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Intersections of Tautological Classes on Blowups of Moduli Spaces of Genus-1 Curves

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1. Introduction

Moduli spaces of stable curves and stable maps play a prominent role in algebraic geometry, symplectic topology, and string theory. Many geometric results have been obtained by using the fact that the moduli space $\overline{\mathfrak{M}}_{0,k}(\mathbb{P}^n,d)$ of degree-d stable maps from genus-0 curves with k marked points into $\mathbb{P}^n$ is a smooth unidimensional orbivariety of the expected dimension. This is not the case for positive-genus moduli spaces $\overline{\mathfrak{M}}_{g,k}(\mathbb{P}^n,d)$. However, if $d \geq 1$, then the closure

$$\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \subset \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$$

of the space $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$ of stable maps with smooth domains is an irreducible orbivariety of the expected dimension. This component of $\overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d)$ contains all the relevant genus-1 information for the purposes of enumerative geometry and, as shown in [LZ] and [Z1], of the Gromov–Witten theory.

For $d \geq 3$, $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ is singular. In [VaZ] the authors construct a desingularization of the space $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$—that is, a smooth orbivariety $\widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ and a map

$$\pi\colon \widetilde{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d),$$

which is biholomorphic onto $\mathfrak{M}_{1,k}^0(\mathbb{P}^n,d)$. Via this desingularization and the classical localization theorem of [ABo], intersections of naturally arising cohomology classes on $\overline{\mathfrak{M}}_{1,k}^0(\mathbb{P}^n,d)$ can be expressed in terms of integrals of certain ψ-classes on moduli spaces of genus-0 and genus-1 stable curves and on blowups of moduli spaces of genus-1 stable curves. The former can be computed using two well-known recursions: string and dilaton equations (see [H+, Sec. 26.3]). In this paper we give three recursions for top intersections of ψ-classes on blowups of moduli spaces of genus-1 curves; see Theorem 1.1. Two of these recursions generalize the genus-1 string and dilaton relations. Together with the standard genus-1 initial condition (i.e., equation (1.2)), the three recursions completely determine the top intersections of ψ-classes on blowups of moduli spaces of genus-1 curves.

Corollary 1.2 of Theorem 1.1 is used in [Z2] and [Z3] to compute the genus-1 GW-invariants of any Calabi–Yau projective hypersurface, verifying the long-standing prediction by [BCOV] for a quintic 3-fold as a special case. The full statement of Theorem 1.1 is used in [Z3] to describe the difference between the

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standard and reduced genus-1 GW-invariants, making it possible to compute the genus-1 GW-invariants of any complete intersection.

For J a finite nonempty set, let $\bar{\mathcal{M}}_{1,J}$ be the moduli space of genus-1 curves with marked points indexed by the set J. Let

$$\mathbb{E} \longrightarrow \bar{\mathcal{M}}_{1,J}$$

be the Hodge line bundle of holomorphic differentials. For each $j \in J$, we denote by

$$L_j \longrightarrow \bar{\mathcal{M}}_{1,J}$$

the universal tangent line bundle for the jth marked point and put

$$\psi_j = c_1(L_j^*) \in H^*(\bar{\mathcal{M}}_{1,J};\mathbb{Q}).$$

If $(c_j)_{j\in J}$ is a tuple of integers, let

$$\langle (c_j)_{j\in J}\rangle_{|J|} = \Big\langle \prod_{j\in J} \psi_j^{c_j}, \bar{\mathcal{M}}_{1,J}\Big\rangle.$$

Let I and J be two finite sets that are not both empty. The inductive procedure of [VaZ, Sec. 2.3], which is reviewed here in Section 2.1, constructs a blowup

$$\pi : \widetilde{\mathcal{M}}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}_{1,I\sqcup J}$$

of $\bar{\mathcal{M}}_{1,I\sqcup J}$ along natural subvarieties and their proper transforms. In addition, it describes $|I|+1$ line bundles

$$\tilde{\mathbb{E}}, \tilde{L}_i \longrightarrow \widetilde{\mathcal{M}}_{1,(I,J)}, \quad i \in I,$$

and $|I|$ nowhere vanishing sections

$$\tilde{s}_i \in \Gamma(\widetilde{\mathcal{M}}_{1,(I,J)}; \tilde{L}_i^* \otimes \tilde{\mathbb{E}}^*), \quad i \in I.$$

These line bundles are obtained by twisting $\mathbb{E}$ and L_i. Since the sections $\tilde{s}_i$ do not vanish, all $|I|+1$ line bundles $\tilde{L}_i$ and $\tilde{\mathbb{E}}^*$ are explicitly isomorphic. They will be denoted by

$$\mathbb{L} \longrightarrow \widetilde{\mathcal{M}}_{1,(I,J)}$$

and will be called the *universal tangent line bundle*. Let

$$\tilde{\psi} = c_1(\mathbb{L}^*) \in H^2(\widetilde{\mathcal{M}}_{1,(I,J)};\mathbb{Q})$$

be the corresponding "ψ-class" on $\widetilde{\mathcal{M}}_{1,(I,J)}$. If $(\tilde{c}, (c_j)_{j\in J})$ is a tuple of integers, we put

$$\langle \tilde{c}; (c_j)_{j\in J}\rangle_{(|I|,|J|)} = \Big\langle \tilde{\psi}^{\tilde{c}} \cdot \prod_{j\in J} \pi^*\psi_j^{c_j}, \widetilde{\mathcal{M}}_{1,(I,J)}\Big\rangle. \tag{1.1}$$

If $\tilde{c} + \sum_{j\in J} c_j \neq |I|+|J|$, or $\tilde{c} < 0$, or $c_j < 0$ for some $j \in J$, then this number is defined to be zero.

THEOREM 1.1. *Suppose I and J are finite sets such that $|I|+|J| \geq 2$, and let $(\tilde{c}, (c_j)_{j\in J})$ be a tuple of integers. Then we have the following recursions.*

(R1) *If* $I \neq \emptyset$ *and* $c_j > 0$ *for all* $j \in J$, *then*

$$\langle \tilde{c};\, (c_j)_{j \in J} \rangle_{(|I|,|J|)} = \langle \tilde{c};\, (c_j)_{j \in J} \rangle_{(|I|-1,|J|+1)}.$$

(R2) *If* $c_{j^*} = 1$ *for some* $j^* \in J$, *then*

$$\langle \tilde{c};\, (c_j)_{j \in J} \rangle_{(|I|,|J|)} = (|I| + |J| - 1)\langle \tilde{c};\, (c_j)_{j \in J - \{j^*\}} \rangle_{(|I|,|J|-1)}.$$

(R3) *If* $c_{j^*} = 0$ *for some* $j^* \in J$, *then*

$$\langle \tilde{c};\, (c_j)_{j \in J} \rangle_{(|I|,|J|)} = |I| \langle \tilde{c} - 1;\, (c_j)_{j \in J - \{j^*\}} \rangle_{(|I|,|J|-1)} + \sum_{j \in J - \{j^*\}} \langle \tilde{c};\, c_j - 1, (c_{j'})_{j' \in J - \{j^*, j\}} \rangle_{(|I|,|J|-1)}.$$

COROLLARY 1.2. *If* I *and* J *are finite sets and* $I \neq \emptyset$, *then*

$$\langle \tilde{\psi}^{|I|+|J|}, \widetilde{\mathcal{M}}_{I,J} \rangle = \frac{1}{24} \cdot |I|^{|J|} \cdot (|I| - 1)!.$$

We recall that

$$\langle \psi_1, \overline{\mathcal{M}}_{1,1} \rangle = \frac{1}{24}. \tag{1.2}$$

Thus, Corollary 1.2 is obtained by applying (R3) $|J|$ times and then (R1) followed by (R3) $|I| - 1$ times.

The recursion (R1) of Theorem 1.1 follows easily from the relevant definitions, which are reviewed in Section 2.1. The reason is that the blowups of $\overline{\mathcal{M}}_{1,I \sqcup J}$ corresponding to the two sides of the relation in (R1) differ by blowups along loci on which $\prod_{j \in J} \psi_j$ vanishes (see the end of Section 2.1).

The $\tilde{c} = 0$ cases of (R2) and (R3) are precisely the standard genus-1 dilaton and string recursions, respectively. The relations (R2) and (R3) are proved in Section 2.2 by an argument similar to the usual proof of the latter. In particular, we consider the forgetful morphism

$$f \colon \overline{\mathcal{M}}_{1,I \sqcup J} \longrightarrow \overline{\mathcal{M}}_{1,I \sqcup (J - \{j^*\})}.$$

By Proposition 2.1, it lifts to a morphism on the blowups,

$$\tilde{f} \colon \widetilde{\mathcal{M}}_{1,(I,J)} \longrightarrow \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})};$$

see the LHS of Figure 1. Each of the blowups is obtained through a sequence of blowups along smooth subvarieties, but the order of the blowups is not unique. We prove Proposition 2.1 in Section 3.3 by fixing an order for blowups on $\overline{\mathcal{M}}_{1,I \sqcup (J - \{j^*\})}$ and then choosing a consistent order for blowups on $\overline{\mathcal{M}}_{1,I \sqcup J}$. We show that f then

$$\begin{array}{ccccccc}
\widetilde{\mathcal{M}}_{1,(I,J)} & \xrightarrow{\tilde{f}} & \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})} & \quad & \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n, d) & \xrightarrow{\tilde{f}} & \widetilde{\mathfrak{M}}^0_{1,k-1}(\mathbb{P}^n, d) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\
\overline{\mathcal{M}}_{1,I \sqcup J} & \xrightarrow{f} & \overline{\mathcal{M}}_{1,I \sqcup (J-\{j^*\})} & & \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n, d) & \xrightarrow{f} & \overline{\mathfrak{M}}_{1,k-1}(\mathbb{P}^n, d)
\end{array}$$

Figure 1 Lifts of forgetful maps

lifts to a morphism between corresponding stages of the two blowup constructions (see Lemma 3.5). Once the existence of the morphism $\tilde{f}$ is established, we compare $\tilde{\psi}$ with $\tilde{f}^*\tilde{\psi}$ and describe their restrictions to the relevant divisors (Lemmas 2.2 and 2.3).

If $k > 0$, there is also a natural forgetful morphism

$$f\colon \overline{\mathfrak{M}}_{1,k}(\mathbb{P}^n,d) \longrightarrow \overline{\mathfrak{M}}_{1,k-1}(\mathbb{P}^n,d).$$

The proof of Proposition 2.1 can be modified in a straightforward way to show that this morphism f lifts to a morphism

$$\tilde{f}\colon \widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d) \longrightarrow \widetilde{\mathfrak{M}}^0_{1,k-1}(\mathbb{P}^n,d);$$

see the RHS of Figure 1. This observation implies that the desingularization $\widetilde{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ of $\overline{\mathfrak{M}}^0_{1,k}(\mathbb{P}^n,d)$ constructed in [VaZ] preserves one of the properties central to the Gromov–Witten theory.

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2. Preliminaries

2.1. Blowup Construction

If I is a finite set, let

$$\mathcal{A}_1(I) = \Big\{(I_P,\{I_k : k\in K\} : K \neq \emptyset;\ I = \bigsqcup_{k\in\{P\}\sqcup K} I_k;\ |I_k| \geq 2\ \forall k\in K\Big\}. \quad (2.1)$$

Here P stands for "principal" (component). If $\rho = (I_P,\{I_k : k\in K\})$ is an element of $\mathcal{A}_1(I)$, we denote by $\mathcal{M}_{1,\rho}$ the subset of $\overline{\mathcal{M}}_{1,I}$ consisting of the stable curves $\mathcal{C}$ such that:

(i) $\mathcal{C}$ is a union of a smooth torus and $|K|$ projective lines, indexed by K;
(ii) each line is attached directly to the torus;
(iii) for each $k\in K$, the marked points on the line corresponding to k are indexed by I_k.

Let $\overline{\mathcal{M}}_{1,\rho}$ be the closure of $\mathcal{M}_{1,\rho}$ in $\overline{\mathcal{M}}_{1,I}$. Figure 2 illustrates this definition, from the points of view of symplectic topology and algebraic geometry. In the first diagram, each circle represents a sphere, or $\mathbb{P}^1$. In the second diagram, the irreducible components of $\mathcal{C}$ are represented by curves, and the integer next to each component shows its genus. It is well known that each space $\overline{\mathcal{M}}_{1,\rho}$ is a smooth subvariety of $\overline{\mathcal{M}}_{1,I}$.

