N$ then the *join of* $X_1,\dots,X_r$ is

$$J(X_1,\dots,X_r) := \overline{\bigcup\{J(x_1,\dots,x_r) \mid x_i \in X_i,\ \{x_1,\dots,x_r\} \text{ independent}\}}.$$

If $X_1 = \cdots = X_r = X$ then we write

$$J(X_1,\dots,X_r) := \mathrm{Sec}_{r-1}(X)$$

and call this the $(r-1)$th *(higher) secant variety of* X.

Joins and secants of projective varieties are important auxiliary varieties that can help us better understand the geometry of the original varieties (see e.g. [Åd; C-J; CGGi1; CGGi2; ChCi; Ci; G; LaMa]). One of the most fundamental questions we can ask about joins and secants is: What are their dimensions?

This is, in general, an extremely difficult question to answer. The famous lemma of Terracini (our Lemma 3.1) is an important observation that will aid us in answering this question.

Lemma 3.1 (Terracini's lemma). *Let $X_1,\dots,X_r$ be reduced subvarieties of $\mathbb{P}^N$, and let $p \in J = J(X_1,\dots,X_r)$ be a generic point of J. Suppose that $p \in J(p_1,\dots,p_r)$. Then the (projectivized) tangent space to J at p (i.e., $T_p(J)$) can be written as*

$$T_p(J) = \langle T_{p_1}(X_1),\dots,T_{p_r}(X_r)\rangle.$$

Consequently,

$$\dim J = \dim\langle T_{p_1}(X_1),\dots,T_{p_r}(X_r)\rangle.$$

We want to apply this lemma in the case that the X_i are all of the form $\mathbb{X}_{\lambda^{(i)}}$, where $\lambda^{(i)} \vdash d$ for $i = 1,\dots,r$. A crucial first step in such an application is, therefore, a calculation of $T_{p_i}(\mathbb{X}_{\lambda^{(i)}})$ when $p_i \in \mathbb{X}_{\lambda^{(i)}}$.

PROPOSITION 3.2. *Let $\lambda \vdash d$ for $\lambda = (\lambda_1, \dots, \lambda_s)$ and let $p \in \mathbb{X}_\lambda$ be a generic point of $\mathbb{X}_\lambda$. Write $p = [F_1 \cdots F_s]$, where* $\deg F_i = \lambda_i$ $(i = 1, \dots, s)$, *and let $I_p \subset S = \mathbb{C}[x_0, \dots, x_n]$ be the ideal defined by*

$$I_p = (F_2 \cdots F_s,\ F_1 F_3 \cdots F_s,\ \dots,\ F_1 \cdots F_{s-1}).$$

Then the tangent space to $\mathbb{X}_\lambda$ at the point p is the projectivization of $(I_p)_d$ and hence has dimension

$$\dim T_p(\mathbb{X}_\lambda) = \dim_{\mathbb{C}}(I_p)_d - 1.$$

Proof. Consider the map of affine spaces

$$\Phi\colon S_{\lambda_1} \times \cdots \times S_{\lambda_s} \to S_d$$

defined by

$$\Phi((A_1, \dots, A_s)) = A_1 \cdots A_s.$$

Let $P \in S_{\lambda_1} \times \cdots \times S_{\lambda_s}$ be the point $P = (F_1, \dots, F_s)$. A tangent direction at P is given by any vector of the form $v = (F_1', \dots, F_s')$, and the line through P in that direction is

$$L_v := (F_1 + \mu F_1', \dots, F_s + \mu F_s'), \quad \mu \in \mathbb{C}.$$

A simple calculation shows that the tangent vector to $\Phi(L_v)$ at the point $\Phi(P) = p$ is exactly $\sum_{i=1}^{s} F_1 \cdots F_i' \cdots F_s$, and this proves the proposition. □

In view of Terracini's lemma, the following corollary is immediate.

COROLLARY 3.3. *Let $\lambda^{(1)}, \dots, \lambda^{(r)}$ be partitions of d, where*

$$\lambda^{(i)} = (\lambda_{i1}, \lambda_{i2}).$$

Let

$$I = (F_{11}, F_{12}, F_{21}, F_{22}, \dots, F_{r1}, F_{r2})$$

be an ideal of S generated by generic forms, where

$$\deg F_{ij} = \lambda_{ij} \quad \text{for } 1 \le i \le r,\ j = 1, 2.$$

If

$$J = J(\mathbb{X}_{\lambda^{(1)}}, \dots, \mathbb{X}_{\lambda^{(r)}}),$$

then

$$\dim J = \dim_{\mathbb{C}} I_d - 1.$$

REMARK 3.4. It is useful to note the following.

