Michigan Math. J. 55 (2007)

Invariant Differential Operators Associated with a Conformal Metric

Seong-A Kim & Toshiyuki Sugawa

1. Introduction

Peschl defined invariant higher-order derivatives of a holomorphic or meromorphic function on the unit disk. Here, the invariance is concerned with the hyperbolic metric of the source domain and the canonical metric of the target domain. Minda and Schippers extended Peschl's invariant derivatives to the case of general conformal metrics. We introduce similar invariant derivatives for smooth functions on a Riemann surface and show a complete analogue of Faà di Bruno's formula for the composition of a smooth function with a holomorphic map with respect to the derivatives. An interpretation of these derivatives in terms of intrinsic geometry and some applications will be also given.

The uniformization theory tells us that an arbitrary Riemann surface has the natural geometry—namely, spherical, Euclidean, or hyperbolic geometry. Standard examples are the Riemann sphere $\hat{\mathbb{C}}$ with the spherical metric $|dz|/(1+|z|^2)$, the complex plane with the Euclidean metric $|dz|$, and the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the hyperbolic (or the Poincaré) metric $|dz|/(1-|z|^2)$. For a unifying treatment, we introduce the notation $\mathbb{C}_\varepsilon$ to designate $\hat{\mathbb{C}}$ for $\varepsilon = 1$, $\mathbb{C}$ for $\varepsilon = 0$, and $\mathbb{D}$ for $\varepsilon = -1$. Unless otherwise stated, we understand that $\mathbb{C}_\varepsilon$ is equipped with the canonical metric $\lambda_\varepsilon(z)|dz| = |dz|/(1+\varepsilon|z|^2)$. Note that λ_ε has constant Gaussian curvature 4ε.

For a holomorphic map $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ ($\delta, \varepsilon = 1, 0, -1$), it is more natural to consider a type of invariant derivatives of $f(z)$ associated with $\mathbb{C}_\delta$ and $\mathbb{C}_\varepsilon$ rather than the usual derivatives $f^{(n)}(z) = d^n f(z)/dz^n$. As such, commonly used is the invariant derivative $D^n f(z)$ due to Peschl [Pe], which is defined by the power series expansion

$$\frac{f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right) - f(z)}{1+\varepsilon\overline{f(z)}\,f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right)} = \sum_{n=1}^{\infty} \frac{D^n f(z)}{n!} \cdot \zeta^n \tag{1.1}$$

around $\zeta = 0$. Note that the group $\mathrm{Isom}^+(\mathbb{C}_\varepsilon)$ of sense-preserving isometries of $\mathbb{C}_\varepsilon$ consists of the maps $L(\zeta) = \eta(\zeta - a)/(1+\varepsilon\bar{a}\zeta)$ for some $a \in \mathbb{C}_\varepsilon$ and $\eta \in \mathbb{C}$ with $|\eta| = 1$, where $L(\zeta) = -\eta/\zeta$ for $\varepsilon = 1$ and $a = \infty$. For example,

Received August 17, 2006. Revision received January 4, 2007.
The first author was supported by the Dongguk University research fund.

$$\begin{aligned}
D^1 f(z) &= \frac{(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2}, \\
D^2 f(z) &= \frac{(1+\delta|z|^2)^2 f''(z)}{1+\varepsilon|f(z)|^2} + \frac{2\delta\bar{z}(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2} \\
&\quad - \frac{2\varepsilon(1+\delta|z|^2)^2\overline{f(z)}f'(z)^2}{(1+\varepsilon|f(z)|^2)^2}, \\
D^3 f(z) &= \frac{(1+\delta|z|^2)^3 f'''(z)}{1+\varepsilon|f(z)|^2} - \frac{6\varepsilon(1+\delta|z|^2)^3\overline{f(z)}f'(z)f''(z)}{(1+\varepsilon|f(z)|^2)^2} \\
&\quad + \frac{6\delta\bar{z}(1+\delta|z|^2)^2 f''(z)}{1+\varepsilon|f(z)|^2} + \frac{6\delta^2\bar{z}^2(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2} \\
&\quad - \frac{12\delta\varepsilon\bar{z}(1+\delta|z|^2)^2\overline{f(z)}f'(z)^2}{(1+\varepsilon|f(z)|^2)^2} + \frac{6\varepsilon^2(1+\delta|z|^2)^3\overline{f(z)}^2 f'(z)^3}{(1+\varepsilon|f(z)|^2)^3}.
\end{aligned}$$

These derivatives are invariant in the sense that $|D^n(L\circ f\circ M)| = |D^n f|\circ M$ for $L\in \operatorname{Isom}^+(\mathbb{C}_\varepsilon)$ and $M\in\operatorname{Isom}^+(\mathbb{C}_\delta)$. Minda [M] and Schippers [S] generalized this for arbitrary conformal metrics. We now give a generalized definition of $D^n f$.

In this introductory section, we consider plane domains with smooth conformal metrics for the sake of simplicity. As we will see in Section 3, the notions given here can be extended for a holomorphic map f between Riemann surfaces with smooth conformal metrics in an obvious manner. See [Su] for examples of useful (but not necessarily smooth) conformal metrics on Riemann surfaces.

We define invariant differential operators ∂_ρ^n acting on the space $C^\infty(V)$ of smooth (complex-valued) functions on a plane domain V with smooth conformal metric $\rho = \rho(z)|dz|$ inductively by

$$\partial_\rho^1\varphi = \partial_\rho\varphi = \frac{1}{\rho(z)}\frac{\partial\varphi(z)}{\partial z} \quad \text{and}$$
$$\partial_\rho^{n+1}\varphi = (\partial_\rho\circ\partial_\rho^n)\varphi - n(\partial_\rho\log\rho)\cdot\partial_\rho^n\varphi,\quad n\ge 1, \tag{1.2}$$

for $\varphi\in C^\infty(V)$. Note that the symbol ∂_ρ^n does not mean an iteration of ∂_ρ. However, when $\rho = |dz|$ (the Euclidean metric), obviously $\partial_\rho^n = \partial^n = (\partial/\partial z)^n$, which is the nth iterate of ∂.

The operator ∂_ρ^2 for $\rho = \lambda_\varepsilon$ appeared in [KM] (see also [KSu]). Note that the quantity $\rho^n\partial_\rho^n\varphi$ appears in some computations of Laplacians for the n-differential on Riemann surfaces with variable conformal metrics (see e.g. [HP]).

Let $f\colon V\to W$ be a holomorphic map between plane domains. If $\rho = \rho(z)|dz|$ and $\sigma = \sigma(w)|dw|$ are smooth conformal metrics on V and W, respectively, then $D^n f = D^n_{\sigma,\rho} f$ is defined on V inductively by

$$D^1 f = \frac{\sigma\circ f}{\rho} f' \quad \text{and}$$
$$D^{n+1} f = [\partial_\rho - n(\partial_\rho\log\rho) + (\partial_\sigma\log\sigma)\circ f\cdot D^1 f]D^n f,\quad n\ge 1. \tag{1.3}$$

Here, D^n does not mean the nth iterate of D^1. It should be noted that the chain rule

$$D^1_{\tau,\rho}(g \circ f) = (D^1_{\tau,\sigma} g) \circ f \cdot D^1_{\sigma,\rho} f$$

is valid for holomorphic maps $f: V \to W$ and $g: W \to X$ and conformal metrics ρ, σ, τ on V, W, X, respectively. This definition looks different from Peschl's one, but it turns out that these are equivalent when $V = \mathbb{C}_\delta$ and $W = \mathbb{C}_\varepsilon$ (see Proposition 7.2).

One of the purposes of this paper is to show that these invariant derivatives satisfy the same rule as do the ordinary derivatives of compositions of smooth functions of a real variable. It is known that higher-order derivatives of the composite function $g \circ f$ of smooth real-valued functions f and g of a real variable are described by Faà di Bruno's formula (cf. [Ri, p. 36] or [C, p. 137])

$$(g \circ f)^{(n)} = \sum_{k=1}^{n} g^{(k)} \circ f \cdot A_{n,k}(f', \dots, f^{(n-k+1)}),$$

where $A_{n,k} = A_{n,k}(x_1, \dots, x_{n-k+1})$ are Bell polynomials (see Section 4 for the definition). The following result is an analogue of Faà di Bruno's formula for our invariant differential operators. We note that, by virtue of transformation rules for these operators (Lemmas 3.2 and 3.6), the result can be extended for Riemann surfaces with smooth conformal metrics via local coordinates (see Section 3).

Theorem 1.1. *Let V and W be plane domains with smooth conformal metrics ρ and σ, respectively, and let $f: V \to W$ be holomorphic. Then, for every function φ in $C^\infty(W)$, the relation*

$$\partial^n_\rho(\varphi \circ f) = \sum_{k=1}^{n} (\partial^k_\sigma \varphi) \circ f \cdot A_{n,k}(D^1 f, \dots, D^{n-k+1} f) \tag{1.4}$$

holds for each $n \geq 1$.

We remark that these relations for $n = 1, 2$ were previously noticed in [KM] when $V = \mathbb{C}_{-1}$ and $W = \mathbb{C}_{+1}$ and in [KSu] when $V = W = \mathbb{C}_{-1}$.

