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Gorenstein Algebras and Hochschild Cohomology

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To Mel Hochster on his 65th birthday

Introduction

Each one of the main classes of commutative Noetherian rings—regular, complete intersection, Gorenstein, and Cohen–Macaulay—is defined by *local* properties that require verification at *every* maximal ideal. It is therefore important to develop for these properties *global* recognition criteria involving only *finitely many* invariants. Finitely generated algebras over fields provide the test case. Our goal is to devise finitistic global tests applicable also in a more general, relative situation.

To fix notation, let K be a commutative Noetherian ring and $\sigma\colon K \to S$ a *flat* homomorphism of rings, which is *essentially of finite type*; σ is said to be Cohen–Macaulay or Gorenstein if its fiber rings have the corresponding property (see Section 2 for details). The following result is taken from Theorem 4.2; recall that $\operatorname{grade}_P S$ is the smallest integer n with $\operatorname{Ext}^n_P(S,P) \neq 0$.

Theorem 1. *Assume that* $\operatorname{Spec} S$ *is connected. Let* $K \to P \to S$ *be a factorization of* σ *with* P *a localization of a polynomial ring* $K[x_1,\dots,x_d]$ *and* S *a finite* P*-module.*

The map σ *is Cohen–Macaulay if for* $g = \operatorname{grade}_P S$ *one has*

$$\operatorname{Ext}^n_P(S,P) = 0 \quad \text{for } g < n \le g+d.$$

Conversely, if σ *is Cohen–Macaulay, then* $\operatorname{Ext}^n_P(S,P) = 0$ *holds for* $n \neq g$.

The homomorphism σ *is Gorenstein if and only if it is Cohen–Macaulay and the* S*-module* $\operatorname{Ext}^g_P(S,P)$ *is invertible.*

Thus, it is easy to recognize Cohen–Macaulay maps, because they are characterized in terms of *vanishing* of cohomology in a *finite* number of *specified* degrees; this can be decided from finite constructions. On the other hand, the condition needed to identify the Gorenstein property involves the *structure* of $\operatorname{Ext}^g_P(S,P)$ as a module over S, which is not determined by finitistic data (see Remark 5.4).

Partly motivated by recent characterizations of regular homomorphisms (recalled in Section 5), we approach the problem by studying the homological properties of S as a module over the *enveloping algebra* $S^{\mathrm{e}} = S \otimes_K S$, which acts on

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S through the *multiplication map* $\mu\colon S\otimes_K S\to S$. As usual, $\dim S$ denotes the *Krull dimension* of S. For a prime ideal $\mathfrak{q}$ of S we let $\operatorname{tr\,deg}_K(S/\mathfrak{q})$ denote the *transcendence degree* of the field of fractions of $S/\mathfrak{q}$ over that of the image of K.

The next theorem characterizes Gorenstein homomorphisms in terms of properties of Ext modules over $S\otimes_K S$, even without assuming that σ is Cohen–Macaulay.

THEOREM 2. *The homomorphism $\sigma\colon K\to S$ is Gorenstein if and only if the S-module $\bigoplus_{n=0}^{\infty}\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})$ is invertible.*

This result is contained in Theorem 2.4, whose proof hinges on properties of quasi-Gorenstein homomorphisms, defined and studied in [5], and on a strengthening of Foxby's criterion for finite G-dimension, obtained in Theorem 1.3. When the K-module S is *projective,* rather then just flat, the theorem can be stated in terms of *Hochschild cohomology* by using the isomorphisms of S-modules

$$\mathrm{HH}^n(S\,|\,K;\,N)\cong\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,N)\quad\text{for all }\ n\in\mathbb{Z}.$$

Two aspects of Theorem 2 present difficulties in applications: All the modules $\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})$ are involved; conditions other than vanishing are imposed on these modules. The next result identifies critical values of n.

THEOREM 3. *For every minimal prime ideal $\mathfrak{q}$ of S one has*

$$\operatorname{Ext}^t_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})_{\mathfrak{q}}\neq 0\quad\textit{with}\ \ t=\operatorname{tr\,deg}_K(S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}}).$$

If $\operatorname{Spec} S$ *is connected and σ is Cohen–Macaulay, then t is independent of $\mathfrak{q}$. If, moreover, σ is Gorenstein, then* $\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})=0$ *for $n\neq t$.*

Theorem 3 is abstracted from Theorems 3.1, 4.5, and 5.1. Their proofs hinge on an expression for the modules $\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})$ in terms of cohomology computed over the ring S. The relevant formula complements the classical technique of reduction to the diagonal; it is proved in [8] and is reproduced in Section 3.3.

We expect that the last statement of Theorem 3 admits a strong converse, as follows.

CONJECTURE. *When* $\operatorname{Spec} S$ *is connected, σ is Cohen–Macaulay, and one has*

$$\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})=0\quad\textit{for}\ \operatorname{tr\,deg}_K(S/\mathfrak{q})<n\le\operatorname{tr\,deg}_K(S/\mathfrak{q})+\max\{\dim S,1\},$$

then the homomorphism σ is Gorenstein.

When K is a field and $\operatorname{rank}_K S$ is finite one has $\dim S=\operatorname{tr\,deg}_K(S/\mathfrak{q})=0$; hence the preceding statement specializes to a conjecture of Asashiba (see [1, Sec. 3]), which strengthens the still open commutative case of a conjecture of Tachikawa: If $\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})=0$ holds for *every* $n\geq 1$, then S is self-injective (see [16, p. 115]). The next result is new even when K is a field and S is reduced; it is proved as Theorem 5.3.

THEOREM 4. *The previous conjecture holds when the ring K is Gorenstein and the ring $S_{\mathfrak{q}}$ is Gorenstein for every minimal prime ideal $\mathfrak{q}$ of S.*

The additional hypotheses allow one to apply a recent characterization of Gorenstein rings from [3] or [14] in order to prove that the ring S is Gorenstein. Classical properties of flat maps then imply that the homomorphism σ is Gorenstein.

1. Gorenstein Dimension

In this paper rings are commutative. For modules *finite* means finitely generated. The results in this section concern an invariant of complexes called G-dimension (for *Gorenstein* dimension), defined for finite modules by Auslander and Bridger [2].

Let R be a commutative ring. We write $\mathsf{D}(R)$ for the derived category of R-modules. Its objects are the complexes of R-modules of the form

$$M = \cdots \longrightarrow M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \longrightarrow \cdots .$$

We write $M \xrightarrow{\simeq} N$ to flag a *quasi-isomorphism*—that is, a morphism of complexes inducing an isomorphism in homology. The notation $M \simeq N$ means that M and N are linked by a chain of quasi-isomorphisms; that is, they are isomorphic in $\mathsf{D}(R)$.

A complex M is *homologically bounded* if $\mathrm{H}_i(M) = 0$ for $|i| \gg 0$; it is said to be *homologically finite* if, in addition, each R-module $\mathrm{H}_i(M)$ is finite.

1.1. Let R be a Noetherian ring.

An R-module G is said to be *totally reflexive* if it satisfies

$$G \cong \mathrm{Hom}_R(\mathrm{Hom}_R(G, R), R) \quad \text{and}$$
$$\mathrm{Ext}^n_R(G, R) = 0 = \mathrm{Ext}^n_R(\mathrm{Hom}_R(G, R), R) \quad \text{for all } n \geq 1.$$

The *G-dimension* of a homologically finite complex M of R-modules is the number

$$\text{G-dim}_R M = \inf_n\{n \geq \sup \mathrm{H}(M) \mid \mathrm{Coker}(\partial^P_{n+1}) \text{ is totally reflexive in some semiprojective resolution } P \xrightarrow{\simeq} M\}.$$

Finite projective modules are totally reflexive, so one has $\text{G-dim}_R M \leq \mathrm{pd}_R M$ (see [17, (3.4), (2.4.1)]). Every finite R-module (equivalently, homologically finite complex) has finite G-dimension if and only if the ring R is Gorenstein (see [2, (4.20)]).

1.2. Foxby [10, (2.2.3)] obtained an alternative characterization of complexes of finite G-dimension: A homologically finite complex of R-modules M has finite G-dimension if and only if the following two conditions hold.

(a) The canonical *biduality map* in $\mathsf{D}(R)$ is an isomorphism:

$$\delta_M \colon M \xrightarrow{\simeq} \mathsf{RHom}_R(\mathsf{RHom}_R(M, R), R).$$

(b) The complex $\mathsf{RHom}_R(M, R)$ is homologically bounded.

Moreover, when these conditions hold one has

$$\text{G-dim}_R M = \sup\{n \mid \mathrm{Ext}^n_R(M, R) \neq 0\}. \tag{1.2.1}$$

Thus, a lower bound for $\operatorname{G-dim}_R M$ is provided by the *grade* of M, defined by

$$\operatorname{grade}_R M = \inf\{n \mid \operatorname{Ext}^n_R(M, R) \neq 0\}. \tag{1.2.2}$$

When M is a module its grade is equal to the maximal length of an R-regular sequence contained in $\operatorname{Ann}_R M$, its annihilator ideal.

We prove a more flexible version of Foxby's characterization of finite Gorenstein dimension. We write $\operatorname{Max} R$ for the set of maximal ideals of R.