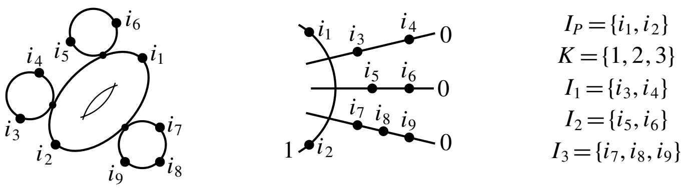

Figure 2 A typical element of $\overline{\mathcal{M}}_{1,\rho}$

We define a partial ordering on the set $\mathcal{A}_1(I) \sqcup \{(I, \emptyset)\}$ by setting

$$\rho' \equiv (I'_P, \{I'_k : k \in K'\}) \prec \rho \equiv (I_P, \{I_k : k \in K\}) \tag{2.2}$$

if $\rho' \neq \rho$ and if there exists a map $\varphi\colon K \to K'$ such that $I_k \subset I'_{\varphi(k)}$ for all $k \in K$. This condition means that the elements of $\mathcal{M}_{1,\rho'}$ can be obtained from the elements of $\mathcal{M}_{1,\rho}$ by moving more points onto the bubble components or combining the bubble components; see Figure 3.

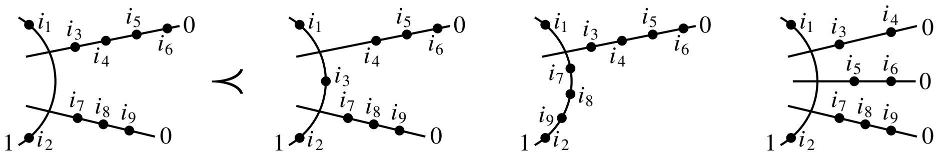

Figure 3 Examples of partial ordering (2.2)

Let I and J be finite sets such that I is not empty and $|I| + |J| \geq 2$. We put

$$\mathcal{A}_1(I, J) = \big\{((I_P \sqcup J_P), \{I_k \sqcup J_k : k \in K\}) \in \mathcal{A}_1(I \sqcup J) : I_k \neq \emptyset \ \forall k \in K\big\}.$$

We note that if $\varrho \in \mathcal{A}_1(I \sqcup J)$ then $\varrho \in \mathcal{A}_1(I, J)$ if and only if every bubble component of an element of $\mathcal{M}_{1,\varrho}$ carries at least one element of I. The partially ordered set $(\mathcal{A}_1(I, J), \prec)$ has a unique minimal element

$$\varrho_{\min} \equiv (\emptyset, \{I \sqcup J\}).$$

Let $<$ be an ordering on $\mathcal{A}_1(I, J)$ extending the partial ordering $\prec$. We denote the corresponding maximal element by $\varrho_{\max}$. For $\varrho \in \mathcal{A}_1(I, J)$ we put

$$\varrho - 1 = \begin{cases} \max\{\varrho' \in \mathcal{A}_1(I, J) : \varrho' < \varrho\} & \text{if } \varrho \neq \varrho_{\min}, \\ 0 & \text{if } \varrho = \varrho_{\min}, \end{cases} \tag{2.3}$$

where the maximum is taken with respect to the ordering $<$.

The starting data for the blowup construction of [VaZ, Sec. 2.3] is given by

$$\bar{\mathcal{M}}^0_{1,(I,J)} = \bar{\mathcal{M}}_{1,I \sqcup J}, \qquad \bar{\mathcal{M}}^0_{1,\varrho} = \bar{\mathcal{M}}_{1,\varrho} \quad \forall \varrho \in \mathcal{A}_1(I, J),$$

$$\mathbb{E}_0 = \mathbb{E} \longrightarrow \bar{\mathcal{M}}^0_{1,(I,J)}, \qquad L_{0,i} = L_i \longrightarrow \bar{\mathcal{M}}^0_{1,(I,J)} \quad \forall i \in I.$$

Suppose $\varrho \in \mathcal{A}_1(I, J)$ and we have constructed:

(I1) a blowup $\pi_{\varrho-1}\colon \bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)} \to \bar{\mathcal{M}}^0_{1,(I,J)}$ of $\bar{\mathcal{M}}^0_{1,(I,J)}$ such that $\pi_{\varrho-1}$ is one-to-one outside of the preimages of the spaces $\bar{\mathcal{M}}^0_{1,\varrho'}$ with $\varrho' \leq \varrho - 1$; and

(I2) line bundles $L_{\varrho-1,i} \to \bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ for $i \in I$ and $\mathbb{E}_{\varrho-1} \to \bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$.

For each $\varrho^* > \varrho - 1$, let $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho^*}$ be the proper transform of $\bar{\mathcal{M}}^0_{1,\varrho^*}$ in $\bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$.

Given $\varrho \in \mathcal{A}_1(I, J)$ as just described, let

$$\tilde{\pi}_\varrho\colon \bar{\mathcal{M}}^{\varrho}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$$

be the blowup of $\bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ along $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}$. We denote by $\bar{\mathcal{M}}^{\varrho}_{1,\varrho}$ the corresponding exceptional divisor. For $\varrho^* > \varrho$, let $\bar{\mathcal{M}}^{\varrho}_{1,\varrho^*} \subset \bar{\mathcal{M}}^{\varrho}_{1,(I,J)}$ be the proper transform of $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho^*}$. If

$$\varrho = ((I_P \sqcup J_P), \{I_k \sqcup J_k : k \in K\}) \in \mathcal{A}_1(I \sqcup J) \quad \text{and} \quad i \in I,$$

then we put

$$L_{\varrho,i} = \begin{cases} \tilde{\pi}_\varrho^* L_{\varrho-1,i} & \text{if } i \notin I_P, \\ \tilde{\pi}_\varrho^* L_{\varrho-1,i} \otimes \mathcal{O}(-\bar{\mathcal{M}}_{1,\varrho}^{\varrho}) & \text{if } i \in I_P; \end{cases}$$
$$\mathbb{E}_\varrho = \tilde{\pi}_\varrho^* \mathbb{E}_{\varrho-1} \otimes \mathcal{O}(\bar{\mathcal{M}}_{1,\varrho}^{\varrho}). \tag{2.4}$$

It is immediate that the requirements (I1) and (I2), with $\varrho - 1$ replaced by ϱ, are satisfied.

We conclude the blowup construction after $|\varrho_{\max}|$ steps. Let

$$\widetilde{\mathcal{M}}_{1,(I,J)} = \bar{\mathcal{M}}_{1,(I,J)}^{\varrho_{\max}}, \quad \tilde{L}_i = L_{\varrho_{\max},i} \ \forall i \in I, \quad \text{and} \quad \tilde{\mathbb{E}} = \mathbb{E}_{\varrho_{\max}}.$$

By [VaZ, Lemma 2.6], the end result of this blowup construction is well-defined—that is, independent of the choice of an ordering $<$ extending the partial ordering $\prec$. The reason is that different extensions of the partial order $\prec$ correspond to different orders of blowups along disjoint subvarieties. By the inductive assumption (I4) in [VaZ, Sec. 2.3], there is a natural isomorphism between the line bundles $\tilde{L}_i$ and $\tilde{\mathbb{E}}^*$. Thus, these line bundles are the same. We denote them by $\mathbb{L}$.

Remark. If $\varrho, \varrho' \in \mathcal{A}_1(I, J)$ are not comparable with respect to $\prec$ and if $\varrho < \varrho'$, then $\bar{\mathcal{M}}_{1,\varrho}^{\varrho-1}$ and $\bar{\mathcal{M}}_{1,\varrho'}^{\varrho-1}$ are disjoint subvarieties in $\bar{\mathcal{M}}_{1,(I,J)}^{\varrho-1}$. However, $\bar{\mathcal{M}}_{1,\varrho}$ and $\bar{\mathcal{M}}_{1,\varrho'}$ need not be disjoint in $\bar{\mathcal{M}}_{1,I\sqcup J}$. For example, if

$$I = \{1,2,3,4\}, \quad J = \emptyset, \quad \varrho_{12} = ((\{3,4\}), \{\{1,2\}\}),$$
$$\varrho_{34} = ((\{1,2\}), \{\{3,4\}\}), \quad \varrho_{12,34} = ((\emptyset), \{\{1,2\}, \{3,4\}\}),$$

then $\bar{\mathcal{M}}_{1,\varrho_{12}}$ and $\bar{\mathcal{M}}_{1,\varrho_{34}}$ intersect at $\bar{\mathcal{M}}_{1,\varrho_{12,34}}$ in $\bar{\mathcal{M}}_{1,4}$, but their proper transforms in the blowup of $\bar{\mathcal{M}}_{1,4}$ along $\bar{\mathcal{M}}_{1,\varrho_{12,34}}$ are disjoint.

We are now ready to verify recursion (R1) of Theorem 1.1. If $i^* \in I$, then

$$\mathcal{A}_1(I - \{i^*\}, J \sqcup \{i^*\}) \subset \mathcal{A}_1(I, J) \quad \text{and}$$

$$\mathcal{A}_1(I, J) - \mathcal{A}_1(I - \{i^*\}, J \sqcup \{i^*\}) \\ = \{\varrho = (I_P \sqcup J_P, \{\{i^*\} \sqcup J_1\} \sqcup \{I_k \sqcup J_k : k \in K'\}) \in \mathcal{A}_1(I \sqcup J)\}.$$

With ϱ as before, we have a natural isomorphism

$$\bar{\mathcal{M}}_{1,\varrho} \approx \bar{\mathcal{M}}_{1,\bar{\varrho}} \times \bar{\mathcal{M}}_{0,\{q,i^*\}\sqcup J_1}, \quad \text{where } \bar{\varrho} = (I_P \sqcup J_P \sqcup \{p\}, \{I_k \sqcup J_k : k \in K'\}).$$

Let

$$\pi_2 \colon \bar{\mathcal{M}}_{1,\varrho} \longrightarrow \bar{\mathcal{M}}_{0,\{q,i^*\}\sqcup J_1}$$

be the projection map. By definition,

$$\psi_j|_{\bar{\mathcal{M}}_{1,\varrho}} = \pi_2^* \psi_j \ \forall j \in J_1 \implies \prod_{j\in J_1} \psi_j|_{\bar{\mathcal{M}}_{1,\varrho}} = \pi_2^* \prod_{j\in J_1} \psi_j = \pi_2^* 0 = 0,$$

since the dimension of $\bar{\mathcal{M}}_{0,\{q,i^*\}\sqcup J_1}$ is $|J_1|-1$. It follows that

$$\prod_{j\in J}\psi_j|_{\bar{\mathcal{M}}_{1,\varrho}}=0 \quad \forall\varrho\in\mathcal{A}_1(I,J)-\mathcal{A}_1(I-\{i^*\},J\sqcup\{i^*\}).$$

Thus, the constructions of $\tilde{\psi}\equiv c_1(\tilde{\mathbb{E}})$ from $\lambda\equiv c_1(\mathbb{E}_0)$ for $\widetilde{\mathcal{M}}_{1,(I-\{i^*\},J\sqcup\{i^*\})}$ and $\widetilde{\mathcal{M}}_{1,(I,J)}$ differ by varieties along which $\prod_{j\in J}\psi_j^{c_j}$ vanishes, as long as $c_j>0$ for all $j\in J$. We conclude that

$$\left\langle\tilde{\psi}^{\tilde{c}}\cdot\prod_{j\in J}\pi^*\psi_j^{c_j},\widetilde{\mathcal{M}}_{1,(I,J)}\right\rangle=\left\langle\tilde{\psi}^{\tilde{c}}\cdot\prod_{j\in J}\pi^*\psi_j^{c_j},\widetilde{\mathcal{M}}_{1,(I-\{i^*\},J\sqcup\{i^*\})}\right\rangle$$

whenever $c_j>0$ for all $j\in J$, as needed.

2.2. *Outline of Proof of Recursions* (R2) *and* (R3) *in Theorem 1.1*

In this section we state three structural descriptions—Proposition 2.1 and Lemmas 2.2 and 2.3—and use them to verify the last two recursions of Theorem 1.1. Proposition 2.1 and Lemmas 2.2 and 2.3 are proved in Section 3.