Theorem 1.1 shows that our invariant derivatives $D^n f$ are natural and enables us to compute higher-order derivatives of functions more easily. For instance, when $V = W = \mathbb{C}_{-1}$ and f maps $\mathbb{C}_{-1}$ conformally onto a hyperbolically convex subdomain Ω of $\mathbb{C}_{-1}$, several characterizations of the domain Ω are given in terms of the invariant derivatives $D^n f$ in [MaM]. Invariant derivatives of $\mu \circ f$ can be computed and related to $D^n f$ by using equation (1.4) for a geometric quantity μ on Ω. In this way, Theorem 1.1 in this special case was used to simplify the involved computations in [KSu].

In order to give a natural interpretation of our invariant derivatives on Riemann surfaces, we need a differential geometric setup. In Section 2, we give basic concepts in differential geometry that we need and introduce necessary notation and terminology. Though the material is standard, an expository account will be given there because we could not find a convenient reference containing all the needed content concisely.

Section 3 will be devoted to an explanation of the way that the invariant derivatives $D^n f = D^n_{\sigma,\rho} f$ arise for a holomorphic map $f: R \to S$ between Riemann surfaces with conformal metrics ρ and σ. Prior to this, we define the operators ∂_ρ^n in a natural way. Since these operators are described as tensors of specific types, it is a routine task to see that they obey certain transformation rules.

Section 4 summarizes basic properties of the (exponential) Bell polynomials as well as a principle leading to Faà di Bruno–type formulas for a sort of differential operators (see Lemma 4.2). This principle plays a decisive role in the proof of Theorem 1.1.

Section 5 gives a proof of Theorem 1.1. Toward this end, we introduce an auxiliary differential operator. A remarkable fact is that the nth iterate of the differential operator can describe our differential operators ∂_ρ^n and $D_{\sigma,\rho}^n$ in simple ways, which makes the proof of Theorem 1.1 dramatically short.

The defining recursive relations (1.3) give apparently complicated expressions of $D^n f$. In Section 6, as an application of Theorem 1.1, we derive another expression of $D^n f$ in terms of $f^{(n)}$ and the lower-order derivatives $D^1 f, \dots, D^{n-1} f$. Moreover, we give concrete forms for $D^n f$ in terms of only the ordinary derivatives $f', \dots, f^{(n)}$ and for $f^{(n)}$ in terms of $D^1 f, \dots, D^n f$.

Section 7 will explore the consequences of the previous sections for the canonical surfaces $\mathbb{C}_\varepsilon$ for $\varepsilon = +1, 0, -1$. Although some of them are known already, we believe that our approach will give a further insight even into the classical invariant derivatives.

Further applications of theorems given in this paper to the study of Schwarzian derivatives will be supplied in forthcoming papers of the authors.

ACKNOWLEDGMENTS. The authors are grateful to Kyung Hyun Cho for his support. They also thank Eric Schippers for valuable information and comments.

2. Connections on Vector Bundles

We recall basic notions used in differential geometry, referring to an excellent book [KoN] by Kobayashi and Nomizu for details. We give a somewhat detailed exposition of the necessary material (for the reader who is not familiar with differential geometry) as well as of the terminology and notation.

Let R be a Riemann surface. Let E be a holomorphic vector bundle over R with projection $\pi: E \to R$, and denote by $\Gamma(E)$ the set of smooth cross-sections of E over R. In what follows, vector bundles will always be holomorphic. The most fundamental vector bundles over R are the (complexified) tangent bundle $T(R) = T_{\mathbb{C}}(R)$ and its dual $T^*(R)$ (over $\mathbb{C}$), the cotangent bundle. An element of $\mathfrak{X}(R) = \Gamma(T(R))$ is called a *vector field* on R and an element of $\Gamma(T^*(R))$ is called a *1-form* on R.

A *connection* on E is a complex linear mapping $\nabla\colon \Gamma(E) \to \Gamma(T^*(R) \otimes E)$ satisfying the Leibniz rule

$$\nabla(\varphi\xi) = d\varphi \otimes \xi + \varphi\nabla\xi, \quad \varphi \in C^\infty(R),\ \xi \in \Gamma(E).$$

Note that the operator ∇ is local. In other words, ∇ naturally operates on $\Gamma(E|_U)$ for an open subset U of R. (This fact enables us to consider ∇ as a sheaf homomorphism of the sheaf of local smooth sections of E, though we do not take this formalism explicitly.) Identifying $\Gamma(T^*(R)\otimes E)$ with $\Gamma(\mathrm{Hom}(T(R),E))$, we can define a linear transformation $\nabla_X\colon \Gamma(E)\to\Gamma(E)$ for $X\in\mathfrak{X}(R)$ by setting $\nabla_X\xi=(\nabla\xi)(X)$ for $\xi\in\Gamma(E)$. We call $\nabla_X\xi$ the *covariant derivative* of ξ with respect to X. Let ω_k^i be the *connection forms* of ∇ with respect to a local frame $(e_1,\dots,e_r)$ of E; namely,

$$\nabla e_k=\sum_{i=1}^r \omega_k^i\otimes e_i,$$

where the ω_k^i are local 1-forms on R. The connection forms then reproduce ∇ by the formula

$$\nabla\xi=\sum_{i=1}^r\left(d\xi^i+\sum_{k=1}^r\xi^k\omega_k^i\right)\otimes e_i,$$

where $\xi=\sum_k\xi^k e_k\in\Gamma(E)$.

Let E and F be vector bundles over R with connections ∇' and ∇'', respectively. Then the tensor product $E\otimes F$ admits a connection ∇ such that

$$\nabla_X(\xi\otimes\eta)=\nabla'_X\xi\otimes\eta+\xi\otimes\nabla''_X\eta,\quad X\in\mathfrak{X}(R),\ \xi\in\Gamma(E),\ \eta\in\Gamma(F).$$

We will write $\nabla=\nabla'\otimes 1_F+1_E\otimes\nabla''$.

Let E be a vector bundle over R with connection ∇. Then a connection, which will be denoted by the same letter ∇, is defined on the dual vector bundle E^* by the rule

$$d\langle\xi^*,\xi\rangle=\langle\nabla\xi^*,\xi\rangle+\langle\xi^*,\nabla\xi\rangle,\quad \xi\in\Gamma(E),\ \xi^*\in\Gamma(E^*),$$

where $\langle\xi^*,\xi\rangle=\xi^*(\xi)$. If ω_k^i are the connection forms of ∇ on E with respect to a local frame $(e_1,\dots,e_r)$ of E, then the connection forms of ∇ on E^* with respect to the dual frame $(e_1^*,\dots,e_r^*)$ of E^* are given by $-\omega_i^k$.

Let $f\colon R\to S$ be a holomorphic map and let F be a vector bundle over S with projection $\pi: F\to S$ and connection ∇. Recall first that the induced bundle f^*F is realized as the fibre product $R\times_S F=\{(p,\xi)\in R\times F : f(p)=\pi(\xi)\}$. In particular, one can define the pullback $f^*\xi$ of $\xi\in\Gamma(F)$ by $f^*\xi(p)=(p,\xi(f(p)))$. The induced connection $f^*\nabla$ on f^*F is defined by the connection forms $f^*\omega_j^k$ with respect to $(f^*e_1,\dots,f^*e_r)$, where ω_j^k are connection forms of ∇ with respect to a local frame $(e_1,\dots,e_r)$.

Let g be a smooth *conformal metric* on a Riemann surface R; that is, g is a Riemannian metric on R written locally in the form $g=\rho(z)^2(dx^2+dy^2)$, where $z=x+\mathrm{i}y\colon U\to U'$ is a local coordinate of R and ρ is a smooth positive function on U'. (We use "i" to denote the imaginary unit $\sqrt{-1}$.) It is a simple exercise to see that a Riemannian metric g on a Riemann surface is conformal if and only if it is Hermitian. Note that a smooth Hermitian metric on a Riemann surface is automatically Kählerian. A conformal metric g is sometimes written in the form $ds=\rho(z)|dz|$ as a line element or in the form $g=\rho(z)^2dzd\bar{z}$ as a Hermitian metric. In what follows, we will refer to the conformal metric as $\rho=\rho(z)|dz|$.