(i) In Proposition 3.2 we are using that $\mathbb{C}$ has characteristic 0. The problem is that the differential is not necessarily generically injective in characteristic p.

(ii) Observe that the generic point in $\mathbb{X}_\lambda$, $\lambda = (\lambda_1, \dots, \lambda_s) \vdash d$, can always be written as the product of s irreducible forms with the property that any ℓ-subset of these s forms $(\ell \le n + 1)$ is a regular sequence.

(iii) The previous observation can be extended easily to joins of varieties of reducible forms. In other words, the generic point in such a join can be written as a sum of elements with the property that each summand is a point that has the property described in (ii). Moreover, every ℓ-subset ($\ell \leq n+1$) of the set of all the irreducible factors of all these summands is also a regular sequence.

(iv) Fröberg [F] made a conjecture about the multiplicative structure of rings S/I, where $S = \mathbb{C}[x_0, \dots, x_n]$ and I is an ideal generated by a set of generic forms; this conjecture gives the Hilbert functions of such rings. However, apart from the cases of $n = 1$ (proved several times by various authors; see [F; GS; IK]) and $n = 2$ (proved in [An]), this conjecture has resisted all attempted proofs. Observe that, in terms of the geometric problem in Corollary 3.3, one need only consider Fröberg's conjecture for a strongly restricted collection of degrees.

We will need some specific information about the Hilbert function of some Artinian complete intersections in polynomial rings. The following lemma summarizes the facts we shall use.

LEMMA 3.5. *Let $r > 1$, and let $F_1, \dots, F_r$ and $G_1, \dots, G_r$ be generic forms in $\mathbb{C}[y_1, \dots, y_{2r-1}]$ with degrees*

$$1 < \deg F_1 = a_1 \leq \deg F_2 = a_2 \leq \cdots \leq \deg F_r = a_r \leq d/2$$

and

$$d/2 \leq \deg G_r = d - a_r \leq \cdots \leq \deg G_1 = d - a_1$$

for a nonnegative integer d.

Consider the quotient

$$A = \mathbb{C}[y_1, \dots, y_{2r-1}]/(F_1, \dots, F_r, G_r, \dots, G_3, G_2)$$

and its Hilbert function H_A. The following statements hold:

(i) *H_A is symmetric with respect to*

$$c = \frac{(r-1)d + a_1 - 2r + 1}{2};$$

(ii) *if $H_A(i) \geq H_A(i+1)$, then $H_A(j)$ is nonincreasing for $j \geq i$;*

(iii) *the multiplication map on A_i given by $\bar{G}_1$ (the class of G_1 in A) has maximal rank.*

Suppose that one of the following holds:

$$\begin{aligned} &r = 2 \quad \text{and} \quad a_1 \geq 5; \quad \text{or} \\ &r = 3 \quad \text{and} \quad a_1 \geq 3; \quad \text{or} \\ &r = 3 \quad \text{and} \quad a_1 = 2,\ d \neq 4; \quad \text{or} \\ &r > 3 \quad \text{and} \quad a_1 \geq 2. \end{aligned}$$

Then we also have:

(iv) *if $i \leq a_1$, then $H_A(i) < H_A(i+1)$;*

(v) *if $a_1 < i \leq c$, then $H_A(a_1) < H_A(i)$;*

(vi) *if $c < i$, then $H_A(a_1) > H_A(i)$ if and only if $c - a_1 < i - c$.*

Proof. Since A is a Gorenstein graded ring, (i) follows immediately; (iii) is a consequence of a theorem of Stanley [St] and Watanabe [W].

To prove (ii) we can use the weak Lefschetz property, where multiplication by a general linear form has maximal rank (see e.g. [MiM-R]). The condition on H_A together with the weak Lefschetz property yields that every element of A_{i+1} is the product of a fixed linear form and a form of degree i. Now consider an element M of A_{i+2}; then, since A is a standard graded algebra, it follows that $M = \sum_{i=1}^{2r-1} y_i C_i$, where y_i is the class of y_i in A and C_i is the class of a form of degree $i+1$. By what we have already seen, $C_i = LD_i$, where L is the form we had earlier and the D_i are forms of degree i. Rewriting yields $M = L\sum_{i=1}^{2r-1} y_i D_i$. But $\sum_{i=1}^{2r-1} y_i D_i$ is in A_{i+1}, so $A_{i+2} = LA_{i+1}$ and hence the dimension cannot increase. Proceeding by induction proves the statement.