THEOREM 1.3. *Let R be a Noetherian ring and M a complex with* $\mathrm{H}(M)$ *finite.*
The complex M has finite G-dimension when the following conditions hold:

(a) *for each maximal ideal* $\mathfrak{m}$ *in* R, *in* $\mathsf{D}(R_{\mathfrak{m}})$ *there exists an isomorphism*

$$M_{\mathfrak{m}} \simeq \mathsf{RHom}_{R_{\mathfrak{m}}}(\mathsf{RHom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}), R_{\mathfrak{m}});$$

(b) *the complex* $\mathsf{RHom}_R(M, R)$ *is homologically bounded or* $\dim R$ *is finite.*

As a corollary we show that for certain modules the finiteness of G-dimension can be read off their cohomological invariants. For any ideal I of R, we set

$$\operatorname{big\,height} I = \max\{\operatorname{height}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ is minimal over } I\}.$$

COROLLARY 1.4. *Let M be a finite R-module.*
If for each $\mathfrak{m} \in \operatorname{Max} R$ *there exists an integer* $d(\mathfrak{m})$, *such that one has*

$$\operatorname{Ext}^n_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}) \cong \begin{cases} M_{\mathfrak{m}} & \text{for } n = d(\mathfrak{m}); \\ 0 & \text{for } n \neq d(\mathfrak{m}), \end{cases}$$

then the following inequalities hold:

$$\operatorname{G-dim}_R M \leq \operatorname{big\,height}(\operatorname{Ann}_R M) < \infty.$$

Proof. The hypothesis implies $\mathsf{RHom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}) \simeq \Sigma^{d(\mathfrak{m})} M_{\mathfrak{m}}$ and hence one has

$$\begin{aligned} \mathsf{RHom}_{R_{\mathfrak{m}}}(\mathsf{RHom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}), R_{\mathfrak{m}}) &\simeq \mathsf{RHom}_{R_{\mathfrak{m}}}(\Sigma^{d(\mathfrak{m})} M_{\mathfrak{m}}, R_{\mathfrak{m}}) \\ &\simeq \Sigma^{-d(\mathfrak{m})} \mathsf{RHom}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, R_{\mathfrak{m}}) \\ &\simeq M_{\mathfrak{m}} \end{aligned}$$

in the derived category $\mathsf{D}(R_{\mathfrak{m}})$. From (1.2.2) and Krull's Principal Ideal Theorem one obtains inequalities

$$d(\mathfrak{m}) = \operatorname{grade}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \leq \operatorname{height}(\operatorname{Ann}_{R_{\mathfrak{m}}} M_{\mathfrak{m}}) \leq \operatorname{big\,height}(\operatorname{Ann}_R M).$$

They imply $\operatorname{Ext}^n_R(M, R) = 0$ for $n > \operatorname{big\,height}(\operatorname{Ann}_R M)$, so $\mathrm{H}(\mathsf{RHom}_R(M, R))$ is bounded. Now one gets $\operatorname{G-dim}_R M < \infty$ from Theorem 1.3, and then $\operatorname{G-dim}_R M \leq \sup\{d(\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Max} R\}$ by (1.2.1). $\square$

The proof of the theorem uses the following result, which is of independent interest.

PROPOSITION 1.5. *Let R be a local ring and X a complex with each R-module $\mathrm{H}_i(X)$ finite. If $\mathrm{H}_i(\mathsf{R}\mathrm{Hom}_R(X,R)) = 0$ for $|i| \gg 0$, then $\mathrm{H}_i(X) = 0$ for $i \ll 0$.*

Proof. Let k be the residue field of R. As $\mathrm{H}_i(\mathsf{R}\mathrm{Hom}_R(X,R)) = 0$ for $i \gg 0$, one has

$$\mathrm{H}_n(\mathsf{R}\mathrm{Hom}_R(k, \mathsf{R}\mathrm{Hom}_R(X,R))) = 0 \quad \text{for } n \gg 0.$$

On the other hand, adjunctions yield the first two isomorphisms below:

$$\begin{aligned}\mathrm{H}_n(\mathsf{R}\mathrm{Hom}_R(k, \mathsf{R}\mathrm{Hom}_R(X,R))) &\cong \mathrm{H}_n(\mathsf{R}\mathrm{Hom}_R(k \otimes^{\mathsf{L}}_R X, R)) \\ &\cong \mathrm{H}_n(\mathsf{R}\mathrm{Hom}_k(k \otimes^{\mathsf{L}}_R X, \mathsf{R}\mathrm{Hom}_R(k,R))) \\ &\cong \mathrm{Hom}_k(\mathrm{H}(k \otimes^{\mathsf{L}}_R X), \mathrm{Ext}_R(k,R))_n \\ &= \prod_{j+i=-n} \mathrm{Hom}_k(\mathrm{H}_i(k \otimes^{\mathsf{L}}_R X), \mathrm{Ext}^j_R(k,R)).\end{aligned}$$

As $\mathrm{Ext}_R(k,R) \neq 0$, it follows that one has $\mathrm{H}_i(k \otimes^{\mathsf{L}}_R X) = 0$ for $i \ll 0$. At this point, we can conclude that $\mathrm{H}_i(X) = 0$ holds for $i \ll 0$ by invoking [12, (4.5)]. What follows is a direct and elementary proof, which exploits an idea from [11, (5.12)].

The first step is to verify that every complex C with $\mathrm{H}(C)$ of finite length has $\mathrm{H}_i(C \otimes_R X) = 0$ for $i \ll 0$. In the case where C is a module, this is achieved by an obvious induction on its length. The general case is settled by induction on the number of nonzero homology modules of C.

Let now C be the Koszul complex on a subset $\boldsymbol{x} = \{x_1, \ldots, x_e\}$ of R. Recall that C is equal to $C' \otimes_R C''$, where C' and C'' are Koszul complexes on $\{x_1, \ldots, x_{e-1}\}$ and x_e, respectively. Thus, one has an exact sequence of complexes

$$0 \longrightarrow C' \otimes_R X \longrightarrow C \otimes_R X \longrightarrow \Sigma(C' \otimes_R X) \longrightarrow 0.$$

Its homology exact sequence yields exact sequences of R-modules

$$0 \longrightarrow \mathrm{H}_i(C' \otimes_R X)/x_e\mathrm{H}_i(C' \otimes_R X) \longrightarrow \mathrm{H}_i(C \otimes_R X) \longrightarrow \mathrm{H}_{i-1}(C' \otimes_R X).$$

By induction on e, one deduces that each R-module $\mathrm{H}_i(C \otimes_R X)$ is finite.

For the final step, choose $\boldsymbol{x}$ to generate the maximal ideal of R. The length $\mathrm{H}(C)$ then is finite, so the first step yields $\mathrm{H}_i(C \otimes_R X) = 0$ for $i \ll 0$. The preceding exact sequence then gives $\mathrm{H}_n(C' \otimes_R X)/x_e\mathrm{H}_n(C' \otimes_R X) = 0$ for $i \ll 0$, which implies $\mathrm{H}_i(C' \otimes_R X) = 0$ for $i \ll 0$, by Nakayama's lemma. Splitting off one element x_j at a time, we arrive at $\mathrm{H}_i(X) = 0$ for $i \ll 0$, as desired. □

Proof of Theorem 1.3. Set $(-)^* = \mathsf{R}\mathrm{Hom}_R(-,R)$. We start with the local case.

Claim. Assume that R is a local ring. If $\mathrm{H}(M)$ is finite and there exits an isomorphism $\mu\colon M \to M^{**}$ in $\mathsf{D}(R)$, then $\text{G-dim}_R M$ is finite.

Indeed, as $\mathrm{H}(M)$ is finite, one has $\mathrm{H}_i(M^*) = 0$ for $i \gg 0$, and the R-module $\mathrm{H}_i(M^*)$ is finite for each i. The isomorphism μ and Proposition 1.5 yield $\mathrm{H}_i(M^*) = 0$ for $i \ll 0$. It thus remains to prove that the biduality morphism $\delta_M\colon M \to M^{**}$ is an isomorphism (see Section 1.2).

The composition $(\delta_M)^*\delta_{M^*}$ is the identity map of M^*, so for each $n \in \mathbb{Z}$ the map $\mathrm{H}_n(\delta_{M^*})\colon \mathrm{H}_n(M^*) \to \mathrm{H}_n(M^{***})$ is a split monomorphism. On the other hand, μ induces an isomorphism $\mathrm{H}_n(M^{***}) \cong \mathrm{H}_n(M^*)$ for each $n \in \mathbb{Z}$. As the R-module $\mathrm{H}_n(M^*) = \mathrm{Ext}_R^{-n}(M, R)$ is finite, we conclude that $\mathrm{H}_n(\delta_{M^*})$ is bijective. Thus, δ_{M^*} is an isomorphism in $\mathsf{D}(R)$ and hence so is $\delta_{M^{**}}$. Since the square

$$
\begin{tikzcd}
M \arrow[r, "\mu", "\simeq"'] \arrow[d, "\delta_M"'] & M^{**} \arrow[d, "\delta_{M^{**}}", "\simeq"'] \\
M^{**} \arrow[r, "\mu^{**}", "\simeq"'] & M^{****}
\end{tikzcd}
$$

in $\mathsf{D}(R)$ commutes, we see that δ_M is an isomorphism, as desired. This completes the proof of the claim.