Let ∇^ρ be the Levi–Civita connection (or the Riemannian connection) on $T(R)$ associated with ρ. For a local coordinate $z=x+\mathrm{i}y$ of R, the (local) vector fields

$e_1 = \partial/\partial z = (1/2)(\partial/\partial x - \mathrm{i}\partial/\partial y)$ and $e_2 = \partial/\partial\bar{z} = (1/2)(\partial/\partial x + \mathrm{i}\partial/\partial y)$ form a local frame (e_1, e_2) of $T(R)$. By using the information in [KoN, Vol. II, Chap. IX, Sec. 5], we obtain the connection forms of ∇^ρ as follows:

$$\omega_1^1 = \overline{\omega_2^2} = \frac{2}{\rho}\frac{\partial\rho}{\partial z}dz = 2\frac{\partial\log\rho}{\partial z}dz, \qquad \omega_1^2 = \omega_2^1 = 0. \tag{2.1}$$

We remark that the Christoffel symbol Γ_{jk}^i is defined to be $\omega_k^i(e_j)$. Note also that the connection forms of ∇^ρ on $T^*(R)$ with respect to the dual frame $(e_1^*, e_2^*) = (dz, d\bar{z})$ are given by $-\omega_i^k$. Thus, for instance,

$$\nabla^\rho(dz) = -2\frac{\partial\log\rho}{\partial z}dz\otimes dz. \tag{2.2}$$

We denote by $T_s^r(R)$ the tensor bundle of type (r,s) over R (so $T_s^r(R) = T(R)^{\otimes r}\otimes T^*(R)^{\otimes s}$) and denote by $\mathfrak{D}_s^r(R)$ the set of smooth tensor fields of type (r,s) on R: $\mathfrak{D}_s^r(R) = \Gamma(T_s^r(R))$. By the operation explained previously, the connection ∇^ρ is defined on $T_s^r(R)$ as well. Note that $\mathfrak{D}_0^0(R) = C^\infty(R)$ and that ∇^ρ acts on it as the exterior differentiation: $\nabla^\rho\varphi = d\varphi$ for $\varphi\in C^\infty(R)$. The direct sum $\mathfrak{D}(R) = \sum_{r,s=0}^\infty \mathfrak{D}_s^r(R)$ has the structure of a bi-graded $C^\infty(R)$-algebra and is called the *mixed tensor algebra* on R.

3. Invariant Higher-Order Derivatives

Let R be a Riemann surface with conformal metric ρ. We define a linear transformation $\Lambda = \Lambda_\rho$ of $\mathfrak{D}(R)$ by

$$\Lambda(\omega) = dz\otimes\nabla^\rho_{\partial/\partial z}(\omega), \quad \omega\in\mathfrak{D}(R),$$

where $z\colon U\to U'$ for $U\subset R$ and $U'\subset\mathbb{C}$ is a local coordinate of R. We see that $\Lambda(\omega)$ does not depend on the choice of local coordinates. Indeed, let w be another local coordinate defined in the same domain as z, and let $w = h(z)$ be the transition function. Since $dw = h'dz$ and $\partial/\partial w = (1/h')\partial/\partial z$, we obtain

$$\begin{aligned} dw\otimes\nabla^\rho_{\partial/\partial w}(\omega) &= (h'dz)\otimes\nabla^\rho_{(1/h')\partial/\partial z}(\omega) \\ &= \frac{1}{h'}\cdot h'dz\otimes\nabla^\rho_{\partial/\partial z}(\omega) = dz\otimes\nabla^\rho_{\partial/\partial z}(\omega) \end{aligned}$$

for a tensor field ω. We observe that, by definition, $\Lambda(\mathfrak{D}_s^r(R))\subset\mathfrak{D}_{s+1}^r(R)$. We also note that $\Lambda(\omega)$ is nothing but the projection of $\nabla^\rho\omega$ to the $(1,0)$-part $T^*_{(1,0)}(R)\otimes E$ of $T^*(R)\otimes E$.

LEMMA 3.1. *Let $\varphi\in C^\infty(R)$ and $z\colon U\to U'$ be a local coordinate of R. The tensor field $\Lambda^n(\varphi) := (\Lambda\circ\cdots\circ\Lambda)(\varphi)$ (n times) can be written in the form*

$$\Lambda^n(\varphi) = \varphi_n(z)dz^n$$

for each $n\ge 0$, where $dz^n = dz\otimes\cdots\otimes dz$ (n times) and φ_n is the smooth function on U' determined by the recurrence relations with initial condition:

$$\varphi_0 = \varphi\circ z^{-1}, \qquad \varphi_{n+1} = \frac{\partial\varphi_n}{\partial z} - 2n\frac{\partial\log\rho}{\partial z}\varphi_n, \quad n\ge 0. \tag{3.1}$$

Proof. We prove the lemma by induction. When $n = 0$, the assertion is trivial. So let $n \geq 0$ and assume that the assertion holds for n, $\Lambda^n(\varphi) = \varphi_n dz^n$. Then

$$\Lambda^{n+1}(\varphi) = \Lambda(\varphi_n dz^n) = \Lambda(\varphi_n) \otimes dz^n + \varphi_n \Lambda(dz^n).$$

Since $\Lambda(dz^n) = n\Lambda(dz) \otimes dz^{n-1} = -2n\frac{\partial}{\partial z}(\log\rho)dz^{n+1}$ by (2.2), it follows that

$$\Lambda^{n+1}(\varphi) = \left\{\frac{\partial \varphi_n}{\partial z} - 2n\frac{\partial(\log\rho)}{\partial z}\varphi_n\right\}dz^{n+1} = \varphi_{n+1}dz^{n+1},$$

which completes the induction argument. $\square$

We now define the operator ∂_ρ^n by

$$\partial_\rho^n \varphi(z) = \rho(z)^{-n}\varphi_n(z) \tag{3.2}$$

on U' for $\varphi \in C^\infty(R)$ and $n \in \mathbb{N}$, where φ_n is given by (3.1). We also write ∂_ρ for ∂_ρ^1. We will call $\partial_\rho^n\varphi$ the nth *ρ-derivative* of φ.

We now show that this definition agrees with that given in Section 1 when R is a plane domain V and $z\colon V \to V$ is the identity. Indeed, differentiating both sides of (3.2), we obtain

$$\begin{aligned}
\frac{\partial}{\partial z}(\partial_\rho^n\varphi) &= \frac{\partial}{\partial z}(\rho^{-n}\varphi_n) = \rho^{-n}\frac{\partial\varphi_n}{\partial z} + \frac{\partial(\rho^{-n})}{\partial z}\varphi_n \\
&= \rho^{-n}\left\{\varphi_{n+1} + 2n\frac{\partial \log\rho}{\partial z}\varphi_n\right\} - n\rho^{-n}\frac{\partial\log\rho}{\partial z}\varphi_n \\
&= \rho\partial_\rho^{n+1}\varphi + n\frac{\partial\log\rho}{\partial z}\partial_\rho^n\varphi,
\end{aligned}$$

where we have used (3.1). We now divide both sides by ρ to obtain the relation (1.2).

Note that $\partial_\rho^n\varphi$ is no longer a function on R, in general. More precisely, $\partial_\rho^n\varphi$ should be understood as $(\Lambda^n\varphi)/\rho^n = \varphi_n(z)dz^n/(\rho(z)^n|dz|^n)$, which is sometimes called an $(n/2, -n/2)$-differential on R because $|dz| = dz^{1/2}d\bar{z}^{1/2}$ formally. At least the modulus $|\partial_\rho^n\varphi|$ can be regarded as a function on R, and that is enough in most applications.

Since a local isometry between Riemann surfaces with conformal metrics can be regarded (at least locally) as a change of local coordinates, we are able to state the invariance property of $(\Lambda^n\varphi)/\rho^n$ as a lemma in the following way.

LEMMA 3.2. *Let V and W be plane domains with smooth conformal metrics ρ and σ, respectively. Suppose that a locally univalent holomorphic map $p\colon V \to W$ is locally isometric. Then*

$$\partial_\sigma^n(\varphi \circ p) = \left(\frac{p'}{|p'|}\right)^n [(\partial_\rho^n\varphi)\circ p], \quad \varphi \in C^\infty(W),$$

for each $n \geq 1$.

Let R and S be Riemann surfaces. Suppose now that a holomorphic map $f\colon R \to S$ is given. The tangent map $Tf\colon T(R) \to T(S)$ can be regarded as an element of $\Gamma(\mathrm{Hom}(T(R), f^*T(S))) = \Gamma(T^*(R) \otimes f^*T(S))$. By using local coordinates $z\colon U \to U'$ of R and $w\colon V \to V'$ of S with $f(U) \subset V$, the section Tf is described by

$$Tf = \tilde{f}'(z)dz \otimes f^*\left(\frac{\partial}{\partial w}\right) + \overline{\tilde{f}'(z)}d\bar{z} \otimes f^*\left(\frac{\partial}{\partial \bar{w}}\right),$$

where $\tilde{f} = w \circ f \circ z^{-1}\colon U' \to V' \subset \mathbb{C}$. We now set

$$\partial f = dz \otimes Tf\left(\frac{\partial}{\partial z}\right) = \tilde{f}'(z)dz \otimes f^*\left(\frac{\partial}{\partial w}\right).$$

It is clear that ∂f does not depend on the choice of local coordinates and thus $\partial f \in \Gamma(T^*(R) \otimes f^*T(S)) = \Gamma(T_1^0(R) \otimes f^*T(S))$.

Suppose next that the Riemann surfaces R and S are equipped with conformal metrics ρ and σ, respectively. We recall that ∇^ρ is defined on $T_s^r(R)$, so that its action on the mixed tensor algebra $\mathfrak{D}(R)$ satisfies the Leibniz rule $\nabla_X^\rho(\xi \otimes \eta) = \nabla_X^\rho \xi \otimes \eta + \xi \otimes \nabla_X^\rho \eta$ for $\xi, \eta \in \mathfrak{D}(R)$ and $X \in \mathfrak{X}(R)$. For a holomorphic map $f\colon R \to S$, let

$$\begin{aligned}\nabla^{\rho,\sigma,f}\colon \Gamma(T_n^0(R) \otimes f^*T(S)) \to \Gamma(\mathrm{Hom}(T(R), T_n^0(R) \otimes f^*T(S)))\\ = \Gamma(T_{n+1}^0(R) \otimes f^*T(S))\end{aligned}$$

be the connection given by $\nabla^\rho \otimes 1 + 1 \otimes f^*(\nabla^\sigma)$. Furthermore, we define a linear operator $\Lambda_f = \Lambda_{\rho,\sigma,f}\colon \Gamma(T_n^0(R) \otimes f^*T(S)) \to \Gamma(T_{n+1}^0(R) \otimes f^*T(S))$ by

$$\Lambda_f \xi = dz \otimes \nabla_{\partial/\partial z}^{\rho,\sigma,f} \xi, \quad \xi \in \Gamma(T_n^0(R) \otimes f^*T(S)),$$

where z is a local coordinate of R. As before, we can check that $\Lambda_f \xi$ does not depend on the choice of z.