As for (iv), it suffices to give the proof for $i = a_1$ because there are no generators of degree smaller than a_1. Let $\bar{A}$ be a quotient obtained when all the forms F_i and G_i have the same degree $a = a_1 = \cdots = a_r = d - a_r = \cdots = d - a_2$. Notice that it is enough to show the result for $\bar{A}$. In fact, whenever we pass from $\bar{A}$ to another quotient A by increasing the degrees of s forms, we obtain

$$\begin{aligned} H_A(a) &= H_{\bar{A}}(a) + s \quad \text{and} \\ H_{\bar{A}}(a+1) + s(2r-1) - s &\le H_A(a+1), \end{aligned} \tag{1}$$

and the inequality $H_{\bar{A}}(a) < H_{\bar{A}}(a+1)$ is preserved; these Hilbert function estimates use that the forms F_i and G_i do not have linear syzygies. Straightforward computations then yield

$$H_{\bar{A}}(a) = \binom{a+2r-2}{a} - 2r + 1,$$

$$H_{\bar{A}}(a+1) = \binom{a+2r-1}{a+1} - (2r-1)^2.$$

Hence the inequality $H_{\bar{A}}(a) < H_{\bar{A}}(a+1)$ is equivalent to

$$\binom{a+2r-2}{a+1} - (2r-1)(2r-2) > 0. \tag{2}$$

Observe that if (2) holds for the pair (a, r) then it holds for all the pairs $(a+i, r)$ with $i \ge 0$. By direct computation we verify that the inequality is satisfied for $(a, r) = (5, 2)$ and $(3, 3)$ as well as for $a = 2$ and $r > 3$. Consequently, (2) holds for:

$$\begin{aligned} &r = 2 \quad \text{and} \quad a \ge 5; \quad \text{or} \\ &r = 3 \quad \text{and} \quad a \ge 3; \quad \text{or} \\ &r > 3 \quad \text{and} \quad a \ge 2. \end{aligned}$$

To complete the proof of (iv) it is enough to evaluate (1) for $r = 3$ and $a = 2$ in the case $d \ne 4$ (i.e., $s > 0$).

To show (v), notice that (ii) implies that if

$$H_A(a_1) > H_A(i), \quad a_1 < i,$$

then H_A is definitely nonincreasing and hence, by (iv), it cannot be symmetric with respect to c.

To obtain (vi), it is enough to use symmetry and (v). $\square$

4. Equivalences

In this section we give some equivalent formulations of our basic question (Q), formulated at the end of Section 2.

Clearly, if $X \subset \mathbb{P}^n$ is a hypersurface of degree d and if $Y \subset X$ is a $\mathrm{CI}(a_1, \dots, a_r)$, then the ideal inclusion $I(X) = (F) \subset I(Y) = (F_1, \dots, F_r)$ yields

$$F = F_1 G_1 + \cdots + F_r G_r$$

for forms G_i of degrees $d - a_i$. However, the converse is not true in general. If $F = F_1 G_1 + \cdots + F_r G_r$ and the forms F_i do not form a regular sequence, then $(F_1, \dots, F_r)$ is not the ideal of a complete intersection. To produce an equivalence we need to use joins, as follows.

LEMMA 4.1. *The following statements are equivalent.*

(i) *A generic degree-d hypersurface of $\mathbb{P}^n$ contains a* $\mathrm{CI}(a_1, \dots, a_r)$, *where $a_i < d$ for all i.*

(ii) *The join of the varieties of reducible forms $\mathbb{X}_{(a_i, d-a_i)}$, $i = 1, \dots, r$, fills the space of degree-d forms in $n+1$ variables; that is,*

$$J(\mathbb{X}_{(a_1, d-a_1)}, \dots, \mathbb{X}_{(a_r, d-a_r)}) = \mathbb{P}(S_d).$$

Proof. The implication (i) $\Rightarrow$ (ii) simply follows from the foregoing ideal inclusion argument, which yields the presentation $F = \sum F_i G_i$ for the generic degree-d form, where $[F_i G_i] \in \mathbb{X}_{(a_i, d-a_i)}$ for all i. The implication (ii) $\Rightarrow$ (i) is easily shown using the description of the generic element of the join; see Remark 3.4(iii). $\square$