At this point, we can conclude that when condition (a) holds, the number $\text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m}$ is finite for each $\mathfrak{m} \in \operatorname{Max} R$. When $\mathsf{RHom}_R(M, R)$ is homologically bounded, it is homologically finite, so one has $(\delta_M)_\mathfrak{m} = \delta_{M_\mathfrak{m}}$ for each $\mathfrak{m} \in \operatorname{Max} R$. As $\delta_{M_\mathfrak{m}}$ is an isomorphism, so is δ_M, that is to say, $\text{G-dim}_R M$ is finite (see Section 1.2).

It remains to prove the theorem when $\dim R$ is finite. Since $\text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m}$ is finite for each $\mathfrak{m} \in \operatorname{Max} R$, from (1.2.1) we get the second equality below:

$$
\begin{aligned}
-\inf \mathrm{H}(\mathsf{RHom}_R(M, R))_\mathfrak{m} &= -\inf \mathrm{H}(\mathsf{RHom}_{R_\mathfrak{m}}(M_\mathfrak{m}, R_\mathfrak{m})) \\
&= \text{G-dim}_{R_\mathfrak{m}} M_\mathfrak{m} \\
&= \operatorname{depth} R_\mathfrak{m} - \operatorname{depth}_{R_\mathfrak{m}} M_\mathfrak{m} \\
&\le \dim R_\mathfrak{m} + \sup \mathrm{H}(M_\mathfrak{m}) \\
&\le \dim R + \sup \mathrm{H}(M).
\end{aligned}
$$

The third one is the Auslander–Bridger equality [10, (2.3.13)] with depth for complexes defined as in [10, Sec. A.6]. The first inequality follows from [10, (A.6.1.1)], and the other inequality is evident. Thus, $\mathrm{H}_i(\mathsf{RHom}_R(M, R))$ vanishes for $i \ll 0$, so condition (a) holds and $\text{G-dim}_R M$ is finite. This completes the proof of the theorem. □

2. Quasi-Gorenstein Homomorphisms

In this section $\sigma\colon K \to S$ denotes a homomorphism of rings.

One says that σ is (*essentially*) *of finite type* if it can be factored as $\sigma = \pi\varkappa$, where $\varkappa$ is the canonical map to a (localization of a) polynomial ring in finitely many indeterminates, and π is a surjective homomorphism of rings.

A homomorphism of rings is *local* if its source and target are local rings, and it maps the unique maximal ideal of the source into that of the target. The *localization* of σ at a prime ideal $\mathfrak{n}$ of S is the induced local homomorphism $\sigma_\mathfrak{n}\colon K_{\mathfrak{n}\cap K} \to S_\mathfrak{n}$.

We say that σ is *Gorenstein at* some $\mathfrak{n} \in \operatorname{Spec} S$ if $\sigma_\mathfrak{n}$ is flat and the local ring $S_\mathfrak{n}/(\mathfrak{n} \cap K)S_\mathfrak{n}$ is Gorenstein; σ is *Gorenstein* when this holds for every $\mathfrak{n} \in \operatorname{Spec} S$.

Recall that φ is called (*essentially*) *smooth* if it is (essentially) of finite type, flat, and with geometrically regular fibers. Such maps are evidently Gorenstein.

A *Gorenstein-by-finite factorization* of σ is an equality $\sigma = \pi\varkappa$, where $\varkappa$ and π are homomorphisms of rings with $\varkappa$ Gorenstein and π finite. In a similar vein, we speak of (essentially) smooth-by-finite factorizations, et cetera. Each homomorphism (essentially) of finite type has (essentially) smooth-by-surjective decompositions.

Finally, we recall some notions and results from [5].

2.1. Let σ be a homomorphism essentially of finite type. We say that it has *finite G-dimension* at some $\mathfrak{n} \in \operatorname{Spec} S$, and write $\operatorname{G-dim} \sigma_{\mathfrak{n}} < \infty$, if for some Gorenstein-by-surjective factorization $K_{\mathfrak{n}\cap K} \to P' \to S_{\mathfrak{n}}$ of $\sigma_{\mathfrak{n}}$ one has the inequality $\operatorname{G-dim}_{P'} S_{\mathfrak{n}} < \infty$. This property does not depend on the choice of factorization (see [5, (4.3)]). It holds, for instance, when $\sigma_{\mathfrak{n}}$ is flat or when $K_{\mathfrak{m}\cap K}$ is Gorenstein (see [5, (4.4.1), (4.4.2)]).

When $\operatorname{G-dim}_{P'} S_{\mathfrak{n}}$ is finite the complex $\mathsf{R}\mathrm{Hom}_{P'}(S_{\mathfrak{n}}, P')$ does not depend on factorizations, up to isomorphism and shift in $\mathsf{D}(S_{\mathfrak{n}})$ (see [5, (6.6), (6.7), (5.5)]).

We say that σ is *quasi-Gorenstein* at $\mathfrak{n}$ if it has finite G-dimension at $\mathfrak{n}$, and $\mathsf{R}\mathrm{Hom}_{P'}(S_{\mathfrak{n}}, P')$ is isomorphic in $\mathsf{D}(S_{\mathfrak{n}})$ to some shift of $S_{\mathfrak{n}}$. When this holds at each $\mathfrak{n} \in \operatorname{Spec} S$ we say that σ is *quasi-Gorenstein* (see [5, (7.8)]). When $\sigma_{\mathfrak{n}}$ is flat, it is quasi-Gorenstein at $\mathfrak{n}$ if and only if it is Gorenstein at $\mathfrak{n}$ (see [5, (8.1)]).

We give new characterizations of (quasi-)Gorenstein homomorphisms.

THEOREM 2.2. *Let K be a Noetherian ring, $\sigma\colon K \to S$ a homomorphism of rings, and $K \to P \to S$ a Gorenstein-by-finite factorization.*

The homomorphism σ is quasi-Gorenstein if and only if the graded S-module $\operatorname{Ext}_P(S, P)$ *is invertible; when it is, one has* $\operatorname{G-dim}_P S \le \operatorname{big\,height}(\operatorname{Ann}_P S)$.

The notion of invertible graded module is explained further in what follows.

COROLLARY 2.3. *When* Spec S *is connected, the map σ is quasi-Gorenstein if and only if the S-module* $\operatorname{Ext}_P^n(S, P)$ *is invertible for $n =$* $\operatorname{grade}_P S$ *and* $\operatorname{Ext}_P^n(S, P) = 0$ *for $n \neq$* $\operatorname{grade}_P S$.

Set $S^{\mathsf{e}} = S \otimes_K S$ and let $\mu\colon S^{\mathsf{e}} \to S$ denote the multiplication map, $\mu(a \otimes b) = ab$.

THEOREM 2.4. *Let K be a Noetherian ring and let $\sigma\colon K \to S$ be a flat, essentially of finite type homomorphism of rings. The following conditions are then equivalent.*

- (i) *The homomorphism σ is Gorenstein.*
- (i′) *The homomorphism $S \otimes_K \sigma\colon S \to S^{\mathsf{e}}$ is Gorenstein at each* $\mathfrak{m} \in \operatorname{Supp}_{S^{\mathsf{e}}}(S)$.
- (ii) *The homomorphism $\mu\colon S^{\mathsf{e}} \to S$ is quasi-Gorenstein.*
- (iii) *The S^{e}-module S has finite G-dimension.*
- (iv) *The graded S-module* $\operatorname{Ext}_{S^{\mathsf{e}}}(S, S^{\mathsf{e}})$ *is invertible.*

The preceding results are proved at the end of this section, following some technical preparation. Condition 2.4(iv) is refined in Theorem 5.1.

Let L be a finite S-module. Recall that L is projective if and only if for each $\mathfrak{n} \in \operatorname{Spec} S$ the $S_{\mathfrak{n}}$-module $L_{\mathfrak{n}}$ is free. For such a module L the function $\mathfrak{n} \mapsto \operatorname{rank}_{S_{\mathfrak{n}}} L_{\mathfrak{n}}$ on $\operatorname{Spec} S$ is upper semicontinuous and hence constant on each connected component. When $\operatorname{rank}_{S_{\mathfrak{n}}} L_{\mathfrak{n}} = d$ for each $\mathfrak{p} \in \operatorname{Spec} S$ one says that L has *rank d*. Projective modules of rank 1 are called *invertible* modules.

We say that a graded module $(E^n)_{n\in\mathbb{Z}}$ is projective (resp. invertible) if the module $\bigoplus_{n\in\mathbb{Z}} E^n$ is projective (resp. invertible).

LEMMA 2.5. *Let S be a Noetherian ring.*

A graded S-module $(E^n)_{n\in\mathbb{Z}}$ is invertible if and only if E^n is finite over S for each $n \in \mathbb{Z}$ and $E^n_{\mathfrak{n}} \cong S_{\mathfrak{n}}$ holds for all $\mathfrak{n} \in \operatorname{Max} S \cap \operatorname{Supp} E^n$.

When this is the case one has $S = \bigoplus_{i=1}^{q} J_i$ with $J_i = \operatorname{Ann}_S\big(\bigoplus_{j\neq i} E^{n_j}\big)$, where $\{n_1, \dots, n_q\} = \{n \in \mathbb{Z} \mid E^n \neq 0\}$.