Let I be a finite set of which i,j are distinct elements, and let

$$\rho_{ij}=(I-\{i,j\},\{\{i,j\}\})\in\mathcal{A}_1(I).$$

Then there is a natural decomposition

$$\bar{\mathcal{M}}_{1,\varrho_{ij}}=\bar{\mathcal{M}}_{1,(I-\{i,j\})\sqcup\{p\}}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}, \tag{2.5}$$

where the second component is a one-point space. Let

$$\pi_P,\pi_B\colon\bar{\mathcal{M}}_{1,\varrho_{ij}}\longrightarrow\bar{\mathcal{M}}_{1,(I-\{i,j\})\sqcup\{p\}},\bar{\mathcal{M}}_{0,\{q,i,j\}} \tag{2.6}$$

be the two projection maps. Here P and B stand for "principal" and "bubble" (components). It is immediate that

$$\lambda|_{\bar{\mathcal{M}}_{1,\varrho_{ij}}}=\pi_P^*\lambda; \tag{2.7}$$

$$\psi_{j'}|_{\bar{\mathcal{M}}_{1,\varrho_{ij}}}=\begin{cases}\pi_P^*\psi_{j'} & \text{if } j'\neq i,j,\\ \pi_B^*\psi_{j'}=0 & \text{if } j'=i,j,\end{cases}\quad\forall j'\in I. \tag{2.8}$$

When $j'=i,j$ the restriction of $\psi_{j'}$ vanishes because the second component is zero-dimensional.

If I is a finite set, $|I|\geq 2$, and $j^*\in I$, then there is a natural forgetful morphism

$$f\colon\bar{\mathcal{M}}_{1,I}\longrightarrow\bar{\mathcal{M}}_{1,I-\{j^*\}},$$

which is obtained by dropping the marked point j^* from every element of $\bar{\mathcal{M}}_{1,I}$ and then contracting the unstable components of the resulting curve. It is straightforward to check that

$$\lambda=f^*\lambda \quad\text{and} \tag{2.9}$$

$$\psi_j=f^*\psi_j+\bar{\mathcal{M}}_{1,\varrho_{jj^*}}\implies f^*\psi_j|_{\bar{\mathcal{M}}_{1,\varrho_{jj^*}}}=\pi_P^*\psi_p\quad\forall j\in I-\{j^*\} \tag{2.10}$$

(see e.g. [H+, Chap. 25]). Using (2.8), (2.10), and induction on c_j, we find that

$$\begin{aligned}\psi_j^{c_j} &= \psi_j^{c_j-1}(f^*\psi_j + \bar{\mathcal{M}}_{1,\varrho_{jj^*}})\\ &= f^*\psi_j^{c_j} + (\pi_P^*\psi_p^{c_j-1}) \cap \bar{\mathcal{M}}_{1,\varrho_{jj^*}} \quad \forall j \in I - \{j^*\},\ c_j > 0. \end{aligned} \tag{2.11}$$

If I and J are finite sets, $i \in I$, and $j \in J$, then $\bar{\mathcal{M}}_{1,\varrho_{ij}}$ is a divisor in $\bar{\mathcal{M}}_{1,I\sqcup J}$. Thus, in the notation of Section 2.1,

$$\bar{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}} = \bar{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}-1}.$$

Since ϱ_{ij} is a maximal element of $(A_1(I,J),\prec)$, the blowup loci—at the stages of the construction described in Section 2.1—that follow the blowup along $\bar{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}-1}$ are disjoint from $\bar{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}}$. Thus, we can view $\bar{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{ij}}$ as a divisor in $\widetilde{\mathcal{M}}_{1,(I,J)}$. We denote this divisor by $\widetilde{\mathcal{M}}_{1,\varrho_{ij}}$. If $i, j \in J$, then $\bar{\mathcal{M}}_{1,\varrho_{ij}}$ is also a divisor in $\bar{\mathcal{M}}_{1,I\sqcup J}$. Hence, its proper transform $\bar{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho}$ in $\bar{\mathcal{M}}_{1,(I,J)}^{\varrho}$ is a divisor for every $\varrho \in \mathcal{A}_1(I,J)$. Let

$$\widetilde{\mathcal{M}}_{1,\varrho_{ij}} = \bar{\mathcal{M}}_{1,\varrho_{ij}}^{\varrho_{\max}} \subset \widetilde{\mathcal{M}}_{1,(I,J)}.$$

PROPOSITION 2.1. *Suppose I and J are finite sets such that $|I| + |J| \geq 2$ and $j^* \in J$. If*

$$\pi : \widetilde{\mathcal{M}}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}_{1,I\sqcup J} \quad \textit{and} \quad \pi : \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})} \longrightarrow \bar{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}$$

are blowups as in Section 2.1, then the forgetful map

$$f : \bar{\mathcal{M}}_{1,I\sqcup J} \longrightarrow \bar{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}$$

lifts to a morphism

$$\tilde{f} : \widetilde{\mathcal{M}}_{1,(I,J)} \longrightarrow \widetilde{\mathcal{M}}_{1,(I,J-\{j^*\})}$$

(*see the LHS of Figure 1*). *Furthermore,*

$$\tilde{\psi} = \tilde{f}^*\tilde{\psi} + \sum_{i\in I} \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}. \tag{2.12}$$

LEMMA 2.2. *With notation as in Proposition 2.1, for all $i \in I$ we have*

$$\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} = \widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})} \times \bar{\mathcal{M}}_{0,\{q,i,j^*\}} \quad \textit{and}$$

$$\pi_P \circ \pi = \pi \circ \pi_P : \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} \longrightarrow \bar{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\})\sqcup(J-\{j^*\})},$$

where

$$\pi_P : \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} \longrightarrow \widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}$$

is again the projection onto the first component. Furthermore, if $\tilde{\psi}$ denotes the universal ψ-class and if $\tilde{f}$ is as in Proposition 2.1, then

$$\tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = 0 \quad \textit{and} \quad (\tilde{f}^*\tilde{\psi})|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}} = \pi_P^*\tilde{\psi}. \tag{2.13}$$

LEMMA 2.3. *With notation as in Proposition 2.1, for all $j \in J - \{j^*\}$ we have*

$$\pi^{-1}(\bar{\mathcal{M}}_{1,\varrho_{jj^*}}) = \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \approx \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \bar{\mathcal{M}}_{0,\{q,j,j^*\}} \quad \textit{and}$$

$$\pi_P \circ \pi = \pi \circ \pi_P : \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \longrightarrow \bar{\mathcal{M}}_{1,I\sqcup((J-\{j,j^*\})\sqcup\{p\})},$$

where

$$\pi_P : \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \longrightarrow \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}$$

is again the projection onto the first component. Furthermore, if $\tilde{\psi}$ *denotes the universal* ψ*-class on* $\widetilde{\mathcal{M}}_{1,(I,J)}$ *and on* $\widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}$, *then*

$$\tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = (\tilde{f}^*\tilde{\psi})|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi_P^*\tilde{\psi}.$$

We now verify recursion (R2) of Theorem 1.1. Since $c_{j^*} \neq 0$ it follows—by the $j = j' = j^*$ case of (2.8), the first identity in (2.10), and (2.12)—that

$$\tilde{\psi}^{\tilde{c}} \cdot \prod_{j\in J} \pi^*\psi_j^{c_j} = \tilde{f}^*\Big(\psi^{\tilde{c}} \cdot \prod_{j\in J-j^*} \pi^*\psi_j^{c_j}\Big)\psi_{j^*}^{c_{j^*}}.$$

Since $c_{j^*} = 1$, it follows that

$$\langle \tilde{c};\, (c_j)_{j\in J}\rangle_{(|I|,|J|)} = \langle \tilde{c};\, (c_j)_{j\in J-j^*}\rangle_{(|I|,|J|-1)} \cdot \langle \psi_{j^*}, F\rangle, \tag{2.14}$$

where F is a general fiber of the morphism $\tilde{f}$ or (equivalently) of the morphism f. By the standard dilaton equation,

$$\langle \psi_{j^*}, F\rangle = |I| + |J| - 1; \tag{2.15}$$

this relation can also be seen directly from the definition of ψ_{j^*}. The recursion (R2) follows immediately from (2.14) and (2.15).

We now verify recursion (R3). We can assume that $\tilde{c} \neq 0$; otherwise, (R3) reduces to the standard genus-1 string equation. Note that if $i_1, i_2 \in I$ and $i_1 \neq i_2$, then

$$\overline{\mathcal{M}}_{1,\varrho_{i_1 j^*}} \cap \overline{\mathcal{M}}_{1,\varrho_{i_2 j^*}} = \emptyset \implies \widetilde{\mathcal{M}}_{1,\varrho_{i_1 j^*}} \cap \widetilde{\mathcal{M}}_{1,\varrho_{i_2 j^*}} = \emptyset. \tag{2.16}$$

Thus, by (2.12) and (2.13) applied repeatedly,

$$\tilde{\psi}^{\tilde{c}} = \tilde{\psi}^{\tilde{c}-1}\Big(\tilde{f}^*\psi + \sum_{i\in I} \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}\Big) = \tilde{f}^*\tilde{\psi}^{\tilde{c}} + \sum_{i\in I} (\pi_P^*\tilde{\psi}^{\tilde{c}-1}) \cap \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}. \tag{2.17}$$

On the other hand, by (2.11) and Lemma 2.3,

$$\pi^*\psi_j^{c_j} = \tilde{f}^*\pi^*\psi_j^{c_j} + (\pi_P^*\pi^*\psi_p^{c_j-1}) \cap \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} \quad \forall j \in J - \{j^*\}. \tag{2.18}$$

If $c_j = 0$ then we define the last term in (2.18) to be zero. Similarly to (2.16), we have

$$\overline{\mathcal{M}}_{1,\varrho_{ij^*}} \cap \overline{\mathcal{M}}_{1,\varrho_{jj^*}} = \emptyset \implies \widetilde{\mathcal{M}}_{1,\varrho_{ij^*}} \cap \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}} = \emptyset \qquad \forall j \in J - \{j^*\},\ i \in I \sqcup J - \{j, j^*\}. \tag{2.19}$$

Thus, by (2.17), (2.18), and Lemmas 2.2 and 2.3,

$$
\begin{aligned}
&\langle \tilde c;\,(c_j)_{j\in J-\{j^*\}}\rangle_{(|I|,|J|)}\\
&\equiv \Big\langle \tilde\psi^{\tilde c}\cdot \prod_{j\in J-\{j^*\}} \pi^*\psi_j^{c_j},\, \widetilde{\mathcal M}_{1,(I,J)}\Big\rangle\\
&= \Big\langle \tilde f^*\Big(\tilde\psi^{\tilde c}\cdot \prod_{j\in J-\{j^*\}} \pi^*\psi_j^{c_j}\Big),\, \widetilde{\mathcal M}_{1,(I,J)}\Big\rangle\\
&\quad + \sum_{i\in I}\Big\langle \pi_P^*\Big(\tilde\psi^{\tilde c-1}\cdot \prod_{j\in J-\{j^*\}} \pi^*\psi_j^{c_j}\Big),\, \widetilde{\mathcal M}_{1,\varrho_{ij^*}}\Big\rangle\\
&\quad + \sum_{j\in J-\{j^*\}}\Big\langle \pi_P^*\Big(\tilde\psi^{\tilde c}\cdot \pi^*\psi_p^{c_j-1}\cdot \prod_{j'\in J-\{j^*,j\}} \pi^*\psi_{j'}^{c_{j'}}\Big),\, \widetilde{\mathcal M}_{1,\varrho_{jj^*}}\Big\rangle\\
&= 0 + \sum_{i\in I}\Big\langle \tilde\psi^{\tilde c-1}\cdot \prod_{j\in J-\{j^*\}} \pi^*\psi_j^{c_j},\, \widetilde{\mathcal M}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}\Big\rangle\\
&\quad + \sum_{j\in J-\{j^*\}}\Big\langle \tilde\psi^{\tilde c}\cdot \pi^*\psi_p^{c_j-1}\cdot \prod_{j'\in J-\{j^*,j\}} \pi^*\psi_{j'}^{c_{j'}},\, \widetilde{\mathcal M}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})}\Big\rangle\\
&\equiv |I|\langle \tilde c-1;\,(c_j)_{j\in J-\{j^*\}}\rangle_{(|I|,|J|-1)} + \sum_{j\in J-\{j^*\}} \langle \tilde c;\, c_j-1,(c_{j'})_{j'\in J-\{j^*,j\}}\rangle_{(|I|,|J|-1)},
\end{aligned}
$$

as claimed.