REMARK 4.2. Notice that there is an equality of varieties

$$\mathbb{X}_{(i,j)} = \mathbb{X}_{(j,i)}$$

for all nonnegative integers i and j. Hence, by Lemma 4.1, the condition

$$J(\mathbb{X}_{(a_1, d-a_1)}, \dots, \mathbb{X}_{(a_r, d-a_r)}) = \mathbb{P}(S_d)$$

is equivalent to the statement

> *a generic degree-d hypersurface of $\mathbb{P}^n$ contains a* $\mathrm{CI}(b_1, \dots, b_r)$,
> *where $b_i = a_i$ or $b_i = d - a_i$ for all i.*

It follows from these observations that we can further restrict the range of the degrees in our basic question (Q); in other words, it is enough to consider

$$a_1 \le \cdots \le a_r \le d/2.$$

Now we exploit Terracini's lemma and the tangent space description given in Corollary 3.3 in order to produce another equivalent formulation of question (Q).

LEMMA 4.3. *The following statements are equivalent.*

(i) *The generic degree-d hypersurface of* $\mathbb{P}^n$ *contains a* $\mathrm{CI}(a_1, \dots, a_r)$, *where* $a_i < d$ *for all* i.

(ii) *Let* $F_1, \dots, F_r$ *and* $G_1, \dots, G_r$ *be generic forms in* $S = \mathbb{C}[x_0, \dots, x_n]$ *of degrees* $a_1 \leq \cdots \leq a_r < d$ *and* $d - a_1, \dots, d - a_r$, *respectively*; *then*

$$H(S/(F_1, \dots, F_r, G_r, \dots, G_1), d) = 0,$$

where $H(\cdot, d)$ *denotes the Hilbert function in degree* d *of the ring.*

Proof. The condition on the join in Lemma 4.1 can be read, in terms of tangent spaces, as being equivalent to

$$\langle T_{P_1}(\mathbb{X}_{(a_1, d-a_1)}), \dots, T_{P_r}(\mathbb{X}_{(a_r, d-a_r)}) \rangle = \mathbb{P}(S_d)$$

for generic points $P_1 = [F_1 G_1]$, …, $P_r = [F_r G_r]$. Using the description of the tangent space to the variety of reducible forms, this is equivalent to saying that

$$(F_1, G_1)_d + \cdots + (F_r, G_r)_d = S_d,$$

where S_d is the degree-d piece of the polynomial ring S and where the forms F_i and G_i are generic of degrees a_i and $d - a_i$, respectively. □

As a straightforward application, we have the following result.

PROPOSITION 4.4. *If* $a_i < d$ *for all* i, *then the generic degree-d hypersurface of* $\mathbb{P}^n$ *contains no* $\mathrm{CI}(a_1, \dots, a_r)$ *when* $2r < n + 1$.

Proof. We shall use Lemma 4.3. In $S = \mathbb{C}[x_0, \dots, x_n]$, consider the generic forms $F_1, \dots, F_r$ and $G_1, \dots, G_r$ of degrees a_i and $d - a_i$, respectively. Letting I be the ideal $(F_1, \dots, F_r, G_1, \dots, G_r)$, we want to show that $H(S/I, d) \neq 0$; for this it is enough to show that S/I is not an Artinian ring. Since I has height $2r$ and since $2r < n + 1$, the quotient cannot be zero-dimensional and so the conclusion follows. □

REMARK 4.5. Using Lemma 4.3, we can also recover many classical results in an elegant and simple way. More precisely, we can easily study the existence of complete intersection curves (e.g., lines and conics) on hypersurfaces.

EXAMPLE 4.6. As an example we prove the following claim without using Schubert calculus.

Claim. The generic hypersextic of $\mathbb{P}^5$ *contains a line.*

Proof. Let $S = \mathbb{C}[x_0, \dots, x_5]$ and consider the ideal

$$I = (L_1, \dots, L_4, G_1, \dots, G_4),$$

where the forms L_i are linear forms and the forms G_i have degree 5. We want to show that $H(S/I, 6) = 0$. Clearly

$$S/I \simeq \mathbb{C}[x_0, x_1]/(\bar{G}_1, \dots, \bar{G}_4).$$

It is well known [F; GS; IK] that four general binary forms of degree 5 generate $\mathbb{C}[x_0, x_1]_6$, and we are done. □

For more on this topic, see Remark 5.5.