Let us find a concrete expression of $\Lambda_{\rho,\sigma,f}$. For a pair of local coordinates $z\colon U \to U'$ of R and $w\colon V \to V'$ of S with $f(U) \subset V$, we set $\tilde{f} = w \circ f \circ z^{-1}\colon U' \to V'$. We now consider a local section ξ of $T_n^0(R) \otimes f^*T(S)$ in the form $\xi = \varphi(z)dz^n \otimes f^*(\partial/\partial w)$, where $\varphi \in C^\infty(U')$. Recall that the connection forms ω_k^i of ∇^σ with respect to the local frame $(e_1, e_2) = (\partial/\partial w, \partial/\partial \bar{w})$ are given by $\omega_1^1 = 2\partial(\log\sigma)dw = \overline{\omega_2^2}$ and $\omega_1^2 = \omega_2^1 = 0$; see (2.1). Thus, by definition, $(f^*\nabla^\sigma)(f^*e_1) = f^*\omega_1^1 \otimes f^*e_1$ and so

$$(f^*\nabla^\sigma)f^*\left(\frac{\partial}{\partial w}\right) = 2(\partial\log\sigma) \circ \tilde{f} \cdot \tilde{f}'dz \otimes f^*\left(\frac{\partial}{\partial w}\right).$$

By (2.2), we obtain

$$\begin{aligned}\nabla^{\rho,\sigma,f}\xi &= d\varphi \otimes dz^n \otimes f^*\left(\frac{\partial}{\partial w}\right) - 2n(\partial\log\rho)\varphi dz^{n+1} \otimes f^*\left(\frac{\partial}{\partial w}\right)\\ &\quad + 2(\partial\log\sigma) \circ \tilde{f} \cdot \tilde{f}'\varphi dz^{n+1} \otimes f^*\left(\frac{\partial}{\partial w}\right)\\ &= \{[\partial\varphi - 2n(\partial\log\rho)\varphi + 2(\partial\log\sigma) \circ \tilde{f} \cdot \tilde{f}'\varphi]dz + \bar{\partial}\varphi d\bar{z}\}\\ &\quad \otimes dz^n \otimes f^*\left(\frac{\partial}{\partial w}\right).\end{aligned}$$

Therefore,

$$\Lambda_{\rho,\sigma,f}\left[\varphi dz^n \otimes f^*\left(\frac{\partial}{\partial w}\right)\right]$$
$$= [\partial\varphi - 2n(\partial\log\rho)\varphi + 2(\partial\log\sigma)\circ\tilde{f}\cdot\tilde{f}'\varphi]dz^{n+1}\otimes f^*\left(\frac{\partial}{\partial w}\right). \quad (3.3)$$

For the local description $\tilde{f}$ of $f\colon R\to S$, we define f_n inductively for $n\geq 1$ by

$$f_1 = \tilde{f}', \qquad f_{n+1} = \partial f_n - 2n(\partial\log\rho)f_n + 2(\partial\log\sigma)\circ\tilde{f}\cdot\tilde{f}'f_n, \quad n\geq 1. \quad (3.4)$$

It is easy to show a result analogous to Lemma 3.1 by induction.

LEMMA 3.3. *For $n\geq 1$, $\Lambda^{n-1}_{\rho,\sigma,f}(\partial f) := (\Lambda_{\rho,\sigma,f}\circ\cdots\circ\Lambda_{\rho,\sigma,f})(\partial f)$ ($n-1$ times) is of the form $f_n dz^n\otimes f^*(\partial/\partial w)$, where f_n is defined in* (3.4).

We define $D^n f = D^n_{\sigma,\rho} f$ for the coordinates z and w by

$$D^n f(z) = \frac{\sigma(\tilde{f}(z))f_n(z)}{\rho(z)^n}, \quad z\in U'. \quad (3.5)$$

As in the proof of the equivalence of (1.2) and (3.2), it can be checked that this definition is the same as (1.3) when R and S are plane domains with the identity as local coordinates.

REMARK 3.4. When $S=\mathbb{C}$ with the Euclidean metric $\sigma=\lambda_0$, $D^n_{\lambda_0,\rho}f$ coincides with $\partial^n_\rho f$.

The definition of $D^n f$ depends on the choice of the coordinates z and w. Let us observe the effect of a change of coordinates on $D^n f$. Let $\hat{z}$ and $\hat{w}$ be other local coordinates of R and S, respectively, and write $\rho = \hat{\rho}(\hat{z})|d\hat{z}|$ and $\sigma = \hat{\sigma}(\hat{w})|d\hat{w}|$. We set $\hat{f} = \hat{w}\circ f\circ\hat{z}^{-1}$ and write $z = g(\hat{z})$, $\hat{w} = h(w)$, $\Lambda^{n-1}_{\rho,\sigma,f}(\partial f) = \hat{f}_n d\hat{z}^n\otimes f^*(\partial/\partial\hat{w})$, and $\hat{D}^n f = \hat{\sigma}\circ\hat{f}\cdot\hat{f}_n/\hat{\rho}^n$. Since

$$\Lambda^{n-1}_{\rho,\sigma,f}(\partial f) = f_n(z)\cdot(g'd\hat{z})^n\otimes(h'\circ\tilde{f})f^*\left(\frac{\partial}{\partial\hat{w}}\right)$$
$$= (g'(\hat{z}))^n(h'(\tilde{f}(z)))f_n(z)d\hat{z}^n\otimes f^*\left(\frac{\partial}{\partial\hat{w}}\right),$$

it follows that $\hat{f}_n(\hat{z}) = (g'(\hat{z}))^n(h'(\tilde{f}(z)))f_n(z)$. In view of $\rho(z)|g'(\hat{z})| = \hat{\rho}(\hat{z})$ and $\sigma(w) = \hat{\sigma}(\hat{w})|h'(w)|$, we obtain

$$\hat{D}^n f = \left(\frac{h'}{|h'|}\right)\circ\tilde{f}\circ g\cdot(D^n f)\circ g\cdot\left(\frac{g'}{|g'|}\right)^n. \quad (3.6)$$

In particular, it turns out that $|D^n f|$ does not depend on the choice of local coordinates and thus can be regarded as a global function on the Riemann surface R.

REMARK 3.5. By the transformation rule just described, we see that the quotient $D^n f/D^m f$ is independent of the choice of the local coordinate w. Therefore, one can regard it as an $((n-m)/2,(m-n)/2)$-differential on R.

We now reformulate the preceding computation as an invariance property of D^n.

LEMMA 3.6. *Let $V, \hat{V}, W, \hat{W}$ be plane domains with smooth conformal metrics $\rho, \hat{\rho}, \sigma, \hat{\sigma}$, respectively. Suppose that locally isometric holomorphic maps $g\colon \hat{V} \to V$ and $h\colon W \to \hat{W}$ are given. Then, for a holomorphic map $f\colon V \to W$, the formula*

$$D^n_{\hat{\sigma},\hat{\rho}}(h \circ f \circ g) = \left(\frac{h'}{|h'|}\right) \circ f \circ g \cdot (D^n_{\sigma,\rho} f) \circ g \cdot \left(\frac{g'}{|g'|}\right)^n$$

is valid on $\hat{V}$.

4. Bell Polynomials

In this section we give a definition and some properties of the Bell polynomials. As usual, we denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the set of positive integers. Consider the (commutative) polynomial ring of indeterminates x_j $(j \in \mathbb{N})$ with coefficients in $\mathbb{Z}$:

$$P = \mathbb{Z}[x_1, x_2, \dots].$$

Let $\mathcal{D}\colon P \to P$ be the derivation determined by $\mathcal{D}x_j = x_{j+1}$ for each $j \in \mathbb{N}$. In other words, $\mathcal{D}$ can be written in a formal way by

$$\mathcal{D} = \sum_{j=1}^{\infty} x_{j+1} \frac{\partial}{\partial x_j}.$$

The degree and the weight of a monomial $x_{j_1} \cdots x_{j_k}$ are defined to be the numbers k and $j_1 + \cdots + j_k$, respectively. Let P_k and Q_n be the sub-$\mathbb{Z}$-modules of P generated by monomials of degree k and by monomials of weight n, respectively. It is easy to see that $P = \sum_{k=0}^{\infty} P_k$ becomes a graded ring as well as $\sum_{n=0}^{\infty} Q_n$. By definition, $\mathcal{D}$ maps Q_n into Q_{n+1} while $\mathcal{D}$ preserves P_k.