3. Proofs of Main Structural Results

3.1. Proof of Lemma 2.2

Suppose that I is a finite set of which i, j are distinct elements. It is well known that the normal bundle $\mathcal N_{\bar{\mathcal M}_{1,I}}\bar{\mathcal M}_{1,\varrho_{ij}}$ of $\bar{\mathcal M}_{1,\varrho_{ij}}$ in $\bar{\mathcal M}_{1,I}$ is given by

$$\mathcal N_{\bar{\mathcal M}_{1,I}}\bar{\mathcal M}_{1,\varrho_{ij}} = \pi_P^*L_p\otimes \pi_B^*L_q = \pi_P^*L_p, \tag{3.1}$$

where π_P and π_B are as in (2.6) and where

$$L_p \longrightarrow \bar{\mathcal M}_{1,(I-\{i,j\})\sqcup\{p\}} \quad\text{and}\quad L_q \longrightarrow \bar{\mathcal M}_{0,\{q,i,j\}}$$

are the universal tangent line bundles for the marked points p and q (see e.g. [P]). The last equality in (3.1) follows because $\bar{\mathcal M}_{0,\{q,i,j\}}$ consists of one point.

Suppose in addition that

$$\varrho \equiv (I_P, \{I_k : k\in K\}) \in \mathcal A_1(I) \quad\text{and}\quad \varrho \prec \varrho_{ij}. \tag{3.2}$$

Then, by the definition of the partial ordering $\prec$ in (2.2),

$$\{i,j\}\subset I_k \quad\text{for some}\ \ k\in K.$$

Let $\mu_{ij}(\varrho)\in \mathcal A_1((I-\{i,j\})\sqcup\{p\})$ be obtained from ϱ by removing the element k from K and adding an element p to I_P if $I_k=\{i,j\}$ and by replacing $\{i,j\}$ in I_k with p otherwise:

$$\mu_{ij}(\varrho)=\begin{cases}(I_P\sqcup\{p\},\{I_{k'}:k'\in K-\{k\}\}) & \text{if } I_k=\{i,j\},\\ (I_P,\{(I_k-\{i,j\})\sqcup\{p\}\}\sqcup\{I_{k'}:k'\in K-\{k\}\}) & \text{if } I_k\supsetneq\{i,j\}.\end{cases}\tag{3.3}$$

It is straightforward to see that

$$\bar{\mathcal{M}}_{1,\varrho_{ij}}\cap\bar{\mathcal{M}}_{1,\varrho}=\bar{\mathcal{M}}_{1,\mu_{ij}(\varrho)}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}\subset\bar{\mathcal{M}}_{1,(I-\{i,j\})\sqcup\{p\}}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}.\tag{3.4}$$

LEMMA 3.1. *If I and J are finite sets and if $i\in I$ and $j\in J$, then the map*

$$\mu_{ij}:\{\varrho\in\mathcal{A}_1(I,J):\varrho\prec\varrho_{ij}\}\longrightarrow\mathcal{A}_1((I-\{i\})\sqcup\{p\},J-\{j\})\tag{3.5}$$

is an isomorphism of partially ordered sets.

This lemma follows easily from (2.2) and (3.3). It implies that, given an order $<$ on

$$\mathcal{A}_1((I-\{i\})\sqcup\{p\},J-\{j\})$$

that extends the partial ordering $\prec$, we can choose an order $<$ on $A_1(I,J)$ that extends the partial ordering $\prec$ such that

$$\varrho_1,\varrho_2\prec\varrho_{ij},\ \mu_{ij}(\varrho_1)<\mu_{ij}(\varrho_2)\implies\varrho_1<\varrho_2.$$

We shall refer to the constructions of Section 2.1 for the sets

$$\mathcal{A}_1((I-\{i\})\sqcup\{p\},J-\{j\})\quad\text{and}\quad\mathcal{A}_1(I,J)$$

corresponding to such compatible orders $<$. We extend the map μ_{ij} of (3.5) to $\{0\}\sqcup\mathcal{A}_1(I,J)$ by setting

$$\mu_{ij}(\varrho)=\begin{cases}\mu_{ij}(\max\{\varrho'<\varrho:\varrho'\prec\varrho_{ij}\}) & \text{if }\exists\varrho'<\varrho\text{ s.t. }\varrho'\prec\varrho_{ij},\\ 0 & \text{otherwise.}\end{cases}$$

LEMMA 3.2. *Suppose I and J are finite sets with $i\in I$ and $j\in J$. If $\varrho\in\mathcal{A}_1(I,J)$ and $\varrho<\varrho_{ij}$, then (with notation as in Section 2.1 and (2.5)) we have*

$$\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\}),J-\{j\})}\times\bar{\mathcal{M}}_{0,\{q,i,j\}},$$

$$\mathbb{E}_{\varrho}|_{\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}}=\pi_P^*\mathbb{E}_{\mu_{ij}(\varrho)},\quad\textit{and}\quad\mathcal{N}_{\bar{\mathcal{M}}^{\varrho}_{1,(I,J)}}\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\pi_P^*L_{\mu_{ij}(\varrho),p},$$

where

$$\pi_P:\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}\longrightarrow\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}$$

is the projection map onto the first component.

By (2.5), (2.7), and (3.1), Lemma 3.2 holds for $\varrho=0$. Suppose $\varrho\in\mathcal{A}_1(I,J)$, $\varrho<\varrho_{ij}$, and the three claims hold for $\varrho-1$. If $\varrho\not\prec\varrho_{ij}$, then

$$\begin{aligned}\mu_{ij}(\varrho)=\mu_{ij}(\varrho-1)\implies\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}&=\bar{\mathcal{M}}^{\mu_{ij}(\varrho-1)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})},\\ \mathbb{E}_{\mu_{ij}(\varrho)}&=\mathbb{E}_{\mu_{ij}(\varrho-1)},\\ L_{\mu_{ij}(\varrho),p}&=L_{\mu_{ij}(\varrho-1),p}.\end{aligned}\tag{3.6}$$

On the other hand, since ϱ and ϱ_{ij} are not comparable with respect to $\prec$, it follows that the blowup locus $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ in $\bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ is disjoint from $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ (see Section 2.1 and [VaZ, Lemma 2.6]). Thus,

$$\begin{gathered}\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}},\quad \mathbb{E}_{\varrho}|_{\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}}=\mathbb{E}_{\varrho-1}|_{\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}},\\ \mathcal{N}_{\bar{\mathcal{M}}^{\varrho}_{1,(I,J)}}\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\mathcal{N}_{\bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}}\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}.\end{gathered}\tag{3.7}$$

By (3.6), (3.7), and the inductive assumptions, the three claims hold for ϱ.

Suppose that $\varrho\prec\varrho_{ij}$. Since all varieties $\bar{\mathcal{M}}_{1,\varrho'}$ intersect properly in $\bar{\mathcal{M}}_{1,I\sqcup J}$ in the sense of [VaZ, Sec. 2.1], so do their proper transforms $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho'}$ in $\bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$. Furthermore,

$$\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\cap\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}\subset\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\subset\bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$$

is the proper transform of

$$\bar{\mathcal{M}}_{1,\varrho_{ij}}\cap\bar{\mathcal{M}}_{1,\varrho}\subset\bar{\mathcal{M}}_{1,\varrho_{ij}}\subset\bar{\mathcal{M}}_{1,I\sqcup J}.$$

Since $\varrho\prec\varrho_{ij}$, we have $\mu_{ij}(\varrho-1)=\mu_{ij}(\varrho)-1$. Thus, by (3.4) and the inductive assumptions,

$$\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\cap\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}=\bar{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,\mu_{ij}(\varrho)}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}\subset\bar{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}.$$

Since $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ and $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ intersect properly, the proper transform of $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ in $\bar{\mathcal{M}}^{\varrho}_{1,(I,J)}$ (i.e., the blowup of $\bar{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ along $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}$) is the blowup of $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ along $\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\cap\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}$; see [VaZ, Sec. 2.1]. Thus, $\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}$ is the blowup of

$$\bar{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}$$

along $\bar{\mathcal{M}}^{\mu_{ij}(\varrho)-1}_{1,\mu_{ij}(\varrho)}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}$. By the construction of Section 2.1, this blowup is

$$\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,((I-\{i\})\sqcup\{p\},J-\{j\})}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}.$$

Furthermore, by (2.4) and the inductive assumptions,

$$\begin{aligned}\mathbb{E}_{\varrho}|_{\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}}&=(\tilde{\pi}^*_{\varrho}\mathbb{E}_{\varrho-1}+\bar{\mathcal{M}}^{\varrho}_{1,\varrho})|_{\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}}=\tilde{\pi}^*_{\varrho}\pi^*_P\mathbb{E}_{\mu_{ij}(\varrho)-1}+\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)}\times\bar{\mathcal{M}}_{0,\{q,i,j\}}\\ &=\pi^*_P(\tilde{\pi}^*_{\mu_{ij}(\varrho)}\mathbb{E}_{\mu_{ij}(\varrho)-1}+\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)})=\pi^*_P\mathbb{E}_{\mu_{ij}(\varrho)}.\end{aligned}$$

We have thus verified the first two claims of Lemma 3.2.

It remains to determine the normal bundle $\mathcal{N}_{\bar{\mathcal{M}}^{\varrho}_{1,(I,J)}}\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}$ of $\bar{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}$ in $\bar{\mathcal{M}}^{\varrho}_{1,(I,J)}$. We note that, by (2.4) and (3.3),

$$L_{\mu_{ij}(\varrho),p}=\begin{cases}\tilde{\pi}^*_{\mu_{ij}(\varrho)-1}L_{\mu_{ij}(\varrho)-1,p}\otimes\mathcal{O}(-\bar{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)}) & \text{if } I_k=\{i,j\},\\ \tilde{\pi}^*_{\mu_{ij}(\varrho)-1}L_{\mu_{ij}(\varrho)-1,p} & \text{if } I_k\supsetneq\{i,j\},\end{cases}\tag{3.8}$$

if ϱ is as in (3.2). Furthermore, if $I_k=\{i,j\}$ then

$$\bar{\mathcal{M}}_{1,\varrho}\subset\bar{\mathcal{M}}_{1,\varrho_{ij}}\implies\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho}\subset\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}.$$

Thus, by [VaZ, Sec. 3.1],

$$\begin{aligned}
&\mathcal{N}_{\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}\\
&\quad=\tilde{\pi}^*_{\varrho}\mathcal{N}_{\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\otimes\mathcal{O}(-\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}\cap\overline{\mathcal{M}}^{\varrho}_{1,\varrho})\\
&\quad=\tilde{\pi}^*_{\varrho}\mathcal{N}_{\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\otimes\pi_P^*\mathcal{O}(-\overline{\mathcal{M}}^{\mu_{ij}(\varrho)}_{1,\mu_{ij}(\varrho)})\quad\text{if }\ I_k=\{i,j\}.
\end{aligned}\tag{3.9}$$

On the other hand, if $I_k \supsetneq \{i,j\}$ then $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ and $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}$ intersect transversally in $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$, since $\overline{\mathcal{M}}_{1,\varrho}$ and $\overline{\mathcal{M}}_{1,\varrho_{ij}}$ intersect transversally in $\overline{\mathcal{M}}_{1,I\sqcup J}$. Therefore,

$$\mathcal{N}_{\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{ij}}=\tilde{\pi}^*_{\varrho}\mathcal{N}_{\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij}}\quad\text{if }\ I_k\supsetneq\{i,j\}.\tag{3.10}$$

The last statement of Lemma 3.2 now follows from the corresponding inductive assumption for $\varrho-1$ along with (3.8)–(3.10). This completes the proof of Lemma 3.2.

We now finish the proof of Lemma 2.2. By the paragraph preceding Proposition 2.1 and the first statement of Lemma 3.2,

$$\begin{aligned}
\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}=\overline{\mathcal{M}}^{\varrho_{ij^*}-1}_{1,\varrho_{ij^*}}&=\overline{\mathcal{M}}^{\mu_{ij^*}(\varrho_{ij^*}-1)}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}\times\overline{\mathcal{M}}_{0,\{q,i,j^*\}}\\
&=\widetilde{\mathcal{M}}_{1,((I-\{i\})\sqcup\{p\},J-\{j^*\})}\times\overline{\mathcal{M}}_{0,\{q,i,j^*\}},
\end{aligned}$$

since $\mu_{ij^*}(\varrho_{ij^*}-1)$ is the largest element of

$$(\mathcal{A}_1((I-\{i\})\sqcup\{p\},J-\{j^*\}),<)$$

according to Lemma 3.1.