5. The Theorem

We are now ready to prove the main theorem of this paper, a description of all the possible complete intersections of codimension r that can be found on a general hypersurface of degree d in $\mathbb{P}^n$ when $2r \le n + 2$.

THEOREM 5.1. *Let $X \subset \mathbb{P}^n$ be a generic degree-d hypersurface with $n, d > 1$. Then X contains a $\mathrm{CI}(a_1, \dots, a_r)$, with $2r \le n + 2$ and the a_i all less than d, in the following (and only in the following) instances.*

- *$n = 2$: then $r = 2$, d is arbitrary, and a_1 and a_2 can assume any value less than d.*
- *$n = 3$, $r = 2$: for $d \le 3$ we have that a_1 and a_2 can assume any value less than d.*
- *$n = 4$, $r = 3$: for $d \le 5$ we have that a_1, a_2, and a_3 can assume any value less than d.*
- *$n = 6$, $r = 4$ or $n = 8$, $r = 5$: for $d \le 3$ we have that $a_1, \dots, a_r$ can assume any value less than d.*
- *$n = 5$ or 7 or $n > 8$, $2r = n + 1$ or $2r = n + 2$: we have only linear spaces on quadrics; that is, $d = 2$ and $a_1 = \dots = a_r = 1$.*

Proof. Given Lemma 2.1 and Remark 4.2, it is sufficient to consider the existence of a $\mathrm{CI}(a_1, \dots, a_r)$ on the generic hypersurface of degree d when $a_1 \le \dots \le a_r \le d/2$.

When $2r < n + 1$, by Proposition 4.4 we know that no complete intersection exists. Hence we need only consider the cases $2r = n + 1$ and $2r = n + 2$.

In order to use Lemma 4.3, we consider the generic forms $F_1, \dots, F_r$ and $G_1, \dots, G_r$ of degrees a_i and $d - a_i$, respectively. Putting $S = \mathbb{C}[x_0, \dots, x_n]$ and $I = (F_1, \dots, F_r, G_r, \dots, G_1)$, we want to check whether or not $H(S/I, d) = 0$.

If $2r = n + 1$, then S/I is an Artinian Gorenstein ring and $e = r(d - 2) + 1$ is the first place where one has $H(S/I, e) = 0$. Thus, the generic degree-d hypersurface contains a $\mathrm{CI}(a_1, \dots, a_r)$ if and only if $H(S/I, d) = 0$; this is equivalent to the inequality

$$d \ge r(d - 2) + 1,$$

which is never satisfied unless $d = 2$ and $a_1 = \dots = a_r = 1$.

The case $2r = n + 2$ will be proved using Lemma 3.5. In order to do this, we divide the proof into four parts as follows.

(i) *The hyperplane case*: $a_1 = 1$ for any r.
(ii) *The plane case*: $a_1 = 2, 3, 4$ for $r = 2$ and hence $n = 2$.
(iii) *The* 4-*space case*: $a_1 = a_2 = a_3 = 2$ and $d = 4$ for $r = 3$, so $n = 4$.
(iv) *The general case*:

$$\begin{aligned} &r = 2 \quad \text{and} \quad a_1 \geq 5; \quad \text{or} \\ &r = 3 \quad \text{and} \quad a_1 \geq 3; \quad \text{or} \\ &r = 3 \quad \text{and} \quad a_1 = 2 \quad \text{and} \quad d \neq 4; \quad \text{or} \\ &r > 3 \quad \text{and} \quad a_1 \geq 2. \end{aligned} \tag{3}$$

The hyperplane case. We need to study $\mathrm{CI}(1, a_2, \dots, a_r)$ on the generic degree-d hypersurface of $\mathbb{P}^{2r-2}$. Because one of the generators of the complete intersection is a hyperplane, we can reduce to a smaller-dimensional case. In algebraic terms, for a generic linear form L we consider the surjective quotient map

$$S \longrightarrow S/(L)$$

to obtain the following:

> *the generic element of S_d can be decomposed as a product of forms of degrees $1, a_2, \dots, a_r$; in other words, it has the form $\sum_{i=1}^r F_i G_i$ with* $\deg F_1 = 1$ *and* $\deg F_i = a_i$, $i = 2, \dots, r$,

if and only if

> *the generic element of $(S/(L))_d \simeq (\mathbb{C}[x_0, \dots, x_{n-1}])_d$ can be decomposed as a product of forms of degrees $a_2, \dots, a_r$; that is, it has the form $\sum_{i=2}^r \bar{F}_i \bar{G}_i$ with* $\deg \bar{F}_i = a_i$, $i = 2, \dots, r$.