We define the Bell polynomials $A_{n,k}$ $(n \in \mathbb{N}, k \in \mathbb{Z})$ in P inductively by

$$A_{1,k} = \delta_{1,k} x_1 \quad \text{and}$$
$$A_{n+1,k} = \mathcal{D}A_{n,k} + x_1 A_{n,k-1}, \quad n \geq 1, \tag{4.1}$$

where $\delta_{1,k} = 1$ when $k = 1$ and $\delta_{1,k} = 0$ otherwise. By induction, we can easily check the following.

LEMMA 4.1. *The Bell polynomials $A_{n,k}$ have nonnegative coefficients. Moreover, we have*

(i) $A_{n,k} = 0$ *unless* $1 \leq k \leq n$,
(ii) $A_{n,k} \in P_k$,
(iii) $A_{n,k} \in Q_n$,
(iv) $A_{n,k} \in \mathbb{Z}[x_1, \dots, x_{n-k+1}]$ *for* $1 \leq k \leq n$,
(v) $A_{n,1} = x_n$, *and*
(vi) $A_{n,n} = x_1^n$.

We remark that (iv) follows also from (ii) and (iii) because $P_k \cap Q_n \subset \mathbb{Z}[x_1, \dots, x_{n-k+1}]$. Since $A_{n,k} \in \mathbb{Z}[x_1, \dots, x_{n-k+1}]$, we sometimes write $A_{n,k} = A_{n,k}(x_1, \dots, x_{n-k+1})$ for $1 \leq k \leq n$.

The Bell polynomials have a certain universal property, which subsumes Faà di Bruno's formula as a special case. In this section, let V and W be just sets and let $\mathcal{F}(V)$ and $\mathcal{F}(W)$ be $\mathbb{C}$-subalgebras of the algebra of complex-valued functions on V and W, respectively.

LEMMA 4.2. *Let d_V and d_W be $\mathbb{C}$-derivations on $\mathcal{F}(V)$ and $\mathcal{F}(W)$, respectively, and let f be a map of V into W such that $\varphi \circ f \in \mathcal{F}(V)$ for every $\varphi \in \mathcal{F}(W)$. Suppose there exists $\delta f \in \mathcal{F}(V)$ satisfying*

$$d_V(\varphi \circ f) = (d_W \varphi) \circ f \cdot \delta f, \quad \varphi \in \mathcal{F}(W).$$

Then

$$d_V^n(\varphi \circ f) = \sum_{k=1}^{n} (d_W^k \varphi) \circ f \cdot A_{n,k}(\delta f, d_V(\delta f), \ldots, d_V^{n-k}(\delta f))$$

for $\varphi \in \mathcal{F}(W)$. Here d_V^n and d_W^k do mean iterations of d_V and d_W, respectively.

Proof. We prove this by means of induction. When $n = 1$, the assertion is trivial. Assume that the assertion is valid up to n. Taking the derivation d_V of the assertion for n, we obtain

$$\begin{aligned}
d_V^{n+1}(\varphi \circ f) &= \sum_{k=1}^{n} d_V[(d_W^k \varphi) \circ f] \cdot A_{n,k}(\delta f, d_V(\delta f), \ldots, d_V^{n-k}(\delta f)) \\
&\quad + \sum_{k=1}^{n} (d_W^k \varphi) \circ f \cdot d_V[A_{n,k}(\delta f, d_V(\delta f), \ldots, d_V^{n-k}(\delta f))] \\
&= \sum_{k=1}^{n} (d_W^{k+1} \varphi \circ f) \cdot \delta f A_{n,k}(\delta f, d_V(\delta f), \ldots, d_V^{n-k}(\delta f)) \\
&\quad + \sum_{k=1}^{n} (d_W^k \varphi) \circ f \cdot \sum_{j=1}^{n-k+1} \frac{\partial A_{n,k}}{\partial x_j}(\delta f, d_V(\delta f), \ldots, d_V^{n-k}(\delta f)) d_V^j(\delta f) \\
&= \sum_{k=1}^{n+1} (d_W^k \varphi \circ f) \cdot [x_1 A_{n,k-1} + \mathcal{D} A_{n,k}](\delta f, d_V(\delta f), \ldots, d_V^{n-k+1}(\delta f)) \\
&= \sum_{k=1}^{n+1} (d_W^k \varphi \circ f) \cdot A_{n+1,k}(\delta f, d_V(\delta f), \ldots, d_V^{n-k+1}(\delta f))
\end{aligned}$$

by (4.1). Hence, the assertion is valid also for $n + 1$. $\square$

The following property will be crucial in the proof of Theorem 1.1.

LEMMA 4.3. *Let $a, b \in \mathbb{C}$ and $1 \leq k \leq n$. Then*

$$A_{n,k}(abx_1, ab^2 x_2, \ldots, ab^{n-k+1} x_{n-k+1}) = a^k b^n A_{n,k}(x_1, x_2, \ldots, x_{n-k+1}).$$

Proof. The relation

$$A_{n,k}(ax_1, \ldots, ax_{n-k+1}) = a^k A_{n,k}(x_1, \ldots, x_{n-k+1})$$

follows immediately from Lemma 4.1(ii). Similarly, the relation

$$A_{n,k}(bx_1, \dots, b^{n-k+1}x_{n-k+1}) = b^n A_{n,k}(x_1, \dots, x_{n-k+1})$$

is a consequence of Lemma 4.1(iii). □

For instance, we have:

$$A_{1,1} = x_1;$$

$$A_{2,2} = x_1^2, \quad A_{2,1} = x_2;$$

$$A_{3,3} = x_1^3, \quad A_{3,2} = 3x_1x_2, \quad A_{3,1} = x_3;$$

$$A_{4,4} = x_1^4, \quad A_{4,3} = 6x_1^2x_2, \quad A_{4,2} = 3x_2^2 + 4x_1x_3, \quad A_{4,1} = x_4.$$

It is also known that $A_{n,k}$ is given explicitly by

$$A_{n,k} = \sum_{\substack{j_1+2j_2+\cdots+nj_n=n \\ j_1+j_2+\cdots+j_n=k}} \frac{n!}{j_1!\cdots j_n!}\left(\frac{x_1}{1!}\right)^{j_1}\cdots\left(\frac{x_n}{n!}\right)^{j_n}.$$

5. Proof of Theorem 1.1

We are now ready to prove Theorem 1.1. Let us introduce an auxiliary differential operator. Define a linear transformation d_ρ of $C^\infty(V)$ by

$$d_\rho\varphi = \rho^{-1}\partial_\rho\varphi = \rho^{-2}\partial\varphi, \quad \varphi \in C^\infty(V). \tag{5.1}$$

Set also

$$\delta f = \frac{\sigma \circ f}{\rho} D^1_{\sigma,\rho} f = \frac{\sigma^2 \circ f}{\rho^2} f'.$$

We note that the relation

$$d_\rho(\varphi \circ f) = (d_\sigma\varphi) \circ f \cdot \delta f \tag{5.2}$$

holds for $\varphi \in C^\infty(W)$. Indeed, we have

$$\partial_\rho(\varphi \circ f) = \rho^{-1} \cdot \partial\varphi \circ f \cdot f' = [\sigma^{-1}\partial\varphi] \circ f \cdot \frac{\sigma \circ f}{\rho} f' = (\partial_\sigma\varphi) \circ f \cdot D^1_{\sigma,\rho} f. \tag{5.3}$$

Multiplying both sides by ρ^{-1} yields (5.2).

LEMMA 5.1. *For $n \geq 1$,*

$$d^n_\rho\varphi = \rho^{-n}\partial^n_\rho\varphi \quad \text{and} \quad d^{n-1}_\rho(\delta f) = \frac{\sigma \circ f}{\rho^n} D^n_{\sigma,\rho} f.$$

Here d^n_ρ denotes the nth *iterate of the transformation d_ρ.*

Proof. We use induction to show the second relation only; the first relation can be shown similarly (see also Remark 3.4).

The second relation is trivial for $n = 1$, so assume that it holds for all values up to n. Taking the d_ρ-derivative of the logarithm of both sides, we obtain

$$\frac{d^n_\rho(\delta f)}{d^{n-1}_\rho(\delta f)} = \rho^{-1}\left[(\partial_\sigma \log\sigma) \circ f \cdot D^1 f - n\partial_\rho \log\rho + \frac{\partial_\rho D^n f}{D^n f}\right] = \frac{\rho^{-1}D^{n+1}f}{D^n f},$$

where we have used (5.3) and (1.3). Hence

$$d_\rho^n(\delta f) = \frac{\sigma \circ f}{\rho^{n+1}} D_{\sigma,\rho}^{n+1} f,$$

which completes the induction. □

Proof of Theorem 1.1. Since the δf just described satisfies the relation (5.2) for the $\mathbb{C}$-derivations d_ρ and d_σ on $C^\infty(V)$ and $C^\infty(W)$, Lemma 4.2 yields the formula

$$d_\rho^n(\varphi \circ f) = \sum_{k=1}^{n} (d_\sigma^k \varphi) \circ f \cdot A_{n,k}(\delta f, d_\rho^1(\delta f), \ldots, d_\rho^{n-k}(\delta f))$$

for $\varphi \in C^\infty(W)$ and $n \geq 1$, where the $A_{n,k}$ are Bell polynomials.