Because ϱ_{ij^*} is a maximal element of $(\mathcal{A}_1(I,J),\prec)$, we have

$$\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{ij^*}}\cap\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}=\emptyset\quad\forall\varrho\in\mathcal{A}_1(I,J),\ \varrho>\varrho_{ij^*}.$$

Thus, by (2.4) and the second statement of Lemma 3.2,

$$\tilde{\mathbb{E}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}=\mathbb{E}_{\varrho_{ij^*}-1}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}+\sum_{\varrho\geq\varrho_{ij^*}}\overline{\mathcal{M}}^{\varrho}_{1,\varrho}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}=\pi_P^*\tilde{\mathbb{E}}+\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}.\tag{3.11}$$

By the third statement of Lemma 3.2,

$$\begin{aligned}
\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}=\mathcal{N}_{\widetilde{\mathcal{M}}_{1,(I,J)}}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}&=\mathcal{N}_{\overline{\mathcal{M}}^{\varrho_{ij^*}-1}_{1,(I,J)}}\overline{\mathcal{M}}^{\varrho_{ij^*}-1}_{1,\varrho_{ij^*}}\\
&=\pi_P^*L_{\mu_{ij^*}(\varrho_{ij^*}-1),p}=-\pi_P^*\tilde{\mathbb{E}}.
\end{aligned}\tag{3.12}$$

The first identity in (2.13) now follows from (3.11) and (3.12).

Finally, by the last statement of Proposition 2.1, the first identity in (2.13), (2.16), and (3.12), it follows that

$$(\tilde{f}^*\tilde{\psi})|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}=\tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}-\sum_{i'\in I}\widetilde{\mathcal{M}}_{1,\varrho_{i'j^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}=0-\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}}=\pi_P^*\tilde{\psi}.$$

This concludes the proof of Lemma 2.2.

3.2. *Proof of Lemma 2.3*

The proof of Lemma 2.3 is analogous to that for Lemma 2.2. If I is a finite set and j, j^* are distinct elements of I, let

$$\begin{aligned}\mathcal{A}_1(I;\, jj^*)&\\ &= \{\varrho\in\mathcal{A}_1(I)-\{\varrho_{jj^*}\} : \bar{\mathcal{M}}_{1,\varrho_{jj^*}}\cap\bar{\mathcal{M}}_{1,\varrho}\neq\emptyset\}\\ &= \{(I_P,\{I_k : k\in K\})\in\mathcal{A}_1(I)-\{\varrho_{jj^*}\} : \{j,j^*\}\subset I_k \text{ for some } k\in\{P\}\sqcup K\}.\end{aligned}$$

For each $\varrho\in\mathcal{A}_1(I;\, jj^*)$ as just described, let $\eta_{jj^*}(\varrho)\in\mathcal{A}_1((I-\{j,j^*\})\sqcup\{p\})$ be obtained from ϱ by replacing $\{j,j^*\}\subset I_k$ with p if $k=P$ or $\{j,j^*\}\subsetneq I_k$ and by dropping k from K and adding p to I_P otherwise:

$$\eta_{jj^*}(\varrho)=\begin{cases}((I_P-\{j,j^*\})\sqcup\{p\},\{I_{k'} : k'\in K\}) & \text{if } I_P\supset\{j,j^*\},\\ \big(I_P,\{(I_k-\{j,j^*\})\sqcup\{p\}\}\sqcup\{I_{k'} : k'\in K-\{k\}\}\big) & \text{if } I_k\supsetneq\{j,j^*\},\\ (I_P\sqcup\{p\},\{I_{k'} : k'\in K-\{k\}\}) & \text{if } I_k=\{j,j^*\}.\end{cases}\tag{3.13}$$

It is straightforward to see that

$$\bar{\mathcal{M}}_{1,\varrho_{jj^*}}\cap\bar{\mathcal{M}}_{1,\varrho} = \bar{\mathcal{M}}_{1,\eta_{jj^*}(\varrho)}\times\bar{\mathcal{M}}_{0,\{q,j,j^*\}}\subset\bar{\mathcal{M}}_{1,(I-\{j,j^*\})\sqcup\{p\}}\times\bar{\mathcal{M}}_{0,\{q,j,j^*\}}.\tag{3.14}$$

LEMMA 3.3. *If I and J are finite sets, $j,j^*\in J$, and $j\neq j^*$, then the map*

$$\eta_{jj^*}\colon\mathcal{A}_1(I,J)\cap\mathcal{A}_1(I\sqcup J;\, jj^*)\longrightarrow\mathcal{A}_1\big((I,(J-\{j,j^*\})\sqcup\{p\}\big)\tag{3.15}$$

is an isomorphism of partially ordered sets.

This lemma follows easily from (2.2) and (3.13). Note, however, that it is essential that $j,j^*\in J$ and thus the third case in (3.13) does not occur if

$$\varrho\in\mathcal{A}_1(I,J)\cap\mathcal{A}_1(I\sqcup J;\, jj^*).$$

Lemma 3.3 implies that, given an order $<$ on

$$\mathcal{A}_1\big((I,(J-\{j,j^*\})\sqcup\{p\}\big)$$

that extends the partial ordering $\prec$, we can choose an order $<$ on $A_1(I,J)$ that extends the partial ordering $\prec$ such that

$$\varrho_1,\varrho_2\in\mathcal{A}_1(I,J)\cap\mathcal{A}_1(I\sqcup J;\, jj^*),\ \eta_{jj^*}(\varrho_1)<\eta_{jj^*}(\varrho_2)\implies\varrho_1<\varrho_2.$$

We shall refer again to the constructions of Section 2.1 for the sets

$$\mathcal{A}_1\big((I,(J-\{j,j^*\})\sqcup\{p\}\big)\quad\text{and}\quad\mathcal{A}_1(I,J)$$

corresponding to such compatible orders $<$. We extend the map η_{jj^*} of (3.15) to $\{0\}\sqcup\mathcal{A}_1(I,J)$ by setting

$$\eta_{jj^*}(\varrho)=\begin{cases}\eta_{jj^*}(\max\{\varrho'<\varrho : \varrho'\in\mathcal{A}_1(I\sqcup J;\, jj^*)\})\\ \qquad\qquad\text{if } \exists\varrho'<\varrho \text{ s.t. } \varrho'\in\mathcal{A}_1(I\sqcup J;\, jj^*),\\ 0 \qquad\qquad\text{otherwise.}\end{cases}$$

LEMMA 3.4. *Suppose I and J are finite sets with $j,j^*\in J$ and $j\neq j^*$. If $\varrho\in\mathcal{A}_1(I,J)$, then* (*with notation as in Section 2.1 and* (2.5)) *we have*

$$\pi_\varrho^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}) = \overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}} = \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}},$$
$$\mathbb{E}_\varrho\big|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}} = \pi_P^*\mathbb{E}_{\eta_{jj^*}(\varrho)},$$

where

$$\pi_P : \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}} \longrightarrow \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})}$$

is the projection map onto the first component.

By (2.5) and (2.7), Lemma 3.4 holds for $\varrho = 0$. Suppose $\varrho \in \mathcal{A}_1(I,J)$ and the three claims hold for $\varrho-1$. If $\varrho \notin \mathcal{A}_1(I \sqcup J;\, jj^*)$, then

$$\eta_{jj^*}(\varrho) = \eta_{jj^*}(\varrho-1) \implies \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})} = \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho-1)}_{1,((I,(J-\{j,j^*\})\sqcup\{p\})},$$
$$\mathbb{E}_{\eta_{jj^*}(\varrho)} = \mathbb{E}_{\eta_{jj^*}(\varrho-1)}. \tag{3.16}$$

On the other hand, $\overline{\mathcal{M}}_{1,\varrho_{jj^*}} \cap \overline{\mathcal{M}}_{1,\varrho} = \emptyset$ and so the blowup locus $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ in $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ is disjoint from $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$. Therefore,

$$\pi_\varrho^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}) = \pi_{\varrho-1}^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}), \quad \overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}} = \overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}},$$
$$\mathbb{E}_\varrho\big|_{\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}} = \mathbb{E}_{\varrho-1}\big|_{\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}}. \tag{3.17}$$

By (3.16), (3.17), and the inductive assumptions, the three claims of Lemma 3.4 hold for ϱ.

Suppose that $\varrho \in \mathcal{A}_1(I \sqcup J;\, jj^*)$. Since all varieties $\overline{\mathcal{M}}_{1,\varrho'}$ intersect properly in $\overline{\mathcal{M}}_{1,I\sqcup J}$, so do their proper transforms $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho'}$ ($\varrho' > \varrho-1$) in $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$. Since $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ is not contained in the divisor $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$, it follows that $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ and $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ intersect transversally. Thus, using the first statement of the lemma with ϱ replaced by $\varrho-1$, we obtain

$$\pi_\varrho^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}) = \tilde{\pi}_\varrho^{-1}\pi_{\varrho-1}^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}) = \tilde{\pi}_\varrho^{-1}(\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}) = \overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}.$$

Furthermore,

$$\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}} \cap \overline{\mathcal{M}}^{\varrho-1}_{1,\varrho} \subset \overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}} \subset \overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$$

is the proper transform of

$$\overline{\mathcal{M}}_{1,\varrho_{jj^*}} \cap \overline{\mathcal{M}}_{1,\varrho} \subset \overline{\mathcal{M}}_{1,\varrho_{jj^*}} \subset \overline{\mathcal{M}}_{1,I\sqcup J}.$$

Since $\varrho \in \mathcal{A}_1(I \sqcup J;\, jj^*)$, we have $\eta_{jj^*}(\varrho-1) = \eta_{jj^*}(\varrho)-1$. Thus, by (3.14) and the inductive assumptions,

$$\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}} \cap \overline{\mathcal{M}}^{\varrho-1}_{1,\varrho} = \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,\eta_{jj^*}(\varrho)} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}} \subset \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}.$$

Since $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ and $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$ intersect properly, the proper transform of $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ in $\overline{\mathcal{M}}^{\varrho}_{1,(I,J)}$ (i.e., the blowup of $\overline{\mathcal{M}}^{\varrho-1}_{1,(I,J)}$ along $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$) is the blowup of $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}}$ along $\overline{\mathcal{M}}^{\varrho-1}_{1,\varrho_{jj^*}} \cap \overline{\mathcal{M}}^{\varrho-1}_{1,\varrho}$; see [VaZ, Sec. 2.1]. Thus, $\overline{\mathcal{M}}^{\varrho}_{1,\varrho_{jj^*}}$ is the blowup of

$$\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}$$

along $\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)-1}_{1,\eta_{jj^*}(\varrho)} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}$. By the construction of Section 2.1, this blowup is

$$\overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}.$$

Furthermore, by (2.4) and the inductive assumptions,

$$\begin{aligned}\mathbb{E}_\varrho|_{\overline{\mathcal{M}}^\varrho_{1,\varrho_{jj^*}}} &= (\tilde{\pi}_\varrho^*\mathbb{E}_{\varrho-1} + \overline{\mathcal{M}}^\varrho_{1,\varrho})|_{\overline{\mathcal{M}}^\varrho_{1,\varrho_{jj^*}}}\\ &= \tilde{\pi}_\varrho^*\pi_P^*\mathbb{E}_{\eta_{jj^*}(\varrho)-1} + \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,\eta_{jj^*}(\varrho)} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}\\ &= \pi_P^*(\tilde{\pi}^*_{\eta_{jj^*}(\varrho)}\mathbb{E}_{\eta_{jj^*}(\varrho)-1} + \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho)}_{1,\eta_{jj^*}(\varrho)}) = \pi_P^*\mathbb{E}_{\eta_{jj^*}(\varrho)}.\end{aligned}$$

We have thus verified the three claims of Lemma 3.4.

We now finish the proof of Lemma 2.3. By Lemma 3.3, $\eta_{jj^*}(\varrho_{\max})$ is the largest element of

$$(\mathcal{A}_1(I,(J-\{j,j^*\})\sqcup\{p\}),<).$$

Hence, by the first two statements of Lemma 3.4,

$$\begin{aligned}\pi^{-1}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}) = \pi^{-1}_{\varrho_{\max}}(\overline{\mathcal{M}}_{1,\varrho_{jj^*}}) = \overline{\mathcal{M}}^{\varrho_{\max}}_{1,\varrho_{jj^*}} &= \widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}\\ &= \overline{\mathcal{M}}^{\eta_{jj^*}(\varrho_{\max})}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}\\ &= \widetilde{\mathcal{M}}_{1,(I,(J-\{j,j^*\})\sqcup\{p\})} \times \overline{\mathcal{M}}_{0,\{q,j,j^*\}}.\end{aligned}$$

By the last statement of Lemma 3.4, we have

$$\tilde{\mathbb{E}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \mathbb{E}_{\varrho_{\max}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi_P^*\mathbb{E}_{\eta_{jj^*}(\varrho_{\max})} = \pi_P^*\tilde{\mathbb{E}}.$$

Finally, by the last statement of Proposition 2.1 and (2.19),

$$(\tilde{f}^*\tilde{\psi})|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \tilde{\psi}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} - \sum_{i\in I}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}}|_{\widetilde{\mathcal{M}}_{1,\varrho_{jj^*}}} = \pi_P^*\tilde{\psi} - 0.$$

3.3. *Proof of Proposition 2.1*

In this section we prove Proposition 2.1. In fact, we show that there is a lift of the forgetful map f of Proposition 2.1 to morphisms between corresponding stages of the blowup construction of Section 2.1 for $\overline{\mathcal{M}}_{1,I\sqcup J}$ and for $\overline{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}$ (see Lemma 3.5).