Proof. The "only if" part is clear. For the converse, only the finiteness of the S-module $E = \bigoplus_{n\in\mathbb{Z}} E^n$ is at issue. Let $\mathfrak{q}$ be a prime ideal of S and $\mathfrak{n}$ a maximal ideal containing $\mathfrak{q}$. Since $S_{\mathfrak{q}}$ is indecomposable as a module over itself, the isomorphisms

$$\bigoplus_{n\in\mathbb{Z}} E^n_{\mathfrak{q}} \cong E_{\mathfrak{q}} \cong (E_{\mathfrak{n}})_{\mathfrak{q}S_{\mathfrak{n}}} \cong S_{\mathfrak{q}}$$

provide a unique integer $n(\mathfrak{q})$ with the property $E_{\mathfrak{q}} = E^{n(\mathfrak{n})}_{\mathfrak{q}}$. They also imply an equality $n(\mathfrak{q}) = n(\mathfrak{n})$, so the function $\mathfrak{q} \mapsto n(\mathfrak{q})$ is constant on each connected component of $\operatorname{Spec} S$. One has $E^n = 0$ unless n is equal to $n(\mathfrak{q})$ for some $\mathfrak{q}$, so $E = \bigoplus_{n\in\mathbb{Z}} E^n$ has only finitely many nonzero summands.

The finite decomposition of E produces a disjoint union

$$\operatorname{Spec} S = \bigsqcup_{i=1}^{q} \operatorname{Supp}_S E^{n_i}.$$

It yields $S = \bigoplus_{i=1}^{q} J_i$ because one has $\operatorname{Supp}_S E^{n_i} = \{\mathfrak{q} \in \operatorname{Spec} S \mid \mathfrak{q} \supseteq J_i\}$. □

REMARK 2.6. If X is a complex of S-modules with $\mathrm{H}(X)$ graded projective, then in $\mathsf{D}(S)$ there is an isomorphism $X \simeq \mathrm{H}(X)$.

Indeed, using the projectivity of $\mathrm{H}(X)$, choose a section of the canonical homomorphism of graded S-modules $\mathrm{Z}(X) \to \mathrm{H}(X)$. Composing this section with the inclusion $\mathrm{Z}(X) \to X$ one gets a quasi-isomorphism of complexes $\mathrm{H}(X) \to X$.

Proof of Theorem 2.2. Since P is a finite P-module, $\operatorname{Ext}^n_P(S, P)$ is finite over S for each $i \in \mathbb{Z}$ and for every $\mathfrak{n} \in \operatorname{Max} S$ one has isomorphisms of a graded S-module

$$\mathrm{H}(\mathsf{R}\mathrm{Hom}_{P_{\mathfrak{n}\cap P}}(S_{\mathfrak{n}}, P_{\mathfrak{n}\cap P})) \cong \operatorname{Ext}_{P_{\mathfrak{n}\cap P}}(S_{\mathfrak{n}}, P_{\mathfrak{n}\cap P}) \cong \operatorname{Ext}_P(S, P)_{\mathfrak{n}}.$$

Thus, if σ is quasi-Gorenstein, then $\operatorname{Ext}_P(S, P)$ is invertible by Lemma 2.5.

When $\operatorname{Ext}_P(S, P)$ is invertible one has $\mathsf{R}\mathrm{Hom}_P(S, P) \simeq \operatorname{Ext}_P(S, P)$ in $\mathsf{D}(S)$ (see Remark 2.6). For every $\mathfrak{n} \in \operatorname{Spec} S$ this yields an integer $d(\mathfrak{n})$ and an isomorphism $\mathsf{R}\mathrm{Hom}_{P_{\mathfrak{n}\cap P}}(S_{\mathfrak{n}}, P_{\mathfrak{n}\cap P}) \simeq \Sigma^{d(\mathfrak{n})} S_{\mathfrak{n}}$ in $\mathsf{D}(S_{\mathfrak{n}})$. Corollary 1.4 implies that

$\operatorname{G-dim}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$ is finite, so $P \to S$ is quasi-Gorenstein at $\mathfrak{n}$ (see Section 2.1). The same corollary also yields $\operatorname{G-dim}_P S \leq \operatorname{big\,height}(\operatorname{Ann}_P S)$. As $K \to P$ is Gorenstein, it is also quasi-Gorenstein, hence so is σ (see [5, (8.9)]). □

Proof of Corollary 2.3. One has $\operatorname{Ext}_P^g(S, P) \neq 0$ for $g = \operatorname{grade}_P S$, by definition. Thus, when $\operatorname{Spec} S$ is connected, the graded S-module $\operatorname{Ext}_P(S, P)$ is invertible if and only if $\operatorname{Ext}_P^n(S, P)$ is invertible for $n = g$ and zero otherwise (see Lemma 2.5). The desired result now follows from Theorem 2.2. □

For $\mathfrak{n} \in \operatorname{Spec} S$, we write $k(\mathfrak{n})$ for the field of fractions of $S/\mathfrak{n}$.

Proof of Theorem 2.4. Set $Q = S^{\mathrm{e}}$ and $\psi = S \otimes_K \sigma : S \to Q$. This map is flat along with σ, and hence so is $\psi_{\mathfrak{m}}$ for each $\mathfrak{m} \in \operatorname{Spec} Q$; in particular, $\operatorname{G-dim} \psi_{\mathfrak{m}}$ is finite.

(i) $\Rightarrow$ (i′). Setting $\mathfrak{n} = \mathfrak{m} \cap S$, note the isomorphism of rings

$$k(\mathfrak{n}) \otimes_S Q \cong k(\mathfrak{n}) \otimes_{k(\mathfrak{n}\cap K)} (k(\mathfrak{n} \cap K) \otimes_K S).$$

It shows that if the ring $k(\mathfrak{n} \cap K) \otimes_K S$ is Gorenstein, then so is $k(\mathfrak{n}) \otimes_S Q$, hence also $(k(\mathfrak{n}) \otimes_S Q)_{\mathfrak{n}}$, which is isomorphic to $k(\mathfrak{n}) \otimes_{S_{\mathfrak{n}}} Q_{\mathfrak{m}}$.

(i′) $\Rightarrow$ (i). Every prime ideal $\mathfrak{n} \in \operatorname{Spec} S$ is the contraction of the prime ideal $\mathfrak{m} = \mu^{-1}(\mathfrak{n})$ of S^{e}, which contains $\operatorname{Ker}(\mu)$. Thus, when (i′) holds the local homomorphism $\psi_{\mathfrak{m}} : S_{\mathfrak{n}} \to Q_{\mathfrak{m}}$ is flat with Gorenstein closed fiber. The proof of [4, (6.6)] shows that then so is the local homomorphism $\sigma_{\mathfrak{n}} : K_{\mathfrak{n}\cap K} \to S_{\mathfrak{n}}$.

(iv) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). Apply Theorem 2.2 to $K = P = S^{\mathrm{e}}$ and $\varkappa = \mathrm{id}^{S^{\mathrm{e}}}$.

(iii) $\Rightarrow$ (i′) $\Rightarrow$ (ii). These assertions follow from the Decomposition Theorem [4, (8.10)] applied to the evidently quasi-Gorenstein composition $\mu\psi = \mathrm{id}^S$. □

3. Bigrade of a Homomorphism

Invariants provided by Hochschild cohomology reflect the structure of an algebra $\sigma : K \to S$ as a *bimodule* over itself. We define the *bigrade* of σ by the formula

$$\operatorname{bigrade}(\sigma) = \inf\{n \in \mathbb{Z} \mid \operatorname{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \neq 0\}.$$

Applications of this invariant are given in the following sections. Here we examine its formal properties and compare it to other invariants of the K-algebra S. For each $\mathfrak{n} \in \operatorname{Spec} S$ we define the *residual transcendence degree* of σ at $\mathfrak{n}$ as the number

$$\operatorname{tr\,deg}_\sigma k(\mathfrak{n}) = \operatorname{tr\,deg}_{k(\mathfrak{n}\cap K)} k(\mathfrak{n}).$$

When $\varkappa : K \to P$ is an essentially smooth homomorphism of commutative rings the P-module of Kähler differentials $\Omega_{P|K}$ is finite and projective; see [13, Sec. 16.10]. We say that $\varkappa$ has *relative dimension d* if this projective module has rank d (see Section 2). An example is given by the canonical map $K \to U^{-1}K[x_1, \ldots, x_d]$, where $x_1, \ldots, x_d$ are indeterminates and U is any multiplicatively closed set. Thus, every homomorphism (essentially) of finite type has an (essentially) smooth-by-surjective factorization of finite relative dimension.

THEOREM 3.1. *Let K be a Noetherian ring, $\sigma\colon K \to S$ a flat homomorphism, and $K \xrightarrow{\varkappa} P \to S$ an essentially smooth-by-finite factorization of relative dimension d.*

(1) *For every minimal prime ideal $\mathfrak{q}$ of S there are (in)equalities*

$$0 \le d - \operatorname{pd}_P S \le \operatorname{bigrade}(\sigma) \le \operatorname{bigrade}(\sigma_{\mathfrak{q}}) = \operatorname{tr}\deg_\sigma k(\mathfrak{q}) \le d.$$

(2) *The following conditions are equivalent*:
  (i) $\operatorname{bigrade}(\sigma) = d - \operatorname{pd}_P S$;
  (ii) $\operatorname{Ass} S \cap \operatorname{Supp}_S \operatorname{Ext}_P^p(S, P) \neq \emptyset$ *for* $p = \operatorname{pd}_P S$.

(3) *When S and P are integral domains one has*

$$\operatorname{tr}\deg_\sigma(0) = d - \operatorname{grade}_P S.$$

The hypotheses and notation of the theorem stay in force for the rest of this section. Its proof is presented following that of Lemma 3.9.