By Lemma 5.1, we can rewrite this in the form

$$\rho^{-n}\partial_\rho^n(\varphi \circ f) = \sum_{k=1}^{n} (\sigma^{-k}\partial_\sigma^k \varphi) \circ f \cdot A_{n,k}\left(\frac{\sigma \circ f}{\rho} D^1 f, \ldots, \frac{\sigma \circ f}{\rho^{n-k+1}} D^{n-k+1} f\right).$$

We can now use Lemma 4.3 to establish the validity of (1.4). □

6. Another Expression for $D^n f$

Let V and W be plane domains with smooth conformal metrics ρ and σ, respectively, and let $f: V \to W$ be holomorphic. In (1.3) we defined the invariant derivative $D^n f$ recursively in terms of $D^{n-1} f$ and its ρ-derivative. It is, however, natural to find an expression of $D^n f$ in terms of the ordinary derivative $f^{(k)}$ without using the ρ-derivative of $D^{n-1} f$. Toward this end, we introduce a double sequence of auxiliary functions associated with the metric ρ.

Define the double sequence $a_{n,k} = a_{n,k}^\rho$ $(n = 1, 2, \ldots, k = 0, \pm 1, \pm 2, \ldots)$ of functions in $C^\infty(V)$ inductively by

$$\begin{aligned} a_{1,k} &= \delta_{1,k} \quad \text{and} \\ a_{n+1,k} &= a_{n,k-1} + \partial a_{n,k} - 2n(\partial \log \rho) \cdot a_{n,k}, \quad n \geq 1. \end{aligned} \tag{6.1}$$

Note that $a_{n,k} = 0$ unless $1 \leq k \leq n$. Here is a short listing of some $a_{n,k}$ values:

$$\begin{aligned}
&a_{1,1} = 1; \\
&a_{2,2} = 1, \quad a_{2,1} = -2\partial \log \rho; \\
&a_{3,3} = 1, \quad a_{3,2} = -6\partial \log \rho, \quad a_{3,1} = -2\partial^2 \log \rho + 8(\partial \log \rho)^2; \\
&a_{4,4} = 1, \quad a_{4,3} = -12\partial \log \rho, \quad a_{4,2} = -8\partial^2 \log \rho + 44(\partial \log \rho)^2; \\
&a_{4,1} = -2\partial^3 \log \rho + 28\partial \log \rho \cdot \partial^2 \log \rho - 48(\partial \log \rho)^3.
\end{aligned}$$

Let φ_n be the function given in (3.1) for $\varphi \in C^\infty(V)$. We are now able to express φ_n in terms of $a_{n,k}$ as follows.

Lemma 6.1.

$$\varphi_n = \sum_{k=1}^{n} a_{n,k}\partial^k \varphi = \sum_{k=-\infty}^{\infty} a_{n,k}\partial^k \varphi, \quad n \geq 1.$$

Proof. We show the assertion by induction. For $n = 1$ it is trivial, so suppose that the assertion is valid for all values up to n. By definition, we compute

$$\begin{aligned}\varphi_{n+1} &= \frac{\partial}{\partial z}\sum_{k=-\infty}^{\infty} a_{n,k}\partial^k\varphi - 2n(\partial\log\rho)\sum_{k=-\infty}^{\infty} a_{n,k}\partial^k\varphi \\ &= \sum_{k=-\infty}^{\infty}(\partial a_{n,k}\cdot\partial^k\varphi + a_{n,k}\partial^{k+1}\varphi) - 2n(\partial\log\rho)\sum_{k=-\infty}^{\infty} a_{n,k}\partial^k\varphi \\ &= \sum_{k=-\infty}^{\infty}(\partial a_{n,k} + a_{n,k-1} - 2n(\partial\log\rho)\cdot a_{n,k})\partial^k\varphi \\ &= \sum_{k=-\infty}^{\infty} a_{n+1,k}\partial^k\varphi.\end{aligned}$$

Thus the assertion is valid also for $n + 1$. □

We define the double sequence $B_{n,k}$ ($n \in \mathbb{N}$, $k \in \mathbb{Z}$) of polynomials in P inductively by

$$\begin{aligned} B_{1,k} &= \delta_{1,k} \quad \text{and} \\ B_{n+1,k} &= B_{n,k-1} + \mathcal{D}B_{n,k} - 2nx_1B_{n,k}, \quad n \geq 1. \end{aligned}$$

By definition, $B_{n,k} = 0$ unless $1 \leq k \leq n$. For instance:

$B_{1,1} = 1$;

$B_{2,2} = 1$, $B_{2,1} = -2x_1$;

$B_{3,3} = 1$, $B_{3,2} = -6x_1$, $B_{3,1} = -2x_2 + 8x_1^2$;

$B_{4,4} = 1$, $B_{4,3} = -12x_1$, $B_{4,2} = -8x_2 + 44x_1^2$, $B_{4,1} = -2x_3 + 28x_1x_2 - 48x_1^3$.

We summarize properties of $B_{n,k}$ and the relation with $a_{n,k}^{\rho}$ in the next lemma.

LEMMA 6.2.

(i) $B_{n,k} \in Q_{n-k}$ for $1 \leq k \leq n$; *in particular,* $B_{n,k} \in \mathbb{Z}[x_1, \dots, x_{n-k}]$.

(ii) $B_{n,n} = 1$.

(iii) $a_{n,k}^{\rho} = B_{n,k}(\partial\log\rho, \dots, \partial^{n-k}\log\rho)$.

As a consequence of Theorem 1.1, we obtain the following expression of $D^n f$.

PROPOSITION 6.3. *Let $f: V \to W$ be a holomorphic map between plane domains V and W with smooth conformal metrics ρ and σ, respectively. The invariant derivative $D^n f = D^n_{\sigma,\rho} f$ can be expressed by*

$$\begin{aligned} D^n f &= \sigma\circ f\cdot\rho^{-n}\sum_{k=1}^{n} a_{n,k}^{\rho} f^{(k)} \\ &\qquad - \sum_{k=2}^{n}\{\sigma^{1-k}a_{k,1}^{\sigma}\}\circ f\cdot A_{n,k}(D^1 f, \dots, D^{n-k+1} f), \end{aligned} \tag{6.2}$$

where $A_{n,k}$ is the Bell polynomial given in (4.1) *and $a^{\rho}_{n,k}$ and $a^{\sigma}_{n,k}$ are defined in* (6.1) *for ρ and σ, respectively.*

Proof. For brevity, we write $A^f = A(f', \dots, f^{(n)})$ and $A^{Df} = A(D^1 f, \dots, D^n f)$ for $A \in \mathbb{Z}[x_1, \dots, x_n]$. We will express both sides of (1.4) in terms of $\partial^k \varphi$ ($k = 1, 2, \dots, n$) with the aid of Lemma 6.1. We begin with the left-hand side:

$$\partial^n_\rho(\varphi \circ f) = \sum_{k=1}^{n} \rho^{-n} a^{\rho}_{n,k} \partial^k(\varphi \circ f) = \sum_{k=1}^{n} \rho^{-n} a^{\rho}_{n,k} \sum_{l=1}^{k} (\partial^l \varphi) \circ f \cdot A^f_{k,l};$$

here we have used Lemma 4.2 with $d_V = d_W = \partial$. On the other hand, the right-hand side can be written as

$$\sum_{k=1}^{n} (\partial^k_\sigma \varphi) \circ f \cdot A^{Df}_{n,k} = \sum_{k=1}^{n} \sum_{l=1}^{k} (\sigma^{-k} a^{\sigma}_{k,l} \partial^l \varphi) \circ f \cdot A^{Df}_{n,k}.$$

Equating both sides, we obtain the relation

$$\sum_{l=1}^{n} (\partial^l \varphi) \circ f \sum_{k=l}^{n} [\rho^{-n} a^{\rho}_{n,k} \cdot A^f_{k,l} - (\sigma^{-k} a^{\sigma}_{k,l}) \circ f \cdot A^{Df}_{n,k}] = 0.$$

For each l ($1 \le l \le n$), we can choose $\varphi \in C^\infty(W)$ such that $\partial^l \varphi \neq 0$ yet $\partial^m \varphi = 0$ for all m with $1 \le m < l$ at a given point in W; hence we conclude that

$$\sum_{k=l}^{n} [\rho^{-n} a^{\rho}_{n,k} \cdot A^f_{k,l} - (\sigma^{-k} a^{\sigma}_{k,l}) \circ f \cdot A^{Df}_{n,k}] = 0 \tag{6.3}$$

holds for every $l = 1, 2, \dots, n$. In particular, by letting $l = 1$ we obtain the required relation. □

As an application of Proposition 6.3, we give the first three of the invariant derivatives of f:

$$\begin{aligned} D^1 f &= \frac{\sigma \circ f}{\rho} f', \\ D^2 f &= \frac{\sigma \circ f}{\rho^2} f'' - 2\partial_\rho \log \rho \cdot D^1 f + 2(\partial_\sigma \log \sigma) \circ f \cdot (D^1 f)^2, \\ D^3 f &= \sigma \circ f [\rho^{-3} f''' - 6\partial_\rho \log \rho \cdot \rho^{-2} f'' - 2(\partial^2_\rho \log \rho - 2(\partial_\rho \log \rho)^2)\rho^{-1} f'] \\ &\qquad + 6(\partial_\sigma \log \sigma) \circ f \cdot D^1 f D^2 f + 2(\partial^2_\sigma \log \sigma - 2(\partial_\sigma \log \sigma)^2) \circ f \cdot (D^1 f)^3. \end{aligned}$$

In light of (6.3), we may obtain an expression for the invariant derivative $D^n f$ in terms of only the ordinary derivatives $f^{(k)}$, $1 \le k \le n$, as follows.