First, we define a forgetful map

$$f\colon \mathcal{A}_1(I,J) \longrightarrow \bar{\mathcal{A}}_1(I,J-\{j^*\}) \equiv \mathcal{A}_1(I,J-\{j^*\}) \sqcup \{(I\sqcup(J-\{j^*\}),\emptyset)\}.$$

If $\varrho = (I_P\sqcup J_P,\{I_k\sqcup J_k : k\in K\})$, we put

$$f(\varrho) = \begin{cases} (I_P\sqcup(J_P-\{j^*\}),\{I_k\sqcup J_k : k\in K\}) & \text{if } j^*\in J_P,\\ \big(I_P\sqcup J_P,\{I_k\sqcup(J_k-\{j^*\})\}\sqcup\{I_{k'}\sqcup J_{k'} : k'\in K-\{k\}\}\big) & \\ & \text{if } j^*\in J_k,\ |I_k|+|J_k|>2,\\ \big((I_P\sqcup\{i\})\sqcup J_P,\{I_{k'}\sqcup J_{k'} : k'\in K-\{k\}\}\big) & \\ & \text{if } I_k\sqcup J_k=\{ij^*\}.\end{cases}$$

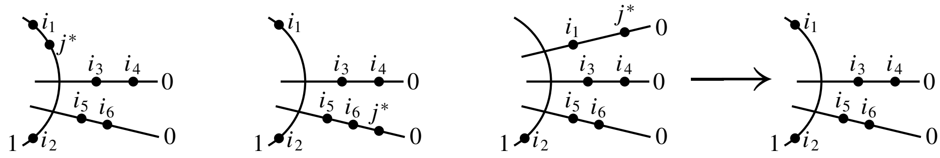

Figure 4 Images under the forgetful map

These three cases are represented in the LHS of Figure 4. We note that for all $\rho \in \mathcal{A}_1(I, J - \{j^*\})$,

$$f^{-1}(\bar{\mathcal{M}}_{1,\rho}) = \bigcup_{\varrho \in f^{-1}(\rho)} \bar{\mathcal{M}}_{1,\varrho}.$$

Furthermore,

$$\rho_1, \rho_2 \in \bar{\mathcal{A}}_1(I, J - \{j^*\}),\ \rho_1 \neq \rho_2,\ \varrho_1 \in f^{-1}(\rho_1),\ \varrho_2 \in f^{-1}(\rho_2),\ \varrho_1 \prec \varrho_2 \implies \rho_1 \prec \rho_2.$$

Thus, given an order $<$ on $\mathcal{A}_1(I, J - \{j^*\})$ extending the partial ordering $\prec$, we can choose an order $<$ on $\mathcal{A}_1(I, J)$ extending $\prec$ such that

$$\rho_1, \rho_2 \in \bar{\mathcal{A}}_1(I, J - \{j^*\}),\ \rho_1 < \rho_2,\ \varrho_1 \in f^{-1}(\rho_1),\ \varrho_2 \in f^{-1}(\rho_2) \implies \varrho_1 < \varrho_2.$$

In the sequel we refer to the blowup constructions of Section 2.1 for $\bar{\mathcal{M}}_{1,I\sqcup J}$ and $\bar{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})}$ corresponding to such compatible orders. Now, for each $\rho \in \mathcal{A}_1(I, J - \{j^*\})$, let

$$\rho^+ = \max f^{-1}(\rho) \in \mathcal{A}_1(I, J), \qquad \rho^- = \min f^{-1}(\rho) - 1 \in \{0\} \sqcup \mathcal{A}_1(I, J).$$

If ρ is not the minimal element of $\mathcal{A}_1(I, J - \{j^*\})$, then $\rho^- = (\rho - 1)^+$.

Lemma 3.5. *Suppose I, J, and f are as in Proposition 2.1. Then, for each $\rho \in \mathcal{A}_1(I, J - \{j^*\})$, f lifts to a morphism*

$$f_\rho : \bar{\mathcal{M}}^{\rho^+}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$$

over the projection maps

$$\pi_{\rho^+} : \bar{\mathcal{M}}^{\rho^+}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}_{1,I\sqcup J} \quad \text{and} \quad \pi_\rho : \bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \longrightarrow \bar{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})};$$

see the LHS of Figure 5. Furthermore,

$$f_\rho^{-1}(\bar{\mathcal{M}}^{\rho}_{1,\rho^*}) = \bigcup_{\varrho \in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+}_{1,\varrho}\ \forall \rho^* > \rho \quad \text{and} \quad \mathbb{E}_{\rho^+} = f_\rho^* \mathbb{E}_\rho. \tag{3.18}$$

Proposition 2.1 follows easily from Lemma 3.5 by taking $\rho = \rho_{\max}$, where $\rho_{\max}$ is the maximal element of $\mathcal{A}_1(I, J - \{j^*\})$. We note that

$$\begin{array}{ccc} \bar{\mathcal{M}}^{\rho^+}_{1,(I,J)} & \xrightarrow{f_\rho} & \bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \\ {\scriptstyle \pi_{\rho^+}}\Big\downarrow & & {\scriptstyle \pi_\rho}\Big\downarrow \\ \bar{\mathcal{M}}_{1,I\sqcup J} & \xrightarrow{f} & \bar{\mathcal{M}}_{1,I\sqcup(J-\{j^*\})} \end{array} \qquad \begin{array}{ccc} \bar{\mathcal{M}}^{\rho^+}_{1,(I,J)} & \xrightarrow{f_\rho} & \bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \\ {\scriptstyle \tilde{\pi}_{\rho^-+1}\circ\cdots\circ\tilde{\pi}_{\rho^+}}\Big\downarrow & & {\scriptstyle \tilde{\pi}_\rho}\Big\downarrow \\ \bar{\mathcal{M}}^{\rho^-}_{1,(I,J)} & \xrightarrow{f_{\rho-1}} & \bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})} \end{array}$$

Figure 5 Main statement of Lemma 3.5 and inductive step in the proof

$$\begin{aligned}\{\varrho\in\mathcal{A}_1(I,J):\varrho>\rho^+_{\max}\} &= \{\varrho\in\mathcal{A}_1(I,J):f(\varrho)=(I\sqcup(J-\{j^*\}),\emptyset)\}\\ &= \{\varrho_{ij^*}:i\in I\}.\end{aligned}$$

Since $\bar{\mathcal{M}}_{1,\varrho_{ij^*}}\subset\bar{\mathcal{M}}_{1,I\sqcup J}$ is a divisor for every $i\in I$, so is

$$\bar{\mathcal{M}}^{\rho^+_{\max}}_{1,\varrho_{ij^*}}\subset\bar{\mathcal{M}}^{\rho^+_{\max}}_{1,I\sqcup J}.$$

Thus, by the construction of Section 2.1,

$$\widetilde{\mathcal{M}}_{1,I\sqcup J}\equiv\bar{\mathcal{M}}^{\varrho_{\max}}_{1,I\sqcup J}=\bar{\mathcal{M}}^{\rho^+_{\max}}_{1,I\sqcup J}\quad\text{and}$$

$$\begin{aligned}\mathbb{E}\equiv\mathbb{E}_{\varrho_{\max}} &= \mathbb{E}_{\rho^+_{\max}}+\sum_{i\in I}\bar{\mathcal{M}}^{\rho^+_{\max}}_{1,\varrho_{ij^*}}\\ &= f^*_{\rho_{\max}}\mathbb{E}_{\rho_{\max}}+\sum_{i\in I}\bar{\mathcal{M}}^{\rho^+_{\max}}_{1,\varrho_{ij^*}}=\tilde{f}^*\mathbb{E}+\sum_{i\in I}\widetilde{\mathcal{M}}_{1,\varrho_{ij^*}},\end{aligned}$$

where $\tilde{f}=f_{\rho_{\max}}$.

Lemma 3.5 will be proved by induction on ρ. It holds for $\rho=0\in\{0\}\cup\mathcal{A}_1(I,J-\{j^*\})$ if we define $0^+=0$. Suppose that

$$\rho=(I_P\sqcup J_P,\{I_k\sqcup J_k:k\in K\})\in\mathcal{A}_1(I,J-\{j^*\})$$

and that the lemma holds for

$$\rho-1\in\{0\}\sqcup\mathcal{A}_1(I,J-\{j^*\}).$$

The elements of $f^{-1}(\rho)\subset\mathcal{A}_1(I,J)$ can be described as follows. The largest element is

$$\rho^+=(I_P\sqcup(J_P\sqcup\{j^*\}),\{I_k\sqcup J_k:k\in K\}).$$

Furthermore, for each $k\in K$ and $i\in I_P$:

$$\begin{aligned}\rho_k(j^*) &\equiv \big(I_P\sqcup J_P,\{I_k\sqcup(J_k\sqcup\{j^*\})\}\sqcup\{I_{k'}\sqcup J_{k'}:k'\in K-\{k\}\}\big)\in f^{-1}(\rho);\\ \rho_i(j^*) &\equiv \big((I_P-\{i\})\sqcup J_P,\{\{i,j\}\}\sqcup\{I_{k'}\sqcup J_{k'}:k'\in K\}\big)\in f^{-1}(\rho).\end{aligned}$$

It is straightforward to see that

$$f^{-1}(\rho)=\{\rho_k(j^*):k\in K\}\sqcup\{\rho_i(j^*):i\in I_P\}\sqcup\{\rho^+\}.$$

Furthermore, ρ^+ is the largest element of $(f^{-1}(\rho),\prec)$, and no two elements of the form $\rho_k(j^*)$ and/or $\rho_i(j^*)$ are comparable with respect to $\prec$. Thus,

$$\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)} = \emptyset \quad \forall i,k \in I_P \sqcup K,\ i \neq k$$

(see Section 2.1). In fact,

$$\bar{\mathcal{M}}_{1,\rho_k(j^*)} \cap \bar{\mathcal{M}}_{1,\rho_i(j^*)} = \emptyset \quad \forall i,k \in I_P \sqcup K,\ i \neq k$$

(see the proof of Lemma 2.6 in [VaZ]). On the other hand, $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)} \subset \bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$ for $i \in I_P$, while $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and $\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$ intersect at a divisor (divisor inside each of them) if $k \in K$.

We will show that every point

$$p \in f^{-1}_{\rho-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho}) \subset \bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$$

has a neighborhood $\tilde{U}$ such that $f_{\rho-1}$ lifts to a morphism f_ρ from the preimage of $\tilde{U}$ in $\bar{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ to $\bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$. Since $\bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$ is the blowup of $\bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$ along $\bar{\mathcal{M}}^{\rho-1}_{1,\rho}$, this implies that $f_{\rho-1}$ lifts to a morphism

$$f_\rho\colon \bar{\mathcal{M}}^{\rho^+}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}.$$

We will consider four cases as follows.

Case 1. $p \in \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)} - \bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$ and thus $p \notin \bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)}$ for all $i \in I_P \sqcup K - k$.

Case 2. $p \in \bar{\mathcal{M}}^{\rho^-}_{1,\rho^+} - \bigcup_{k\in K} \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$:

Case 2a. $p \notin \bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)}$ for all $i \in I_P$;

Case 2b. $p \in \bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)}$ for some $i \in I_P$ and thus

$p \notin \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ for all $k \in I_P \sqcup K - i$.

Case 3. $p \in \bar{\mathcal{M}}^{\rho^-}_{1,\rho^+} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and thus $p \notin \bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)}$ for all $i \in I_P \sqcup K - k$.