The first inequality in Theorem 3.1(1) is a consequence of the following general result, which tracks homological dimensions along smooth homomorphisms.

3.2. For every finite S-module M one has (in)equalities

$$\operatorname{fd}_K M \le \operatorname{pd}_P M \le \operatorname{fd}_K M + d.$$

See [8, 5.6] for a proof. In particular, $\operatorname{pd}_P M$ and $\operatorname{fd}_K M$ are finite simultaneously.

The next result, proved in [8, 5.1], is a critical ingredient in later arguments.

3.3. For every $n \in \mathbb{Z}$ one has an isomorphism of S-modules

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \cong \operatorname{Ext}^{n-d}_S\big(\mathsf{R}\operatorname{Hom}_P\big(S, \textstyle\bigwedge^d_P \Omega_{P|K}\big), S\big). \tag{3.3.1}$$

The next remark is useful in applications of the reduction formula (3.3.1).

3.4. The following subsets of $\operatorname{Spec} S$ are equal:

$$\operatorname{Supp}_S \operatorname{Ext}^n_P\big(S, \textstyle\bigwedge^d_P \Omega_{P|k}\big) = \operatorname{Supp}_S \operatorname{Ext}^n_P(S, P). \tag{3.4.1}$$

Indeed, set $V = \bigwedge^d_P \Omega_{P|k}$. Since the P-module V is invertible, for each $n \in \mathbb{Z}$ and every $\mathfrak{n} \in \operatorname{Spec} S$ there are isomorphisms of $S_{\mathfrak{n}}$-modules

$$\begin{aligned}
\operatorname{Ext}^n_P\big(S, \textstyle\bigwedge^d_P \Omega_{P|k}\big)_{\mathfrak{n}} &\cong \operatorname{Ext}^n_P\big(S, \textstyle\bigwedge^d_P \Omega_{P|k}\big) \otimes_S S_{\mathfrak{n}} \\
&\cong \textstyle\bigwedge^d_P \Omega_{P|k} \otimes_P \operatorname{Ext}^n_P(S, P) \otimes_S S_{\mathfrak{n}} \\
&\cong \textstyle\bigwedge^d_P \Omega_{P|k} \otimes_P (\operatorname{Ext}^n_P(S, P)_{\mathfrak{n}}) \\
&\cong \big(\textstyle\bigwedge^d_P \Omega_{P|k}\big)_{\mathfrak{n}\cap P} \otimes_{P_{\mathfrak{n}\cap P}} \operatorname{Ext}^n_P(S, P)_{\mathfrak{n}} \\
&\cong \operatorname{Ext}^n_P(S, P)_{\mathfrak{n}}.
\end{aligned} \tag{3.4.2}$$

LEMMA 3.5. *There is an inequality* $\operatorname{bigrade}(\sigma) \ge d - \operatorname{pd}_P S$. *Equality holds if and only if* $\operatorname{Ass} S \cap \operatorname{Supp}_S \operatorname{Ext}^p_P(S, P) \neq \emptyset$.

Proof. Set $D = \mathsf{RHom}_P\big(S, \bigwedge_P^d \Omega_{P|k}\big)$ and $C = \mathrm{H}_{-p}(D)$, where $p = \mathrm{pd}_P\, S$.

From (3.4.1) one gets $\mathrm{H}_n(D) = 0$ for $n < -p$. This implies the second isomorphism of S-modules below, while formula (3.3.1) gives the first:

$$\mathrm{Ext}^{n+d}_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \cong \mathrm{Ext}^n_S(D, S) \cong \begin{cases} 0 & \text{for } n < -p, \\ \mathrm{Hom}_S(C, S) & \text{for } n = -p. \end{cases}$$

These isomorphisms yield $\mathrm{bigrade}(\sigma) \geq d - p$ and show that equality is equivalent to $\mathrm{Hom}_S(C, S) \neq 0$. Referring to a standard formula and to (3.4.1) we obtain

$$\mathrm{Ass}_S \mathrm{Hom}_S(C, S) = \mathrm{Ass}\, S \cap \mathrm{Supp}_S\, C = \mathrm{Ass}\, S \cap \mathrm{Supp}_S \mathrm{Ext}^p_P(S, P).$$

Thus, $\mathrm{Hom}_S(C, S) \neq 0$ is equivalent to $\mathrm{Ass}\, S \cap \mathrm{Supp}_S \mathrm{Ext}^p_P(S, P) \neq \emptyset$. □

Before continuing, we recall another canonical isomorphism.

3.6. Fix a prime ideal $\mathfrak{n}$ of S and set $(S_{\mathfrak{n}})^{\mathrm{e}} = S_{\mathfrak{n}} \otimes_{K_{\mathfrak{n}\cap K}} S_{\mathfrak{n}}$. For each $n \in \mathbb{Z}$ there is an isomorphism of $S_{\mathfrak{n}}$-modules

$$\mathrm{Ext}^n_{(S_{\mathfrak{n}})^{\mathrm{e}}}(S_{\mathfrak{n}}, (S_{\mathfrak{n}})^{\mathrm{e}}) \cong \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})_{\mathfrak{n}}.$$

Indeed, let $\lambda\colon S \to S_{\mathfrak{n}}$ and $\kappa\colon K \to K_{\mathfrak{n}\cap K}$ denote the localization maps. The homomorphism of rings $\lambda \otimes_\kappa \lambda\colon S^{\mathrm{e}} \to (S_{\mathfrak{n}})^{\mathrm{e}}$ is flat, and there is an isomorphism $S_{\mathfrak{n}} \cong (S_{\mathfrak{n}})^{\mathrm{e}} \otimes_{S^{\mathrm{e}}} S$ of $(S_{\mathfrak{n}})^{\mathrm{e}}$-modules, whence the first isomorphism below:

$$\begin{aligned} \mathrm{Ext}^n_{(S_{\mathfrak{n}})^{\mathrm{e}}}(S_{\mathfrak{n}}, (S_{\mathfrak{n}})^{\mathrm{e}}) &\cong \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \otimes_{S^{\mathrm{e}}} (S_{\mathfrak{n}})^{\mathrm{e}} \\ &\cong \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \otimes_S S_{\mathfrak{n}}. \end{aligned}$$

For the second one note that S^{e} acts on $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})$ through S.

Lemma 3.7. *The following equality holds*:

$$\mathrm{bigrade}(\sigma) = \inf\{\mathrm{bigrade}(\sigma_{\mathfrak{n}}) \mid \mathfrak{n} \in \mathrm{Spec}\, S\}.$$

Proof. Set $g = \mathrm{bigrade}(\sigma)$. From the isomorphisms in Section 3.6 one reads off an inequality $\mathrm{bigrade}(\sigma_{\mathfrak{n}}) \geq g$, which becomes an equality when $\mathfrak{n}$ is in $\mathrm{Supp}_S \mathrm{Ext}^g_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})$. □

Lemma 3.8. *For each prime ideal* $\mathfrak{m}$ *in* P, *one has* $\mathrm{tr\,deg}_\varkappa\, k(\mathfrak{m}) \leq d$. *Equality holds when* P *is a domain and* $\mathfrak{m} = (0)$.

Proof. Set $k = k(\mathfrak{m} \cap K)$ and $P' = (k \otimes_K P)_{\mathfrak{m}}$.

The composed homomorphism $\varkappa'\colon K \to k \otimes_K P \to P'$ is essentially of finite type. Its fibers are among those of $\varkappa$, so $\varkappa'$ is essentially smooth and the canonical isomorphism $\Omega_{P'|k} \cong (k \otimes_K \Omega_{P|K})_{\mathfrak{m}}$ shows that it has relative dimension d.

The surjection $P' \to k(\mathfrak{m})$ induces a surjection $\omega\colon k(\mathfrak{m}) \otimes_{P'} \Omega_{P'|k} \to \Omega_{k(\mathfrak{m})|k}$. This gives the second inequality below, and [15, (26.10)] provides the first one:

$$\mathrm{tr\,deg}_k\, k(\mathfrak{m}) \leq \mathrm{rank}_{k(\mathfrak{m})}\, \Omega_{k(\mathfrak{m})|k} \leq d.$$

When P is a domain and $\mathfrak{m} = (0)$ one has $P' = k(\mathfrak{m})$. In particular, ω is an isomorphism, and thus the second inequality above becomes an equality. The first inequality also does, as the homomorphism $k \to k(\mathfrak{m})$ is essentially smooth. $\square$

Let Min S denote the set of minimal prime ideals of Spec S.

LEMMA 3.9. *For all prime ideals* $\mathfrak{q} \in \operatorname{Min} S$ *and* $\mathfrak{n} \in \operatorname{Spec} S$ *with* $\mathfrak{q} \subseteq \mathfrak{n}$ *one has*

$$\operatorname{bigrade}(\sigma) \leq \operatorname{bigrade}(\sigma_{\mathfrak{n}}) \leq \operatorname{bigrade}(\sigma_{\mathfrak{q}}) = \operatorname{tr}\deg_{\sigma} k(\mathfrak{q}) \leq d.$$

Proof. Both inequalities on the left come from Lemma 3.7, because one has $\sigma_{\mathfrak{q}} = (\sigma_{\mathfrak{n}})_{\mathfrak{q}S_{\mathfrak{n}}}$.