Hence, we must study $\mathrm{CI}(a_2, \dots, a_r)$ on the generic degree-d hypersurface of $\mathbb{P}^{2r-3}$ (i.e., codimension $r' = r - 1$ complete intersections in $\mathbb{P}^{n'}$, $n' = 2r - 3$). That $2r' = n' + 1$ means this situation was treated before, and the only case where complete intersections exist is for $d = 2$ and $a_i = 1$ for all i.

The plane case. We must study $\mathrm{CI}(a_1, a_2)$ on the generic degree-d curve of $\mathbb{P}^2$ for $a_1 = 2, 3, 4$ and any a_2 and d such that $a_1 \leq a_2 \leq d$. Now, $S = \mathbb{C}[x_0, x_1, x_2]$ and we consider forms F_1, F_2, G_2, and G_1 of respective degrees a_1, a_2, $d - a_2$, and $d - a_1$. We want to study the ring

$$A = S/(F_1, F_2, G_2)$$

and to compare $H(A, a_1)$ and $H(A, d)$; this will enable us to apply Lemma 3.5(iii) to show that

$$H(A/(\bar{G}_1), d) = 0.$$

Using Lemma 3.5(i), we see that the last nonzero value of H_A occurs for

$$d + a_1 - 3.$$

In particular, for $a_1 = 2$ we have $H(A,d) = 0$ and that a $\mathrm{CI}(2,a_2)$ exists for any a_2 and d, $2 \le a_2 \le d$. If $a_1 = 3$, then $H(A,d) = 1$ and the same conclusion holds for $\mathrm{CI}(3,a_2)$. Finally, if $a_1 = 4$, then $H(A,d) = H(A,1)$ and it is easy to see that $H(A,1) \le H(A,4)$. Hence, for $a_1 = 2,3,4$ and any a_2, d such that $a_1 \le a_2 \le d$, the generic degree-d plane curve contains a $\mathrm{CI}(a_1,a_2)$.

The 4-*space case*. We address the case of $\mathrm{CI}(2,2,2)$ on the generic degree-4 3-fold in $\mathbb{P}^4$. Hence, we consider $S = \mathbb{C}[x_0,\dots,x_4]$ and generic quadratic forms F_1, F_2, F_3, G_3, G_2, and G_1. Let A be the quotient ring

$$S/(F_1,F_2,F_3,G_3,G_2)$$

and observe that, by the vanishing of the left-hand side of (2) in the proof of Lemma 3.5, we have $H(A,2) = H(A,3)$. Applying Lemma 3.5(ii) yields $H(A,2) = H(A,4)$ and hence the required complete intersection exists.

The general case. Consider the ring

$$A = \mathbb{C}[x_0,\dots,x_{2r-2}]/(F_1,\dots,F_r,G_r,\dots,G_3,G_2)$$

and the multiplication map given by the form $\bar{G}_1$ of degree $d - a_1$:

$$m\colon A_{a_1} \to A_d.$$

Clearly, with this notation one has that the generic degree-d hypersurface contains a $\mathrm{CI}(a_1,\dots,a_r)$ if and only if

$$H(S/I,d) = H(A/(\bar{G}_1),d) = 0,$$

and this is equivalent to the surjectivity of m. We also recall that, by Lemma 3.5(iii), m has maximal rank. Hence, to study the surjectivity we need only compare $H(A,a_1) = \dim A_{a_1}$ and $H(A,d) = \dim A_d$.

When $d = 2$, all the degrees a_i are equal to 1, and this was treated in the hyperplane case.