Case 1. Since all varieties $\bar{\mathcal{M}}_{1,\varrho^*}$ are smooth and intersect properly in $\bar{\mathcal{M}}_{1,I\sqcup J}$ in the sense of [VaZ, Sec. 2.1], all varieties $\bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*}$ with $\varrho^* > \rho^-$ are also smooth and intersect properly in $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$. Thus, we can choose a neighborhood $\tilde{U}$ of p in $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$, a neighborhood U of $f_{\rho-1}(p)$ in $\bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$, and coordinates (z,v,t) on $\tilde{U}$ such that:

(i) $U = f_{\rho-1}(\tilde{U})$;
(ii) $U = \{(z,v) \in \mathbb{C}^{|I|+|J|-|K|-1} \times \mathbb{C}^K\}$;
(iii) $\bar{\mathcal{M}}^{\rho-1}_{1,\rho} \cap U = \{(z,v) \in U : v = 0\}$;
(iv) $\tilde{U} = \{(z,v,t) \in \mathbb{C}^{|I|+|J|-|K|-1} \times \mathbb{C}^K \times \mathbb{C}\}$ and $f_{\rho-1}(z,v,t) = (z,v)$.

These assumptions imply that

$$\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)} \cap \tilde{U} = \{(z,v,t) \in \tilde{U} : v = 0\}.$$

Since $\bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$ is the blowup of $\bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$ along $\bar{\mathcal{M}}^{\rho-1}_{1,\rho}$, the preimage of U in $\bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$ under the projection map is

$$V = \{(z,v;\ell) \in U \times \mathbb{P}(\mathbb{C}^K) : v \in \ell\}.$$

Since $\bar{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ is the blowup of $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$ along $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and subvarieties that do not contain p, the preimage of $\tilde{U}$ in $\bar{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ under the projection map is

$$\tilde{V} = \{(z,v,t;\ell) \in \tilde{U} \times \mathbb{P}(\mathbb{C}^K) : v \in \ell\},$$

provided $\tilde{U}$ is sufficiently small. Hence the morphism $f_{\rho-1}\colon \tilde{U} \to U$ lifts to a morphism $f_\rho\colon \tilde{V} \to V$, and this lift is defined by

$$f_\rho(z,v,t;\ell) = (z,v;\ell). \tag{3.19}$$

Case 2. We can choose a neighborhood $\tilde{U}$ of p in $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$, a neighborhood U of $f_{\rho-1}(p)$ in $\bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$, and coordinates (z,v,t) on $\tilde{U}$ such that conditions (i)–(iv) are satisfied with $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ replaced by $\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$.

Case 2a. The desired conclusion is obtained as in Case 1.

Case 2b. Since $\bar{\mathcal{M}}_{1,\rho_i(j^*)} \subset \bar{\mathcal{M}}_{1,\rho^+}$ is of codimension 1,

$$\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)} \subset \bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$$

is also of codimension 1. We can thus choose local coordinates so that

$$\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)} \cap \tilde{U} = \{(z,v,t) \in \tilde{U} : v = 0,\ t = 0\}.$$

Since $\bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ is the blowup of $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$ along $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)}$ and subvarieties that do not contain p, the preimage of $\tilde{U}$ in $\bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ under the projection map is

$$\tilde{V} = \{(z,v,t;\ell') \in \tilde{U} \times \mathbb{P}(\mathbb{C}^K \times \mathbb{C}) : (v,t) \in \ell'\},$$

provided $\tilde{U}$ is sufficiently small. It is then immediate that

$$\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+} \cap \tilde{V} = \{(z,0,t;[\alpha,\beta]) \in \tilde{U} \times \mathbb{P}(\mathbb{C}^K \times \mathbb{C}) : \alpha = 0\},$$

where $\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+} \subset \bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ is the proper transform of $\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$. A neighborhood of $\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+} \cap \tilde{V}$ is given by

$$\tilde{V}' = \{(z,u,t) \in \mathbb{C}^{|I|+|J|-|K|-1} \times \mathbb{C}^K \times \mathbb{C}\}, \quad (z,u,t) \longleftrightarrow (z,ut,t;[u,1]) \in \tilde{V}.$$

Since $\bar{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ is the blowup of $\bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ along $\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+}$, the preimage of $\tilde{U}$ in $\bar{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ under the projection map is

$$\widetilde{W} = (\{(z,u,t;\ell) \in \tilde{V}' \times \mathbb{P}(\mathbb{C}^K) : u \in \ell\} \cup \{(z,v,t;[\alpha,\beta]) \in \tilde{V} : \alpha \neq 0\})/\sim,$$
$$(z,u,t;\ell) \sim (z,ut,t;[u,1]).$$

Thus, the morphism $f_{\rho-1}\colon \tilde{U} \to U$ lifts to a morphism $f_\rho\colon \widetilde{W} \to V$. This lift is defined by

$$f_\rho(z,u,t;\ell) = (z,ut;\ell) \quad \text{and} \quad f_\rho(z,v,t;[\alpha,\beta]) = (z,v;[\alpha]) \tag{3.20}$$

on the two charts on $\widetilde{W}$. Observe that if $u \neq 0$ then $[u] = \ell \in \mathbb{P}(\mathbb{C}^K)$. Hence f_ρ agrees on the overlap of the two charts.

Case 3. Since the varieties $\bar{\mathcal{M}}_{1,\varrho^*}$ intersect properly in $\bar{\mathcal{M}}_{1,I\sqcup J}$, it follows that $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and $\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$ intersect properly in $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$ and that $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$ is the proper transform of $\bar{\mathcal{M}}_{1,\rho_k(j^*)} \cap \bar{\mathcal{M}}_{1,\rho^+}$. Thus, $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$ is a divisor in $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and in $\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$. We can therefore choose a neighborhood $\tilde{U}$ of p in $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$, a neighborhood U of $f_{\rho-1}(p)$ in $\bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$, and coordinates (z,v,w_k,w_+) on $\tilde{U}$ such that:

(i) $U = f_{\rho-1}(\tilde{U})$;
(ii) $U = \{(z,v,w) \in \mathbb{C}^{|I|+|J|-|K|-1} \times \mathbb{C}^{K-\{k\}} \times \mathbb{C}\}$;
(iii) $\bar{\mathcal{M}}^{\rho-1}_{1,\rho} \cap U = \{(z,v,w) \in U : v = 0,\ w = 0\}$;
(iv) $\tilde{U} = \{(z,v,w_k,w_+) \in \mathbb{C}^{|I|+|J|-|K|-1} \times \mathbb{C}^{K-\{k\}} \times \mathbb{C} \times \mathbb{C}\}$, $f_{\rho-1}(z,v,w_k,w_+) = (z,v,w_k w_+)$;
(v) $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)} \cap \tilde{U} = \{(z,v,w_k,w_+) \in \tilde{U} : v = 0,\ w_+ = 0\}$;
(vi) $\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+} \cap \tilde{U} = \{(z,v,w_k,w_+) \in \tilde{U} : v = 0,\ w_k = 0\}$.

As before, the preimage of U in $\bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})}$ under the projection map is

$$V = \{(z,v,w;\ell) \in U \times \mathbb{P}(\mathbb{C}^{K-\{k\}} \times \mathbb{C}) : (v,w) \in \ell\}.$$

Since $\bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ is the blowup of $\bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}$ along $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and subvarieties that do not contain p, the preimage of $\tilde{U}$ in $\bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ under the projection map is

$$\tilde{V} = \{(z,v,w_k,w_+;\ell_k) \in \tilde{U} \times \mathbb{P}(\mathbb{C}^{K-\{k\}} \times \mathbb{C}) : (v,w_+) \in \ell_k\},$$

provided $\tilde{U}$ is sufficiently small. It is immediate that

$$\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+} \cap \tilde{V} = \{(z,0,0,w_+;[\alpha,\beta]) \in \tilde{U} \times \mathbb{P}(\mathbb{C}^{K-\{k\}} \times \mathbb{C}) : \alpha = 0\},$$

where $\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+} \subset \bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ is the proper transform of $\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}$. A neighborhood of $\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+} \cap \tilde{V}$ is given by

$$\tilde{V}' = \{(z,u,u_k,w_+) \in \mathbb{C}^{|I|+|J|-|K|-1} \times \mathbb{C}^{K-\{k\}} \times \mathbb{C} \times \mathbb{C}\},$$

$$(z,u,u_k,w_+) \longleftrightarrow (z,uw_+,u_k,w_+;[u,1]) \in \tilde{V}.$$

Since $\bar{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ is the blowup of $\bar{\mathcal{M}}^{\rho^+-1}_{1,(I,J)}$ along $\bar{\mathcal{M}}^{\rho^+-1}_{1,\rho^+}$, the preimage of $\tilde{U}$ in $\bar{\mathcal{M}}^{\rho^+}_{1,(I,J)}$ under the projection map is

$$\begin{aligned}\widetilde{W} = \big(&\{(z,u,u_k,w_+;\ell) \in \tilde{V}' \times \mathbb{P}(\mathbb{C}^{K-\{k\}} \times \mathbb{C}) : (u,u_k) \in \ell\}\\ &\cup \{(z,v,w_k,w_+;[\alpha,\beta]) \in \tilde{V} : \alpha \neq 0\}\big)/\sim,\\ &(z,u,u_k,w_+;\ell) \sim (z,uw_+,u_k,w_+;[u,1]).\end{aligned}$$

Thus $f_{\rho-1}\colon \tilde{U} \to U$ lifts to a morphism $f_\rho\colon \widetilde{W} \to V$, and this lift is defined by

$$\begin{aligned} f_\rho(z,u,u_k,w_+;\ell) &= (z,uw_+,u_k w_+;\ell) \quad \text{and}\\ f_\rho(z,v,w_k,w_+;[\alpha,\beta]) &= (z,v,w_k w_+;[\alpha,w_k\beta]) \end{aligned} \tag{3.21}$$

on the two charts on $\widetilde{W}$. It is immediate that f_ρ is well-defined on the overlap of the two charts.

REMARK. The first identity in (3.18) should be viewed as incorporating the previous information concerning the local structure of the projection map. It is straightforward to see from the verification (to follow) of the first equality in (3.18) that this additional information is also preserved by the inductive step.

It remains to verify the two identities in (3.18). Let

$$\pi_{\rho,\rho-1}\colon \bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \longrightarrow \bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})} \quad \text{and}$$
$$\pi_{\rho^+,\varrho}\colon \bar{\mathcal{M}}^{\rho^+}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}^{\varrho}_{1,(I,J)}, \quad \varrho \in \{\rho^-\}\cup f^{-1}(\rho),$$

be the projection maps. By the construction of the line bundles $\mathbb{E}_\varrho$ in Section 2.1,

$$\mathbb{E}_\rho = \pi^*_{\rho,\rho-1}\mathbb{E}_\rho + \bar{\mathcal{M}}^{\rho}_{1,\rho} \quad \text{and} \tag{3.22}$$

$$\begin{aligned}\mathbb{E}_{\rho^+} &= \pi^*_{\rho^+,\rho^-}\mathbb{E}_{\rho^-} + \sum_{\varrho\in f^{-1}(\rho)} \pi^*_{\rho^+,\varrho}\bar{\mathcal{M}}^{\varrho}_{1,\varrho} \\ &= \pi^*_{\rho^+,\rho^-}\mathbb{E}_{\rho^-} + \sum_{\varrho\in f^{-1}(\rho)} \pi^{-1}_{\rho^+,\varrho}(\bar{\mathcal{M}}^{\varrho}_{1,\varrho}),\end{aligned} \tag{3.23}$$

where

$$\bar{\mathcal{M}}^{\rho}_{1,\rho} = \pi^{-1}_{\rho,\rho-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho}) \subset \bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \quad \text{and} \quad \bar{\mathcal{M}}^{\varrho}_{1,\varrho} \subset \pi^{-1}_{\varrho,\varrho-1}(\bar{\mathcal{M}}^{\varrho-1}_{1,\varrho})$$

are the exceptional divisors for the blowups at the steps ρ and ϱ. Since all divisors $\pi^{-1}_{\rho^+,\varrho}(\bar{\mathcal{M}}^{\varrho}_{1,\varrho})$ are distinct, we have

$$\begin{aligned}\sum_{\varrho\in f^{-1}(\rho)} \pi^{-1}_{\rho^+,\varrho}(\bar{\mathcal{M}}^{\varrho}_{1,\varrho}) &= \pi^{-1}_{\rho^+,\rho^-}\Bigg(\bigcup_{\varrho\in f^{-1}(\rho)} \bar{\mathcal{M}}^{\rho^-}_{1,\varrho}\Bigg) = \pi^{-1}_{\rho^+,\rho^-}(f^{-1}_{\rho-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho})) \\ &= f_\rho^{-1}\pi^{-1}_{\rho,\rho-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho}) = f_\rho^{-1}(\bar{\mathcal{M}}^{\rho}_{1,\rho}) = f_\rho^{*}(\bar{\mathcal{M}}^{\rho}_{1,\rho}).\end{aligned} \tag{3.24}$$

The second equality in (3.18) follows from the same equality with ρ replaced by $\rho-1$ along with (3.22)–(3.24).