Set $\mathfrak{p} = \mathfrak{q} \cap K$. The rings $S_{\mathfrak{q}}$ and $K_{\mathfrak{p}}$ are Artinian, the first one because the ideal $\mathfrak{q}$ is minimal, the second because $\sigma_{\mathfrak{q}}\colon K_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is a flat local homomorphism.

Set $k = k(\mathfrak{p})$, $l = k(\mathfrak{q})$, and $t = \operatorname{tr}\deg_k l$. Choose in $S_{\mathfrak{q}}$ elements $y_1, \dots, y_t$ that map to a transcendence basis of l over k. Let $x_1, \dots, x_t$ be indeterminates over $K_{\mathfrak{p}}$ and Q the localization of $K_{\mathfrak{p}}[x_1, \dots, x_t]$ at the prime ideal $\mathfrak{p}K_{\mathfrak{p}}[x_1, \dots, x_t]$; this is a local ring with maximal ideal $\mathfrak{p}Q$ and residue field $k' = k(x_1, \dots, x_t)$.

The homomorphism of $K_{\mathfrak{p}}$-algebras $K_{\mathfrak{p}}[x_1, \dots, x_t] \to S_{\mathfrak{q}}$ sending x_i to y_i for $i = 1, \dots, t$ induces a local homomorphism $\varphi\colon Q \to S_{\mathfrak{q}}$. A length count yields

$$\operatorname{length}_Q(S_{\mathfrak{q}}) = \operatorname{length}_{S_{\mathfrak{q}}}(S_{\mathfrak{q}}) \operatorname{length}_{k'}(l) < \infty,$$

so φ is finite. Let κ denote the composition $K_{\mathfrak{p}} \to K_{\mathfrak{p}}[x_1, \dots, x_t] \to Q$. It is local, flat, and essentially of finite type, and the fiber $Q \otimes_{K_{\mathfrak{p}}} k$ is equal to k'.

One has $\sigma_{\mathfrak{q}} = \varphi\kappa$, so this is a smooth-by-finite factorization of relative dimension t by the foregoing discussion. The finite Q-module $S_{\mathfrak{q}}$ has finite projective dimension by Lemma 3.2, so it is free because Q is Artinian. By the same token, one has $\operatorname{Supp}_{S_{\mathfrak{q}}} \operatorname{Ext}^0_Q(S_{\mathfrak{p}}, Q) = \mathfrak{q}S_{\mathfrak{q}} = \operatorname{Ass} S_{\mathfrak{q}}$, so Lemma 3.5 yields $\operatorname{bigrade}(\sigma_{\mathfrak{q}}) = t$.

Finally, set $\mathfrak{m} = \mathfrak{q} \cap Q$. As the field extension $k(\mathfrak{m}) \subseteq k(\mathfrak{q})$ is finite, one gets $t = \operatorname{tr}\deg_k k(\mathfrak{m})$. On the other hand, Lemma 3.8 yields $t \leq d$. $\square$

Proof of Theorem 3.1. (1) The first inequality comes from Lemma 3.2, the second from Lemma 3.5, the remaining relations from Lemma 3.9.

(2) The desired assertion is part of Lemma 3.5.

(3) Assume that S and P are integral domains and set $\mathfrak{m} = \operatorname{Ker}(P \to S)$. For each $\mathfrak{n} \in \operatorname{Spec} S$, the projective dimension of $S_{\mathfrak{n}}$ over $P_{\mathfrak{n} \cap P}$ is finite (see Section 3.2). Thus [6, (2.5)] yields $\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \dim P_{\mathfrak{m}}$, so one obtains

$$\operatorname{grade}_P S = \inf\{\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} \mid \mathfrak{n} \in \operatorname{Spec} S\} = \dim P_{\mathfrak{m}}.$$

Set $S' = S_{(0)}$ and $K' = K_{(0S)\cap K}$. As S' is the residue field of $P_{\mathfrak{m}}$, and $K' \to S'$ is a flat local homomorphism, one sees that K' is a field. The local domain $P_{\mathfrak{m}}$ is the localization of some finitely generated K'-algebra P' at a prime ideal $\mathfrak{m}'$, so one has

$$\dim P_{\mathfrak{m}} = \operatorname{height}(\mathfrak{m}') = \dim P' - \dim(P'/\mathfrak{m}') = \operatorname{tr}\deg_{K'} P'_{(0S)\cap P'} - \operatorname{tr}\deg_{K'} S'.$$

To finish the proof, note that Lemma 3.8 yields $d = \operatorname{tr}\deg_{K'} P'$. $\square$

Formal properties of Hochschild cohomology have implications for bigrade, as the following remarks show.

REMARK 3.10. For any homomorphism of rings $K \to K'$ we identify K' and $K \otimes_K K'$ via the canonical isomorphism, set $S' = S \otimes_K K'$, and note that $\sigma \otimes_K K' \colon K' \to S'$ is (essentially) of finite type, or flat, along with σ. Also, set $S'^{\mathrm{e}} = S' \otimes_{K'} S'$.

When $K \to K'$ is flat so is $S^{\mathrm{e}} \to S'^{\mathrm{e}}$, owing to the canonical isomorphism of K'-algebras $S'^{\mathrm{e}} \cong S^{\mathrm{e}} \otimes_K K'$. Thus, for each $n \in \mathbb{Z}$ one gets isomorphisms

$$\mathrm{Ext}^n_{S'^{\mathrm{e}}}(S', S'^{\mathrm{e}}) \cong \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S'^{\mathrm{e}}) \cong \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \otimes_S S'.$$

As a consequence, for every flat homomorphism $K \to K'$ one obtains

$$\mathrm{bigrade}(\sigma) \leq \mathrm{bigrade}(\sigma \otimes_K K');$$

equality holds when K' is faithfully flat over K.

REMARK 3.11. Let $\tau \colon K \to T$ be a flat homomorphism essentially of finite type. For $R = S \times T$, and for each $n \in \mathbb{Z}$ there are canonical isomorphisms

$$\begin{aligned}
\mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R^{\mathrm{e}}) &\cong (S \otimes_R \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R^{\mathrm{e}})) \oplus (T \otimes_R \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R^{\mathrm{e}})),\\
S \otimes_R \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R^{\mathrm{e}}) &\cong \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}),\\
T \otimes_R \mathrm{Ext}^n_{R^{\mathrm{e}}}(R, R^{\mathrm{e}}) &\cong \mathrm{Ext}^n_{T^{\mathrm{e}}}(T, T^{\mathrm{e}})
\end{aligned}$$

of R-modules, S-modules, and T-modules, respectively.

In particular, for the diagonal map $\delta \colon K \to K \times K$ the following equality holds:

$$\mathrm{bigrade}((\sigma \times \tau)\delta) = \min\{\mathrm{bigrade}(\sigma), \mathrm{bigrade}(\tau)\}.$$

We illustrate Theorem 3.1 in a concrete situation. When every associated prime ideal $\mathfrak{q}$ of S satisfies $\dim(S/\mathfrak{q}) = \dim S$ we say that S is *equidimensional.*

PROPOSITION 3.12. *Let K be a field, and S an $\mathbb{N}$-graded K-algebra, generated by finitely many elements in $S_{\geq 1}$. For the inclusion $\sigma \colon K \to S$ one then has*

$$\mathrm{depth}\, S \leq \mathrm{bigrade}(\sigma) \leq \dim S.$$

The first inequality is strict when S is equidimensional, but not Cohen–Macaulay.

Proof. Set $d = \dim S$, and let P be the K-subalgebra of S generated by some homogeneous system of parameters. Thus, P is a polynomial ring in d variables and $K \to P \to S$ is a smooth-by-finite factorization of relative dimension d.

Set $p = \mathrm{pd}_P\, S$. The Auslander–Buchsbaum equality gives $d - p = \mathrm{depth}\, S$, so Theorem 3.1(1) implies the desired inequalities. When S is equidimensional and $\mathfrak{q}$ is in $\mathrm{Ass}\, S$ one has $\dim(S/\mathfrak{q}) = d = \dim P$. Since $S/\mathfrak{q}$ is finite over P and since P is a domain, this implies $\mathfrak{q} \cap P = (0)$. When S is not Cohen–Macaulay one has $p > 0$, hence

$$\begin{aligned}
\mathrm{Ext}^p_P(S, P)_{\mathfrak{q}} &\cong (\mathrm{Ext}^p_P(S, P) \otimes_P P_{(0)}) \otimes_{P_{(0)}} S_{\mathfrak{q}}\\
&\cong \mathrm{Ext}^p_{P_{(0)}}(S_{(0)}, P_{(0)}) \otimes_{P_{(0)}} S_{\mathfrak{q}}\\
&= 0.
\end{aligned}$$

From Theorem 3.1(2) we now conclude that $\mathrm{bigrade}(\sigma) > d - p$ holds. $\square$

Next we show that the second inequality in the proposition can be strict as well.

EXAMPLE 3.13. When K is a field of characteristic 0, the subring

$$S = K[x^3, x^2y, x^2z, xy^2, xz^2, y^3, y^2z, yz^2, z^3]$$

of a polynomial ring $K[x, y, z]$ and the inclusion map $\sigma\colon K \to S$ satisfy

$$\operatorname{bigrade}(\sigma) = 2 < 3 = \dim S.$$

Indeed, for the equality on the right note that the field of fractions of S is equal to $K(x, y, z)$. The one on the left results from the isomorphisms

$$\operatorname{Ext}^n_{S^{\mathsf{e}}}(S, S^{\mathsf{e}}) \cong \begin{cases} 0 & \text{for } n \le 1, \\ K & \text{for } n = 2. \end{cases}$$

For $K = \mathbb{Q}$ these isomorphisms are established through a computation with Macaulay 2. The general case follows from here and Remark 3.10.