Now we consider the $d > 2$ case; as before, we let

$$c = \frac{(r-1)d + a_1 - 2r + 1}{2}.$$

If $d \le c$, then $\dim A_{a_1} < \dim A_d$ by Lemma 3.5(v) and so m cannot be surjective. Standard computations yield, for $r > 3$,

$$d \le c \iff 2d \le (r-1)d + a_1 - 2r + 1 \iff 2 + \frac{5-a_1}{r-3} \le d;$$

for $r = 2$, the inequality $d \le c$ never holds. Thus, when one of the conditions (3) holds, we have that m is not surjective if

$$r > 3 \quad \text{and} \quad d > 7$$

or

$$r = 3 \quad \text{and} \quad a_1 > 5.$$

In the case $c < d$ we need to be more careful, and the distances $\alpha = c - a_1$ and $\beta = d - c$ must be considered. When one of the conditions (3) holds, by 3.5(vi) it follows that m is surjective if and only if $\alpha \le \beta$. Thus we solve the inequality $\beta - \alpha \ge 0$. This is equivalent to

$$d \le 2 + \frac{3}{r-2} \quad \text{for } r > 2;$$

for $r = 2$, we always have $\beta - \alpha \ge 0$.

Summing up all these facts allows us to make the following statements when one of the conditions (3) holds.

$r = 2$: m is surjective.

$r = 3$: $d > 5$, m is not surjective; $d \le 5$, m is surjective.

$r = 4, 5$: $d > 3$, m is not surjective; $d \le 3$, m is surjective.

$r \ge 6$: $d > 2$, m is not surjective; $d \le 2$, m is surjective.

Then, using the previous treatment of the hyperplane, the plane, and the 4-space cases, we obtain the final result. □

REMARK 5.2. That a general hypersurface of degree $d \ge 6$ in $\mathbb{P}^4$ cannot contain a complete intersection of any type with $a_1, a_2, a_3 < d$ is also a consequence of a result concerning vector bundles that was proved by Mohan Kumar, Rao, and Ravindra [MoRRa].

In $\mathbb{P}^n$, the existence statement for $d = 2$ is classical. The $d = 3$ cases in $\mathbb{P}^6$ and $\mathbb{P}^8$ can be obtained using [H, Thm. 12.8] (see also Proposition 5.6 in this paper). In $\mathbb{P}^4$, for $d = 3$ and $d = 4$ the existence also follows from the analysis of arithmetically Cohen–Macaulay rank-2 bundles on hypersurfaces (see [ACo; Mad]).

In $\mathbb{P}^4$, for the case $d = 5$ and when $\min\{a_i\} = 2$, the result also follows from the existence of a canonical curve on the generic quintic 3-fold of $\mathbb{P}^4$; this was essentially proved in [Kl].

REMARK 5.3. For $n = 2$, Theorem 5.1 states that the generic degree-d plane curve contains a $\mathrm{CI}(a, b)$ for any $a, b < d$, but it does not say that this is a set of ab points. The complete intersection scheme could very well not be reduced. Actually, we can show reducedness and hence the following holds:

the generic degree-d plane curve contains ab complete intersection points for any $a, b < d$.

REMARK 5.4. In the case $n = 2$, if $a_1 = a_2 = a$ then Theorem 5.1 states that $\mathrm{Sec}_1(\mathbb{X}_{(a,d-a)})$ is the whole space. Now, quite generally, the points of the variety of secant lines either lie on a true secant line or on a tangent line to $\mathbb{X}_{(a,d-a)}$. We claim that Proposition 3.2 allows us to conclude that the points of the tangent lines are already on the true secant lines. In fact, if $p = [FG] \in \mathbb{X}_{(a,d-a)}$, then any point q of a tangent line to p can be written as $[\alpha FG' + \beta F'G]$ for forms G'

and F' of respective degrees $d-a$ and a and for scalars α and β. Thus, q lies on the secant line to $\mathbb{X}_{(a,d-a)}$ joining $[FG']$ and $[GF']$. In conclusion, we can rephrase the equality $\mathrm{Sec}_1(\mathbb{X}_{(a,d-a)}) = \mathbb{P}(S_d)$ in terms of polynomial decompositions as follows:

> *If $a < d$, then any degree-d form in three variables F can be written as $F = F_1G_1 + F_2G_2$ for suitable forms F_i of degree a and G_i of degree $d-a$.*

This answers a question raised during correspondence between Zinovy Reichstein and the first author.

REMARK 5.5. The restriction $2r \leq n+2$ in Theorem 5.1 is related to the fact that Fröberg's conjecture is known to be true only when the number of forms does not exceed one more than the number of variables. However, there are other partial results on this conjecture that we can use to extend our theorem. For example, Hochester and Laksov [HoL] showed that a piece of Fröberg's conjecture holds. More precisely, they showed that if an ideal is generated by generic forms of the same degree d then the size of that ideal in degree $d+1$ is exactly as predicted by Fröberg's conjecture. Using this, we can prove the following result.