Suppose next that $\rho^* > \rho$. Since

$$\pi_{\rho,\rho-1}\circ f_\rho = f_{\rho-1}\circ \pi_{\rho^+,\rho^-}$$

and since $\bar{\mathcal{M}}^{\rho}_{1,\rho^*}$ is the proper transform of $\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*}$ and $\bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*}$ is the proper transform of $\bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*}$, it follows that

$$f_\rho^{-1}(\bar{\mathcal{M}}^{\rho}_{1,\rho^*}) \supset \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*}$$

by the first equation in (3.18) with ρ replaced by $\rho-1$. We will next verify the opposite inclusion.

Suppose

$$q \in \bar{\mathcal{M}}^{\rho}_{1,\rho^*}, \quad \tilde{p} \in f_\rho^{-1}(q), \quad \text{and}$$
$$p = \pi_{\rho^+,\rho^-}(\tilde{p}) \in f^{-1}_{\rho-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*}) = \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*} \subset \bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}.$$

If $\pi_{\rho,\rho-1}(q) \notin \bar{\mathcal{M}}^{\rho-1}_{1,\rho}$, then q and $f_\rho^{-1}(q)$ lie away from the blowup loci for the blowups

$$\bar{\mathcal{M}}^{\rho}_{1,(I,J-\{j^*\})} \longrightarrow \bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})} \quad \text{and} \quad \bar{\mathcal{M}}^{\rho^+}_{1,(I,J)} \longrightarrow \bar{\mathcal{M}}^{\rho^-}_{1,(I,J)}.$$

Therefore,

$$\begin{aligned} f_\rho^{-1}(q) &= f_{\rho-1}^{-1}(\pi_{\rho,\rho-1}(q)) \\ &\subset \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*} - \bigcup_{\varrho\in f^{-1}(\rho)} \bar{\mathcal{M}}^{\rho^-}_{1,\varrho} \subset \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*}, \end{aligned}$$

as needed. If

$$\pi_{\rho,\rho-1}(q) \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)} \equiv \bar{\mathcal{M}}^{\rho-1}_{1,\rho} \cap \bar{\mathcal{M}}^{\rho-1}_{1,\rho^*},$$

we consider separately the same four cases for p as in the proof of the first statement of Lemma 3.5.

Case 1. Since $\bar{\mathcal{M}}^{\rho-1}_{1,\rho}$ and $\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*}$ intersect properly in $\bar{\mathcal{M}}^{\rho-1}_{1,(I,J-\{j^*\})}$, we can choose local coordinates (z,v,t) near p as before such that, for some $K_{\rho^*}\subset K$,

(v) $\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*} \cap U = \{(z,v)\in U : z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v\in\mathbb{C}^{K_{\rho^*}}\}$.

This assumption implies that

$$\bar{\mathcal{M}}^{\rho}_{1,\rho} \cap \bar{\mathcal{M}}^{\rho}_{1,\rho^*} \cap V = \{(z,0;\ell)\in V : z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}})\}. \tag{3.25}$$

In addition, by (iv) on page 555 and the structure of $f_{\rho-1}$,

$$\begin{aligned} \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*} \cap \tilde{U} &= f_{\rho-1}^{-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*}) \cap \tilde{U} \\ &= \{(z,v,t)\in\tilde{U} : z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v\in\mathbb{C}^{K_{\rho^*}}\}. \end{aligned}$$

Since $\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}$ and $\bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*}$ intersect properly, it follows that

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*} \cap \bar{\mathcal{M}}^{\rho^+}_{1,\rho_k(j^*)} \cap \tilde{V} = \{(z,0,t;\ell)\in\tilde{V};\ z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}})\}.$$

Using (3.19), we conclude that

$$\tilde{p}\in\{f_\rho|_{\tilde{V}}\}^{-1}(\bar{\mathcal{M}}^{\rho}_{1,\rho^*}\cap\bar{\mathcal{M}}^{\rho}_{1,\rho}) = \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*} \cap \bar{\mathcal{M}}^{\rho^+}_{1,\rho_k(j^*)} \cap \tilde{V},$$

as needed.

Case 2a. The argument is exactly the same as in Case 1 but with $\rho_k(j^*)$ replaced by ρ^+.

Case 2b. We can again choose $K_{\rho^*}\subset K$ such that (v) is satisfied. With notation as in the construction of the map f_ρ in this case,

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+-1}_{1,\varrho^*} \cap \tilde{V} = \{(z,v,t;\ell')\in\tilde{V} : z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell'\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\},$$

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+-1}_{1,\varrho^*} \cap \tilde{V}' = \{(z,u,t)\in\tilde{V}' : z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ u\in\mathbb{C}^{K_{\rho^*}}\},$$

$$\begin{aligned} \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*} \cap \pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)}) \cap \widetilde{W} &= \{(z,u,0;\ell)\in\widetilde{W} : z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}})\} \\ &\quad \cup\{(z,0,0;\ell')\in\widetilde{W} : z\in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell'\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\}. \end{aligned}$$

Using (3.20) and (3.25), we conclude that

$$\tilde{p} \in \{f_\rho|_{\pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)})\cap\widetilde{W}}\}^{-1}(\bar{\mathcal{M}}^{\rho}_{1,\rho^*}\cap\bar{\mathcal{M}}^{\rho}_{1,\rho}) = \bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*}\cap\pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)})\cap\widetilde{W}.$$

Note that the map $f_\rho|_{\pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho_i(j^*)})\cap\widetilde{W}}$ is a $\mathbb{P}^1$-fibration and that the map $f_\rho|_{\tilde{V}}$ from Case 1 is a $\mathbb{C}$-fibration.

Case 3. With notation as in the corresponding case in the construction of the map f_ρ and with a good choice of local coordinates, we have two subcases to consider. There exists a $K_{\rho^*}\subset K-\{k\}$ such that:

Case 3a. $\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*}\cap U=\{(z,v,w)\in U : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v\in\mathbb{C}^{K_{\rho^*}}\}$;
Case 3b. $\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*}\cap U=\{(z,v,w)\in U : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v\in\mathbb{C}^{K_{\rho^*}}, w=0\}$.

Case 3a. In this case,

$$\bar{\mathcal{M}}^{\rho}_{1,\rho^*}\cap\bar{\mathcal{M}}^{\rho}_{1,\rho}\cap V=\{(z,0,0;\ell)\in V : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\} \tag{3.26}$$

and

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*}\cap\tilde{U}=f^{-1}_{\rho-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*})\cap\tilde{U} =\{(z,v,w_k,w_+)\in\tilde{U} : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v\in\mathbb{C}^{K_{\rho^*}}\}.$$

It follows that

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\bar{\mathcal{M}}^{\rho^+-1}_{1,\varrho^*}\cap\tilde{V} =\{(z,v,w_k,w_+;\ell_k)\in\tilde{V} : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\},$$

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\bar{\mathcal{M}}^{\rho^+-1}_{1,\varrho^*}\cap\tilde{V}'=\{(z,u,u_k,w_+)\in\tilde{V}' : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ u\in\mathbb{C}^{K_{\rho^*}}\},$$

and

$$\begin{aligned}\bigcup_{\varrho^*\in f^{-1}(\rho^*)}&\bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*}\cap\pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)})\cap\widetilde{W}\\ &=\{(z,u,0,0;\ell)\in\widetilde{W} : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\}\\ &\quad\cup\{(z,0,0,0;\ell_k)\in\widetilde{W} : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times\mathbb{C})\}.\end{aligned}$$

Thus, by (3.21) and (3.26),

$$\tilde{p}\in\{f_\rho|_{\pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)})\cap\widetilde{W}}\}^{-1}(\bar{\mathcal{M}}^{\rho}_{1,\rho^*}\cap\bar{\mathcal{M}}^{\rho}_{1,\rho}) =\bigcup_{\varrho^*\in f^{-1}(\rho^*)}\bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*}\cap\pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+}\cap\bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)})\cap\widetilde{W}. \tag{3.27}$$

Case 3b. In this case,

$$\bar{\mathcal{M}}^{\rho}_{1,\rho^*}\cap\bar{\mathcal{M}}^{\rho}_{1,\rho}\cap V=\{(z,0,0;\ell)\in V : z\in\bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell\in\mathbb{P}(\mathbb{C}^{K_{\rho^*}}\times 0)\} \tag{3.28}$$

and

$$\bigcup_{\varrho^*\in f^{-1}(\rho^*)} \bar{\mathcal{M}}^{\rho^-}_{1,\varrho^*} \cap \tilde{U} = f^{-1}_{\rho-1}(\bar{\mathcal{M}}^{\rho-1}_{1,\rho^*}) \cap \tilde{U} = \tilde{\mathcal{Z}}^{\rho^-}_k \cup \tilde{\mathcal{Z}}^{\rho^-}_+,$$

where

$$\tilde{\mathcal{Z}}^{\rho^-}_{\circledast} = \{(z,v,w_k,w_+) \in \tilde{U} : z \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ v \in \mathbb{C}^{K_{\rho^*}},\ w_{\circledast} = 0\}, \quad \circledast = k,+.$$

We denote by $\tilde{\mathcal{Z}}^{\rho^+-1}_k$ and $\tilde{\mathcal{Z}}^{\rho^+-1}_+$ the proper transforms of $\tilde{\mathcal{Z}}^{\rho^-}_k$ and $\tilde{\mathcal{Z}}^{\rho^-}_+$ in $\tilde{V}$ and by $\tilde{\mathcal{Z}}^{\rho^+}_k$ and $\tilde{\mathcal{Z}}^{\rho^+}_+$ the proper transforms of $\tilde{\mathcal{Z}}^{\rho^-}_k$ and $\tilde{\mathcal{Z}}^{\rho^-}_+$ in $\widetilde{W}$. Then

$$\tilde{\mathcal{Z}}^{\rho^+-1}_k = \{(z,v,0,w_+;\ell_k) \in \tilde{V} : z \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k \in \mathbb{P}(\mathbb{C}^{K_{\rho^*}} \times \mathbb{C})\},$$

$$\tilde{\mathcal{Z}}^{\rho^+-1}_k \cap \tilde{V}' = \{(z,u,0,w_+) \in \tilde{V}' : z \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ u \in \mathbb{C}^{K_{\rho^*}}\},$$

and

$$\begin{aligned}\tilde{\mathcal{Z}}^{\rho^+}_k &\cap \pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}) \\ &= \{(z,u,0,0;\ell) \in \widetilde{W} : z \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell \in \mathbb{P}(\mathbb{C}^{K_{\rho^*}} \times 0)\} \\ &\quad \cup \{(z,0,0,0;\ell_k) \in \widetilde{W} : z \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k \in \mathbb{P}(\mathbb{C}^{K_{\rho^*}} \times \mathbb{C})\}. \end{aligned} \tag{3.29}$$

Similarly:

$$\tilde{\mathcal{Z}}^{\rho^+-1}_+ = \{(z,v,w_k,0;\ell_k) \in \tilde{V} : z \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k \in \mathbb{P}(\mathbb{C}^{K_{\rho^*}} \times 0)\};$$

$$\tilde{\mathcal{Z}}^{\rho^+-1}_+ \cap \tilde{V}' = \emptyset;$$

$$\begin{aligned}\tilde{\mathcal{Z}}^{\rho^+}_+ &\cap \pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}) \\ &= \{(z,0,0,0;\ell_k) \in \widetilde{W} : z \in \bar{\mathcal{M}}^{\rho-1}_{(\rho,\rho^*)};\ \ell_k \in \mathbb{P}(\mathbb{C}^{K_{\rho^*}} \times 0)\}. \end{aligned} \tag{3.30}$$

Since

$$\begin{aligned}\bigcup_{\varrho^*\in f^{-1}(\rho^*)} &\bar{\mathcal{M}}^{\rho^+}_{1,\varrho^*} \cap \pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}) \cap \widetilde{W} \\ &= (\tilde{\mathcal{Z}}^{\rho^+}_k \cap \tilde{\mathcal{Z}}^{\rho^+}_+) \cap \pi^{-1}_{\rho^+,\rho^-}(\bar{\mathcal{M}}^{\rho^-}_{1,\rho^+} \cap \bar{\mathcal{M}}^{\rho^-}_{1,\rho_k(j^*)}),\end{aligned}$$

we conclude from (3.21) and (3.28)–(3.30) that (3.27) holds in this case as well.

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