4. Cohen–Macaulay Homomorphisms

Recall that a flat homomorphism $K \to S$ is said to be *Cohen–Macaulay at a prime ideal* $\mathfrak{n}$ of S if the local ring $S_{\mathfrak{n}}/(\mathfrak{n} \cap K)S_{\mathfrak{n}}$ is Cohen–Macaulay; the homomorphism is *Cohen–Macaulay* if it has this property at each $\mathfrak{n} \in \operatorname{Spec} S$.

We begin by fixing notation and hypotheses for the rest of the section.

4.1. Let K be a Noetherian ring, $\sigma\colon K \to S$ a flat homomorphism essentially of finite type, and $K \to P \to S$ an essentially smooth-by-surjective factorization of σ of relative dimension d. We assume that $\operatorname{Spec} S$ is connected (but see Remark 3.11).

THEOREM 4.2. *The following conditions are equivalent*:

(i) *σ is Cohen–Macaulay*;
(ii) $\operatorname{grade}_P S = \operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{pd}_P S$ *for every* $\mathfrak{n} \in \operatorname{Spec} S$;
(iii) $\operatorname{grade}_P S = \operatorname{pd}_P S$;
(iv) $\operatorname{Ext}^n_P(S, P) = 0$ *for* $\operatorname{grade}_P S < n \le d$.

The homomorphism σ is Gorenstein if and only if it is Cohen–Macaulay and the S-module $\operatorname{Ext}^g_P(S, P)$ *is invertible.*

The theorem is proved after some reminders on localizations of homomorphisms.

4.3. For each $\mathfrak{n} \in \operatorname{Spec} S$ the following inequalities hold:

$$\operatorname{grade}_P S \le \operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} \le \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} \le \operatorname{pd}_P S \le d.$$

The first three are standard, the last one comes from Section 3.2.

4.4. The homomorphism σ is Cohen–Macaulay if and only if for every $\mathfrak{n} \in \operatorname{Spec} S$ one has $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$; this follows from [6, (3.5) and (3.7)].

Proof of Theorem 4.2. The flat homomorphism σ is Gorenstein if and only if it is quasi-Gorenstein (see [5, (8.1)]), so the desired criterion for Gorensteinness is a special case of Corollary 2.3. The rest of the proof is devoted to establishing the equivalence of the conditions in Theorem 4.2. Set $g = \operatorname{grade}_P S$ and $p = \operatorname{pd}_P S$.

(i) $\Rightarrow$ (ii). We start by proving that, for all $\mathfrak{n}, \mathfrak{n}' \in \operatorname{Spec} S$ one has

$$\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{grade}_{P_{\mathfrak{n}'\cap P}} S_{\mathfrak{n}'} = \operatorname{pd}_{P_{\mathfrak{n}'\cap P}} S_{\mathfrak{n}'}.$$

For the first and last equalities it suffices to remark that $\sigma_{\mathfrak{n}}$ and $\sigma_{\mathfrak{n}'}$ are Cohen–Macaulay along with σ and then refer to Section 4.4. When $\mathfrak{n}$ is contained in $\mathfrak{n}'$, the equality in the middle is obtained by applying the chain of inequalities in Section 4.3 to $\sigma_{\mathfrak{n}'}$. Since Spec S is connected, when $\mathfrak{n}$ and $\mathfrak{n}'$ are arbitrary one can find in Spec S a path

$$\mathfrak{n} = \mathfrak{n}_0 \supseteq \mathfrak{n}_1 \subseteq \mathfrak{n}_2 \subseteq \cdots \supseteq \mathfrak{n}_{j-1} \subseteq \mathfrak{n}_j = \mathfrak{n}'.$$

The already treated case of embedded prime ideals shows that the invariants we are tracking remain constant on each segment of such a path.

When $\mathfrak{n}$ ranges over Spec S the infimum of $\operatorname{grade}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$ equals g, and the supremum of $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}$ equals p, so one gets $g = p$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These implications are evident.

(iv) $\Rightarrow$ (iii). From Section 3.2 one gets $p \leq d$. Since P is Noetherian and S is a finite P-module, one has $\operatorname{Ext}_P^p(S, P) \neq 0$. The definition of grade and the hypothesis imply that for $n \in [0, d]$ one has $\operatorname{Ext}_P^n(S, P) \neq 0$ only when $n = g$, so $p = g$ holds.

(iii) $\Rightarrow$ (ii). This follows from Section 4.3.

(ii) $\Rightarrow$ (i). This follows from Section 4.4. $\square$

In the next result we collect some properties of the S-module $\operatorname{Ext}_P\big(S, \bigwedge_P^d \Omega_{P|K}\big)$, for use in the next section. The setup is as in Section 4.1.

Theorem 4.5. *Assume that σ is Cohen–Macaulay and set $p = \operatorname{pd}_P S$. For all $\mathfrak{q} \in \operatorname{Min} S$ and $\mathfrak{n} \in \operatorname{Spec} S$ there are equalities*

$$\operatorname{bigrade}(\sigma) = \operatorname{bigrade}(\sigma_{\mathfrak{n}}) = \operatorname{tr}\deg_{\sigma} k(\mathfrak{q}). \tag{4.5.1}$$

The S-module $C = \operatorname{Ext}_P^p\big(S, \bigwedge_P^d \Omega_{P|K}\big)$ has the following properties.

(1) *There is an equality* $\operatorname{Ass}_S C = \operatorname{Ass} S$.

(2) *For each $n \in \mathbb{Z}$, there is an isomorphism of S-modules*

$$\operatorname{Ext}_{S^{\mathrm{e}}}^n(S, S^{\mathrm{e}}) \cong \operatorname{Ext}_S^{n-b}(C, S), \quad \textit{where } b = \operatorname{bigrade}(\sigma).$$

(3) *If K is Gorenstein, then $C_{\mathfrak{n}}$ is a canonical module for $S_{\mathfrak{n}}$ for each* $\mathfrak{n} \in \operatorname{Spec} S$.

Proof. Set $V = \bigwedge_P^d \Omega_{P|K}$. We deal with $C = \operatorname{Ext}_P^p(S, V)$ first.

Let $F \xrightarrow{\simeq} S$ be a resolution with each F_i finite projective over P and $F_i = 0$ for $i \notin [0, p]$. Theorem 4.2 yields $\operatorname{grade}_P S = p$, so one has $\operatorname{Ext}_P^n(S, V) = 0$ for $n \neq p$; see (3.4.1). Thus, one gets quasi-isomorphisms of complexes of P-modules,

$$\operatorname{RHom}_P(S, V) \simeq \operatorname{Hom}_P(F, V) \simeq \Sigma^{-p} C. \tag{4.5.2}$$

Each $\operatorname{Hom}_P(F_i, V)$ is finite projective, and one has isomorphisms of complexes

$$\begin{aligned}\operatorname{Hom}_P(\operatorname{Hom}_P(F,V),V) &\cong \operatorname{Hom}_P(\operatorname{Hom}_P(F,P)\otimes_P V,V)\\ &\cong \operatorname{Hom}_P(\operatorname{Hom}_P(F,P),\operatorname{Hom}_P(V,V))\\ &\cong \operatorname{Hom}_P(\operatorname{Hom}_P(F,P),P)\\ &\cong F.\end{aligned}$$

These computations localize. In particular, for each $\mathfrak{n}\in\operatorname{Spec} S$ one gets

$$\operatorname{pd}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}} = \operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}}. \tag{4.5.3}$$

(1) An ideal $\mathfrak{n}\in\operatorname{Spec} S$ is associated to C if and only if $\operatorname{depth}_{S_{\mathfrak{n}}} C_{\mathfrak{n}} = 0$. The finiteness of $S_{\mathfrak{n}}$ as a $P_{\mathfrak{n}\cap P}$-module and the Auslander–Buchsbaum equality together yield

$$\operatorname{depth}_{S_{\mathfrak{n}}} C_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} P_{\mathfrak{n}\cap P} - \operatorname{pd}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}}.$$

Thus, $\mathfrak{n}\in\operatorname{Ass}_S C$ is equivalent to $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} C_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} P_{\mathfrak{n}\cap P}$. Similarly, $\mathfrak{n}\in\operatorname{Ass} S$ amounts to $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = \operatorname{depth}_{P_{\mathfrak{n}\cap P}} P_{\mathfrak{n}\cap P}$. Now (4.5.3) gives $\operatorname{Ass}_S C = \operatorname{Ass} S$.