COROLLARY 6.4. *Let $f\colon V \to W$ be a holomorphic map between plane domains V and W with smooth conformal metrics ρ and σ, respectively. Then*

$$D^n f = (\sigma \circ f) \cdot \begin{vmatrix} b_1 & a^\sigma_{2,1} \circ f & a^\sigma_{3,1} \circ f & a^\sigma_{4,1} \circ f & \dots & a^\sigma_{n,1} \circ f \\ b_2 & 1 & a^\sigma_{3,2} \circ f & a^\sigma_{4,2} \circ f & \dots & a^\sigma_{n,2} \circ f \\ b_3 & 0 & 1 & a^\sigma_{4,3} \circ f & \dots & a^\sigma_{n,3} \circ f \\ b_4 & 0 & 0 & 1 & \dots & a^\sigma_{n,4} \circ f \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & 0 & 0 & 0 & \dots & 1 \end{vmatrix},$$

where

$$b_l = \sum_{k=l}^{n} \rho^{-n} a^\rho_{n,k} \cdot A_{k,l}(f', f'', \dots, f^{(k-l+1)})$$

for $l = 1, 2, \dots, n$.

Proof. The equation (6.3) can be interpreted as a system of equations for $A_k = \sigma^{-k} \circ f \cdot A^{Df}_{n,k}$, $1 \le k \le n$:

$$\sum_{k=l}^{n} (a^\sigma_{k,l} \circ f) \cdot A_k = b_l, \quad l = 1, 2, \dots, n.$$

Thus $D^n f = (\sigma \circ f) \cdot A_1$ is obtained as before. □

Conversely, by using (6.2) we can also express the ordinary nth-order derivative $f^{(n)}$ of f in terms of the invariant derivatives $D^1 f, \dots, D^n f$ as follows.

COROLLARY 6.5. *Let $f : V \to W$ be a holomorphic map between plane domains V and W with smooth conformal metrics ρ and σ, respectively. Then*

$$f^{(n)} = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 & p_1 \\ a^\rho_{2,1} & 1 & 0 & \dots & 0 & p_2 \\ a^\rho_{3,1} & a^\rho_{3,2} & 1 & \dots & 0 & p_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a^\rho_{n-1,1} & a^\rho_{n-1,2} & a^\rho_{n-1,3} & \dots & 1 & p_{n-1} \\ a^\rho_{n,1} & a^\rho_{n,2} & a^\rho_{n,3} & \dots & a^\rho_{n,n-1} & p_n \end{vmatrix},$$

where

$$p_l = \rho^l \cdot \sum_{k=1}^{l} (\sigma^{-k} a^\sigma_{k,1}) \circ f \cdot A_{l,k}(D^1 f, \dots, D^{l-k+1} f)$$

for $l = 1, 2, \dots, n$.

Proof. From (6.2) we obtain the system of equations

$$\sum_{k=1}^{l} a^\rho_{l,k} \cdot f^{(k)} = p_l, \quad l = 1, 2, \dots, n.$$

Solving this system yields the $f^{(n)}$ given in the corollary. □

7. Case of Canonical Metric

In this section we consider the special case where $R = \mathbb{C}_\delta$, $S = \mathbb{C}_\varepsilon$, and ρ and σ are their canonical metrics λ_δ and λ_ε, respectively, where $\delta, \varepsilon = +1, 0, -1$. We begin by remarking on a special nature of the metric λ_ε: since

$$\partial_{\lambda_\varepsilon} \log \lambda_\varepsilon(z) = -\varepsilon \bar{z}, \quad z \in \mathbb{C}_\varepsilon, \tag{7.1}$$

many computations become simple. For instance, we have the following.

LEMMA 7.1. *Define $\alpha_{n,k}$ $(n \in \mathbb{N}, k \in \mathbb{Z})$ by*

$$\alpha_{n,k} = (-1)^{n-k} \frac{n!\,(n-1)!}{k!\,(k-1)!\,(n-k)!} = (-1)^{n-k} \frac{n!}{k!} \binom{n-1}{k-1}$$

if $1 \le k \le n$ and by $\alpha_{n,k} = 0$ otherwise. Then

$$a_{n,k}^{\lambda_\varepsilon} = \alpha_{n,k} (\partial \log \lambda_\varepsilon)^{n-k}, \quad n \in \mathbb{N},\ k \in \mathbb{Z}.$$

Proof. We prove the lemma by induction on n. The assertion is trivial for $n = 1$. Assuming the assertion for n, we prove it for $n+1$. First, taking the ∂-derivative of both sides of (7.1) yields

$$\partial^2 \log \lambda_\varepsilon = (\partial \log \lambda_\varepsilon)^2. \tag{7.2}$$

By the defining relation (6.1) of $a_{n,k} = a_{n,k}^{\lambda_\varepsilon}$, we then compute

$$\begin{aligned} a_{n+1,k} &= \alpha_{n,k-1} (\partial \log \lambda_\varepsilon)^{n-k+1} + \alpha_{n,k} \partial (\partial \log \lambda_\varepsilon)^{n-k} \\ &\quad - 2n \partial \log \lambda_\varepsilon \cdot \alpha_{n,k} (\partial \log \lambda_\varepsilon)^{n-k} \\ &= [\alpha_{n,k-1} + (n-k)\alpha_{n,k} - 2n\alpha_{n,k}] (\partial \log \lambda_\varepsilon)^{n-k+1} \\ &= \alpha_{n+1,k} (\partial \log \lambda_\varepsilon)^{n-k+1}. \end{aligned}$$

Thus the assertion is valid for $n+1$, too. □

Concerning the invariant derivative $D^n f = D^n_{\lambda_\varepsilon, \lambda_\delta} f$, the definition (1.1) due to Peschl looks different from our definition (1.3). Schippers [S] gives a brief explanation for the coincidence based on recurrence relations. For the reader's convenience, we give another explanation as an application of Proposition 6.3.

PROPOSITION 7.2. *Let $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ be a holomorphic map. Then Peschl's invariant derivative $D^n f$ defined by* (1.1) *satisfies the recurrence relations* (1.3) *for $\rho = \lambda_\delta$ and $\sigma = \lambda_\varepsilon$.*

Proof. For clarity, in this proof only we write $D_n f$ for $D^n_{\lambda_\varepsilon, \lambda_\delta} f$ and write $D^n f$ for Peschl's derivative. Now we show that $D^n f = D_n f$. It is straightforward to check that D^n obeys the same transformation rule for isometries as D_n does (Lemma 3.6). Because $\mathrm{Isom}^+(\mathbb{C}_\varepsilon)$ acts on $\mathbb{C}_\varepsilon$ transitively, it is enough to show that $D^n f(0) = D_n f(0)$ for a holomorphic map $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ with $f(0) = 0$. Then, by

(1.1), we have $D^n f(0) = f^{(n)}(0)$. We now compute $D_n f(0)$. Apply Lemma 7.1 to see that

$$a_{n,k}^{\lambda_\varepsilon}(0) = \alpha_{n,n}\delta_{n,k} = \delta_{n,k}.$$

Substituting $z = 0$ and $f(0) = 0$ into the expression of $D_n f$ in Proposition 6.3, we find $D_n f(0) = f^{(n)}(0) = D^n f(0)$. □

Our definition of D^n gives, in turn, the following relations for Peschl's invariant derivatives.

COROLLARY 7.3. *Let $f\colon \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ be holomorphic. Then*

$$(1+\delta|z|^2)\partial(D^n f)(z) = D^{n+1}f(z) - \delta n\bar{z}D^n f(z) + \varepsilon\overline{f(z)}D^1 f(z)D^n f(z)$$

for $n \ge 1$.

REMARK 7.4. These relations appeared in [W, p. 7] for the cases $n = 1, 2$. They also follow from the identity

$$(1-\delta\bar{z}\zeta)\frac{\partial W}{\partial\zeta} - (1+\delta|z|^2)\frac{\partial W}{\partial z} = (1-\varepsilon\overline{f(z)}W)D^1 f(z),$$

where W is the left-hand side of (1.1).

Since $\alpha_{k,1}^{\lambda_\varepsilon} = (-1)^{k-1}k!$ for $k \ge 1$, Proposition 6.3 now gives the following result.