PROPOSITION 5.6. *The generic hypersurface of degree $d > 2$ in $\mathbb{P}^n$ contains a complete intersection of type $(a_1, \ldots, a_r)$, where $a_i = 1$ or $a_i = d-1$ for all i, if and only if*

$$\binom{n-r+d}{d} \leq (n-r+1)r.$$

When $a_1 = \cdots = a_r = 1$, this is the well-known result on the nonemptyness of the Fano variety of $(n-r)$-planes on the generic degree-d hypersurface of $\mathbb{P}^n$ (see e.g. [H, Thm. 12.8]).

Proof of Proposition 5.6. Using Lemma 4.3, we must show the vanishing, in degree d, of the Hilbert function of

$$A = \mathbb{C}[x_0, \ldots, x_n]/(L_1, \ldots, L_r, F_1, \ldots, F_r)$$

for generic linear forms L_i and generic forms F_i of degree $d-1$. Clearly, since the linear forms are generic, it follows that

$$A \simeq \mathbb{C}[x_0, \ldots, x_{n-r}]/(\bar{F}_1, \ldots, \bar{F}_r).$$

Hence $A_d = 0$ if and only if $(\bar{F}_1, \ldots, \bar{F}_r)$ contains all the degree-d forms. Using the result of Hochester and Laksov, this is equivalent to

$$\binom{n-r+d}{d} \leq (n-r+1)r. \qquad \square$$

EXAMPLE 5.7. The variety $\mathbb{X}_{(1,3)}$ of reducible quartic hypersurfaces of $\mathbb{P}^3$ and its secant line variety provide interesting examples for several reasons.

First note that $\mathbb{X}_{(1,3)} \subset \mathbb{P}^{34}$ is a variety of dimension $3+19=22$. From Corollary 3.3 it is easy to deduce that $\dim \mathrm{Sec}_1(\mathbb{X}_{(1,3)}) = 33$. Thus, $\mathbb{X}_{(1,3)}$ is a defective variety whose virtual defect e is

$$e = 2\dim \mathbb{X}_{(1,3)} + 1 - \dim \mathrm{Sec}_1(\mathbb{X}_{(1,3)}) = 12.$$

(1) Consider the Noether–Lefschetz locus of quartic hypersurfaces in $\mathbb{P}^3$ with Picard group $\neq \mathbb{Z}$. The quartic hypersurfaces that contain a line are clearly in the Noether–Lefschetz locus. If ℓ is a line defined by the linear forms L_1, L_2, then the degree-4 form F defines a hypersurface containing ℓ if and only if

$$F = L_1G_1 + L_2G_2, \quad \text{where } \deg G_i = 3,$$

that is, if and only if $[F] \in \mathrm{Sec}_1(\mathbb{X}_{(1,3)})$. Because

$$\dim \mathrm{Sec}_1(\mathbb{X}_{(1,3)}) = 33$$

(as we have already observed), this forces the secant variety to be a component of the Noether–Lefschetz locus.

We wonder how often joins of other varieties of reducible forms give components of the appropriate Noether–Lefschetz locus.

(2) Since $\mathbb{X}_{(1,3)}$ is defective for secant lines, it follows by a theorem of [ChCi] that, for every two points on $\mathbb{X}_{(1,3)}$, there is a subvariety Σ containing those two points and whose linear span has dimension $\leq 2\dim \Sigma + 1 - e$, where e is the defect of $\mathbb{X}_{(1,3)}$. We now give a description of such Σs for the variety $\mathbb{X}_{(1,3)}$.

Let $[H_1F_1]$ and $[H_2F_2]$ be two points of $\mathbb{X}_{(1,3)}$, and let ℓ be the line in $\mathbb{P}^3$ defined by $H_1 = 0 = H_2$. Consider $\Sigma \subset \mathbb{X}_{(1,3)}$, the subvariety of reducible quartics whose linear components contain ℓ. Clearly, $\dim \Sigma = 1 + 19 = 20$. Now $\langle\Sigma\rangle$, the linear span of Σ, is contained in the subvariety of all quartics containing ℓ, and that variety has dimension $34 - 5 = 29$. Thus,

$$\dim\langle\Sigma\rangle \leq 29 = 2(20) + 1 - 12 = 2\dim \Sigma + 1 - e,$$

as we wanted to show.

Observe that the existence of Σ, as described here, gives another proof of the defectivity of $\mathbb{X}_{(1,3)}$.

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