(2) From (4.5.2) and Section 3.3, for each $n\in\mathbb{Z}$ one gets

$$\operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}}) \cong \operatorname{Ext}^{n-d+p}_S(C,S).$$

We have just proved $\operatorname{Ass} S\cap \operatorname{Supp}_S \operatorname{Ext}^p_P(S,P)\neq\emptyset$, so Theorem 3.1(2) implies

$$\operatorname{bigrade}(\sigma) = d-p. \tag{4.5.4}$$

Now we can prove (4.5.1). Pick $\mathfrak{n}$ in $\operatorname{Spec} S$. The induced homomorphisms $K_{\mathfrak{n}\cap K}\to P_{\mathfrak{n}\cap P}\to S_{\mathfrak{n}}$ provide a smooth-by-surjective factorization of relative dimension d of $\sigma_{\mathfrak{n}}\colon K_{\mathfrak{n}\cap K}\to S_{\mathfrak{n}}$. We have already proved that $\sigma_{\mathfrak{n}}$ is Cohen–Macaulay, and $\operatorname{pd}_{P_{\mathfrak{n}\cap P}} S_{\mathfrak{n}} = p$ holds, so we get $\operatorname{bigrade}(\sigma) = \operatorname{bigrade}(\sigma_{\mathfrak{n}})$ from formula (4.5.4) applied to $\sigma_{\mathfrak{n}}$. This proves the first equality. Theorem 3.1 gives the second one.

(3) When K is Gorenstein the rings P and S are Gorenstein and Cohen–Macaulay, respectively, as they are flat over K with fibers of the corresponding type. For $\mathfrak{n}\in\operatorname{Spec} S$ and $\mathfrak{m}=\mathfrak{n}\cap P$ Theorem 4.2(1) gives $\operatorname{pd}_{P_{\mathfrak{m}}} S_{\mathfrak{n}} = p$ and (3.4.2) gives $C_{\mathfrak{n}}\cong \operatorname{Ext}^p_{P_{\mathfrak{m}}}(S_{\mathfrak{n}},P_{\mathfrak{m}})$, so $C_{\mathfrak{n}}$ is a canonical module for $S_{\mathfrak{n}}$ by [9, (3.3.7)]. □

5. Gorenstein Homomorphisms

Combining earlier results we get a "structure theorem" for Gorenstein algebras.

THEOREM 5.1. *If K is a Noetherian ring and $\sigma\colon K\to S$ is a Gorenstein homomorphism essentially of finite type, then for some $q\geq 0$ one has*

$$\{n\in\mathbb{Z}\mid \operatorname{Ext}^n_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})\neq 0\} = \{n_1,\dots,n_q\}.$$

Furthermore, there is an isomorphism of K-algebras

$$S\cong\prod_{i=1}^q S_i \quad \textit{with}\ \ S_i = S/\operatorname{Ann}(\operatorname{Ext}^{n_i}_{S^{\mathrm{e}}}(S,S^{\mathrm{e}})),$$

for each i the map $K \to S \to S_i$ is Gorenstein, $n_i = \operatorname{tr}\deg_\sigma k(\mathfrak{q})$ for every $\mathfrak{q}$ in $\operatorname{Min} S_i$*, the S_i-module $\operatorname{Ext}^{n_i}_{S_i^{\mathrm{e}}}(S_i, S_i^{\mathrm{e}})$ is invertible, and $\operatorname{Ext}^{n}_{S_i^{\mathrm{e}}}(S_i, S_i^{\mathrm{e}}) = 0$ for $n \neq n_i$.*

Proof. The graded S-module $\operatorname{Ext}_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})$ is invertible by Theorem 2.4. In particular, $\operatorname{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})$ is not zero for finitely many values of n, say $n_1, \dots, n_q$. Lemma 2.5 provides the decomposition of S and Remark 3.11 an equivariant isomorphism

$$\operatorname{Ext}_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \cong \bigoplus_{i=1}^{q} \operatorname{Ext}_{S_i^{\mathrm{e}}}(S_i, S_i^{\mathrm{e}})$$

of graded modules. It implies that the S_i-module $\operatorname{Ext}^n_{S_i^{\mathrm{e}}}(S_i, S_i^{\mathrm{e}})$ is invertible for each $n \in \{n_1, \dots, n_q\}$ and is zero otherwise. Theorem 2.4 now shows that every composition $K \to S \to S_i$ is Gorenstein. In particular, it is Cohen–Macaulay, so Theorem 4.5 yields $n_i = \operatorname{tr}\deg_\sigma k(\mathfrak{q})$ for every $\mathfrak{q} \in \operatorname{Min} S_i$. □

Next we search for a converse, in the spirit of the conjecture stated in the Introduction.

One says that S is *generically Gorenstein* if for each $\mathfrak{q} \in \operatorname{Min} S$ the ring $S_\mathfrak{q}$ is Gorenstein. The next result was proved independently in [3, (2.1)] and [14, (2.2)].

5.2. If S is a generically Gorenstein, Cohen–Macaulay local ring with canonical module C, and $\operatorname{Ext}^n_S(C, S) = 0$ holds for $1 \leq n \leq \dim S$, then S is Gorenstein.

When K is a Gorenstein ring, a homomorphism $\sigma\colon K \to S$ with S Noetherian is Gorenstein if and only if the ring S is Gorenstein (see [15, (23.4)]). Thus, our next result is a reformulation of Theorem 4 from the Introduction.

THEOREM 5.3. *Let K be a Gorenstein ring, S a Cohen–Macaulay ring with connected spectrum, and $K \to S$ a flat homomorphism essentially of finite type.*

If S is generically Gorenstein, and for some minimal prime ideal $\mathfrak{q}$ of S one has

$$\operatorname{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) = 0 \quad \textit{for } \operatorname{tr}\deg_\sigma k(\mathfrak{q}) < n \leq \operatorname{tr}\deg_\sigma k(\mathfrak{q}) + \dim S,$$

then the ring S is Gorenstein.

Proof. Note that σ is Cohen–Macaulay because it is flat and its target is a Cohen–Macaulay ring; see [15, (23.3), Cor.]. Theorem 4.5(3) then yields a finite S-module C, such that $C_\mathfrak{n}$ is a canonical module for the Cohen–Macaulay local ring $S_\mathfrak{n}$ for each $\mathfrak{n} \in \operatorname{Spec} S$. Theorem 4.5(2) and formula (4.5.1) translate our hypothesis into equalities $\operatorname{Ext}^n_{S_\mathfrak{n}}(C_\mathfrak{n}, S_\mathfrak{n}) = 0$ for $1 \leq n \leq \dim S$. It remains to invoke Section 5.2. □

It is instructive to compare the criterion that we just proved with the one afforded by Theorem 4.2: The ring S is Gorenstein if for some essentially smooth-by-surjective factorization $K \to P \to S$ of σ of relative dimension d the S-module $\operatorname{Ext}^n_P(S, P)$ is zero for $\operatorname{grade}_P S < n \leq d$ and is projective for $n = \operatorname{grade}_P S$.

On the face of it, the difference lies only in the condition on the S-module structure of a single, finitely generated module $\operatorname{Ext}^n_P(S, P)$. However, this structure is induced through an inherently infinite construction. We elaborate in the next remark.

REMARK 5.4. The action of any single *element* $s \in S$ on every $\operatorname{Ext}^i_P(S, P)$ can be computed from a free resolution of S over P, as the result of applying the map

$$\mathrm{H}^i(\operatorname{Hom}_R(s\,\mathrm{id}^F, P)\colon \mathrm{H}^i(\operatorname{Hom}_R(F, P)) \longrightarrow \mathrm{H}^i(\operatorname{Hom}_R(F, P)).$$

Each F_j can be chosen finite free, so to compute such a map it suffices to know the $i+1$ matrices with elements in P that describe the differentials ∂^F_j for $j = 1, \dots, i+1$.

However, the S-module structure of $\operatorname{Ext}^i_P(S, P)$ comes through an isomorphism $\operatorname{Ext}^i_P(S, P) \cong \mathrm{H}^i(\operatorname{Hom}_P(S, I))$, where I is an injective resolution of P over itself. The information needed to construct I is not finite in two distinct ways: (1) the module I^i is a direct sum of injective envelopes of $P/\mathfrak{m}$ for *every* $\mathfrak{m} \in \operatorname{Spec} S$ of P with $\operatorname{Ext}^i_P(P/\mathfrak{m}, P)_{\mathfrak{m}} \neq 0$, and for $1 \leq i < \dim P$ *infinitely many* distinct $\mathfrak{m}$ satisfy this property; (2) injective envelopes are *not finitely generated,* unless $\mathfrak{m}$ is minimal.

To finish, we take another look at Theorem 2.4 and Theorem 5.3 this time against backdrops provided by results coming from three different directions:

(1) characterizations of regular local rings $(R, \mathfrak{m}, k)$ by the finiteness of $\operatorname{pd}_R k$, or the vanishing of $\operatorname{Ext}^n_R(k, k)$ for all (respectively, for some) $n > \dim R$ (see [15]);
(2) characterizations of Gorenstein local rings $(R, \mathfrak{m}, k)$ by the finiteness of $\operatorname{id}_R R$, or the vanishing of $\operatorname{Ext}^n_R(k, R)$ for all (respectively, for some) $n > \operatorname{depth} R$ (see [15]);
(3) characterizations of smooth K-algebras S essentially of finite type by the finiteness of $\operatorname{pd}_{S^{\mathrm{e}}} S$, or the vanishing of $\operatorname{Ext}^n_{S^{\mathrm{e}}}(S, S)$ for all $n \geq m$ (respectively, for $n \in [m, m + \dim S]$) when $\Omega_{S\,|\,K}$ can be generated by $m-1$ elements (see [7]).

Comparing the statements of these results one will observe that the homological properties of a flat algebra of finite type, viewed as a bimodule over itself, encode information about all its fibers, and that the code is similar to the one that translates properties of a local ring into homological data on its residue field.

This is the reasoning behind the conjecture stated in the Introduction.

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