COROLLARY 7.5. *Let $f\colon \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ be holomorphic. Then*

$$\begin{aligned} D^n f &= \lambda_\varepsilon \circ f \cdot \lambda_\delta^{-n}\sum_{k=1}^{n} a_{n,k}^{\lambda_\delta} f^{(k)} \\ &\quad - \sum_{k=2}^{n}\{\lambda_\varepsilon^{1-k}\cdot a_{k,1}^{\lambda_\varepsilon}\}\circ f \cdot A_{n,k}(D^1 f, \dots, D^{n-k+1}f) \\ &= \sum_{k=1}^{n}\alpha_{n,k}\frac{(-\delta\bar{z})^{n-k}(1+\delta|z|^2)^k f^{(k)}(z)}{1+\varepsilon|f(z)|^2} \\ &\quad - \sum_{k=2}^{n} k!\,(\varepsilon\overline{f(z)})^{k-1}A_{n,k}(D^1 f, \dots, D^{n-k+1}f), \end{aligned}$$

where $A_{n,k}$ is the Bell polynomial given in (4.1), *$a_{n,k}^{\lambda_\varepsilon}$ is defined in* (6.1) *for the canonical metric λ_ε, and $\alpha_{n,k}$ is given in Lemma 7.1.*

The special case when $\varepsilon = 0$ was previously proved by Gong [G1] (see also [G2, p. 133]). Using Corollary 7.5, we can express the ordinary derivative $f^{(n)}$ in terms of the invariant derivatives $D^k f$ $(k = 1, 2, \dots, n)$ as follows. Hereafter, we set $z^0 = 1$ regardless of what the complex number z is.

COROLLARY 7.6. *Let $f\colon \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ be holomorphic. Then*

$$\frac{(1+\delta|z|^2)^n}{1+\varepsilon|f(z)|^2}\frac{f^{(n)}(z)}{n!} = \sum_{k=1}^{n}\binom{n-1}{k-1}(-\delta\bar{z})^{n-k}\cdot c_k,$$

where

$$c_k = \sum_{l=1}^{k} \frac{l!}{k!}(\varepsilon\overline{f(z)})^{l-1} A_{k,l}(D^1 f, \dots, D^{k-l+1} f).$$

Proof. Let $f_k = (1+\delta|z|^2)^k f^{(k)}(z)/(k!\,(1+\varepsilon|f(z)|^2))$. Then, putting the explicit form of $\alpha_{n,k}$ given in Lemma 7.1 into the formula in Corollary 7.5, we obtain

$$\sum_{k=1}^{n} \binom{n-1}{k-1} (\delta\bar{z})^{n-k} \cdot f_k = c_n.$$

It is not difficult to invert this relation and so express f_n in terms of c_k $(1 \le k \le n)$, as given in the assertion of the corollary. $\square$

For the case when $\delta = -1$ and $\varepsilon = 0$, Ruscheweyh [R2] derived the relation in Corollary 7.6. We end this section with an application to a rough estimate of higher-order derivatives of a holomorphic map $f\colon \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$.

THEOREM 7.7. *Let $\delta, \varepsilon \in \{0, 1, -1\}$ and write $D^n = D^n_{\lambda_\varepsilon,\lambda_\delta}$. Suppose that a holomorphic map $f\colon \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ satisfies the inequality $|D^1 f| \le M$ on $\mathbb{C}_\delta$ for a positive constant M. Then, for each $n \in \mathbb{N}$, there exists a positive constant M_n (depending only on n and M) such that $|D^n f| \le M_n$ on $\mathbb{C}_\delta$. Moreover, the inequality*

$$\frac{(1+\delta|z|^2)^n}{1+\varepsilon|f(z)|^2} \frac{|f^{(n)}(z)|}{n!} \le \sum_{k=1}^{n} \binom{n-1}{k-1} |\delta z|^{n-k} \cdot C_k \tag{7.3}$$

holds for $z \in \mathbb{C}_\delta$, where

$$C_k = \sum_{l=1}^{k} \frac{l!}{k!} |\varepsilon f(z)|^{l-1} A_{k,l}(M_1, M_2, \dots, M_{k-l+1}).$$

REMARK 7.8. It is well known that a holomorphic map $f\colon \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ must be constant whenever $\delta > \varepsilon$. The Schwarz–Pick lemma implies the inequality $|D^1 f| \le 1$ for any holomorphic map $f\colon \mathbb{C}_{-1} \to \mathbb{C}_{-1}$. On the other hand, a meromorphic function $f\colon \mathbb{C}_{-1} \to \mathbb{C}_{+1}$ is called *normal* if $D^1 f$ is bounded (cf. [LV]).

Proof of Theorem 7.7. We first show that $|D^n f| \le M_n$ holds for a constant M_n depending only on n and M. Since the relevant conditions are invariant under isometries, it suffices to show that $|D^n f(0)| = |f^{(n)}(0)| \le M_n$ for f with $f(0) = 0$. Let $d_\varepsilon(z, w)$ denote the distance induced by the metric λ_ε. Note that $d_1(z, 0) = \arctan|z|$, $d_0(z, 0) = |z|$, and $d_{-1}(z, 0) = \operatorname{arctanh}|z|$. Since

$$f^*\lambda_\varepsilon(z) = \frac{|f'(z)|}{1+\varepsilon|f(z)|^2} \le \frac{M}{1+\delta|z|^2} = M\lambda_\delta(z),$$

we obtain the estimate

$$d_\varepsilon(f(z), 0) \le M d_\delta(z, 0). \tag{7.4}$$

In particular, for a fixed positive number r in $\mathbb{C}_\delta$, there exists a number N depending only on M, δ, and ε such that $|f(z)| \le N$ for $|z| \le r$. Cauchy's theorem now yields

$$\begin{aligned}|f^{(n)}(0)| &= \left|\frac{(n-1)!}{2\pi i}\int_{|z|=r}\frac{f'(z)\,dz}{z^n}\right| \\ &\le \frac{(n-1)!}{2\pi r^n}\int_{|z|=r}\frac{M(1+\varepsilon|f(z)|^2)|dz|}{1+\delta r^2} \\ &\le \frac{M(n-1)!\,(1+\max\{\varepsilon N^2,0\})}{(1+\delta r^2)r^{n-1}}.\end{aligned}$$

The last term depends only on n and M (and δ,ε), so we have shown an inequality of the form $|D^n f|\le M_n$. Now (7.3) follows from Corollary 7.6 together with $|D^n f|\le M_n$ and Lemma 4.1. □

REMARK 7.9. When $\varepsilon\le 0$, in the proof we can choose

$$M_n = \inf_{r\in\mathbb{C}_\delta,\, r>0}\frac{M(n-1)!}{(1+\delta r^2)r^{n-1}}.$$

When $\delta=\varepsilon=-1$, by Theorem 7.7 we obtain an estimate of the form

$$\frac{(1-|z|^2)^n}{1-|f(z)|^2}|f^{(n)}(z)|\le K_n$$

for a holomorphic map $f\colon\mathbb{C}_{-1}\to\mathbb{C}_{-1}$ with some constant K_n. Note that Ruscheweyh [R1] showed the same inequality with the sharp constant $K_n = 2^{n-1}n!$.

References

[C] L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions,* Reidel, Dordrecht, 1974.

[G1] S. Gong, *Coefficient inequalities and geometric inequalities,* Chinese Sci. Bull. 32 (1987), 732–736.

[G2] ———, *The Bieberbach conjecture,* AMS/IP Stud. Adv. Math., 12, Amer. Math. Soc., Providence, RI, 1999.

[HP] E. d'Hoker and D. H. Phong, *On determinants of Laplacians on Riemann surfaces,* Comm. Math. Phys. 104 (1986), 537–545.

[KM] S. Kim and D. Minda, *The hyperbolic metric and spherically convex regions,* J. Math. Kyoto Univ. 41 (2001), 297–314.

[KoN] S. Kobayashi and K. Nomizu, *Foundations of differential geometry,* 2 vols., Wiley, New York, 1969.

[KSu] S. Kim and T. Sugawa, *Characterizations of hyperbolically convex regions,* J. Math. Anal. Appl. 309 (2005), 37–51.

[LV] O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions,* Acta Math. 97 (1957), 47–65.

[MaM] W. Ma and D. Minda, *Hyperbolically convex functions,* Ann. Polon. Math. 60 (1994), 81–100.

[M] D. Minda, unpublished notes.

[Pe] E. Peschl, *Les invariants différentiels non holomorphes et leur rôle dans la théorie des fonctions,* Rend. Sem. Mat. Messina Ser. I (1955), 100–108.

[Ri] J. Riordan, *An introduction to combinatorial analysis,* Wiley, New York, 1958.

[R1] S. Ruscheweyh, *Über einige Klassen im Einheitskreis holomorpher Funktionen,* Ber. Math.-Stat. Forsch. Graz 7 (1974), 1–12.
[R2] ———, *Two remarks on bounded analytic functions,* Serdica Math. J. 11 (1985), 200–202.
[S] E. Schippers, *Conformal invariants and higher-order Schwarz lemmas,* J. Anal. Math. 90 (2003), 217–241.
[Su] T. Sugawa, *Unified approach to conformally invariant metrics on Riemann surfaces,* Proceedings of the Second ISAAC Congress, vol. 2 (Fukuoka, 1999), Int. Soc. Anal. Appl. Comput., 8, pp. 1117–1127, Kluwer, Dordrecht, 2000.
[W] K.-J. Wirths, *Verallgemeinerungen eines Maximumprinzips,* Bonner Math. Schriften, 51, pp. 1–60, Univ. Bonn, 1971.

S.-A Kim
Department of Mathematics
Dongguk University
Kyongju, 780-714
Korea

sakim@dongguk.ac.kr

T. Sugawa
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima, 739-8526
Japan

sugawa@math.sci.hiroshima-u.ac.jp